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NEW ESTIMATES AND TESTS OF INDEPENDENCE IN SOME COPULA MODELS

SALIM BOUZEBDA* & AMOR KEZIOU**

ABSTRACT. *We introduce new estimates and tests of independence in copula models with unknown margins using ϕ -divergences and the duality technique. The asymptotic laws of the estimates and the test statistics are established both when the parameter is an interior point or not.*

Key words: Dependence function ; Multivariate rank statistics; Semiparametric inference; Boundary value parameter.

1. INTRODUCTION AND MOTIVATIONS

Parametric models for copulas with unknown margins have been intensively investigated during the last decades. *Copulas* have become popular in applied statistics, because of the fact that they constitute a flexible and robust way to model dependence between the margins of random vectors. In this framework, semiparametric inference methods, based on *pseudo-likelihood*, have been applied to copulas by a number of authors (see, e.g., Shih and Louis (1995), Wang and Ding (2000), Tsukahara (2005) and the references therein). Throughout the available literature, investigations on the asymptotic properties of parametric estimators, as well as the relevant test statistics, have privileged the case where the parameter is an interior point of the admissible domain. However, for most parametric copula models of interest, the boundaries of the admissible parameter spaces include some important parameter values, typically among which, that corresponding to the independence of margins. We find in Joe (1997) many examples of parametric copulas, for which marginal independence is verified for some specific values of the parameter θ , on the boundary $\partial\Theta$ of the admissible parameter set $\Theta \subseteq \mathbb{R}^d$, $d \geq 1$. This paper concentrates on this specific problem. We aim, namely, to investigate parametric inference procedures,

in the case where the parameter belongs to the boundary of the admissible domain. In particular, it will become clear, that the usual limit laws both for parametric copula estimators and test statistics become invalid under these limiting cases, and, in particular, under marginal independence. Motivated by this observation, we will introduce a new semiparametric inference procedure based on ϕ -divergences and the *duality* technique. We will show that the proposed estimators remain asymptotically normal, even under the marginal independence assumption for appropriate choice of the divergence. This will allow us to introduce test statistics of independence, whose study will be made, both under the null and the alternative hypotheses. Let $F(x_1, x_2) := P(X_1 \leq x_1, X_2 \leq x_2)$ be a 2-dimensional distribution function, and $F_i(x_i) := P(X_i \leq x_i)$, $i = 1, 2$ the marginal distribution of F . It is well known since the work of Sklar (1959) that there exists a distribution function C on $[0, 1]^2$ with uniform marginals such that

$$C(u_1, u_2) := P\{F_1(X_1) \leq u_1, F_2(X_2) \leq u_2\}.$$

See also Deheuvels (1979b), Deheuvels (1980, 1981), Moore and Spruill (1975) and Schweizer (1991). Formally, copulas can be defined as follows.

Definition 1.1. *An p -dimensional copula is a function $C : [0, 1]^p \rightarrow [0, 1]$ with the following properties*

- (1) C is grounded, i.e., for every $\mathbf{u} = (u_1, \dots, u_d)$, $C(\mathbf{u}) = 0$ if at least one coordinate $u_i = 0$, $i = 1, \dots, p$;
- (2) C , is p -increasing, i.e., for every $\mathbf{u} \in [0, 1]^p$ and $\mathbf{v} \in [0, 1]^p$ such that $\mathbf{u} \leq \mathbf{v}$, the C -volume $V_C[\mathbf{u}, \mathbf{v}]$ of the box $[\mathbf{u}, \mathbf{v}]$ is non negative (see Nelsen (1999));
- (3) $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$ for all $u_i \in [0, 1]^p$, $u_i = 0$, $i = 1, \dots, p$.

Many useful multivariate models for dependence between X_1 and X_2 turn out to be generated by *parametric* families of copulas of the form $\{C_\theta; \theta \in \Theta\}$, typically indexed by a vector valued parameter $\theta \in \Theta \subseteq \mathbb{R}^d$ (see, e.g., Kimeldorf and Sampson (1975a), Kimeldorf and Sampson (1975b), Nelsen (1999), and Joe (1993) among others). In the sequel, we assume that $C_\theta(\cdot, \cdot)$ admits a density $c_\theta(\cdot, \cdot)$ with respect to the Lebesgue measure λ on \mathbb{R}^2 . The nonparametric approach to copula estimation has been initiated

by Deheuvels (1979b), who introduced and investigated the *empirical copula process*. In addition, Deheuvels (1980, 1981) described the limiting behavior of this empirical process (see, also Fermanian *et al.* (2004) and the references therein). In this paper, we consider the estimation and test problems for semiparametric copula models with unknown general margins. Let (X_{1k}, X_{2k}) $k = 1, \dots, n$ be a bivariate sample with distribution function $F_{\theta_T}(\cdot, \cdot) = C_{\theta_T}(F_1(\cdot), F_2(\cdot))$ where $\theta_T \in \Theta$ is used to denote the true unknown value of the parameter. In order to estimate θ_T , some semiparametric estimation procedures, based on the maximization, on the parameter space Θ , of properly chosen *pseudo-likelihood* criterion, have been proposed by Oakes (1994), Shih and Louis (1995), Liang and Self (1996), Wang and Ding (2000) and Tsukahara (2005) among others. In each of these papers, some asymptotic normality properties are established for $\sqrt{n}(\tilde{\theta} - \theta_T)$, where $\tilde{\theta} = \tilde{\theta}_n$ denotes a properly chosen estimator of θ_T . This is achieved, provided that θ_T lies in the *interior*, denoted by $\overset{\circ}{\Theta}$, of the parameter space $\Theta \subseteq \mathbb{R}^d$. On the other hand, the case where $\theta_T \in \partial\Theta := \overline{\Theta} - \overset{\circ}{\Theta}$ is a *boundary value* of Θ , has not been studied in a systematical way until present. Moreover, it turns out that, for the above-mentioned estimators, the asymptotic normality of $\sqrt{n}(\tilde{\theta} - \theta_T)$, may fail to hold for $\theta_T \in \partial\Theta$. In the framework of full parametric models with i.i.d. data, Self and Liang (1987) summarized all the earlier work and provided the exact limiting distributions which are complicated by the number of unknown parameters, how many of them are on the boundary and the correlation between the components of the estimator proposed. Our approach is novel in this setting and it will become clear later on from our results, that the asymptotic normality of the estimate based on ϕ -divergences holds, even under the independence assumption, when, either, θ_0 is an interior, or a boundary point of Θ , independently of the dimension of the parameter space. The proposed test statistics of independence using ϕ -divergences are also studied, under the null hypothesis \mathcal{H}_0 of independence, as well as under the alternative hypothesis. The asymptotic distributions of the test statistics under the alternative hypothesis are used to derive an approximation to the power functions. An application of the forthcoming results will allow us to evaluate the sample size necessary to guarantee a pre-assigned power level, with respect to a specified alternative. To establish our results, we use similar arguments as those developed by Tsukahara (2005) in connection with the instrumental statements

on rank statistics established by Ruymgaart *et al.* (1972) and Rüschemdorf (1976) among others, combined with a new technique (based on the law of iterated logarithm given in Lemma 6.1 below) to show both existence and consistency of our estimates and test statistics. All mathematical developments are relegated to the appendix.

2. A NEW INFERENCE PROCEDURE

Recall that the ϕ -divergences between a bounded signed measure Q , and a probability P on \mathcal{D} , when $Q \ll P$ is absolutely continuous with respect to P , is defined by

$$D_\phi(Q, P) := \int_{\mathcal{D}} \phi \left(\frac{dQ}{dP} \right) dP,$$

where ϕ is a proper closed convex function from $] - \infty, \infty[$ to $[0, \infty[$ with $\phi(1) = 0$ and such that the domain $\text{dom}\phi := \{x \in \mathbb{R} : \phi(x) < \infty\}$ is an interval with end points $a_\phi < 1 < b_\phi$. The Kullback-Leibler, modified Kullback-Leibler, χ^2 , modified χ^2 , Hellinger and L^1 divergences are examples of ϕ -divergences; they are obtained respectively for $\phi(x) = x \log x - x + 1$, $\phi(x) = -\log x + x - 1$, $\phi(x) = \frac{1}{2}(x - 1)^2$, $\phi(x) = \frac{1}{2} \frac{(x-1)^2}{x}$, $\phi(x) = 2(\sqrt{x} - 1)^2$ and $\phi(x) = |x - 1|$. We refer to Liese and Vajda (1987) for a systematic theory of divergences. In the sequel, for all θ , we denote by $D_\phi(\theta, \theta_T)$ the ϕ -divergences between C_θ and C_{θ_T} , i.e.,

$$D_\phi(\theta, \theta_T) := \int_I \phi \left(\frac{dC_\theta}{dC_{\theta_T}} \right) dC_{\theta_T}(u_1, u_2) = \int_I \phi \left(\frac{c_\theta}{c_{\theta_T}} \right) dC_{\theta_T}(u_1, u_2). \quad (2.1)$$

Denote C_n the empirical copula associated to the data, i.e.,

$$C_n(u_1, u_2) := \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{F_{1n}(X_{1k}) \leq u_1\}} \mathbb{1}_{\{F_{2n}(X_{2k}) \leq u_2\}}, \quad (u_1, u_2) \in I, \quad (2.2)$$

and $F_{jn}(t) := \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{]-\infty, t]}(X_{jk})$, $j = 1, 2$. In order to estimate the divergences $D_\phi(\theta, \theta_T)$ for a given $\theta \in \Theta$ in particular for $\theta = \theta_0$, and the parameter θ_T , we will make use of the dual representation of ϕ -divergences obtained by Broniatowski and Keziou (2006) Theorem 4.4 and Keziou (2003) Theorem 2.3. By this, we readily obtain that $D_\phi(\theta_0, \theta_T)$ can be rewritten into

$$D_\phi(\theta_0, \theta_T) := \sup_{f \in \mathcal{F}} \left\{ \int_I f dC_{\theta_0} - \int_I \phi^*(f) dC_{\theta_T} \right\}, \quad (2.3)$$

where ϕ^* is used to denote the convex conjugate of ϕ , namely, the function defined by

$$\phi^* : t \in \mathbb{R} \mapsto \phi^*(t) := \sup_{x \in \mathbb{R}} \{tx - \phi(x)\},$$

and \mathcal{F} is an arbitrary class of measurable functions fulfilling the following two conditions: $\forall f \in \mathcal{F}$, $\int |f| dC_{\theta_0}$ is finite and $\phi'(dC_{\theta_0}/dC_{\theta_T}) \in \mathcal{F}$.

Furthermore, the sup in the above display is unique and is achieved at $f = \phi'(dC_{\theta_0}/dC_{\theta_T})$.

Note that the plug-in estimate

$$\int_I \phi \left(\frac{dC_{\theta}}{dC_n} \right) dC_n(u_1, u_2)$$

of $D_{\phi}(\theta, \theta_T)$ is not well defined since C_{θ} is not absolutely continuous with respect to C_n ; the use of the dual representation, as we will show, avoids this problem. By the above statement, taking the class of functions

$$\mathcal{F} = \{u \in I \mapsto \phi'(1/c_{\theta}); \theta \in \Theta\},$$

we obtain the formula

$$D_{\phi}(\theta_0, \theta_T) = \sup_{\theta \in \Theta} \left\{ \int_I \phi' \left(\frac{1}{c_{\theta}} \right) du_1 du_2 - \int_I \left[\frac{1}{c_{\theta}} \phi' \left(\frac{1}{c_{\theta}} \right) - \phi \left(\frac{1}{c_{\theta}} \right) \right] dC_{\theta_T}(u_1, u_2) \right\}, \quad (2.4)$$

whenever $\int_I |\phi'(1/c_{\theta})| du_1 du_2$ is finite for all $\theta \in \Theta$. Furthermore, the sup is unique and reached at $\theta = \theta_T$. Hence, the divergence $D_{\phi}(\theta_0, \theta_T)$ and the parameter θ_T can be estimated respectively by

$$\sup_{\theta \in \Theta} \left\{ \int_I \phi' \left(\frac{1}{c_{\theta}} \right) du_1 du_2 - \int_I \left[\frac{1}{c_{\theta}} \phi' \left(\frac{1}{c_{\theta}} \right) - \phi \left(\frac{1}{c_{\theta}} \right) \right] dC_n(u_1, u_2) \right\} \quad (2.5)$$

and

$$\arg \sup_{\theta \in \Theta} \left\{ \int_I \phi' \left(\frac{1}{c_{\theta}} \right) du_1 du_2 - \int_I \left[\frac{1}{c_{\theta}} \phi' \left(\frac{1}{c_{\theta}} \right) - \phi \left(\frac{1}{c_{\theta}} \right) \right] dC_n(u_1, u_2) \right\}, \quad (2.6)$$

in which C_{θ_T} is replaced by C_n . Note that this class of estimates contains the maximum pseudo-likelihood (MPL) estimator proposed by Oakes (1994); it is obtained for the KL_m -divergence taking $\phi(x) = -\log(x) + x - 1$. Under some regularity conditions, we can prove that these estimates are consistent and asymptotically normal in the same way as the MPL estimate when the parameter θ_T is an interior point of the parameter space Θ . The interest of divergence remains in the fact that a properly choice of the divergence may ameliorate the MPL one in terms of efficiency-robustness. The results in Bouzebda and Keziou

(2008) show that, for $\Theta = [\theta_0, \infty)$, and when the true value θ_T of the parameter is equal to θ_0 (corresponding to the independence assumption), the classical asymptotic normality property of the MPL estimate is no more satisfied. To overcome this difficulty, in what follows, we enlarge the parameter space Θ into a wider space $\Theta_e \supset \Theta$. This is tailored to let θ_0 become an interior point of Θ_e . More precisely, set

$$\Theta_e := \left\{ \theta \in \mathbb{R}^d \text{ such that } \int |\phi'(1/c_\theta(u_1, u_2))| du_1 du_2 < \infty \right\}. \quad (2.7)$$

So, applying (2.3), with the class of functions

$$\mathcal{F} := \{(u_1, u_2) \mapsto \phi'(1/c_\theta(u_1, u_2)); \theta \in \Theta_e\},$$

we obtain

$$D_\phi(\theta_0, \theta_T) = \sup_{\theta \in \Theta_e} \left\{ \int_I \phi' \left(\frac{1}{c_\theta} \right) du_1 du_2 - \int_I \left[\frac{1}{c_\theta} \phi' \left(\frac{1}{c_\theta} \right) - \phi \left(\frac{1}{c_\theta} \right) \right] dC_{\theta_T}(u_1, u_2) \right\}. \quad (2.8)$$

Furthermore, the sup in this display is unique and reached in $\theta = \theta_T$. Hence, we propose to estimate $D_\phi(\theta_0, \theta_T)$ by

$$\widehat{D}_\phi(\theta_0, \theta_T) := \sup_{\theta \in \Theta_e} \int_I m(\theta, u_1, u_2) dC_n(u_1, u_2), \quad (2.9)$$

and to estimate the parameter θ_T by

$$\widehat{\theta}_n := \arg \sup_{\theta \in \Theta_e} \left\{ \int_I m(\theta, u_1, u_2) dC_n(u_1, u_2) \right\}, \quad (2.10)$$

where

$$m(\theta, u_1, u_2) := \int_I \phi' \left(\frac{1}{c_\theta(u_1, u_2)} \right) du_1 du_2 - \left\{ \phi' \left(\frac{1}{c_\theta(u_1, u_2)} \right) \frac{1}{c_\theta(u_1, u_2)} - \phi \left(\frac{1}{c_\theta(u_1, u_2)} \right) \right\}.$$

In the sequel we denote by $\frac{\partial}{\partial \theta} m(\theta, u_1, u_2)$ the d -dimensional vector with entries $\frac{\partial}{\partial \theta_i} m(\theta, u_1, u_2)$ and $\frac{\partial^2}{\partial \theta^2} m(\theta, u_1, u_2)$ the $d \times d$ -matrix with entries $\frac{\partial^2}{\partial \theta_i \partial \theta_j} m(\theta, u_1, u_2)$.

Remark 2.1. *Divergences measures have been intensively used in estimation and test in the framework of the discret parametric models with independent identically distributed data; the estimates of the divergences and the parameter are obtained by the plug-in method; see Liese and Vajda (1987), Pardo (2006) and the references therein. For continuous parametric models the plug-in procedure does not lead to well defined estimates; Keziou (2003), Liese and Vajda (2006), Broniatowski and Keziou (2008) introduce new*

estimates and tests, using the dual representation of divergences, extending the maximum likelihood procedure.

3. THE ASYMPTOTIC BEHAVIOR OF THE ESTIMATES

In this section, we provide both weak and strong consistency of the estimates (2.9) and (2.10). We also state their asymptotic normality and evaluate their limiting variance. Statistics of the form

$$\Psi_n := \int_I \psi(u_1, u_2) dC_n(u_1, u_2),$$

belong to the general class of *multivariate rank statistics*. Their asymptotic properties have been investigated at length by a number of authors, among whom we may cite Ruymgaart *et al.* (1972), Ruymgaart (1974) and Rüschemdorf (1976). In particular, the previous authors have provided regularity conditions, imposed on $\psi(\cdot, \cdot)$, which imply the asymptotic normality of Ψ_n . The corresponding arguments have been modified by Genest *et al.* (1995), as to establish almost sure convergence of the estimators that they consider (see, e.g., Genest *et al.* (1995) Proposition A.1). In the same spirit, the limiting behavior, as n tends to the infinity, of the estimators and test statistics which we will introduce later on, will make an instrumental use of the general theory of multivariate rank statistics, and rely, in particular, on Proposition A.1 in Genest *et al.* (1995). The existence and consistency of our estimators will be established through an application of the law of the iterated logarithm for empirical copula processes, in combination with general arguments from multivariate rank statistics theory (we refer to Deheuvels (1979a), Fermanian *et al.* (2004) and references therein). We will use the following notations

$$K_1(\theta, u_1, u_2) := \phi' \left(\frac{1}{c_\theta(u_1, u_2)} \right)$$

and

$$K_2(\theta, u_1, u_2) := \left\{ \phi' \left(\frac{1}{c_\theta(u_1, u_2)} \right) \frac{1}{c_\theta(u_1, u_2)} - \phi \left(\frac{1}{c_\theta(u_1, u_2)} \right) \right\}.$$

Definition 3.1. (i) Let \mathcal{Q} be the set of continuous functions q on $[0, 1]$ which are positive on $(0, 1)$, symmetric about $1/2$, increasing on $[0, 1/2]$ and satisfy $\int_0^1 \{q(t)\}^{-2} dt < \infty$.

- (ii) A function $r : (0, 1) \rightarrow (0, \infty)$ is called *u-shaped* if it is symmetric about $1/2$ and increasing on $(0, 1/2]$.
- (iii) For $0 < \beta < 1$ and *u-shaped* function r , we define

$$r_\beta(t) = \begin{cases} r(\beta t) & \text{if } 0 < t \leq 1/2; \\ r\{1 - \beta(1 - t)\} & \text{if } 1/2 < t \leq 1/2. \end{cases}$$

If for $\beta > 0$ in a neighborhood of 0, there exists a constant M_β , such that $r_\beta \leq M_\beta r$ on $(0, 1)$, then r is called a *reproducing u-shaped function*. We denote by \mathcal{R} the set of *reproducing u-shaped functions*.

We make use of the following conditions.

- (C.1) There exists a neighborhood $N(\theta_T) \subset \Theta_e$ of θ_T such that the first and the second partial derivatives with respect to θ of $K_1(\theta, u_1, u_2)$ are dominated on $N(\theta_T)$ by some λ -integrable functions;
- (C.2) There exists a neighborhood $N(\theta_T)$ of θ_T , such that for all $\theta \in N(\theta_T)$, the function $\frac{\partial}{\partial \theta} m(\theta, u_1, u_2) : (0, 1)^2 \rightarrow \mathbb{R}$ is continuously differentiable and there exist functions $r_i \in \mathcal{R}$, $\tilde{r}_i \in \mathcal{R}$ and $q \in \mathcal{Q}$ ($i, j = 1, 2$, $i \neq j$ and $\ell, \ell' = 1, \dots, m$) with
- (i) $|\frac{\partial}{\partial \theta_\ell} m(\theta, u_1, u_2)| \leq r_1(u_1)r_2(u_2)$,
 - $|\frac{\partial^2}{\partial \theta_\ell \partial u_i} m(\theta, u_1, u_2)| \leq \tilde{r}_i(u_i)r_j(u_j)$;
 - (ii) $|m^2(\theta, u_1, u_2)| \leq \tilde{r}_i(u_i)r_j(u_j)$;
 - (iii) $|\frac{\partial^3}{\partial \theta^3} K_2(\theta, u_1, u_2)| \leq \tilde{r}_i(u_i)r_j(u_j)$;
 - (iv) $|m(\theta, u_1, u_2)| \leq \tilde{r}_i(u_i)r_j(u_j)$;
 - (v) $|\frac{\partial}{\partial \theta_\ell} m(\theta, u_1, u_2)|^2 \leq \tilde{r}_i(u_i)r_j(u_j)$,
 - $|\frac{\partial^2}{\partial \theta_\ell \partial \theta_{\ell'}} m(\theta, u_1, u_2)| \leq \tilde{r}_i(u_i)r_j(u_j)$;
 - (vi) $|\frac{\partial}{\partial u_i} m(\theta, u_1, u_2)| \leq \tilde{r}_i(u_i)r_j(u_j)$
- and $\int_I \{r_1(u_1)r_2(u_2)\}^2 dC_{\theta_T}(u_1, u_2) < \infty$, $\int_I \{q_i(u_i)\tilde{r}_i(u_i)r_j(u_j)\} dC_{\theta_T}(u_1, u_2) < \infty$;
- (C.3) The matrix $\int (\partial^2 / \partial^2 \theta) m(\theta, u_1, u_2) dC_{\theta_T}(u_1, u_2)$ is non singular;
- (C.4) The function $(u_1, u_2) \in I \mapsto \frac{\partial}{\partial \theta} m(\theta_T, u_1, u_2)$ is of bounded variation on I .

Theorem 3.1. *Assume that conditions (C.1-3) hold.*

- (1) Let $B(\theta_T, n^{-1/3}) := \{\theta \in \Theta_e, |\theta - \theta_T| \leq n^{-1/3}\}$, then as n tends to infinity, with probability one, the function $\theta \mapsto \int m(\theta, u_1, u_2) dC_n(u_1, u_2)$ attains its maximum value at some point $\widehat{\theta}_n$ in the interior of $B(\theta_T, n^{-1/3})$, which implies that the estimate $\widehat{\theta}_n$ is consistent and satisfies

$$\int \frac{\partial}{\partial \theta} m(\widehat{\theta}_n, u_1, u_2) dC_n(u_1, u_2) = 0.$$

- (2) $\sqrt{n}(\widehat{\theta}_n - \theta)$ converges in distribution to a centered multivariate normal random variable with covariance matrix

$$\Xi_\phi = \mathbf{S}^{-1} \mathbf{M} \mathbf{S}^{-1}, \quad (3.1)$$

with

$$\mathbf{S} := - \int \frac{\partial^2}{\partial \theta^2} m(\theta_T, u_1, u_2) dC_{\theta_T}(u_1, u_2), \quad (3.2)$$

and

$$\mathbf{M} := \text{Var} \left[\frac{\partial}{\partial \theta} m(\theta_T, F_1(X_1), F_2(X_2)) + W_1(\theta_T, X_1) + W_2(\theta_T, X_2) \right], \quad (3.3)$$

where

$$W_i(\theta_T, X_i) := \int_I \mathbf{1}_{\{F_i(X_i) \leq u_i\}} \frac{\partial^2}{\partial \theta \partial u_i} m(\theta_T, u_1, u_2) c_{\theta_T}(u_1, u_2) du_1 du_2, \quad i = 1, 2.$$

The proof of Theorem 3.1 is postponed to section 6.

4. NEW TESTS OF INDEPENDENCE

One of our motivation is to build a statistical test of independence, based on ϕ -divergence. In the framework of the parametric copula model, the null hypothesis, i.e., the independence case $C_\theta(u_1, u_2) = u_1 u_2$ corresponds to $\mathcal{H}_0 : \theta_T = \theta_0$. We consider the composite alternative hypothesis $\mathcal{H}_1 : \theta_T \neq \theta_0$. Since, θ_0 is a boundary value of the parameter space Θ , we can see that the convergence in distribution of the corresponding pseudo-likelihood ratio statistic to a χ^2 random variable does not hold; see Bouzebda and Keziou (2008). We give now a solution to this problem. We propose the following statistics

$$\mathbf{T}_n := \frac{2n}{\phi''(1)} \widehat{D}_\phi(\theta_0, \theta_T). \quad (4.1)$$

We will see that the proposed statistic converges in distribution, under the null hypothesis \mathcal{H}_0 , to a χ^2 random variable with d degrees of freedom, which permits to build a test of \mathcal{H}_0 against \mathcal{H}_1 asymptotically of level α . The limit law of \mathbf{T}_n is given also under the alternative hypothesis \mathcal{H}_1 . We will use the following additional conditions

(C.5) We have

$$\lim_{\theta \rightarrow \theta_0} \frac{\partial^2}{\partial \theta_\ell \partial u_i} m(\theta, u_1, u_2) = 0,$$

and there exist $M_1 > 0$ and $\delta_1 > 0$ such that, for all θ in some neighborhood of θ_0 , one has, for $i = 1, 2$,

$$\left| \frac{\partial^2}{\partial \theta_\ell \partial u_i} m(\theta, u_1, u_2) c_{\theta_T}(u_1, u_2) \right| < M_1 r(u_i)^{-1.5+\delta_1} r(u_{3-i})^{0.5+\delta_1},$$

where $r(u) := u(1-u)$ for $u \in (0, 1)$.

Remark 4.1. *When $\theta_T = \theta_0$, under the conditions (C.1) and (C.5) we can see that \mathbf{S} and \mathbf{M} can be written as*

$$\mathbf{S} = \mathbf{M} = \int \left[\frac{\partial}{\partial \theta} m(\theta_T, u_1, u_2) \right] \left[\frac{\partial}{\partial \theta} m(\theta_T, u_1, u_2) \right]^\top dC_{\theta_T}(u_1, u_2).$$

Theorem 4.1. (1) *Assume that conditions (C.1-5) hold. If $\theta_T = \theta_0$, then the statistic \mathbf{T}_n converges in distribution to a χ^2 variable with d degrees of freedom.*

(2) *Assume that conditions (C.1-4) hold. If $\theta_T \neq \theta_0$, then $\sqrt{n} \left(\widehat{D}_\phi(\theta_0, \theta_T) - D_\phi(\theta_0, \theta_T) \right)$ converges in distribution to a centered normal variable with variance*

$$\sigma_\phi^2(\theta_0, \theta_T) := \text{Var} [m(\theta_T, F_1(X_1), F_2(X_2)) + Y_1(\theta_T, X_1) + Y_2(\theta_T, X_2)], \quad (4.2)$$

where

$$Y_i(\theta_T, X_i) := \int_I \mathbf{1}_{\{F_i(X_i) \leq u_i\}} \frac{\partial}{\partial u_i} m(\theta_T, u_1, u_2) c_{\theta_T}(u_1, u_2) du_1 du_2, \quad i = 1, 2.$$

The proof of Theorem 4.1 is postponed to section 6.

Remark 4.2. *An application of Theorem 4.1, leads to reject the null hypothesis $\mathcal{H}_0 : \theta_T = \theta_0$, whenever the value of the statistic \mathbf{T}_n exceeds $q_{1-\alpha}$, namely, the $(1-\alpha)$ -quantile of the χ^2 law with d degrees of freedom. The corresponding test is then, asymptotically of level α , when $n \rightarrow \infty$. The critical region is, accordingly, given by*

$$CR := \{\mathbf{T}_n > q_{1-\alpha}\}.$$

The fact that this test is consistent follows from Theorem 4.1. Further, this theorem can be used to give an approximation to the power function $\theta_T \in \Theta \mapsto \beta(\theta_T) := P_{\theta_T} \{CR\}$ in a similar way to Keziou and Leoni-Aubin (2008). We so obtain that

$$\beta(\theta_T) \approx 1 - \Phi \left(\frac{\sqrt{n}}{\sigma_\phi(\theta_0, \theta_T)} \left(\frac{q_{1-\alpha}}{2n} - D_\phi(\theta_0, \theta_T) \right) \right), \quad (4.3)$$

where Φ denotes, as usual, the cumulative distribution function of a $N(0, 1)$ standard normal random variable. A useful consequence of (4.3) is the possibility of computing an approximate value of the sample size ensuring a specified power $\beta(\theta_T)$, with respect to some pre-assigned alternative $\theta_T \neq \theta_0$. Let n_0 be the positive root of the equation

$$\beta = 1 - \Phi \left(\frac{\sqrt{n}}{\sigma_\phi(\theta_0, \theta_T)} \left(\frac{q_{1-\alpha}}{2n} - D_\phi(\theta_0, \theta_T) \right) \right),$$

which can be rewritten into

$$n_0 = \frac{(a + b) - \sqrt{a(a + 2b)}}{2D_\phi(\theta_0, \theta_T)^2},$$

where $a := \sigma_\phi(\theta_0, \theta_T) (\Phi^{-1}(1 - \beta))^2$ and $b := q_{1-\alpha} D_\phi(\theta_0, \theta_T)$. The sought-after approximate value of the sample size is then given

$$n^* := \lfloor n_0 \rfloor + 1,$$

where $\lfloor u \rfloor$ denote the integer part of u .

Remark 4.3. From Theorem 3.1 and 4.1, it is clear that an asymptotic $1 - \alpha$ confidence interval or region, \mathcal{R}_α about θ can be easily constructed using the Intersection method as described in Feng and McCulloch (1992).

Remark 4.4. The above regularity conditions are satisfied by a large number of parametric families of bivariate copulas; see for instance Tsukahara (2005).

Remark 4.5. The parameters (3.2) and (3.3) may be consistently estimated respectively by the sample mean of

$$\frac{\partial^2}{\partial \theta^2} m(\hat{\theta}_n, F_{1n}(X_{1,k}), F_{2n}(X_{2,k})), \quad k = 1, \dots, n, \quad (4.4)$$

and the sample variance of

$$\frac{\partial}{\partial \theta} m \left(\widehat{\theta}_n, F_{1n}(X_{1,k}), F_{2n}(X_{2,k}) \right) + W_1(\widehat{\theta}_n, X_{1,k}) + W_2(\widehat{\theta}_n, X_{2,k}), \quad k = 1, \dots, n, \quad (4.5)$$

as was done in Genest et al. (1995). The asymptotic variance (4.2) can be consistently estimated in the same way.

5. CONCLUDING REMARKS

We have introduced a new estimation and test procedure in parametric copula models with unknown margins. The method is based on divergences between copulas and the duality technique. It generalizes the maximum pseudo-likelihood one, and applies both when the parameter is an interior or a boundary value, in particular for testing the null hypothesis of independence. It will be interesting to investigate the problem of the choice of the divergence which leads to an “optimal” (in some sense) estimate or test in terms of efficiency and robustness.

6. APPENDIX

First we give a technical Lemma which we will use to prove our results.

Lemma 6.1. *Let F_{θ_T} have a continuous margins and let C_{θ_T} have continuous partial derivatives. Assume that j is a continuous function, with bounded variation. Then*

$$\int_I j(u_1, u_2) d(C_n(u_1, u_2) - C(u_1, u_2)) = O(n^{-1/2}(\log \log n)^{1/2}) \quad (a.s.). \quad (6.1)$$

Proof of Lemma 6.1. Recall that the *modified empirical copula* C_n , is slightly different from the *empirical copula* $\mathbb{C}_n(u_1, u_2)$, introduced by Deheuvels (1979a), and defined by

$$\mathbb{C}_n(u_1, u_2) = F_n \left(F_{1n}^{-1}(u_1), F_{2n}^{-1}(u_2) \right) \quad \text{for } (u_1, u_2) \in (0, 1)^2, \quad (6.2)$$

where $F_{1n}^{-1}(\cdot)$ and $F_{2n}^{-1}(\cdot)$ denote the empirical quantile functions, associated with $F_{1n}(x_1) = F_n(x_1, \infty)$ and $F_{2n}(x_2) = F_n(\infty, x_2)$, respectively, and defined by

$$F_{jn}^{-1}(t) := \inf\{x \in \mathbb{R} \mid F_{jn}(x) \geq t\}, \quad j = 1, 2. \quad (6.3)$$

Here, $F_n(\cdot, \cdot)$ denotes the joint empirical distribution function, associated with the sample $\{(X_{1k}, X_{2k}); k = 1, \dots, n\}$, defined by

$$F_n(x_1, x_2) = \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{X_{1k} \leq x_1\}} \mathbb{1}_{\{X_{2k} \leq x_2\}}, \quad -\infty < x_1, x_2 < \infty. \quad (6.4)$$

We know that \mathbb{C}_n and C_n coincide on the grid $\{(i/n, j/n), 1 \leq i \leq j \leq n\}$. The subtle difference lies in the fact that \mathbb{C}_n is left-continuous with right-hand limits, whereas C_n on the other hand is right continuous with left-hand limits. The difference between \mathbb{C}_n and C_n , however, is small

$$\sup_{\mathbf{u} \in I} |\mathbb{C}_n(\mathbf{u}) - C_n(\mathbf{u})| \leq \max_{1 \leq i, j \leq n} \left| \mathbb{C}_n \left(\frac{i}{n}, \frac{j}{n} \right) - C_n \left(\frac{i-1}{n}, \frac{j-1}{n} \right) \right| \leq \frac{2}{n}. \quad (6.5)$$

Using integration by parts, as in Fermanian *et al.* (2004), we see that

$$\begin{aligned} \sqrt{n} \int_I j(u_1, u_2) d(C_n - C)(u_1, u_2) &= \int_I \sqrt{n}(C_n - C)(u_1, u_2) dj(u_1, u_2) \\ &\quad - \int_I \sqrt{n}(C_n - C)(u_1, 1) dj(u_1, u_2) - \int_I \sqrt{n}(C_n - C)(1, u_2) dj(u_1, u_2) \\ &\quad - \int_{[0,1]} \sqrt{n}(C_n(u_1, 1) - u_1) dj(u_1, 1) - \int_{[0,1]} \sqrt{n}(C_n(1, u_2) - u_2) dj(1, u_2). \end{aligned}$$

Hence,

$$\left| \sqrt{n} \int_I j(u_1, u_2) d(C_n - C)(u_1, u_2) \right| \leq 5\sqrt{n} \sup_{\mathbf{u} \in I} |(C_n - C)(\mathbf{u})| \int_I d|j(\mathbf{u})|.$$

From this and (6.5), applying Theorem 3.1 in Deheuvels (1979a), we obtain the following result

$$\int_I j(u_1, u_2) d(C_n - C)(u_1, u_2) = O(n^{-1/2}(\log \log n)^{1/2}) \quad (a.s.).$$

■

Proof of Theorem 3.1 (1) Under the Assumption (C.1) and (C.2.iii), a simple calculation gives

$$\int_I \frac{\partial}{\partial \theta} m(\theta, u_1, u_2) dC_{\theta_T}(u_1, u_2) = 0, \quad (6.6)$$

and

$$\int_I \frac{\partial^2}{\partial \theta^2} m(\theta, u_1, u_2) dC_{\theta_T}(u_1, u_2) = - \int \phi'' \left(\frac{1}{c_{\theta_T}} \right) \frac{\dot{c}_{\theta_T} \dot{c}_{\theta_T}^\top}{c_{\theta_T}^3} d\lambda = -\mathbf{S} \quad (6.7)$$

We see that the matrix \mathbf{S} is symmetric and positive using the fact that the second derivative $\phi''(\cdot)$ is nonnegative by the assumption that the function $\phi(\cdot)$ is convex. Hence, \mathbf{S} is positive definite by (C.3). Introduce the statistic $\Phi_n(\theta_T)$ defined by

$$\Phi_n(\theta_T) := \int_I \frac{\partial}{\partial \theta} m(\theta, u_1, u_2) dC_n(u_1, u_2), \quad (6.8)$$

and combine (6.6) with Theorem 2.1 in Ruymgaart *et al.* (1972) to show that, as $n \rightarrow \infty$

$$\sqrt{n}\Phi_n(\theta_T) \xrightarrow{d} N(0, \mathbf{M}), \quad (6.9)$$

where \mathbf{M} is defined in (3.3). Denote

$$\Upsilon_n(\theta_T) := \int_I \frac{\partial^2}{\partial \theta^2} m(\theta, u_1, u_2) dC_n(u_1, u_2), \quad (6.10)$$

we make use of (6.7) and (C.2.v) in combinaison with Proposition A.1 of Genest *et al.* (1995), one finds

$$\Upsilon_n(\theta_T) \rightarrow -\mathbf{S}, \quad (a.s.). \quad (6.11)$$

We recall that \mathbf{S} is in (3.2). Now, for any $\theta = \theta_T + vn^{-1/3}$ with $|v| \leq 1$, consider a Taylor expansion of $\int m(\theta, u_1, u_2) dC_n(u_1, u_2)$ in θ around θ_T , and use (6.11), and (C.2.iii) to obtain

$$\begin{aligned} & n \int_I m(\theta, u_1, u_2) dC_n(u_1, u_2) - n \int_I m(\theta_T, u_1, u_2) dC_n(u_1, u_2) = \\ & n^{2/3} v^\top \Phi_n(\theta_T) + 2^{-1} n^{1/3} v \mathbf{S} v^\top + O(1) \quad (a.s.) \end{aligned} \quad (6.12)$$

uniformly in v with $|v| \leq 1$. On the other hand, since $\int \frac{\partial}{\partial \theta_\ell} m(\theta_T, u_1, u_2)^2 dC_{\theta_T}(u_1, u_2) < \infty$, and $\frac{\partial}{\partial \theta_\ell} m(\theta, \cdot, \cdot)$ is of bounded variation by assumption (C.4) ($\ell = 1, \dots, d$), using Lemma 6.1 we can show that

$$\int_I \frac{\partial}{\partial \theta_\ell} m(\theta_T, u_1, u_2) dC_n(u_1, u_2) = O(n^{-1/2} (\log \log n)^{1/2}) \quad (a.s.). \quad (6.13)$$

Therefore, using (6.12) and (6.13), we obtain for any $\theta = \theta_T + vn^{-1/3}$ with $|v| = 1$:

$$\begin{aligned} & n \int_I m(\theta, u_1, u_2) dC_n(u_1, u_2) - n \int_I m(\theta_T, u_1, u_2) dC_n(u_1, u_2) \\ & \leq O(n^{1/6} (\log \log n)^{1/2}) - 2^{-1} v \mathbf{S} v^\top + O(1) \quad (a.s.), \end{aligned} \quad (6.14)$$

where ϑ is the smallest eigenvalue of the matrix \mathbf{S} . Observe that ϑ is positive since \mathbf{S} is symmetric, positive and non singular by assumption (C.3) Using (6.14) and the fact that the function $\theta \mapsto \int_I m(\theta_T, u_1, u_2) dC_n(u_1, u_2)$ is continuous, we conclude that as $n \rightarrow \infty$, with probability one, $\theta \mapsto \int_I m(\theta_T, u_1, u_2) dC_n(u_1, u_2)$ reaches its maximum value at some point $\hat{\theta}_n$ fulfills $\int_I \frac{\partial}{\partial \theta} m(\hat{\theta}_n, u_1, u_2) dC_n(u_1, u_2) = 0$ and $|\hat{\theta}_n - \theta_T| = O(n^{-1/3})$.

(2) Using the first part of Theorem 3.1, by a Taylor expansion of

$$\int_I \frac{\partial}{\partial \theta} m(\hat{\theta}_n, u_1, u_2) dC_n(u_1, u_2)$$

in $\hat{\theta}_n$ around θ_T , we obtain

$$\begin{aligned} 0 &= \int_I \frac{\partial}{\partial \theta} m(\hat{\theta}_n, u_1, u_2) dC_n(u_1, u_2) \\ &= \int_I \frac{\partial}{\partial \theta} m(\theta_T, u_1, u_2) dC_n(u_1, u_2) + (\hat{\theta}_n - \theta_T)^\top \int_I \frac{\partial^2}{\partial \theta^2} m(\theta_T, u_1, u_2) dC_n(u_1, u_2) + o(n^{-1/2}). \end{aligned}$$

Hence,

$$\sqrt{n}(\hat{\theta}_n - \theta_T) = -[\mathbf{\Upsilon}_n(\theta_T)]^{-1} \sqrt{n} \mathbf{\Phi}_n(\theta_T) + o_P(1) \quad (6.15)$$

Using (6.9) and (6.11), by Slutsky theorem, we conclude then

$$\sqrt{n}(\hat{\theta}_n - \theta_T) \rightarrow N(0, \mathbf{\Xi}_\phi), \quad (6.16)$$

where we recall that $\mathbf{\Xi}_\phi$ is defined in (3.1). ■

Proof of Theorem 4.1

(1) Assume that $\theta_T = \theta_0$. Hence, from (6.15), using (6.7), we obtain

$$\sqrt{n}(\hat{\theta}_n - \theta_T) = -\mathbf{S}^{-1} \sqrt{n} \mathbf{\Phi}_n(\theta_T) + o_P(1). \quad (6.17)$$

On the other hand, expanding in Taylor series

$$\frac{2n}{\phi''(1)} \widehat{D}_\phi(\theta_0, \hat{\theta}_n) = \frac{2n}{\phi''(1)} \int_I m(\hat{\theta}_n, u_1, u_2) dC_n(u_1, u_2)$$

in $\hat{\theta}_n$ around θ_T , in combination with the fact that $\int_I m(\theta_T, u_1, u_2) dC_n(u_1, u_2) = 0$, we get

$$\frac{2n}{\phi''(1)} \widehat{D}_\phi(\theta_0, \hat{\theta}_n) = \frac{2n}{\phi''(1)} \mathbf{\Phi}_n(\theta_T) (\hat{\theta}_n - \theta_T) - \frac{n}{\phi''(1)} (\hat{\theta}_n - \theta_T)^\top \mathbf{\Upsilon}_n(\theta_T) (\hat{\theta}_n - \theta_T) + o_P(1). \quad (6.18)$$

Using (6.7), (6.17) and the fact that $\mathbf{S} = \phi''(1)I_{\theta_T}$ (I_{θ_T} denotes the Fisher information matrix) when $\theta_T = \theta_0$ to obtain

$$\frac{2n}{\phi''(1)}\widehat{D}_\phi(\theta_0, \widehat{\theta}_n) = \frac{1}{\phi''(1)}\sqrt{n}\Phi_n(\theta_T)I_{\theta_T}^{-1}\sqrt{n}\Phi_n(\theta_T). \quad (6.19)$$

Finally, use the convergence in (6.9) and the fact that $\mathbf{M} = \phi''(1)I_{\theta_T}$ when $\theta_T = \theta_0$, to conclude that $[\frac{2n}{\phi''(1)}]\widehat{D}_\phi(\theta_0, \theta_T)$ converge in distribution to a χ^2 variable with d degrees of freedom when $\theta_T = \theta_0$. ■

(2) Assume that $\theta_T \neq \theta_0$, using Taylor expansion again of

$$\widehat{D}_\phi(\theta_T, \theta_0) = \int_I m(\widehat{\theta}_n, u_1, u_2) dC_n(u_1, u_2)$$

in $\widehat{\theta}_n$ around θ_T , combined with the fact that $\int_I \frac{\partial}{\partial \theta} m(\theta_T, u_1, u_2) dC_{\theta_T}(u_1, u_2) = 0$, we obtain from part (2) of Theorem 3.1

$$\int_I m(\widehat{\theta}_n, u_1, u_2) dC_n(u_1, u_2) = \int_I m(\theta_T, u_1, u_2) dC_n(u_1, u_2) + o_P(n^{-1/2}).$$

Hence,

$$\begin{aligned} \sqrt{n} \left(\widehat{D}_\phi(\theta_0, \theta_T) - D_\phi(\theta_0, \theta_T) \right) = \\ \sqrt{n} \left(\int_I m(\theta_T, u_1, u_2) dC_n(u_1, u_2) - \int_I m(\theta_T, u_1, u_2) dC_{\theta_T}(u_1, u_2) \right) + o_P(1), \end{aligned}$$

which under assumption (C.2.ii) and (C.2.vi) by Theorem 2.1 in Ruymgaart *et al.* (1972) once more, converges to a centred normal variable with variance given in (4.2). ■

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