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POWER SUMS OF HECKE EIGENVALUES AND APPLICATION

J. WU

ABSTRACT. We sharpen some estimates of Rankin on power sums of Hecke eigenvalues, by using Kim & Shahidi's recent results on higher order symmetric powers. As an application, we improve Kohnen, Lau & Shparlinski's lower bound for the number of Hecke eigenvalues of same signs.

1. INTRODUCTION

Let $k \geq 2$ be an even integer and $N \geq 1$ be squarefree. Denote by $H_k^*(N)$ the set of all normalized Hecke primitive eigencuspforms of weight k for the congruence modular group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Here the normalization is taken to have $\lambda_f(1) = 1$ in the Fourier series of $f \in H_k^*(N)$ at the cusp ∞ ,

$$(1.1) \quad f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \quad (\Im m z > 0).$$

Inherited from the Hecke operators, the normalized Fourier coefficient $\lambda_f(n)$ satisfies the following relation

$$(1.2) \quad \lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right)$$

for all integers $m \geq 1$ and $n \geq 1$. In particular, $\lambda_f(n)$ is multiplicative.

Following Deligne [3], for any prime number p there are two complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$(1.3) \quad \begin{cases} \alpha_f(p) = \varepsilon_f(p)p^{-1/2}, \beta_f(p) = 0 & \text{if } p \mid N \\ |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1 & \text{if } p \nmid N \end{cases}$$

and

$$(1.4) \quad \lambda_f(p^\nu) = \frac{\alpha_f(p)^{\nu+1} - \beta_f(p)^{\nu+1}}{\alpha_f(p) - \beta_f(p)}$$

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for all integers $\nu \geq 1$, where $\varepsilon_f(p) = \pm 1$. Hence $\lambda_f(n)$ is real and verifies Deligne's inequality

$$(1.5) \quad |\lambda_f(n)| \leq d(n)$$

for all integers $n \geq 1$, where $d(n)$ is the divisor function. In particular for each prime number $p \nmid N$ there is $\theta_f(p) \in [0, \pi]$ such that

$$(1.6) \quad \lambda_f(p) = 2 \cos \theta_f(p).$$

See e.g. [7] for basic analytic facts about modular forms.

Positive real moments of Hecke eigenvalues were firstly studied by Rankin ([14], [15]). For $f \in H_k^*(N)$ and $r \geq 0$, consider the sum of the $2r$ th power of $|\lambda_f(n)|$:

$$(1.7) \quad S_f^*(x; r) := \sum_{n \leq x} |\lambda_f(n)|^{2r}.$$

The method of Rankin [15] illustrates how to obtain optimally the lower and upper bounds for $S_f^*(x; r)$ if we only know that the associated Dirichlet series

$$(1.8) \quad F_r(s) := \sum_{n \geq 1} |\lambda_f(n)|^{2r} n^{-s} \quad (\Re s > 1)$$

is invertible for $\Re s \geq 1$ (i.e. holomorphic and nonzero for $\Re s \geq 1$) when $r = 1, 2$. (The invertibility of these two cases are known by Moreno & Shahidi [13].) Rankin's result ([15], Theorem 1) reads that

$$(1.9) \quad x(\log x)^{\delta_r^\mp} \ll S_f^*(x; r) \ll x(\log x)^{\delta_r^\pm} \quad (r \in \mathcal{R}^\mp)$$

for $x \geq x_0(f, r)$, where

$$\mathcal{R}^- := [0, 1] \cup [2, \infty), \quad \mathcal{R}^+ := [1, 2],$$

and

$$\delta_r^- := 2^{r-1} - 1, \quad \delta_r^+ := \frac{2^{r-1}}{5}(2^r + 3^{2-r}) - 1.$$

The implied constants in (1.9) depend on f and r .

On the other hand, if the Sato-Tate conjecture holds for newform f , then

$$(1.10) \quad S_f^*(x; r) \sim C_r(f) x(\log x)^{\theta_r} \quad (x \rightarrow \infty),$$

where $C_r(f)$ is a positive constant depending on f, r and

$$\theta_r := \frac{4^r \Gamma(r + \frac{1}{2})}{\sqrt{\pi} \Gamma(r + 2)} - 1.$$

Very recently, Tenenbaum [20] improved Rankin's exponent $\delta_{1/2}^+ = 0.0651 \dots$ to $\rho_{1/2}^+ = 0.1185 \dots$ (see (1.13) below for the definition of ρ_r^+), as an application of his general result on the mean values of multiplicative functions and the fact that $F_3(s)$ and $F_4(s)$ are invertible for $\Re s \geq 1$, proven in the excellent work of Kim & Shahidi [9]. Although the result ([20], Corollary) is stated only for Ramanujan's τ -function, it is apparent that Tenenbaum's method applies to establish the upper bound for $S_f^*(x; r)$ in (1.11) below. It should be pointed out that Tenenbaum's approach is different from that of Rankin and does not give a lower bound for $S_f^*(x; r)$.

The first aim of this paper is to improve the lower and upper bounds in (1.9), by generalizing Rankin's method to incorporate the aforementioned results of Kim & Shahidi on $F_3(s)$ and $F_4(s)$.

Theorem 1. *For any $f \in H_k^*(N)$, we have*

$$(1.11) \quad x(\log x)^{\rho_r^\mp} \ll S_f^*(x; r) \ll x(\log x)^{\rho_r^\pm} \quad (r \in \mathcal{R}^\mp)$$

for $x \geq x_0(f, r)$, where

$$(1.12) \quad \mathcal{R}^- := [0, 1] \cup [2, 3] \cup [4, \infty), \quad \mathcal{R}^+ := [1, 2] \cup [3, 4],$$

and

$$(1.13) \quad \begin{cases} \rho_r^- := \frac{3^{r-1} - 1}{2}, \\ \rho_r^+ := \frac{102 + 7\sqrt{21}}{210} \left(\frac{6 - \sqrt{21}}{5} \right)^r + \frac{102 - 7\sqrt{21}}{210} \left(\frac{6 + \sqrt{21}}{5} \right)^r + \frac{4^r}{35} - 1. \end{cases}$$

The implied constants in (1.11) depend on f and r .

The upper bound part in (1.11) are essentially due to Tenenbaum [20], since his method with a minuscule modification allows to obtain this result. The lower bound part is new. The following table illustrates progress against Rankin's (1.9) and the difference from the conjectured values (1.10).

r	0	0.5	1	1.5	2	2.5	3	3.5	4
δ_r^-	-0.5	-0.2929	0	0.4142	1	1.8284	3	4.6569	7
ρ_r^-	-0.3333	-0.2113	0	0.3660	1	2.0981	4	7.2945	13
θ_r	0	-0.1512	0	0.3581	1	2.1043	4	7.2781	13
ρ_r^+	0	-0.1185	0	0.3502	1	2.1112	4	7.2576	13
δ_r^+	0	-0.0652	0	0.2899	1	2.5266	5.6667	12.0177	24.7778

In order to detect sign changes or cancellations among $\lambda_f(n)$, it is natural to study summatory function

$$(1.14) \quad S_f(x) := \sum_{n \leq x} \lambda_f(n)$$

and compare it with (1.11). There is a long history on the investigation of the upper estimate for $S_f(x)$. In 1927, Hecke [6] showed

$$S_f(x) \ll_f x^{1/2}$$

for all $f \in H_k^*(N)$ and $x \geq 1$. Subsequent improvements came with the use of the identity:

$$\frac{1}{\Gamma(r+1)} \sum_{n \leq x} (x-n)^r a_f(n) = \frac{1}{(2\pi)^3} \sum_{n \geq 1} \left(\frac{x}{n} \right)^{(k+3)/2} a_f(n) J_{k+3}(4\pi\sqrt{nx}),$$

where $a_f(n) := \lambda_f(n)n^{(k-1)/2}$ and $J_k(t)$ is the first kind Bessel functions. Such an identity was first given by Wilton [22] in which only the case of Ramanujan's τ -function was stated, and later generalized by Walfisz [21] to other forms. Let ϑ be the constant satisfying

$$|\lambda_f(n)| \ll n^\vartheta \quad (n \geq 1).$$

Walfisz proved that

$$(1.15) \quad S_f(x) \ll_f x^{(1+\vartheta)/3} \quad (x \geq 1).$$

Inserting the values of ϑ in the historical record into (1.15) yields

$$S_f(x) \ll_{f,\varepsilon} \begin{cases} x^{11/24+\varepsilon} & \text{Kloosterman [10]} \\ x^{4/9+\varepsilon} & \text{Davenport [1], Salié [17]} \\ x^{5/12+\varepsilon} & \text{Weil [23]} \\ x^{1/3+\varepsilon} & \text{Deligne [3]} \end{cases}$$

for any $\varepsilon > 0$. Hafner & Ivić ([5], Theorem 1) removed the factor x^ε of Deligne's result. On the other hand, by combining Walfisz' method with his idea in the study of (1.7), Rankin [16] showed that

$$(1.16) \quad S_f(x) \ll_{f,\varepsilon} x^{1/3} (\log x)^{\delta_{1/2}^+ + \varepsilon}$$

for any $\varepsilon > 0$ and $x \geq 2$.

Here we propose a better bound, by combining Walfisz' method [21] and Tenenbaum's approach [20]. It is worthy to point out that Tenenbaum's method is not only to improve $\delta_{1/2}^+$ to $\rho_{1/2}^+$ but also remove the ε in (1.16).

Theorem 2. *For $f \in H_k^*(N)$, we have*

$$(1.17) \quad S_f(x) \ll x^{1/3} (\log x)^{\rho_{1/2}^+}$$

for $x \geq 2$, where the implied constant depends on f .

In the opposite direction, Hafner & Ivić ([5], Theorem 2) proved that there is a positive constant D such that

$$S_f(x) = \Omega_\pm \left(x^{1/4} \exp \left\{ \frac{D(\log_2 x)^{1/4}}{(\log_3 x)^{3/4}} \right\} \right),$$

where \log_r denotes the r -fold iterated logarithm.

As an application of Theorems 1 and 2, we consider the quantities

$$(1.18) \quad \mathcal{N}_f^\pm(x) := \sum_{\substack{n \leq x \\ \lambda_f(n) \geq 0}} 1.$$

Very recently Kohnen, Lau & Shparlinski ([11], Theorem 1) proved

$$(1.19) \quad \mathcal{N}_f^\pm(x) \gg_f \frac{x}{(\log x)^{17}}$$

for $x \geq x_0(f)$.[†]

Here we propose a better bound.

[†]It is worthy to indicate that they gave explicit values for the implied constant in \ll and $x_0(f)$.

Corollary 1. *For any $f \in H_k^*(N)$, we have*

$$\mathcal{N}_f^\pm(x) \gg \frac{x}{(\log x)^{1-1/\sqrt{3}}}$$

for $x \geq x_0(f)$, where the implied constant depends on f . If we assume Sato-Tate's conjecture, the exponent $1 - 1/\sqrt{3} \approx 0.422$ can be improved to $2 - 16/(3\pi) \approx 0.302$.

In a joint paper with Lau [12], we shall remove the logarithmic factor by a completely different method.

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2. METHOD OF RANKIN

Let $k \geq 2$ be an even integer, $N \geq 1$ be squarefree, $f \in H_k^*(N)$ and $r > 0$. Following Rankin's idea [15], we shall find two optimal multiplicative functions $\lambda_{f,r}^\pm(n)$ such that

$$(2.1) \quad \lambda_{f,r}^\mp(p^\nu) \leq |\lambda_f(p^\nu)|^{2r} \leq \lambda_{f,r}^\pm(p^\nu) \quad (r \in \mathcal{R}^\mp)$$

for all primes p and integers $\nu \geq 1$, and furthermore, their associated Dirichlet series $\Lambda_{f,r}^\pm(s)$ (see (2.8) below) in the half-plane $\Re s \geq 1$ is controlled by $F_j(s)$ for $j = 1, \dots, 4$. Then we can apply Tauberian theorems to obtain the asymptotic behaviour of the summatory functions of $\lambda_{f,r}^\pm(n)$.

2.1. Construction of $\lambda_{f,r}^\pm(n)$. For $\mathbf{a} := (a_1, \dots, a_4) \in \mathbb{R}^4$ and $r > 0$, consider the function

$$(2.2) \quad h_r(t; \mathbf{a}) := t^r - a_1 t - a_2 t^2 - a_3 t^3 - a_4 t^4 \quad (0 \leq t \leq 1)$$

and let

$$(2.3) \quad \kappa_- := \frac{1}{4}, \quad \eta_- := \frac{3}{4}, \quad \kappa_+ := \frac{6-\sqrt{21}}{20}, \quad \eta_+ := \frac{6+\sqrt{21}}{20}.$$

In Subsection 2.3, we shall explain the reason behind this choice.

Lemma 2.1. *If the function $h_r(t; \mathbf{a})$ defined by (2.2) satisfies*

$$h_r'(\kappa_-; \mathbf{a}) = h_r'(\eta_-; \mathbf{a}) = h_r(\kappa_-; \mathbf{a}) = h_r(\eta_-; \mathbf{a}) = 0,$$

then

$$(2.4) \quad a_j = a_j^- := \frac{P_j^-(\kappa_-, \eta_-) - P_j^-(\eta_-, \kappa_-)}{(\kappa_- - \eta_-)^3}$$

for $1 \leq j \leq 4$, where

$$\begin{aligned} P_1^-(\kappa, \eta) &:= \{(4-r)\kappa + (r-2)\eta\} \kappa^{r-1} \eta^2, \\ P_2^-(\kappa, \eta) &:= \{(2r-8)\kappa^2 + (1-r)\kappa\eta + (1-r)\eta^2\} \kappa^{r-2} \eta, \\ P_3^-(\kappa, \eta) &:= \{(4-r)\kappa^2 + (4-r)\kappa\eta + 2(r-1)\eta^2\} \kappa^{r-2}, \\ P_4^-(\kappa, \eta) &:= \{(r-3)\kappa + (1-r)\eta\} \kappa^{r-2}. \end{aligned}$$

Proof. This can be done by routine calculation. \square

Lemma 2.2. *If the function $h_r(t; \mathbf{a})$ defined by (2.2) is such that*

$$\begin{cases} h'_r(\kappa_+; \mathbf{a}) = h'_r(\eta_+; \mathbf{a}) = 0, \\ h_r(\kappa_+; \mathbf{a}) = h_r(\eta_+; \mathbf{a}) = h_r(1; \mathbf{a}), \end{cases}$$

then

$$(2.5) \quad a_j = a_j^+ := \frac{P_j^+(\kappa_+, \eta_+) - P_j^+(\eta_+, \kappa_+)}{(\kappa_+ - 1)^2(\eta_+ - 1)^2(\kappa_+ - \eta_+)^3}$$

for $1 \leq j \leq 4$, where

$$\begin{aligned} P_1^+(\kappa, \eta) &:= r\kappa^{r-1}\eta(\kappa - 1)(\eta - \kappa)(\kappa\eta + 2\kappa + \eta)(\eta - 1)^2 \\ &\quad + 2(\kappa^r - 1)\kappa\eta(\eta - 1)^2(2\kappa\eta + 4\kappa - \eta^2 - 2\eta - 3), \\ P_2^+(\kappa, \eta) &:= r\kappa^{r-1}(\kappa - 1)(\kappa - \eta)(\eta - 1)^2(2\kappa\eta + \kappa + \eta^2 + 2\eta) \\ &\quad + (\eta^r - 1)(\kappa - 1)^2(8\kappa\eta^2 + 4\eta^2 - \eta\kappa^2 - 2\kappa\eta - 3\eta - \kappa^3 - 2\kappa^2 - 3\kappa), \\ P_3^+(\kappa, \eta) &:= r\kappa^{r-1}(\kappa - 1)(\kappa + 2\eta + 1)(\eta - \kappa)(\eta - 1)^2 \\ &\quad + 2(\kappa^r - 1)(2\kappa^2 + 2\kappa\eta - \eta^2 - 2\eta - 1)(\eta - 1)^2, \\ P_4^+(\kappa, \eta) &:= r\kappa^{r-1}(\kappa - 1)(\kappa - \eta)(\eta - 1)^2 + (\eta^r - 1)(\kappa - 1)^2(3\eta - \kappa - 2). \end{aligned}$$

Proof. This is done by routine calculation as well. \square

Lemma 2.3. *Let $\mathbf{a}^\pm := (a_1^\pm, \dots, a_4^\pm)$, where each a_i^\pm is given by the value in Lemmas 2.1-2.2, respectively. Then for $0 \leq t \leq 1$ we have*

$$h_r(t; \mathbf{a}^-) \geq 0 \quad \text{and} \quad h_r(t; \mathbf{a}^+) \leq h_r(1; \mathbf{a}^+) \quad \text{for } r \in \mathcal{R}^\mp.$$

Proof. We have

$$h_r^{(4)}(t; \mathbf{a}^-) = r(r-1)(r-2)(r-3)t^{r-4} - 24a_4^-,$$

so $h_r^{(4)}(t; \mathbf{a}^-)$ has at most one zero for $t > 0$ and $h_r^{(i)}(t; \mathbf{a}^-)$ has at most $5 - i$ zeros for $t > 0$ ($i = 3, 2, 1, 0$). Since $h_r(\kappa_-; \mathbf{a}^-) = h_r(\eta_-; \mathbf{a}^-) = h_r(0; \mathbf{a}^-)$, it follows that $h'_r(\xi_-; \mathbf{a}^-) = h'_r(\xi'_-; \mathbf{a}^-) = 0$ for some $\xi_- \in (0, \kappa_-)$ and $\xi'_- \in (\kappa_-, \eta_-)$. Therefore ξ_- , κ_- , ξ'_- and η_- are the only zeros of $h'_r(t; \mathbf{a}^-)$ in $(0, 1)$.

Now

$$h_r''(\kappa_-; \mathbf{a}^-) = 8 \cdot 4^{-r}(2r^2 - 2r + 3 + 2r3^{r-2} - 11 \cdot 3^{r-2})$$

and

$$h_r''(\eta_-; \mathbf{a}^-) = 8 \cdot 4^{-r}(2r^2 - 6r - 3 - 2r3^r + 43 \cdot 3^{r-2}).$$

From these, it is easy to verify that

$$h_r''(\kappa_-; \mathbf{a}^-), h_r''(\eta_-; \mathbf{a}^-) \begin{cases} \geq 0 & \text{if } r \in \mathring{\mathcal{R}}^\mp, \\ = 0 & \text{if } r = 1, 2, 3, 4, \end{cases}$$

where $\mathring{\mathcal{R}}^\mp$ denotes the interior of \mathcal{R}^\mp . Hence $h_r(t; \mathbf{a}^-)$ takes its minimum (maximum, respectively) values in $[0, 1]$ at $0, \kappa_-, \eta_-$ when $r \in \mathring{\mathcal{R}}^-$ ($r \in \mathring{\mathcal{R}}^+$, respectively). Moreover, $h_r(t; \mathbf{a}^-)$ has local maxima (minima, respectively) at ξ_-, ξ'_- when $r \in \mathring{\mathcal{R}}^-$ ($r \in \mathring{\mathcal{R}}^+$, respectively). This proves the assertion about $h_r(t; \mathbf{a}^-)$.

Similarly we can prove the corresponding result on $h_r(t; \mathbf{a}^+)$. \square

Now we define the multiplicative function $\lambda_{f,r}^\pm(n)$ by

$$(2.6) \quad \lambda_{f,r}^\mp(p^\nu) := \begin{cases} \sum_{0 \leq j \leq 4} 2^{2(r-j)} a_j^\mp \lambda_f(p)^{2j} & \text{if } \nu = 1 \text{ and } r > 0, \\ 0 & \text{if } \nu \geq 2 \text{ and } r \in \mathcal{R}^\mp, \\ |\lambda_f(p^\nu)|^{2r} & \text{if } \nu \geq 2 \text{ and } r \in \mathcal{R}^\pm, \end{cases}$$

where

$$(2.7) \quad a_0^- := 0 \quad \text{and} \quad a_0^+ := 1 - a_1^+ - a_2^+ - a_3^+ - a_4^+.$$

In view of (1.6), we can apply Lemma 2.3 with $t = |\cos \theta_f(p)|$ to deduce that the inequality (2.1) hold for all primes p and integers $\nu \geq 1$. Thanking to the multiplicativity, these inequalities also hold for all integers $n \geq 1$.

2.2. Dirichlet series associated to $\lambda_{f,r}^\pm(n)$. For $f \in \mathbf{H}_k^*(N)$, $r > 0$ and $\Re s > 1$, we define

$$(2.8) \quad \Lambda_{f,r}^\pm(s) := \sum_{n \geq 1} \lambda_{f,r}^\pm(n) n^{-s}.$$

Next we shall study their analytic properties in the half-plane $\Re s \geq 1$ by using the higher order symmetric power L -functions $L(s, \text{sym}^m f)$ associated to $f \in \mathbf{H}_k^*(N)$, due to Gelbart & Jacquet [4] for $m = 2$, Kim & Shahidi ([8], [9]) for $m = 3, 4, 5, 6, 7, 8$. Here the symmetric m th power associated to f is defined as

$$L(s, \text{sym}^m f) := \prod_p \prod_{0 \leq j \leq m} (1 - \alpha_f(p)^{m-j} \beta_f(p)^j p^{-s})^{-1}$$

for $\Re s > 1$, where $\alpha_f(p)$ and $\beta_f(p)$ are given by (1.3) and (1.4). According to the literature mentioned above, it is known that the function $L(s, \text{sym}^m f)$ for $m = 2, 3, \dots, 8$ is invertible for $\Re s \geq 1$.

We start to study $F_1(s)$, $F_2(s)$, $F_3(s)$ and $F_4(s)$.

Lemma 2.4. *Let $k \geq 2$ be an even integer, $N \geq 1$ be squarefree and $f \in \mathbf{H}_k^*(N)$. For $j = 1, 2, 3, 4$ and $\Re s > 1$, we have*

$$(2.9) \quad F_j(s) = \zeta(s)^{m_j} G_j(s) H_j(s),$$

where

$$(2.10) \quad m_1 := 1, \quad m_2 := 2, \quad m_3 := 5, \quad m_4 := 14,$$

and

$$\begin{aligned} G_1(s) &:= L(s, \text{sym}^2 f), \\ G_2(s) &:= L(s, \text{sym}^2 f)^3 L(s, \text{sym}^4 f), \\ G_3(s) &:= L(s, \text{sym}^2 f)^9 L(s, \text{sym}^4 f)^5 L(s, \text{sym}^6 f), \\ G_4(s) &:= L(s, \text{sym}^2 f)^{34} L(s, \text{sym}^4 f)^{20} L(s, \text{sym}^6 f)^7 L(s, \text{sym}^8 f) \end{aligned}$$

are invertible for $\Re s \geq 1$. Here the function $H_j(s)$ admits a Dirichlet series convergent absolutely in $\Re s > \frac{1}{2}$ and $H_j(s) \neq 0$ for $\Re s = 1$.

Proof. Write x for the trace of a local factor of $L(s, f)$ (i.e. $\alpha_f(p) + \beta_f(p)$), and denote by $T_n(x)$ the polynomial for the trace of its symmetric n th power. Then

$$\begin{aligned} T_2 &= x^2 - 1, \\ T_4 &= x^4 - 3x^2 + 1, \\ T_6 &= x^6 - 5x^4 + 6x^2 - 1, \\ T_8 &= x^8 - 7x^6 + 15x^4 - 10x^2 + 1, \end{aligned}$$

from which we deduce

$$\begin{aligned} x^2 &= 1 + T_2, \\ x^4 &= 2 + 3T_2 + T_4, \\ x^6 &= 5 + 9T_2 + 5T_4 + T_6, \\ x^8 &= 14 + 34T_2 + 20T_4 + 7T_6 + T_8. \end{aligned}$$

This implies (2.9). By using results on $L(s, \text{sym}^m f)$ mentioned above, $G_j(s)$ is invertible for $\Re s \geq 1$. This completes the proof. \square

Lemma 2.5. *Let $k \geq 2$ be an even integer, $N \geq 1$ be squarefree and $f \in \mathbf{H}_k^*(N)$. For $r > 0$ and $\Re s > 1$, we have*

$$(2.11) \quad \Lambda_{f,r}^\pm(s) = \zeta(s)^{\rho_r^\pm + 1} H_{f,r}^\pm(s),$$

where

$$(2.12) \quad \rho_r^\pm := 2^{2r-8}(2^8 a_0^\pm + 2^6 a_1^\pm + 2^4 \cdot 2a_2^\pm + 2^2 \cdot 5a_3^\pm + 14a_4^\pm) - 1$$

and $H_{f,r}^\pm(s)$ is invertible for $\Re s \geq 1$.

Proof. By definition (2.6), for $\Re s > 1$ we can write

$$\begin{aligned} \Lambda_{f,r}^-(s) &= \prod_p \left(1 + \sum_{0 \leq j \leq 4} 2^{2(r-j)} a_j^- \lambda_f(p)^{2j} p^{-s} \right) \\ &= \prod_{0 \leq j \leq 4} F_j(s)^{2^{2(r-j)} a_j^-} H_r^-(s) \end{aligned}$$

for $r \in \mathcal{R}^-$, and

$$\begin{aligned} \Lambda_{f,r}^-(s) &= \prod_p \left(1 + \sum_{0 \leq j \leq 4} 2^{2(r-j)} a_j^- \lambda_f(p)^{2j} p^{-s} + \sum_{\nu \geq 2} |\lambda_f(p^\nu)|^{2r} p^{-\nu s} \right) \\ &= \prod_{0 \leq j \leq 4} F_j(s)^{2^{2(r-j)} a_j^-} H_r^-(s) \end{aligned}$$

for $r \in \mathcal{R}^+$, where $F_0(s) = \zeta(s)$ is the Riemann zeta-function and $H_r^-(s)$ is a Dirichlet series absolutely convergent for $\Re s > \frac{1}{2}$ such that $H_r^-(s) \neq 0$ for $\Re s = 1$. Now the desired result with the sign ‘-’ follows from Lemma 2.4. The other part can be treated in the same way. \square

2.3. Optimisation of $\lambda_{f,r}^\pm(p)$ and choice of κ_\pm, η_\pm . If we regard κ_\pm, η_\pm as parameters, the ρ_r^\pm given by (2.12) are functions of these parameters. We choose (κ_\pm, η_\pm) in $(0, 1)^2$ optimally, which can be done by using formal calculation via Maple. Their values are given by (2.3).

3. PROOF OF THEOREM 1

In view of Lemma 2.5 and the classical fact on $\zeta(s)$, we can write

$$(3.1) \quad \Lambda_{f,r}^{\pm}(s) = \frac{H_{f,r}^{\pm}(1)}{(s-1)^{\rho_r^{\pm}+1}} + g_{f,r}^{\pm}(s)$$

in some neighbourhood of $s = 1$ with $\Re s > 1$, where $H_{f,r}^{\pm}(1) \neq 1$ and $g_{f,r}^{\pm}(s)$ is holomorphic at $s = 1$. Since $\lambda_{f,r}^{\pm}(n) \geq 0$, we can apply Delange's tauberian theorem [2] to write

$$(3.2) \quad \sum_{n \leq x} \lambda_{f,r}^{\pm}(n) \sim H_{f,r}^{\pm}(1)x(\log x)^{\rho_r^{\pm}} \quad (x \rightarrow \infty).$$

Now Theorem 1 follows from (2.1) and (3.2).

4. PROOF OF THEOREM 2

By (3.1), it follows that

$$\prod_p \left(1 + \sum_{\nu \geq 1} \frac{\lambda_{f,r}^{\pm}(p^{\nu})}{p^{\nu\sigma}} \right) = \frac{H_{f,r}^{\pm}(1)}{(\sigma-1)^{\rho_r^{\pm}+1}} + g_{f,r}^{\pm}(\sigma)$$

for $\sigma > 1$. From this, (2.6), (2.7) and Deligne's inequality, we deduce that

$$\sum_p \frac{\lambda_{f,r}^{\pm}(p)}{p^{\sigma}} = (\rho_r^{\pm} + 1) \log(\sigma - 1)^{-1} + C_{f,r}^{\pm} + o(1) \quad (\sigma \rightarrow 1+),$$

where $C_{f,r}^{\pm}$ is some constant.

On the other hand, the prime number theorem implies, by a partial integration, that

$$\sum_p p^{-\sigma} = \log(\sigma - 1)^{-1} + C + o(1) \quad (\sigma \rightarrow 1+),$$

where C is an absolute constant. Thus the preceding relation can be written as

$$(4.1) \quad \sum_p \frac{\lambda_{f,r}^{\pm}(p) - (\rho_r^{\pm} + 1)}{p^{\sigma}} = C_{f,r}^{\pm} + (\rho_r^{\pm} + 1)C + o(1) \quad (\sigma \rightarrow 1+).$$

According to Exercise II.7.8 of [19], the formula (4.1) implies

$$\sum_p \frac{\lambda_{f,r}^{\pm}(p) - (\rho_r^{\pm} + 1)}{p} = C_{f,r}^{\pm} + (\rho_r^{\pm} + 1)C.$$

Hence

$$\sum_{p \leq x} \frac{\lambda_{f,r}^{\pm}(p)}{p} = (\rho_r^{\pm} + 1) \log_2 x + C_{f,r}^{\pm} + (\rho_r^{\pm} + 1)C + o(1) \quad (x \rightarrow \infty).$$

Now we apply a well known result of Shiu [18] and (2.1) to write

$$\begin{aligned}
 \sum_{x \leq n \leq x+z} |\lambda_f(n)|^{2r} &\ll \frac{z}{\log x} \exp \left(\sum_{p \leq x} \frac{|\lambda_f(p)|^{2r}}{p} \right) \\
 (4.2) \qquad \qquad \qquad &\ll \frac{z}{\log x} \exp \left(\sum_{p \leq x} \frac{\lambda_{f,r}^+(p)}{p} \right) \\
 &\ll z(\log x)^{\rho_r^+}
 \end{aligned}$$

for $r \in \mathcal{R}^-$, any $\varepsilon > 0$, $x \geq x_0(\varepsilon)$ and $x^{1/4} \leq z \leq x$. Using this with $r = \frac{1}{2}$ in (9) of [16], the first term on the right-hand side of (10) of [16] is replaced by $x^{1/2} z^{-1/2} (\log x)^{\rho_{1/2}^+}$. Applying (4.2) with $r = \frac{1}{2}$ again to the second term on the right-hand side of (10) of [16], it follows that

$$S_f(x) \ll x^{1/2} z^{-1/2} (\log x)^{\rho_{1/2}^+} + z(\log x)^{\rho_{1/2}^+}.$$

Taking $z = x^{1/3}$, we obtain the required result when the level is $N = 1$. The general case can be treated similarly as indicated in [16]. \square

5. PROOF OF COROLLARY 1

By comparing (1.17) and the lower bound part in (1.11) with $r = \frac{1}{2}$, it is easy to deduce that

$$\sum_{\substack{n \leq x \\ \lambda_f(n) \geq 0}} |\lambda_f(n)| \gg_f x(\log x)^{\rho_{1/2}^-}$$

for $x \geq x_0(f)$. Since $\rho_{1/2}^- = -(1 - 1/\sqrt{3})/2$ and $\rho_1^+ = 0$, a simple application of the Cauchy-Schwarz inequality yields the following result.

The second assertion can be obtained by noticing that $\theta_{1/2} = 8/(3\pi) - 1$. \square

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