Power sums of Hecke's eigenvalues and application

Jie Wu

POWER SUMS OF HECKE EIGENVALUES AND APPLICATION

J. WU

Abstract. We sharpen some estimates of Rankin on power sums of Hecke eigenvalues, by using Kim & Shahidi's recent results on higher order symmetric powers. As an application, we improve Kohnen, Lau & Shparlinski's lower bound for the number of Hecke eigenvalues of same signs.

1. Introduction

Let $k \geqslant 2$ be an even ineteger and $N \geqslant 1$ be squarefree. Denote by $\mathrm{H}_k^*(N)$ the set of all normalized Hecke primitive eigencuspforms of weight k for the congruence modular group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \,(\mathrm{mod}\, N) \right\}.$$

Here the normalization is taken to have $\lambda_f(1) = 1$ in the Fourier series of $f \in \mathrm{H}_k^*(N)$ at the cusp ∞,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\Im m z > 0). \tag{1.1}$$

Inherited from the Hecke operators, the normalized Fourier coefficient $\lambda_f(n)$ satisfies the following relation

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right) \tag{1.2}$$

for all integers $m \geqslant 1$ and $n \geqslant 1$. In particular, $\lambda_f(n)$ is multiplicative.

Following Deligne [3], for any prime number p there are two complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\begin{cases} \alpha_f(p) = \varepsilon_f(p) p^{-1/2},\ \beta_f(p) = 0 & \text{if } p \mid N \\ |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1 & \text{if } p \nmid N \end{cases} \tag{1.3}$$

and

$$\lambda_f(p^\nu) = \frac{\alpha_f(p)^{\nu+1} - \beta_f(p)^{\nu+1}}{\alpha_f(p) - \beta_f(p)} \tag{1.4}$$

Date: June 27, 2008.

2000 *Mathematics Subject Classification.* 11F30, 11F66.

Key words and phrases. Fourier coefficients of automorphic forms, Dirichlet series.

for all integers $\nu \geqslant 1$, where $\varepsilon_f(p) = \pm 1$. Hence $\lambda_f(n)$ is real and verifies Deligne's inequality

$$|\lambda_f(n)| \leqslant d(n) \tag{1.5}$$

for all integers $n \geqslant 1$, where $d(n)$ is the divisor function. In particular for each prime number $p \nmid N$ there is $\theta_f(p) \in [0, \pi]$ such that

$$\lambda_f(p) = 2\cos\theta_f(p). \tag{1.6}$$

See e.g. [7] for basic analytic facts about modular forms.

Positive real moments of Hecke eigenvalues were firstly studied by Rankin ([14], [15]). For $f \in \mathrm{H}_k^*(N)$ and $r \geqslant 0$, consider the sum of the $2r$th power of $|\lambda_f(n)|$:

$$S_f^*(x; r) := \sum_{n \leqslant x} |\lambda_f(n)|^{2r}. \tag{1.7}$$

The method of Rankin [15] illustrates how to obtain optimally the lower and upper bounds for $S_f^*(x; r)$ if we only know that the associated Dirichlet series

$$F_r(s) := \sum_{n \geqslant 1} |\lambda_f(n)|^{2r} n^{-s} \quad (\Re e\, s > 1) \tag{1.8}$$

is invertible for $\Re e\, s \geqslant 1$ (i.e. holomorphic and nonzero for $\Re e\, s \geqslant 1$) when $r = 1, 2$. (The invertibility of these two cases are known by Moreno & Shahidi [13].) Rankin's result ([15], Theorem 1) reads that

$$x(\log x)^{\delta_r^{\mp}} \ll S_f^*(x; r) \ll x(\log x)^{\delta_r^{\pm}} \quad (r \in \mathcal{R}^{\mp}) \tag{1.9}$$

for $x \geqslant x_0(f, r)$, where

$$\mathcal{R}^- := [0, 1] \cup [2, \infty), \qquad \mathcal{R}^+ := [1, 2],$$

and

$$\delta_r^- := 2^{r-1} - 1, \qquad \delta_r^+ := \frac{2^{r-1}}{5}(2^r + 3^{2-r}) - 1.$$

The implied constants in (1.9) depend on f and r.

On the other hand, if the Sato-Tate conjecture holds for newform f, then

$$S_f^*(x; r) \sim C_r(f) x(\log x)^{\theta_r} \quad (x \to \infty), \tag{1.10}$$

where $C_r(f)$ is a positive constant depending on f, r and

$$\theta_r := \frac{4^r \Gamma(r + \frac{1}{2})}{\sqrt{\pi}\Gamma(r + 2)} - 1.$$

Very recently, Tenenbaum [20] improved Rankin's exponent $\delta_{1/2}^+ = 0.0651\cdots$ to $\rho_{1/2}^+ = 0.1185\cdots$ (see (1.13) below for the definition of ρ_r^+), as an application of his general result on the mean values of multiplicative functions and the fact that $F_3(s)$ and $F_4(s)$ are invertible for $\Re e\, s \geqslant 1$, proven in the excellent work of Kim & Shahidi [9]. Although the result ([20], Corollary) is stated only for Ramanujan's τ-function, it is apparent that Tenenbaum's method applies to establish the upper bound for $S_f^*(x; r)$ in (1.11) below. It should be pointed out that Tenenbaum's approach is different from that of Rankin and does not give a lower bound for $S_f^*(x; r)$.

The first aim of this paper is to improve the lower and upper bounds in (1.9), by generalizing Rankin's method to incorporate the aforementioned results of Kim & Shahidi on $F_3(s)$ and $F_4(s)$.

Theorem 1. *For any $f \in \mathrm{H}_k^*(N)$, we have*

$$x(\log x)^{\rho_r^{\mp}} \ll S_f^*(x;r) \ll x(\log x)^{\rho_r^{\pm}} \quad (r \in \mathscr{R}^{\mp}) \tag{1.11}$$

for $x \geqslant x_0(f,r)$, where

$$\mathscr{R}^- := [0,1] \cup [2,3] \cup [4,\infty), \qquad \mathscr{R}^+ := [1,2] \cup [3,4], \tag{1.12}$$

and

$$\begin{cases} \rho_r^- := \dfrac{3^{r-1}-1}{2}, \\ \rho_r^+ := \dfrac{102+7\sqrt{21}}{210}\left(\dfrac{6-\sqrt{21}}{5}\right)^r + \dfrac{102-7\sqrt{21}}{210}\left(\dfrac{6+\sqrt{21}}{5}\right)^r + \dfrac{4^r}{35} - 1. \end{cases} \tag{1.13}$$

The implied constants in (1.11) depend on f and r.

The upper bound part in (1.11) are essentially due to Tenenbaum [20], since his method with a minuscule modification allows to obtain this result. The lower bound part is new. The following table illustrates progress against Rankin's (1.9) and the difference from the conjectured values (1.10).

r	0	0.5	1	1.5	2	2.5	3	3.5	4
δ_r^-	-0.5	-0.2929	0	0.4142	1	1.8284	3	4.6569	7
ρ_r^-	-0.3333	-0.2113	0	0.3660	1	2.0981	4	7.2945	13
θ_r	0	-0.1512	0	0.3581	1	2.1043	4	7.2781	13
ρ_r^+	0	-0.1185	0	0.3502	1	2.1112	4	7.2576	13
δ_r^+	0	-0.0652	0	0.2899	1	2.5266	5.6667	12.0177	24.7778

In order to detect sign changes or cancellations among $\lambda_f(n)$, it is natural to study summatory function

$$S_f(x) := \sum_{n \leqslant x} \lambda_f(n) \tag{1.14}$$

and compare it with (1.11). There is a long history on the investigation of the upper estimate for $S_f(x)$. In 1927, Hecke [6] showed

$$S_f(x) \ll_f x^{1/2}$$

for all $f \in \mathrm{H}_k^*(N)$ and $x \geqslant 1$. Subsequent improvements came with the use of the identity:

$$\frac{1}{\Gamma(r+1)} \sum_{n \leqslant x} (x-n)^r a_f(n) = \frac{1}{(2\pi)^3} \sum_{n \geqslant 1} \left(\frac{x}{n}\right)^{(k+3)/2} a_f(n) J_{k+3}\big(4\pi\sqrt{nx}\big),$$

where $a_f(n) := \lambda_f(n)n^{(k-1)/2}$ and $J_k(t)$ is the first kind Bessel functions. Such an identity was first given by Wilton [22] in which only the case of Ramanujan's τ-function was stated, and later generalized by Walfisz [21] to other forms. Let ϑ be the constant satisfying

$$|\lambda_f(n)| \ll n^{\vartheta} \quad (n \geqslant 1).$$

Walfisz proved that

$$S_f(x) \ll_f x^{(1+\vartheta)/3} \quad (x \geqslant 1). \tag{1.15}$$

Inserting the values of ϑ in the historical record into (1.15) yields

$$S_f(x) \ll_{f,\varepsilon} \begin{cases} x^{11/24+\varepsilon} & \text{Kloosterman [10]} \\ x^{4/9+\varepsilon} & \text{Davenport [1], Salié [17]} \\ x^{5/12+\varepsilon} & \text{Weil [23]} \\ x^{1/3+\varepsilon} & \text{Deligne [3]} \end{cases}$$

for any $\varepsilon > 0$. Hafner & Ivić ([5], Theorem 1) removed the factor x^{ε} of Deligne's result. On the other hand, by combining Walfisz' method with his idea in the study of (1.7), Rankin [16] showed that

$$S_f(x) \ll_{f,\varepsilon} x^{1/3}(\log x)^{\delta_{1/2}^{+}+\varepsilon} \tag{1.16}$$

for any $\varepsilon > 0$ and $x \geqslant 2$.

Here we propose a better bound, by combining Walfisz' method [21] and Tenenbaum's approach [20]. It is worthy to point out that Tenenbaum's method is not only to improve $\delta_{1/2}^{+}$ to $\rho_{1/2}^{+}$ but also remove the ε in (1.16).

Theorem 2. *For $f \in \mathrm{H}_k^*(N)$, we have*

$$S_f(x) \ll x^{1/3}(\log x)^{\rho_{1/2}^{+}} \tag{1.17}$$

for $x \geqslant 2$, where the implied constant depends on f.

In the opposite direction, Hafner & Ivić ([5], Theorem 2) proved that there is a positive constant D such that

$$S_f(x) = \Omega_{\pm}\bigg(x^{1/4}\exp\bigg\{\frac{D(\log_2 x)^{1/4}}{(\log_3 x)^{3/4}}\bigg\}\bigg),$$

where $\log_r$ denotes the r-fold iterated logarithm.

As an application of Theorems 1 and 2, we consider the quantities

$$\mathscr{N}_f^{\pm}(x) := \sum_{\substack{n\leqslant x \\ \lambda_f(n)\gtrless 0}} 1. \tag{1.18}$$

Very recently Kohnen, Lau & Shparlinski ([11], Theorem 1) proved

$$\mathscr{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{17}} \tag{1.19}$$

for $x \geqslant x_0(f)$.[†]

Here we propose a better bound.

[†]It is worthy to indicate that they gave explicit values for the implied constant in $\ll$ and $x_0(f)$.

Corollary 1. *For any $f \in \mathrm{H}_k^*(N)$, we have*

$$\mathscr{N}_f^{\pm}(x) \gg \frac{x}{(\log x)^{1-1/\sqrt{3}}}$$

for $x \geqslant x_0(f)$, where the implied constant depends on f. If we assume Sato-Tate's conjecture, the exponent $1-1/\sqrt{3} \approx 0.422$ can be improved to $2-16/(3\pi) \approx 0.302$.

In a joint paper with Lau [12], we shall remove the logarithmic factor by a completely different method.

Acknowledgment. The author would like to thank Winfried Kohnen for the preprint [11] and Yuk Kam Lau for his many suggestions that improved the writting of this paper.

2. Method of Rankin

Let $k \geqslant 2$ be an even integer, $N \geqslant 1$ be squarefree, $f \in \mathrm{H}_k^*(N)$ and $r > 0$. Following Rankin's idea [15], we shall find two optimal multiplicative functions $\lambda_{f,r}^{\pm}(n)$ such that

$$\lambda_{f,r}^{\mp}(p^\nu) \leqslant |\lambda_f(p^\nu)|^{2r} \leqslant \lambda_{f,r}^{\pm}(p^\nu) \quad (r \in \mathscr{R}^{\mp}) \tag{2.1}$$

for all primes p and integers $\nu \geqslant 1$, and furthermore, their associated Dirichlet series $\Lambda_{f,r}^{\pm}(s)$ (see (2.8) below) in the half-plane $\Re e\, s \geqslant 1$ is controlled by $F_j(s)$ for $j = 1, \dots, 4$. Then we can apply Tauberian theorems to obtain the asymptotic behaviour of the summatory functions of $\lambda_{f,r}^{\pm}(n)$.

2.1. Construction of $\lambda_{f,r}^{+}(n)$.

For $\boldsymbol{a} := (a_1, \dots, a_4) \in \mathbb{R}^4$ and $r > 0$, consider the function

$$h_r(t; \boldsymbol{a}) := t^r - a_1 t - a_2 t^2 - a_3 t^3 - a_4 t^4 \qquad (0 \leqslant t \leqslant 1) \tag{2.2}$$

and let

$$\kappa_- := \tfrac{1}{4}, \qquad \eta_- := \tfrac{3}{4}, \qquad \kappa_+ := \tfrac{6-\sqrt{21}}{20}, \qquad \eta_+ := \tfrac{6+\sqrt{21}}{20}. \tag{2.3}$$

In Subsection 2.3, we shall explain the reason behind this choice.

Lemma 2.1. *If the function $h_r(t; \boldsymbol{a})$ defined by (2.2) satisfies*

$$h_r'(\kappa_-; \boldsymbol{a}) = h_r'(\eta_-; \boldsymbol{a}) = h_r(\kappa_-; \boldsymbol{a}) = h_r(\eta_-; \boldsymbol{a}) = 0,$$

then

$$a_j = a_j^- := \frac{P_j^-(\kappa_-, \eta_-) - P_j^-(\eta_-, \kappa_-)}{(\kappa_- - \eta_-)^3} \tag{2.4}$$

for $1 \leqslant j \leqslant 4$, where

$$\begin{aligned}
P_1^-(\kappa, \eta) &:= \{(4-r)\kappa + (r-2)\eta\}\kappa^{r-1}\eta^2, \\
P_2^-(\kappa, \eta) &:= \{(2r-8)\kappa^2 + (1-r)\kappa\eta + (1-r)\eta^2\}\kappa^{r-2}\eta, \\
P_3^-(\kappa, \eta) &:= \{(4-r)\kappa^2 + (4-r)\kappa\eta + 2(r-1)\eta^2\}\kappa^{r-2}, \\
P_4^-(\kappa, \eta) &:= \{(r-3)\kappa + (1-r)\eta\}\kappa^{r-2}.
\end{aligned}$$

Proof. This can be done by routine calculation. □

Lemma 2.2. *If the function $h_r(t;\boldsymbol{a})$ defined by (2.2) is such that*

$$\begin{cases} h_r'(\kappa_+;\boldsymbol{a}) = h_r'(\eta_+;\boldsymbol{a}) = 0, \\ h_r(\kappa_+;\boldsymbol{a}) = h_r(\eta_+;\boldsymbol{a}) = h_r(1;\boldsymbol{a}), \end{cases}$$

then

$$a_j = a_j^+ := \frac{P_j^+(\kappa_+,\eta_+) - P_j^+(\eta_+,\kappa_+)}{(\kappa_+-1)^2(\eta_+-1)^2(\kappa_+-\eta_+)^3} \tag{2.5}$$

for $1 \leqslant j \leqslant 4$, where

$$\begin{aligned}
P_1^+(\kappa,\eta) &:= r\kappa^{r-1}\eta(\kappa-1)(\eta-\kappa)(\kappa\eta+2\kappa+\eta)(\eta-1)^2 \\
&\quad + 2(\kappa^r-1)\kappa\eta(\eta-1)^2(2\kappa\eta+4\kappa-\eta^2-2\eta-3), \\
P_2^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa-\eta)(\eta-1)^2(2\kappa\eta+\kappa+\eta^2+2\eta) \\
&\quad + (\eta^r-1)(\kappa-1)^2(8\kappa\eta^2+4\eta^2-\eta\kappa^2-2\kappa\eta-3\eta-\kappa^3-2\kappa^2-3\kappa), \\
P_3^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa+2\eta+1)(\eta-\kappa)(\eta-1)^2 \\
&\quad + 2(\kappa^r-1)(2\kappa^2+2\kappa\eta-\eta^2-2\eta-1)(\eta-1)^2, \\
P_4^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa-\eta)(\eta-1)^2 + (\eta^r-1)(\kappa-1)^2(3\eta-\kappa-2).
\end{aligned}$$

Proof. This is done by routine calculation as well. □

Lemma 2.3. *Let $\boldsymbol{a}^\pm := (a_1^\pm,\dots,a_4^\pm)$, where each $a_i^\pm$ is given by the value in Lemmas 2.1-2.2, respectively. Then for $0 \leqslant t \leqslant 1$ we have*

$$h_r(t;\boldsymbol{a}^-) \gtrless 0 \quad \textit{and} \quad h_r(t;\boldsymbol{a}^+) \lessgtr h_r(1;\boldsymbol{a}^+) \quad \textit{for} \quad r \in \mathscr{R}^\mp.$$

Proof. We have

$$h_r^{(4)}(t;\boldsymbol{a}^-) = r(r-1)(r-2)(r-3)t^{r-4} - 24a_4^-,$$

so $h_r^{(4)}(t;\boldsymbol{a}^-)$ has at most one zero for $t>0$ and $h_r^{(i)}(t;\boldsymbol{a}^-)$ has at most $5-i$ zeros for $t>0$ ($i=3,2,1,0$). Since $h_r(\kappa_-;\boldsymbol{a}^-) = h_r(\eta_-;\boldsymbol{a}^-) = h_r(0;\boldsymbol{a}^-)$, it follows that $h_r'(\xi_-;\boldsymbol{a}^-) = h_r'(\xi_-';\boldsymbol{a}^-) = 0$ for some $\xi_- \in (0,\kappa_-)$ and $\xi_-' \in (\kappa_-,\eta_-)$. Therefore ξ_-, κ_-, ξ_-' and η_- are the only zeros of $h_r'(t;\boldsymbol{a}^-)$ in $(0,1)$.

Now

$$h_r''(\kappa_-;\boldsymbol{a}^-) = 8\cdot 4^{-r}(2r^2-2r+3+2r3^{r-2}-11\cdot 3^{r-2})$$

and

$$h_r''(\eta_-;\boldsymbol{a}^-) = 8\cdot 4^{-r}(2r^2-6r-3-2r3^r+43\cdot 3^{r-2}).$$

From these, it is easy to verify that

$$h_r''(\kappa_-;\boldsymbol{a}^-),\ h_r''(\eta_-;\boldsymbol{a}^-) \begin{cases} \gtrless 0 & \text{if } r \in \mathring{\mathscr{R}}^\mp, \\ = 0 & \text{if } r = 1,2,3,4, \end{cases}$$

where $\mathring{\mathscr{R}}^\mp$ denotes the interior of $\mathscr{R}^\mp$. Hence $h_r(t;\boldsymbol{a}^-)$ takes its mimimum (maximum, respectively) values in $[0,1]$ at 0, κ_-, η_- when $r \in \mathring{\mathscr{R}}^-$ ($r \in \mathring{\mathscr{R}}^+$, respectively). Moreover, $h_r(t;\boldsymbol{a}^-)$ has local maxima (minima, respectively) at ξ_-, ξ_-' when $r \in \mathring{\mathscr{R}}^-$ ($r \in \mathring{\mathscr{R}}^+$, respectively). This proves the assertion about $h_r(t;\boldsymbol{a}^-)$.

Similarly we can prove the corresponding result on $h_r(t;\boldsymbol{a}^+)$. □

Now we define the multiplicative function $\lambda_{f,r}^{\pm}(n)$ by

$$\lambda_{f,r}^{\mp}(p^\nu) := \begin{cases} \sum_{0\leqslant j\leqslant 4} 2^{2(r-j)} a_j^{\mp}\lambda_f(p)^{2j} & \text{if } \nu = 1 \text{ and } r > 0, \\ 0 & \text{if } \nu \geqslant 2 \text{ and } r \in \mathscr{R}^{\mp}, \\ |\lambda_f(p^\nu)|^{2r} & \text{if } \nu \geqslant 2 \text{ and } r \in \mathscr{R}^{\pm}, \end{cases} \tag{2.6}$$

where

$$a_0^- := 0 \quad \text{and} \quad a_0^+ := 1 - a_1^+ - a_2^+ - a_3^+ - a_4^+. \tag{2.7}$$

In view of (1.6), we can apply Lemma 2.3 with $t = |\cos\theta_f(p)|$ to deduce that the inequality (2.1) hold for all primes p and integers $\nu \geqslant 1$. Thanking to the multiplicativity, these inequalities also hold for all integers $n \geqslant 1$.

2.2. **Dirichlet series associated to $\lambda_{f,r}^{\pm}(n)$.** For $f \in \mathrm{H}_k^*(N)$, $r > 0$ and $\Re e\, s > 1$, we define

$$\Lambda_{f,r}^{\pm}(s) := \sum_{n\geqslant 1} \lambda_{f,r}^{\pm}(n) n^{-s}. \tag{2.8}$$

Next we shall study their analytic properties in the half-plane $\Re e\, s \geqslant 1$ by using the higher order symmetric power L-functions $L(s, \mathrm{sym}^m f)$ associated to $f \in \mathrm{H}_k^*(N)$, due to Gelbart & Jacquet [4] for $m = 2$, Kim & Shahidi ([8], [9]) for $m = 3, 4, 5, 6, 7, 8$. Here the symmetric mth power associated to f is defined as

$$L(s, \mathrm{sym}^m f) := \prod_p \prod_{0\leqslant j\leqslant m} \big(1 - \alpha_f(p)^{m-j}\beta_f(p)^j p^{-s}\big)^{-1}$$

for $\Re e\, s > 1$, where $\alpha_f(p)$ and $\beta_f(p)$ are given by (1.3) and (1.4). According to the literature mentioned above, it is known that the function $L(s, \mathrm{sym}^m f)$ for $m = 2, 3, \ldots, 8$ is invertible for $\Re e\, s \geqslant 1$.

We start to study $F_1(s)$, $F_2(s)$, $F_3(s)$ and $F_4(s)$.

Lemma 2.4. *Let $k \geqslant 2$ be an even integer, $N \geqslant 1$ be squarefree and $f \in \mathrm{H}_k^*(N)$. For $j = 1, 2, 3, 4$ and $\Re e\, s > 1$, we have*

$$F_j(s) = \zeta(s)^{m_j} G_j(s) H_j(s), \tag{2.9}$$

where

$$m_1 := 1, \qquad m_2 := 2, \qquad m_3 := 5, \qquad m_4 := 14, \tag{2.10}$$

and

$$\begin{aligned} G_1(s) &:= L(s, \mathrm{sym}^2 f), \\ G_2(s) &:= L(s, \mathrm{sym}^2 f)^3 L(s, \mathrm{sym}^4 f), \\ G_3(s) &:= L(s, \mathrm{sym}^2 f)^9 L(s, \mathrm{sym}^4 f)^5 L(s, \mathrm{sym}^6 f), \\ G_4(s) &:= L(s, \mathrm{sym}^2 f)^{34} L(s, \mathrm{sym}^4 f)^{20} L(s, \mathrm{sym}^6 f)^7 L(s, \mathrm{sym}^8 f) \end{aligned}$$

are invertible for $\Re e\, s \geqslant 1$. Here the function $H_j(s)$ admits a Dirichlet series convergent absolutely in $\Re e\, s > \frac{1}{2}$ and $H_j(s) \neq 0$ for $\Re e\, s = 1$.

Proof. Write x for the trace of a local factor of $L(s,f)$ (i.e. $\alpha_f(p)+\beta_f(p)$), and denote by $T_n(x)$ the polynomial for the trace of its symmetric nth power. Then

$$\begin{aligned}
T_2 &= x^2-1,\\
T_4 &= x^4-3x^2+1,\\
T_6 &= x^6-5x^4+6x^2-1,\\
T_8 &= x^8-7x^6+15x^4-10x^2+1,
\end{aligned}$$

from which we deduce

$$\begin{aligned}
x^2 &= 1+T_2,\\
x^4 &= 2+3T_2+T_4,\\
x^6 &= 5+9T_2+5T_4+T_6,\\
x^8 &= 14+34T_2+20T_4+7T_6+T_8.
\end{aligned}$$

This implies (2.9). By using results on $L(s,\mathrm{sym}^m f)$ mentioned above, $G_j(s)$ is invertible for $\Re e\, s \geqslant 1$. This completes the proof. □

Lemma 2.5. *Let $k \geqslant 2$ be an even integer, $N \geqslant 1$ be squarefree and $f \in \mathrm{H}_k^*(N)$. For $r>0$ and $\Re e\, s>1$, we have*

$$\Lambda_{f,r}^{\pm}(s) = \zeta(s)^{\rho_r^{\pm}+1} H_{f,r}^{\pm}(s), \tag{2.11}$$

where

$$\rho_r^{\pm} := 2^{2r-8}(2^8 a_0^{\pm} + 2^6 a_1^{\pm} + 2^4\cdot 2a_2^{\pm} + 2^2\cdot 5a_3^{\pm} + 14a_4^{\pm}) - 1 \tag{2.12}$$

and $H_{f,r}^{\pm}(s)$ is invertible for $\Re e\, s \geqslant 1$.

Proof. By definition (2.6), for $\Re e\, s>1$ we can write

$$\begin{aligned}
\Lambda_{f,r}^{-}(s) &= \prod_p \Big(1+\sum_{0\leqslant j\leqslant 4} 2^{2(r-j)} a_j^{-}\lambda_f(p)^{2j}p^{-s}\Big)\\
&= \prod_{0\leqslant j\leqslant 4} F_j(s)^{2^{2(r-j)}a_j^{-}}\, H_r^{-}(s)
\end{aligned}$$

for $r\in\mathscr{R}^{-}$, and

$$\begin{aligned}
\Lambda_{f,r}^{-}(s) &= \prod_p \Big(1+\sum_{0\leqslant j\leqslant 4} 2^{2(r-j)} a_j^{-}\lambda_f(p)^{2j}p^{-s} + \sum_{\nu\geqslant 2}|\lambda_f(p^\nu)|^{2r}p^{-\nu s}\Big)\\
&= \prod_{0\leqslant j\leqslant 4} F_j(s)^{2^{2(r-j)}a_j^{-}}\, H_r^{-}(s)
\end{aligned}$$

for $r\in\mathscr{R}^{+}$, where $F_0(s)=\zeta(s)$ is the Riemann zeta-function and $H_r^{-}(s)$ is a Dirichlet series absolutely convergent for $\Re e\, s>\frac12$ such that $H_r^{-}(s)\neq 0$ for $\Re e\, s=1$. Now the desired result with the sign '$-$' follows from Lemma 2.4. The other part can be treated in the same way. □

2.3. **Optimalisation of $\lambda_{f,r}^{\pm}(p)$ and choice of $\kappa_{\pm},\eta_{\pm}$.** If we regard $\kappa_{\pm},\eta_{\pm}$ as parameters, the $\rho_r^{\pm}$ given by (2.12) are functions of these parameters. We choose $(\kappa_{\pm},\eta_{\pm})$ in $(0,1)^2$ optimally, which can be done by using formal calculation via Maple. Their values are given by (2.3).

3. Proof of Theorem 1

In view of Lemma 2.5 and the classical fact on $\zeta(s)$, we can write

$$\Lambda_{f,r}^{\pm}(s) = \frac{H_{f,r}^{\pm}(1)}{(s-1)^{\rho_r^{\pm}+1}} + g_{f,r}^{\pm}(s) \tag{3.1}$$

in some neighbourhood of $s = 1$ with $\Re e\, s > 1$, where $H_{f,r}^{\pm}(1) \neq 1$ and $g_{f,r}^{\pm}(s)$ is holomorphic at $s = 1$. Since $\lambda_{f,r}^{\pm}(n) \geqslant 0$, we can apply Delange's tauberian theorem [2] to write

$$\sum_{n\leqslant x} \lambda_{f,r}^{\pm}(n) \sim H_{f,r}^{\pm}(1) x (\log x)^{\rho_r^{\pm}} \quad (x \to \infty). \tag{3.2}$$

Now Theorem 1 follows from (2.1) and (3.2).

4. Proof of Theorem 2

By (3.1), it follows that

$$\prod_p \left(1 + \sum_{\nu\geqslant 1} \frac{\lambda_{f,r}^{\pm}(p^\nu)}{p^{\nu\sigma}}\right) = \frac{H_{f,r}^{\pm}(1)}{(\sigma-1)^{\rho_r^{\pm}+1}} + g_{f,r}^{\pm}(\sigma)$$

for $\sigma > 1$. From this, (2.6), (2.7) and Deligne's inequality, we deduce that

$$\sum_p \frac{\lambda_{f,r}^{\pm}(p)}{p^\sigma} = (\rho_r^{\pm}+1)\log(\sigma-1)^{-1} + C_{f,r}^{\pm} + o(1) \quad (\sigma \to 1+),$$

where $C_{f,r}^{\pm}$ is some constant.

On the other hand, the prime number theorem implies, by a partial integration, that

$$\sum_p p^{-\sigma} = \log(\sigma-1)^{-1} + C + o(1) \quad (\sigma \to 1+),$$

where C is an absolute constant. Thus the preceding relation can be written as

$$\sum_p \frac{\lambda_{f,r}^{\pm}(p) - (\rho_r^{\pm}+1)}{p^\sigma} = C_{f,r}^{\pm} + (\rho_r^{\pm}+1)C + o(1) \quad (\sigma \to 1+). \tag{4.1}$$

According to Exercise II.7.8 of [19], the formula (4.1) implies

$$\sum_p \frac{\lambda_{f,r}^{\pm}(p) - (\rho_r^{\pm}+1)}{p} = C_{f,r}^{\pm} + (\rho_r^{\pm}+1)C.$$

Hence

$$\sum_{p\leqslant x} \frac{\lambda_{f,r}^{\pm}(p)}{p} = (\rho_r^{\pm}+1)\log_2 x + C_{f,r}^{\pm} + (\rho_r^{\pm}+1)C + o(1) \quad (x \to \infty).$$

Now we apply a well known result of Shiu [18] and (2.1) to write

$$\begin{aligned}\sum_{x\leqslant n\leqslant x+z}|\lambda_f(n)|^{2r} &\ll \frac{z}{\log x}\exp\bigg(\sum_{p\leqslant x}\frac{|\lambda_f(p)|^{2r}}{p}\bigg)\\ &\ll \frac{z}{\log x}\exp\bigg(\sum_{p\leqslant x}\frac{\lambda_{f,r}^+(p)}{p}\bigg)\\ &\ll z(\log x)^{\rho_r^+}\end{aligned}\tag{4.2}$$

for $r\in\mathscr{R}^-$, any $\varepsilon>0$, $x\geqslant x_0(\varepsilon)$ and $x^{1/4}\leqslant z\leqslant x$. Using this with $r=\frac12$ in (9) of [16], the first term on the right-hand side of (10) of [16] is replaced by $x^{1/2}z^{-1/2}(\log x)^{\rho_{1/2}^+}$. Applying (4.2) with $r=\frac12$ again to the second term on the right-hand side of (10) of [16], it follows that

$$S_f(x)\ll x^{1/2}z^{-1/2}(\log x)^{\rho_{1/2}^+}+z(\log x)^{\rho_{1/2}^+}.$$

Taking $z=x^{1/3}$, we obtain the required result when the level is $N=1$. The general case can be treated similarly as indicated in [16]. □

5. Proof of Corollary 1

By comparing (1.17) and the lower bound part in (1.11) with $r=\frac12$, it is easy to deduce that

$$\sum_{\substack{n\leqslant x\\ \lambda_f(n)\geqslant 0}}|\lambda_f(n)|\gg_f x(\log x)^{\rho_{1/2}^-}$$

for $x\geqslant x_0(f)$. Since $\rho_{1/2}^-=-(1-1/\sqrt{3})/2$ and $\rho_1^+=0$, a simple application of the Cauchy-Schwarz inequality yields the following result.

The second assertion can be obtained by noticing that $\theta_{1/2}=8/(3\pi)-1$. □

References

[1] H. Davenport, *On certain exponential sums*, J. reine angew. Math. **169** (1932), 158–176.

[2] H. Delange, *Généralisation du théorème de Ikehara*, Ann. Sci. Ecole Norm. Sup. **71** (1954), 213–242.

[3] P. Deligne, *La conjecture de Weil, I, II*, Publ. Math. IHES **48** (1974), 273–308, **52** (1981), 313–428.

[4] S. Gelbart & H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542.

[5] J. L. Hafner & A. Ivić, *On sums of Fourier coefficients of cusp forms*, Enseign. Math. (2) **35** (1989), no. 3-4, 375–382.

[6] E. Hecke, *Theorie der Eisensteinsche Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 199–224.

[7] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, Rhode Island, 1997.

[8] H. H. Kim & F. Shahidi, *Functorial products for* $\mathrm{GL}_2\times\mathrm{GL}_3$ *and the symmetric cube for* GL_2. With an appendix by Colin J. Bushnell and Guy Henniart. Ann. of Math. (2) **155** (2002), no. 3, 837–893.

[9] H. H. Kim & F. Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J. **112** (2002), no. 1, 177–197.
[10] H. D. Kloosterman, *Asymptotische Formeln für die Fourier-koeffizienten ganzer Modulformen*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 337–352.
[11] W. Kohnen, Y.-K. Lau & I. E. Shparlinski, *On the number of sign changes of Hecke eigenvalues of newforms*, J. Austral. Math. Soc., to appear.
[12] Y.-K. Lau & J. Wu, *The number of Hecke eigenvalues of same signs*, Preprint 2008.
[13] C. J. Moreno & F. Shahidi, *The fourth moment of Ramanujan τ-function*, Math. Ann. **266** (1983), no. 2, 233–239.
[14] R. A. Rankin, *Sums of powers of cusp form coefficients*, Math. Ann. **263** (1983), no. 2, 227–236.
[15] R. A. Rankin, *Sums of powers of cusp form coefficients, II*, Math. Ann. **272** (1985), no. 4, 593–600.
[16] R. A. Rankin, *Sums of cusp form coefficients*, Automorphic forms and analytic number theory (Montreal, PQ, 1989), 115–121, Univ. Montréal, Montreal, QC, 1990.
[17] H. Salié, *Zur Abschätzung der Fourierkoeffizienten ganzer Modulformen*, Math. Z. **36** (1933), 263–278.
[18] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. reine angew Math. **313** (1980), 161–170.
[19] G. Tenenbaum (en collaboration avec J. Wu), *Exercices corrigés de théorie analytique et probabiliste des nombres*, Cours Spécialisés N° 2. Société Mathématique de France, 1996. xiv+251 pp.
[20] G. Tenenbaum, *Remarques sur les valeurs moyennes de fonctions multiplicatives*, Enseignement Mathématique (2) **53** (2007), 155–178.
[21] A. Walfisz, *Über die Koeffizientensummen einiger Moduformen*, Math. Ann. **108** (1933), 75–90.
[22] J. R. Wilton, *A note on Ramanujan's arithmetical function $\tau(n)$*, Proc. Cambridge Philos. Soc. **25** (1928), 121–129.
[23] A. Weil, *On some exponential sums*, Proc. Acad. Sci. U.S.A. **34** (1948), 204–207.

Institut Elie Cartan Nancy (IECN), Nancy-Université, CNRS, INRIA, Boulevard des Aiguillettes, B.P. 239, 54506 Vandœuvre-lès-Nancy, France
E-mail address: wujie@iecn.u-nancy.fr