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Abstract. We prove that the weakly damped cubic Schrödinger flow in $L^2(\mathbb{T})$ provides a dynamical system that possesses a global attractor. The proof relies on a sharp study of the behavior of the associated flow-map with respect to the weak $L^2(\mathbb{T})$-convergence inspired by [14]. Combining the compactness in $L^2(\mathbb{T})$ of the attractor with the approach developed in [9], we show that the attractor is actually a compact set of $H^2(\mathbb{T})$. This asymptotic smoothing effect is optimal in view of the regularity of the steady states.

1 Introduction

The cubic nonlinear Schrödinger equation (NLS) can be derived as an asymptotic model to describe long waves propagation in different dispersive media. In some physical contexts, an exterior forcing and some damping effects have to be taken into account and this can lead to the following cubic NLS equation

$$u_t + \gamma u + i u_{xx} \mp i|u|^2 u = f, \tag{1}$$

where $\gamma > 0$ is the damping parameter and f is the forcing term. In this paper we focus on the case where $u(t,x)$ is a function from $\mathbb{R}_+ \times \mathbb{T}$ to $\mathbb{C}$, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $f \in L^2(\mathbb{T})$ does not depend on time. Also since the sign in front of the nonlinear term will not play any role in our analysis, we will take the $+$ sign in all this paper.

It is well-known since the work of Bourgain [3] that (1) provides an infinite dimensional dynamical system on $H^s(\mathbb{T})$ for $s \geq 0$. Using an a priori estimate in $H^1(\mathbb{T})$ related to the energy conservation of the classical cubic NLS equation, the existence of a global attractor in $H^1(\mathbb{T})$ can be obtained by a standard method (see for instance [18] or [8] where the additional regularity of the attractor is also proved). This method contains principally two steps. A first step consists in proving the continuity of the flow-map associated with the equation with respect to the weak topology of the phase space. This ensures the existence of a compact global attractor for the weak topology. The second step used the argument of Ball [2] to convert the weak convergence to the attractor into a strong one. The standart way to prove the first step is to use the well-posedness of the equation in a larger function space where the phase space is compactly embedded (cf. for instance [5]). This approach cannot be applied to (1) in $L^2(\mathbb{T})$ or $L^2(\mathbb{R})$ since the well-posedness of this equation is not known below L^2. Actually, the strong ill-posedness of the classical cubic NLS equation below $L^2(\mathbb{T})$ has been even proved (cf. [4], [14]). In [10] Goubet and the author used another approach involving the so called Kato local smoothing effect for (1) on $\mathbb{R}$ to establish the weak continuity of the flow-map in $L^2(\mathbb{R})$. However, in $L^2(\mathbb{T})$, the situation seems more complicated since, as was shown in [14], the flow-map of the classical cubic NLS equation is discontinuous for the weak topology of $L^2(\mathbb{T})$. In this paper, proceeding as in [14], we precise some behaviors of the flow-map with respect to the weak $L^2(\mathbb{T})$-convergence. This information supplemented with the argument of Ball enables to prove the existence of a global attractor in $L^2(\mathbb{T})$. Finally, combining the approach developed in [8]-[9] with the compactness of the attractor, we prove that the global attractor actually belongs to $H^2(\mathbb{T})$ which can be view as an asymptotic smoothing effect. This smoothing effect is optimal since for f belonging to $L^2(\mathbb{T})$ but not to $H^s(\mathbb{T})$, with $s > 0$, the steady state to (1) does not belong to $H^s(\mathbb{T})$ for $s > 2$, .

Our main result states as follows

Theorem 1.1 *The nonlinear semigroup $S(t)$ associated with (1) provides an infinite-dimensional dynamical system in $L^2(\mathbb{T})$ that has a compact global attractor $\mathcal{A}$. Moreover, $\mathcal{A}$ is a compact set of $H^2(\mathbb{T})$.*

Remark 1.2 *Exactly the same proof as in Section 5 below shows that the $L^2(\mathbb{R})$ global attractor to (1) on the line, that was constructed in [10], is actually a compact set of $H^2(\mathbb{R})$.*

The main new ingredient for proving Theorem 1.1 is the following result on the behavior of the flow-map of (1) with respect to the weak $L^2(\mathbb{T})$-convergence.

Theorem 1.3 *Let $\{u_{0,n}\}$ be a sequence of $L^2(\mathbb{T})$ converging weakly to u_0 in $L^2(\mathbb{T})$ and let $\{u_n\}$ be the sequence of the associated emanating solutions of* *the weakly damped cubic Schrödinger equation (1). For any adherence value a_0 of $\{\|u_{0,n}\|_{L^2}^2\}$ there exists a continuous function $t \mapsto a(t)$ from $\mathbb{R}$ to $\mathbb{R}_+$, with $a(0) = a_0$, and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that, for any $t \in \mathbb{R}$, $u_{n_k}(t)$ converges weakly in $L^2(\mathbb{T})$ to $v(t)$ where $v \in C(\mathbb{R}; L^2(\mathbb{T})) \cap L^4_{loc}(\mathbb{R}\times\mathbb{T})$ is the unique solution to*

$$\begin{cases} v_t + iv_{xx} + \gamma v + i|v|^2 v + \dfrac{i}{\pi}\Big(a(\cdot) - \|v(\cdot)\|_{L^2}^2\Big)v = f \\ v(0) = u_0 \end{cases} . \tag{2}$$

Remark 1.4 *It is worth noticing that this theorem ensures that, in sharp contrast with the case on the line (cf. [10]), the flow-map associated with (1) is not continuous for the weak topology of $L^2(\mathbb{T})$. Indeed, following [14], let $u_0 \in L^2(\mathbb{T})$ different from 0 and let $\{\phi_n\} \subset L^2(\mathbb{T})$ be a sequence such that $\phi_n \rightharpoonup 0$ in $L^2(\mathbb{T})$ and $\|\phi_n\|_{L^2}^2 \to 2\pi$ as n goes to infinity (one can take for instance $\phi_n = e^{inx}$). Setting $u_{0,n} = u_0 + \phi_n$, we get that $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$ and $\|u_{0,n}\|_{L^2}^2 \to \|u_0\|_{L^2}^2 + 2\pi$ as $n \to \infty$. On account of Theorem 1.3, the emanating solutions u_n tend weakly in $L^2(\mathbb{T})$ for any fixed $t \in \mathbb{R}$ to v satisfying (2). Observe that $w = v - u$ is solution of*

$$\begin{cases} w_t + iw_{xx} + \gamma w + i\Big(|v|^2 w + (\overline{w}u + w\overline{v})u\Big) = -\dfrac{i}{\pi}\Big(a(\cdot) - \|v(\cdot)\|_{L^2}^2\Big)v \\ w(0) = 0 \end{cases} . \tag{3}$$

Since $v(0) = u_0 \neq 0$ and $a(0) = \|u_0\|_{L^2}^2 + 2\pi \neq \|v(0)\|_{L^2}^2$ we infer that the $L^2(\mathbb{T})$-norm of the right-hand side of (3) cannot vanish for small $t \neq 0$. Hence $w(t) = 0$ is not solution of (3) and thus $v(t) \neq u(t)$ for small $t \neq 0$. Finally, note that, since $L^2(\mathbb{T})$ is compactly embedded in $H^s(\mathbb{T})$ for $s < 0$, this proves that (1) is ill-posed in $H^s(\mathbb{T})$ as soon as $s < 0$.

This paper is organized as follows. In the next section we introduce some notations and the function spaces we will work with. Section 3 is devoted to the proof of Theorem 1.3 and Section 4 is devoted to the existence of the global attractor. Finally in Section 5 we prove the asymptotic smoothing effect.

2 Function spaces and notations

When we affirm that a proposition is valid for $x+$ with $x \in \mathbb{R}$, we mean that there exists a small real number $\epsilon > 0$ such that the proposition is valid for any real number in the interval $]x, x+\epsilon[$. For $(x, y) \in \mathbb{R}^2$, $x \lesssim y$ means that

there exists $C>0$ such that $x\le Cy$. We will also denote by ε any function from $\mathbb{R}_+$ into itself that goes to zero at infinity.
For a 2π-periodic function φ, we define its space Fourier transform by

$$\hat{\varphi}(k) := \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx}\,\varphi(x)\,dx, \quad \forall k\in\mathbb{Z}\ ,$$

and we denote by $P_N\varphi$ and $Q_N\varphi$ the $L^2(\mathbb{T})$ orthogonal projections on respectively the space Fourier modes $|k|\le N$ and $|k|>N$.
We denote by $V(\cdot)$ the free group associated with the linearized Schrödinger equation,

$$\widehat{V(t)\varphi}(k) := e^{-ik^2t}\,\hat{\varphi}(k), \quad k\in\mathbb{Z}\,.$$

The Sobolev spaces $H^s(\mathbb{T})$ for 2π-periodic functions are defined as usually and endowed with

$$\|\varphi\|_{H^s(\mathbb{T})} := \|\langle k\rangle^s\widehat{\varphi}(k)\|_{l^2(\mathbb{Z})} = \|J_x^s\varphi\|_{L^2(\mathbb{T})}\ ,$$

where $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$ and $\widehat{J_x^s\varphi}(k) := \langle k\rangle^s\widehat{\varphi}(k)$.
For a function $u(t,x)$ on $\mathbb{R}\times\mathbb{T}$, we define its space-time Fourier transform by

$$\hat{u}(\tau,\xi) := \mathcal{F}_{t,x}(u)(\tau,\xi) := \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{T}} e^{-i(\tau t+kx)}\,u(t,x)dt\,dx, \quad \forall(\tau,k)\in\mathbb{R}\times\mathbb{Z} \quad .$$

and define the Bourgain spaces $X^{b,s}$ and $\tilde{X}^{b,s}$ of functions on $\mathbb{R}\times\mathbb{T}$ respectively endowed with the norm

$$\|u\|_{X^{b,s}} := \|\langle\tau+k^2\rangle^b\langle k\rangle^s\hat{u}\|_{L^2(\mathbb{R};l^2(\mathbb{Z}))} = \|\langle\tau\rangle^b\langle k\rangle^s\mathcal{F}_{t,x}(V(-t)u)\|_{L^2(\mathbb{R};l^2(\mathbb{Z}))}\ .$$

and

$$\|u\|_{\tilde{X}^{b,s}} := \|\langle\tau-k^2\rangle^b\langle k\rangle^s\hat{u}\|_{L^2(\mathbb{R};l^2(\mathbb{Z}))} = \|\langle\tau\rangle^b\langle k\rangle^s\mathcal{F}_{t,x}(V(t)u)\|_{L^2(\mathbb{R};l^2(\mathbb{Z}))}\ .$$

Finally, for an open interval $I\subset\mathbb{R}$ we define the restriction in time spaces $X_I^{b,s}$ of functions on $I\times\mathbb{T}$ endowed with the norm

$$\|u\|_{X_I^{b,s}} := \inf_{v\in X^{b,s}}\{\|v\|_{X^{b,s}},\ v(\cdot)\equiv u(\cdot)\text{ on } I\,\}\ .$$

It is worth noticing that for $b>1/2$ $X_I^{b,s}$ is continuously embedding in $L^\infty(I;H^s(\mathbb{T}))$ with a constant of continuity that depends on b and on $|I|$ the length of I, i.e.

$$\|u\|_{L^\infty(I;H^s(\mathbb{T}))} \le C(b,|I|)\|u\|_{X_I^{b,s}},\ \forall u\in X_I^{b,s} \tag{4}$$

3 Proof of Theorem 1.3

Theorem 1.3 is based on the observation made in [14] on the cubic NLS equation posed on the one-dimensional torus. We first recall the following well-posedness result due to Bourgain ([3]) for (1). Let us mention that this result was established for the cubic Schrödinger equation without damping and forcing but the adaptations for (1) are straightforward.

Theorem 3.1 *Let $s \geq 0$. For any $u_0 \in H^s(\mathbb{T})$, $f \in H^s(\mathbb{T})$ and any $T > 0$, there exists a unique solution*

$$u \in L^4(]-T,T[\times\mathbb{T})$$

of (1). Moreover $u \in C([-T,T]; H^s(\mathbb{T})) \cap X^{1/2+,s}_{]-T,T[}$ and the map data to solution $u_0 \mapsto u$ is real analytic from $H^s(\mathbb{T})$ to $C([-T,T]; H^s(\mathbb{T}))$.

Let us recall that this theorem principally use the linear estimates in Bourgain'spaces for the free evolution and the retarded Duhamel operator

$$\|V(t)\varphi\|_{X^{b,s}_{]-T,T[}} \leq C(T,b)\|\varphi\|_{H^s} \;, b \in \mathbb{R}, \; s \in \mathbb{R}, \, 0 < T < 1, \tag{5}$$

and for any $0 < \varepsilon << 1$ and $0 < T < 1$,

$$\|\int_0^t V(t-t')f(t')\,dt'\|_{X^{b,s}_{]-T,T[}} \leq C(b,\varepsilon)\,T^{\varepsilon}\|f\|_{X^{b-1+\varepsilon,s}_{]-T,T[}} \;, 1/2 \leq b < 1, \tag{6}$$

as well as the following linear dispersive estimate

$$\|v\|_{L^4(\mathbb{R}\times\mathbb{T})} \lesssim \|v\|_{X^{3/8,0}}, \quad \forall v \in X^{3/8,0}\,. \tag{7}$$

This estimate is proven in [3] for functions on $\mathbb{T}^2$ but also holds for function on $\mathbb{R}\times\mathbb{T}$ (See [13] for a shorter proof that works also clearly on $\mathbb{R}\times\mathbb{T}$). Note that applying (7) with $\overline{v}$ we obtain that (6) also holds with $X^{3/8,0}$ replaced by $\tilde{X}^{3/8,0}$. Moreover, according to [6] and [7], (7) ensures that for $0 < T < 1$ it holds

$$\|V(t)\varphi\|_{L^4(]-T,T[\times\mathbb{T})} \lesssim T^{1/8}\|\varphi\|_{L^2(\mathbb{T})}, \quad \forall \varphi \in L^2(\mathbb{T})\,, \tag{8}$$

which gives directly the existence and uniqueness in $L^4(]-T,T[\times\mathbb{T})$ by classical TT^* arguments. On the other hand, to prove that $u \in X^{1/2+,0}_{]-T,T[}$, one has to notice that (7) yields

$$\|u_1 u_2 \overline{u_3}\|_{X^{-1/2+,0}_{]-T,T[}} \lesssim \prod_{i=1}^{3} \|u_i\|_{X^{1/2,0}_{]-T,T[}}, \quad \forall u_i \in X^{1/2,0}_{]-T,T[} \tag{9}$$

Writing the Duhamel formulation of (1), using (5)-(6) and (9) and choosing some small positive real number ε, one can then eventually derive the key estimate :

$$\|u\|_{X^{1/2+,0}_{]-T,T[}} \lesssim \|u_0\|_{L^2(\mathbb{T})} + T^{0+}\Big[(\|u\|^2_{X^{1/2+,0}_{]-T,T[}} + \gamma)\|u\|_{X^{1/2+,0}_{]-T,T[}} + \|f\|_{L^2(\mathbb{T})}\Big] . \quad (10)$$

This lead to the local existence result in $X^{1/2+,0}_{]-T,T[}$. Finally, the fact that the time of existence in Theorem 3.1 can be chosen arbitrarly large follows from the a priori bound on the $L^2(\mathbb{T})$-norm of the solution (see (12)) .

Now, let $u_0 \in L^2(\mathbb{T})$ and $\{u_{0,n}\} \subset L^2(\mathbb{T})$ be a sequence converging weakly to u_0 in $L^2(\mathbb{T})$. Note that, from Banach Steinhaus'theorem, $\{||u_{0,n}||_{L^2(\mathbb{T})}\}$ is bounded in $\mathbb{R}_+$. It is well-known that the solutions of (1), given by Theorem 3.1, satisfy for all $t \in \mathbb{R}$,

$$\frac{1}{2}\frac{d}{dt}||u||^2_{L^2(\mathbb{T})} + \gamma||u||^2_{L^2(\mathbb{T})} = \mathrm{Re} \int_{\mathbb{T}} f\bar{u}dx \quad (11)$$

By Young inequality and Gronwall lemma, we deduce that the L^2-solutions satisfy for any $t \in \mathbb{R}_+$,

$$||u(t)||^2_{L^2(\mathbb{T})} \le e^{-\gamma t}||u_0||^2_{L^2(\mathbb{T})} + \frac{1-e^{-\gamma t}}{\gamma^2}||f||^2_{L^2(\mathbb{T})}. \quad (12)$$

Performing the change of variables $(t,x) \mapsto (-t,x)$ and proceeding as above we also infer that for any $t \in \mathbb{R}_-$ it holds

$$||u(t)||^2_{L^2(\mathbb{T})} \le e^{3\gamma|t|}||u_0||^2_{L^2(\mathbb{T})} + \frac{e^{3\gamma|t|}-1}{\gamma^2}||f||^2_{L^2(\mathbb{T})} \quad (13)$$

Therefore, from (11), we deduce that for any $(t_0,t_1) \in \mathbb{R}^2$ with $t_1 > t_0$,

$$\begin{aligned} \Big|||u(t_1)||^2_{L^2_x} - ||u(t_0)||^2_{L^2_x}\Big| &= \Big|2\gamma \int_{t_0}^{t_1} ||u(\tau)||^2_{L^2(\mathbb{T})}\, d\tau + 2\mathrm{Re}\int_{t_0}^{t_1}\int_{\mathbb{T}} f\bar{u}(\tau)dx\,\tau\Big| \\ &\le |t_1-t_0|\Big[3\gamma\Big(e^{3\gamma|t_0|}||u_0||^2_{L^2(\mathbb{T})} + \frac{e^{3\gamma|t_0|}-1}{\gamma^2}||f||^2_{L^2(\mathbb{T})}\Big) \\ &\quad +\frac{1}{\gamma}||f||^2_{L^2(\mathbb{T})}\Big] . \quad (14) \end{aligned}$$

Denoting by u_n the solution to (1) associated with the initial data $u_{0,n}$, this last inequality ensures that the sequence $\{t \mapsto ||u_n(t)||_{L^2(\mathbb{T})}\}$ is uniformly equi-continuous on any bounded interval of $\mathbb{R}$. It follows from Ascoli's theorem that there exists a subsequence $\{t \mapsto ||u_{n_k}(t)||_{L^2(\mathbb{T})}\}$ that converges

to some function $t \mapsto a(t)$ in $C([-T,T])$ for any $T>0$. Moreover, from Theorem 3.1 we know that $\{u_{n_k}\}$ is bounded in $X^{1/2+,0}_{]-T,T[}$ and thus, up to the extraction of a subsequence, converges weakly to some v in $X^{1/2+,0}_{]-T,T[}$.
Now, in [14] it is proven that the nonlinear term $|u|^2u$ can be rewritten as

$$|u|^2 u = \frac{1}{\pi}\|u\|^2_{L^2} u + \Lambda_1(u,u,u) + \Lambda_2(u,u,u) \ , \tag{15}$$

where Λ_i are continuous from $(X^{1/2+,0}_1)^3$ to $X^{-7/16,0}_1$ equipped with their respective weak topology. We thus rewrite the Duhamel formulation for u_n in the following way :

$$\begin{aligned} u_n(t) &= V(t)u_{0,n} - i\int_0^t V(t-t')\Big(\Lambda_1(u_n(t')) + \Lambda_2(u_n(t'))\Big)\, dt' \\ &\quad -\frac{i}{\pi}\int_0^t V(t-t')\Big(\|u_n(t')\|^2_{L^2} u_n(t')\Big)\, dt' - \gamma\int_0^t V(t-t')u_n(t')dt' \\ &\quad + \int_0^t V(t-t') f\, dt' \ . \end{aligned} \tag{16}$$

According to the linear estimates (5)-(6), the decomposition (15), the continuity of the Λ_i's for the weak topology and the above convergence results, we can pass to the limit to obtain that

$$\begin{aligned} v(t) &= V(t)u_0 - i\int_0^t V(t-t')\Big(\Lambda_1(v(t')) + \Lambda_2(v(t'))\Big)\, dt' \\ &\quad -\frac{i}{\pi}\int_0^t V(t-t')(a(t')v(t'))\, dt' - \gamma\int_0^t V(t-t')v(t')dt' + \int_0^t V(t-t')f dt' \end{aligned}$$

and v is solution of the following Cauchy problem on $]-1,1[$:

$$\left\{ \begin{array}{l} v_t + v_{xx} + \gamma v + i(\Lambda_1+\Lambda_2)(v) + \dfrac{i}{\pi} a(\cdot) v = f \\ v(0) = u_0 \end{array} \right. . \tag{17}$$

Proceeding exactly as for the cubic Schrödinger equation, it is easy to prove that this Cauchy problem is globally well-posed[1] in $H^s(\mathbb{T})$, $s \geq 0$, with a solution belonging for all $T>0$ to

$$C([-T,T]; H^s(\mathbb{T})) \cap L^4(]-T,T[\times\mathbb{T}) \cap X^{1/2+,0}_{]-T,T[}$$

[1] Note that the L^2-norm is controlled on any bounded interval of $\mathbb{R}$

with uniqueness in $L^4(]-T,T[\times\mathbb{T})$. Therefore, there exists only one possible limit and thus the whole sequence $\{u_{n_k}\}$ converges weakly to v in $X^{1/2+,0}_{]-1,1[}$. Moreover, using the equation satisfied by the u_n and the uniform bound in $L^\infty(]-T,T[;L^2(\mathbb{T}))\cap L^4(]-T,T[\times\mathbb{T})$, it is easy to check that for any smooth 2π-periodic function ϕ, the family $\{t\mapsto (u_{n_k}(t),\phi)_{L^2}\}$ is bounded in $C([-1,1])$ and uniformly equi-continuous on $[-1,1]$. Ascoli's theorem then ensures that (u_{n_k},ϕ) converges to (v,ϕ) on $[-1,1]$ and thus $u_{n_k}(t)\rightharpoonup v(t)$ in $L^2(\mathbb{T})$ for all $t\in[-1,1]$. By direct iteration this clearly also holds for all $t\in\mathbb{R}$.
Finally, according to (15), v can be also characterized as the unique solution in $L^4(]-T,T[\times\mathbb{T})$ to

$$\left\{\begin{array}{l} v_t+iv_{xx}+\gamma v+i|v|^2v+\dfrac{i}{\pi}\Big(a(\cdot)-\|v(\cdot)\|^2_{L^2}\Big)v=f \\ v(0)=u_0 \end{array}\right. . \tag{18}$$

4 Existence of the global attractor

Let us denote by $S(t)$ the nonlinear group associated with (1), i.e.

$$S(t)u_0:=u(t),\ t\in\mathbb{R}\ .$$

On account of Theorem 1.3 and (12), we infer that the ball of $L^2(\mathbb{T})$,

$$X:=\Big\{v\in L^2(\mathbb{T}),\ \|v\|_{L^2(\mathbb{T})}\le M_0:=2\frac{||f||_{L^2(\mathbb{T})}}{\gamma}\ \Big\}$$

is a global absorbing set for the dynamical system under consideration and that $S(t)$ acts continuously on X . To prove that there exists a global attractor it suffices to check the relative compactness in $L^2(\mathbb{T})$ of sequences of the type $\{S(t_n)b_n\}$ with $t_n\uparrow+\infty$ and $\{b_n\}\subset X$. This is the aim of the following proposition.

Proposition 4.1 *For any sequences $\{b_n\}\subset X$ and $\{t_n\}\uparrow+\infty$, the sequence $\{S(t_n)b_n\}$ has an adherence point in $L^2(\mathbb{T})$.*

Proof . We combine Theorem 1.3 with the famous J. Ball's argument (see [2], [18], [12]). Let $\{b_n\}\subset X$ and let $\{t_n\}$ be a sequence of positive real numbers that goes to infinity. From (12) the sequence $\{S(t_n)b_n\}$ remains bounded in $L^2(\mathbb{T})$ and thus, up to the extraction of a subsequence, converges weakly in $L^2(\mathbb{T})$ to some v_0. According to Theorem 1.3 there exists a subsequence $\{S(t_{n_k})b_{n_k}\}$ and a continuous function $t\mapsto a(t)$ from $\mathbb{R}$ to $\mathbb{R}_+$ such that the solutions emanating from $\{S(t_{n_k})b_{n_k}\}$ converge weakly in $L^2(\mathbb{T})$ for all $t \in \mathbb{R}$ to $v(t)$ where v is the unique solution to

$$\begin{cases} v_t + iv_{xx} + \gamma v + i|v|^2 v + \dfrac{i}{\pi}\Big(a(\cdot) - \|v(\cdot)\|^2_{L^2}\Big)v = f \\ v(0) = v_0 \end{cases} . \tag{19}$$

From (11) we infer that for $\tau > 0$ fixed and n_k large enough,

$$\|S(t_{n_k})b_{n_k}\|^2_{L^2(\mathbb{T})} = e^{-2\gamma\tau}\|S(t_{n_k}-\tau)b_{n_k}\|^2_{L^2(\mathbb{T})} - 2\mathrm{Re}\int_0^\tau \int_{\mathbb{T}} e^{-2\gamma s}\overline{f}\, S(t_{n_k}-s)b_{n_k}\, dsdx \tag{20}$$

where $\|S(t_{n_k}-\tau)b_{n_k}\|^2_{L^2(\mathbb{T})} \leq M_0^2$ and, according to the weak convergence and the dominated convergence theorem,

$$\lim_{n_k\to+\infty} 2\mathrm{Re}\int_0^\tau \int_{\mathbb{T}} e^{-2\gamma s}\overline{f}\, S(t_{n_k}-s)b_{n_k} dxds = 2\mathrm{Re}\int_0^\tau \int_{\mathbb{T}} e^{-2\gamma s}\overline{f}\, v(-s) dxds \ . \tag{21}$$

On the other hand, using the energy identity for equation (19) , we get

$$||v_0||^2_{L^2(\mathbb{T})} = e^{-2\gamma\tau}||v(-\tau)||^2_{L^2(\mathbb{T})} - 2\mathrm{Re}\int_0^\tau \int_{\mathbb{T}} e^{-2\gamma s}\overline{f}\, v(-s) dxds. \tag{22}$$

But since $S(t_{n_k}-\tau)b_{n_k} \rightharpoonup v(-\tau)$ in $L^2(\mathbb{T})$, it follows from (12) that

$$\|v(-\tau)\|_{L^2(\mathbb{T})} \leq M_0^2 \ .$$

Gathering the above three equalities, we thus infer that for any fixed $\tau > 0$,

$$\limsup_{n_k\to+\infty} ||S(t_{n_k})b_{n_k}||^2_{L^2(\mathbb{T})} \leq ||v_0||^2_{L^2(\mathbb{T})} + 2e^{-2\gamma\tau}M_0^2, \tag{23}$$

which ensures that $S(t_{n_k})b_{n_k}$ converges actually strongly to v_0 in $L^2(\mathbb{T})$. This completes the proof of Theorem 4.1. $\square$

Proposition 4.1 ensures the existence of a compact global attractor in $L^2(\mathbb{T})$. More precisely, from classical arguments (see for instance the proof of Theorem 1.1 in [16]), it follows that the invariant closed set

$$\mathcal{A} := \omega(X) = \bigcap_{s>0}\overline{\bigcup_{t>s} S(t)X}$$

is non-empty and attracts any bounded set of $L^2(\mathbb{T})$. The compactness of $\mathcal{A}$ follows as well. Indeed, let $\{a_n\} \subset \mathcal{A}$. Taking a sequence $\{t_n\} \uparrow +\infty$ and setting $b_n = S(-t_n)a_n$, we get that $a_n = S(t_n)b_n$ with $\{b_n\} \subset \mathcal{A} \subset X$ and thus $\{a_n\}$ has got an adherence point in $L^2(\mathbb{T})$.

5 Asymptotic smoothing effect

In this section we prove that the global attractor lies actually in $H^2(\mathbb{T})$ and is moreover compact in this space. Following the approach developed in [9], we split the solution $u(t) = S(t)u_0$ emanating from u_0 into two parts by setting[2]

$$v_t + iv_{xx} + \gamma v + |v|^2 v = f - P_N(|u|^2 u) + P_N(|v|^2 v) \tag{24}$$

$$w_t + iw_{xx} + \gamma w + Q_N(|u|^2 u) - Q_N(|v|^2 v) = 0 \tag{25}$$

with initial conditions

$$v(0) = P_N(u_0) \ \text{ and } \ w(0) = Q_N(u_0)\ . \tag{26}$$

In [9], Goubet introduced this decomposition for the weakly damped KdV equation. A first step of his analysis consists in proving that the high frequency part $w(t)$ is decreasing to 0 in $L^2(\mathbb{T})$. This decay of $\|w(t)\|_{L^2(\mathbb{T})}$, which is uniform for all u_0 in the absorbing ball, is obtained by using the dispersive damping effect on the high-high frequencies interactions that occurs for the nonlinear part of the KdV equation above $H^{-1/2}(\mathbb{T})$. This is related to the fact that the associated Cauchy problem is well-posed in $H^s(\mathbb{T})$ for $s \geq -1/2$. For the cubic Schrödinger equation the situation is more delicate since as recalled in the introduction, this equation is ill-posed below $L^2(\mathbb{T})$. Actually, due to some resonant parts in the nonlinear term, there is no damping effect on high-high-high interactions. To overcome this difficulty we will work directly on the global attractor and use in a crucial way that we already proved that it is compact in $L^2(\mathbb{T})$. Note that, the a priori compactness of the global attractor is not required in [9] where the compactness of the attractor can be obtained as a consequence of the asymptotic behavior of v and w.

The second step of the analysis in [9] consists in proving an uniform bound in $H^3(\mathbb{T})$ on v. This uniform estimate follows from an uniform bound in $L^2(\mathbb{T})$ on the time derivative v_t of v. To get this last bound the author uses among others that, in view of the equation satisfied by v, the low frequencies $P_N v_t$ belongs to any $H^s(\mathbb{T})$, $s \in \mathbb{R}$. We will not be able to use this approach here since for $v \in L^2(\mathbb{T})$, $P_N(|v|^2 v)$ does not belong a priori to any $H^s(\mathbb{T})$. Inspired by [17] we will instead introduce the auxiliary function $z := Q_N v - g_N$, where g_N is defined by $\widehat{g}_N(k) := \frac{\widehat{Q_N f}(k)}{-ik^2+\gamma}$, and prove that z

[2]Recall that P_N and Q_N are the projections on respectively the spatial Fourier modes $|k| \leq N$ and $|k| > 0$.

is uniformly bounded in $H^2(\mathbb{T})$.

The key proposition to derive the regularity of the attractor is the following.

Proposition 5.1 *There exist functions h and K : $\mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to+\infty} h(t) = 0$ such that for all $N > 0$ large enough and all $u_0 \in \mathcal{A}$ the function v and w constructed in (24)-(26) satisfy*

$$\|w(t)\|_{L^2(\mathbb{T})} \leq h(t) \quad \textit{and} \ \|v(t)\|_{H^2(\mathbb{T})} \leq K(N)\ , \ \forall t \in \mathbb{R}_+\,. \tag{27}$$

With Proposition 5.1 in hand it is straightforward to check that $\mathcal{A}$ is embedded in $H^2(\mathbb{T})$. Indeed, let $a \in \mathcal{A}$ and $\{t_n\} \uparrow +\infty$. For all $n \in \mathbb{N}$ we can write a as

$$a = S(t_n)S(-t_n)a = S(t_n)b_n$$

with $b_n = S(-t_n)a \in \mathcal{A}$. From Proposition 5.1 it follows that, for any $n \in \mathbb{N}$, a can be decomposed as $a = v_n + w_n$ with $\|v_n\|_{H^2(\mathbb{T})} \leq K$ and $\|w_n\|_{L^2(\mathbb{T})} \to 0$ as $n \to +\infty$. Therefore $a \in H^2(\mathbb{T})$ and $\|a\|_{H^2(\mathbb{T})} \leq K$. Hence, there exists $K > 0$ such that the following uniform bound holds on the attractor :

$$\|a\|_{H^2(\mathbb{T})} \leq K\ , \ \forall a \in \mathcal{A}. \tag{28}$$

5.1 Proof of Proposition 5.1

5.1.1 Preliminaries

The $L^2(\mathbb{T})$-compactness of $\mathcal{A}$ ensures the following uniform bound on the $L^2(\mathbb{T})$-norm of the high frequency part to the elements of $\mathcal{A}$.

Proposition 5.2 *There exists a function ε from $\mathbb{R}_+$ into itself that goes to zero at infinity such that*

$$\|Q_N a\|_{L^2(\mathbb{T})} \leq \varepsilon(N)\, , \ \forall a \in \mathcal{A}\ . \tag{29}$$

Thanks to this remark we will have to prove a damping effect only on terms of the form $P_{N/2}u_1 P_{N/2}u_2\overline{Q_N u_3}$. This is the aim of the following lemma :

Lemma 5.3 *Let $I \subset \mathbb{R}$ be a bounded interval, N be a positive integer and let $u_i \in X_I^{1/2,0}$, $i = 1, 2, 3$. Then it holds*

$$\|P_{N/2}u_1 P_{N/2}u_2\overline{Q_N u_3}\|_{X_I^{-1/2,0}} \lesssim N^{-1/4}\prod_{i=1}^{3}\|u_i\|_{X_I^{1/2,0}} \tag{30}$$

Proof. We take extensions v_i of the u_i's such that $\|v_i\|_{X^{1/2,0}} \le 2\|u_i\|_{X_I^{1/2,0}}$. By duality we have to prove that

$$\sup_{\|w\|_{X^{1/2,0}}=1} \Big| \Big(w, P_{N/2}v_1 P_{N/2}v_2 \overline{Q_N v_3} \Big)_{L^2(\mathbb{R}\times\mathbb{T})} \Big| \lesssim N^{-1/4} \prod_{i=1}^{3} \|v_i\|_{X^{1/2,0}} \ .$$

It thus suffices to estimate

$$J = \int_{\mathbb{R}^3} \sum_{(k_1,k_2,k_3)\in A(N)} |\widehat{w}(\tau,k)||\widehat{v_1}(\tau_1,k_1)||\widehat{v_2}(\tau_2,k_2)||\widehat{\overline{v_3}}(\tau_3,k_3)|\, d\tau_1\, d\tau_2\, d\tau_3$$

where $\tau = \tau_1+\tau_2+\tau_3$, $k = k_1+k_2+k_3$ and

$$A(N) := \{(k_1,k_2,k_3)\in\mathbb{Z}^3,\ |k_1|\le N/2,\ |k_2|\le N/2 \text{ and } |k_3|>N\ \} \ .$$

To do this we will use the famous resonant relation for the Schrödinger equation. Setting $\sigma=\tau+k^2$, $\sigma_1=\tau_1+k_1^2$, $\sigma_2=\tau_2+k_2^2$ and $\tilde{\sigma}_3=\tau_3-k_3^2$, it holds

$$\sigma-\sigma_1-\sigma_2-\tilde{\sigma}_3 = 2(k_3+k_1)(k_3+k_2) \ . \tag{31}$$

This ensures that on $\mathbb{R}^3\times A(N)$,

$$\max(|\sigma|,|\sigma_1|,|\sigma_2|,|\tilde{\sigma}_3|) \gtrsim N^2 \ .$$

Therefore we get, thanks to (7),

$$\begin{aligned} J &\lesssim N^{-1/4} \int_{\mathbb{R}^3} \sum_{(k_1,k_2,k_3)\in A(N)} |\sigma|^{1/8}|\widehat{w}(\tau,k)||\sigma_1|^{1/8}|\widehat{v_1}(\tau_1,k_1)| \\ &\qquad |\sigma_2|^{1/8}|\widehat{v_2}(\tau_2,k_2)||\tilde{\sigma}_3|^{1/8}|\widehat{\overline{v_3}}(\tau_3,k_3)|\, d\tau_1\, d\tau_2\, d\tau_3 \\ &\lesssim N^{-1/4}\| \mathcal{F}^{-1}(|\sigma|^{1/8}|\widehat{w}|)\|_{L^4(\mathbb{R}\times\mathbb{T})} \|\mathcal{F}^{-1}(|\tilde{\sigma}|^{1/8}|\widehat{\overline{v_3}}|)\|_{L^4(\mathbb{R}\times\mathbb{T})} \prod_{i=1}^{2} \|\mathcal{F}^{-1}(|\sigma|^{1/8}|\widehat{v_i}|)\|_{L^4(\mathbb{R}\times\mathbb{T})} \\ &\lesssim N^{-1/4}\|w\|_{X^{1/2,0}} \prod_{i=1}^{3}\|v_i\|_{X^{1/2,0}} \end{aligned}$$

where we used that according to the remark after (7),

$$\|\mathcal{F}^{-1}(|\tilde{\sigma}|^{1/8}|\widehat{\overline{v_3}}|)\|_{L^4(\mathbb{R}\times\mathbb{T})} \lesssim \|\overline{v_3}\|_{\tilde{X}^{1/2,0}} = \|v_3\|_{X^{1/2,0}} \ .$$

This completes the proof of the lemma. □

We are now in position to prove the Proposition 5.1. Let $u_0\in\mathcal{A}$ we decompose $u(t)=S(t)u_0$ by

$$u(t) = v(t) + w(t)$$

where v and w are defined as in (24)-(26).

5.1.2 Decay in time of w

The equation for w can be rewritten as

$$w_t+iw_{xx}+\gamma w+iQ_N(|w|^2w-2|w|^2u-w^2\overline{u})+iQ_N(2|u|^2w+u^2\overline{w})=0\ . \quad (32)$$

Since $u\in X^{1/2+,0}_{]-T,T[}$ for any $T>0$, proceeding as in Theorem 3.1 it is easy to prove that the Cauchy problem for w is locally well-posed in $L^2(\mathbb{T})$. We can thus multiply (32) with $2\overline{w}$ and integrate over $\mathbb{T}$ to obtain

$$\begin{aligned}\frac{d}{dt}\|w\|^2_{L^2(\mathbb{T})}+2\gamma\|w\|^2_{L^2(\mathbb{T})} &\leq \Big|\int_{\mathbb{T}}(2|w|^2u-w^2\overline{u})\overline{w}\Big| \\ &+\Big|\int_{\mathbb{T}}u^2\overline{w}^2\Big|\ . \end{aligned}\quad (33)$$

From Theorem 3.1, (10) and the fact that u belongs to the attractor, we know that for all $t\in\mathbb{R}$,

$$\|u\|_{X^{1/2+,0}_{]t-1,t+1[}}\lesssim M_0 \text{ and } \|Q_Nu\|_{X^{1/2,0}_{t-1,t+1}}\lesssim\varepsilon(N)\,. \quad (34)$$

Therefore, proceeding as in the proof of Theorem 3.1, we get the following estimate on w for all $t\in\mathbb{R}$ and $0<\delta<1$,

$$\|w\|_{X^{1/2+,0}_{]t-\delta,t+\delta[}}\lesssim\|w(t)\|_{L^2(\mathbb{T})}+\delta^{0+}\|w\|_{X^{1/2+,0}_{]t-\delta,t+\delta[}}\Big(\|w\|^2_{X^{1/2+,0}_{]t-\delta,t+\delta[}}+\|u\|^2_{X^{1/2+,0}_{]t-\delta,t+\delta[}+}+1\Big)\ . \quad (35)$$

Assuming that $\|w(t)\|_{L^2(\mathbb{T})}$ is bounded by some constant $A>0$ on $[0,T]$ for some positive time $T>0$, we deduce that there exists $\delta_0=\delta_0(A)>0$ such that for $0<\delta<\delta_0$,

$$\|w\|_{X^{1/2+,0}_{]t-\delta,t+\delta[}}\lesssim\|w(t)\|_{L^2(\mathbb{T})},\quad\forall t\in[0,T]\ . \quad (36)$$

From now on, we fix $0<\delta<\delta_0$ such that (36) holds. From this last inequality and (4) we infer that

$$\inf_{\tau\in]t,t+\delta[}\|w(\tau)\|_{L^2(\mathbb{T})}\gtrsim\|w(t)\|_{L^2(\mathbb{T})},\ \forall t\in[0,T]\ . \quad (37)$$

Integrating (33) with respect to time we obtain the following estimate for any $t\in[0,T]$,

$$\begin{aligned}\|w(t+\delta)\|^2_{L^2(\mathbb{T})} &\leq \|w(t)\|^2_{L^2(\mathbb{T})}e^{-\gamma\delta}-\gamma\int_t^{t+\delta}e^{-\gamma(t-s)}\|w(s)\|^2_{L^2(\mathbb{T})}\,ds \\ &+\int_t^{t+\delta}e^{-\gamma(t-s)}\Big|\int_{\mathbb{T}}(2|w|^2u-w^2\overline{u})\overline{w}\Big|\,ds \\ &+\int_t^{t+\delta}e^{-\gamma(t-s)}\Big|\int_{\mathbb{T}}u^2\overline{w}^2\Big|\,ds\ . \end{aligned}\quad (38)$$

From (37) we infer that,

$$-\gamma\int_t^{t+\delta} e^{-\gamma(t-s)}\|w(s)\|^2_{L^2(\mathbb{T})}\,ds \le -C\,(1-e^{-\gamma\delta})\|w(t)\|^2_{L^2(\mathbb{T})}\ . \tag{39}$$

Let us estimate now the two last time integrals in (38). First, we deduce from (9), (34) and (36) that

$$\begin{aligned}
I_1 &:= \int_t^{t+\delta} e^{-\gamma(t-s)}\Big|\int_{\mathbb{T}}(2|w|^2u-w^2\overline{u})\overline{w}\Big|\,ds \\
&\lesssim \|(2|w|^2u-w^2\overline{u})\|_{X^{-1/2,0}_{]t,t+\delta[}}\|w\|_{X^{1/2,0}_{]t,t+\delta[}} \\
&\lesssim \|w\|^3_{X^{1/2,0}_{]t,t+\delta[}}\Big(\|w\|_{X^{1/2,0}_{]t,t+\delta[}}+\|u\|_{X^{1/2,0}_{]t,t+\delta[}}\Big) \\
&\lesssim \|w(t)\|^3_{L^2(\mathbb{T})}\Big(\|w(t)\|_{L^2(\mathbb{T})}+M_0\Big)
\end{aligned} \tag{40}$$

To estimate the last time integral we split it into two parts in the following way :

$$\begin{aligned}
I_2 &:= \int_t^{t+\delta} e^{-\gamma(t-s)}\Big|\int_{\mathbb{T}}u^2\overline{w}^2\Big|\,ds \\
&= \int_t^{t+\delta} e^{-\gamma(t-s)}\Big|\int_{\mathbb{T}}(Q_{N/2}u(s))(Q_{N/2}u(s)+2P_{N/2}u(s))(\overline{w}(s))^2\Big|\,ds \\
&\quad+\int_t^{t+\delta} e^{-\gamma(t-s)}\Big|\int_{\mathbb{T}}(P_{N/2}u(s))^2(\overline{w}(s))^2\Big|\,ds \\
&= I_{21}+I_{22}\ .
\end{aligned} \tag{41}$$

To estimate I_{21} we proceed as above using (34) and (36) to get

$$\begin{aligned}
I_{21} &\lesssim \|u\|_{X^{1/2,0}_{]t,t+\delta[}}\|Q_{N/2}u\|_{X^{1/2,0}_{]t,t+\delta[}}\|w\|^2_{X^{1/2,0}_{]t,t+\delta[}} \\
&\lesssim M_0\,\varepsilon(N/2)\|w(t)\|^2_{L^2(\mathbb{T})}\ .
\end{aligned} \tag{42}$$

Finally, to estimate I_{22} we use Lemma 5.3 (recall that $w=Q_Nw$) and (36) to obtain

$$\begin{aligned}
I_{22} &\lesssim \|(P_{N/2}v(s))^2\overline{w}\|_{X^{-1/2,0}_{]t,t+\delta[}}\|w\|_{X^{1/2,0}_{]t,t+\delta[}} \\
&\lesssim \frac{M_0^2}{N^{1/4}}\|w\|^2_{X^{1/2,0}_{]t,t+\delta[}} \\
&\lesssim \frac{M_0^2}{N^{1/4}}\|w(t)\|^2_{L^2(\mathbb{T})}\ .
\end{aligned} \tag{43}$$

Gathering (38)-(43) we thus infer that for all $t \in [0,T]$,

$$\begin{aligned}
\|w(t+\delta)\|^2_{L^2(\mathbb{T})} &- \|w(t)\|^2_{L^2(\mathbb{T})} e^{-\gamma\delta} \\
&\leq \Big[C_1 \Big(\|w(t)\|_{L^2(\mathbb{T})}(\|w(t)\|_{L^2(\mathbb{T})} + M_0) + M_0\, \varepsilon(N/2) + \frac{M_0^2}{N^{1/4}}\Big) \\
&\quad -C_2(1-e^{-\gamma\delta})\Big] \|w(t)\|^2_{L^2(\mathbb{T})} \, . \qquad (44)
\end{aligned}$$

Since $w(0) = Q_N(u_0)$, according to Proposition 5.2, we can choose $N > 0$ large enough so that the right-hand side of the above inequality is negative at $t = 0$. By direct iteration in time and (37) we thus infer that

$$\|w(t)\|_{L^2(\mathbb{T})} \lesssim e^{-\gamma t}\|w(0)\|_{L^2(\mathbb{T})} \lesssim e^{-\gamma t}\|Q_N u_0\|_{L^2(\mathbb{T})} \lesssim e^{-\gamma t}\varepsilon(N)\,,\ \forall t \in [0,T]\,. \qquad (45)$$

In particular, $\|w(t)\|_{L^2(\mathbb{T})}$ is bounded by

$$A = C\|w(0)\|_{L^2(\mathbb{T})} \lesssim \|u_0\|_{L^2(\mathbb{T})} \lesssim M_0$$

on $[0,T]$ and thus, by a continuity argument, (45) holds actually for any $T > 0$. This proves the first assertion of Proposition 5.1.

5.1.3 Estimate on $Q_N v$

For N fixed, since by construction $P_N v = P_N u$ and u belongs to the global attractor, we get thanks to (34) that

$$\|P_N v(t)\|_{H^2} \lesssim \|P_N u\|_{X^{1/2+,2}_{]t-1,t+1[}} \leq C(N),\ \forall t \in \mathbb{R}\,. \qquad (46)$$

It thus remains to control the high frequencies of v. Inspired by [17] we introduce the function g_N defined by

$$\hat{g}_N(k) = \frac{\widehat{Q_N f}(k)}{-ik^2 + \gamma} \qquad (47)$$

so that g_N satisfies the equation

$$\partial_t g_N + i\partial_x^2 g_N + \gamma g = Q_N f\ .$$

Therefore, setting $z := Q_N v - g_N$, $z = Q_N z$ and is solution of

$$\left\{\begin{array}{l} z_t + iz_{xx} + \gamma z + iQ_N(|v|^2 v) = 0 \\ z(0) = -g_N \end{array}\right. . \qquad (48)$$

We plan to prove that $z(t)$ is uniformly bounded in $H^2(\mathbb{T})$ for positive times. We will need the following result on the behavior of g_N with respect to N.

Lemma 5.4 $g_N \in H^2(\mathbb{T}) \cap X^{1/2,1}_{]-1,1[}$ *and it holds*

$$\|g_N\|_{H^2(\mathbb{T})} + \|g_N\|_{X^{1/2,1}_{]-1,1[}} \le \varepsilon(N) \tag{49}$$

where $\varepsilon(N) \to 0$ *as* $N \to +\infty$.

Proof. It is clear that

$$\|g_N\|_{H^2(\mathbb{T})} \le \|Q_N f\|_{L^2(\mathbb{T})} \to 0 \text{ as } N \to +\infty \, .$$

Let now $\psi \in C_0^\infty(]-2,2[)$ such that $\psi \equiv 1$ on $[-1,1]$. It holds

$$\begin{aligned} \|g_N\|_{X^{1/2,1}_{]-1,1[}} &\le \|\psi g_N\|_{X^{1/2,1}} = \|\langle \tau + k^2\rangle^{1/2} \langle k \rangle \hat{\psi} \hat{g_N}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \\ &\le \|\langle \tau \rangle^{1/2} \hat{\psi}\|_{L^2(\mathbb{R})} \|\langle k\rangle \hat{g_N}\|_{L^2(\mathbb{Z})} + \|\psi\|_{L^2(\mathbb{R})} \|\langle k\rangle^2 \hat{g_N}\|_{L^2(\mathbb{Z})} \lesssim \|g_N\|_{H^2(\mathbb{T})} \, . \end{aligned}$$

It is worth noticing that combining (29), (49), (34), (36) and (45) it clearly holds

$$\|z(t)\|_{L^2(\mathbb{T})} \le \varepsilon(N) \text{ and } \|z\|_{X^{1/2+,0}_{]t-1,t+1[}} \lesssim M_0, \ \forall t \ge 0, \tag{50}$$

where $\varepsilon(N) \to 0$ as $N \to +\infty$. Therefore, taking $\beta > 0$ small enough, it holds

$$\|z\|_{X^{1/2,0}_{]t-1,t+1[}} \lesssim \|z\|^{\frac{2\beta}{1+2\beta}}_{X^{0,0}_{]t-1,t+1[}} \|z\|^{\frac{1}{1+2\beta}}_{X^{1/2+\beta,0}_{]t-1,t+1[}} \lesssim \varepsilon(N) \tag{51}$$

where $\varepsilon(N) \to 0$ as $N \to +\infty$.

According to the linear estimates (5)-(6), to prove that the equation (48) is globally well-posed in $H^2(\mathbb{T})$, it suffices to prove the following estimate :

Lemma 5.5 *Assuming that* $z \in X_I^{1/2+,2}$ *for some time interval* $I \subset \mathbb{R}$ *with* $|I| \le 1$. *The following estimate holds :*

$$\Big\| Q_N(|v|^2 v) \Big\|_{X_I^{-1/2+\epsilon,2}} \lesssim C(N) + \|z\|_{X_I^{1/2+,2}} \tag{52}$$

Proof. We decompose v as $v = P_N u + z + g_N$ so that we have to estimate

$$\Big\| Q_N\Big(|P_N u + z + g_N|^2 (P_N u + z + g_N)\Big)\Big\|_{X_I^{-1/2+\epsilon,2}} \, .$$

Let us first estimate the expression containing g_N, i.e. terms of the form $\|Q_N(\overline{g_N} w_1 w_2)\|_{X_I^{-1/2+\epsilon,2}}$ or $\|Q_N(\overline{w}_1 g_N w_2)\|_{X_I^{-1/2+\epsilon,2}}$ with $(w_1, w_2) \in \{g_N, P_N u, z\}^2$.

By the triangular inequality we can write

$$\begin{aligned}\|Q_N(\overline{g_N}w_1w_2)\|_{X_I^{-1/2+\epsilon,2}} &\lesssim \|Q_N(\overline{g_N}w_1w_2)\|_{X_I^{-1/2+\epsilon,0}}+\|Q_N(\overline{g_N}D_x^2\tilde{w}_1w_2)\|_{X_I^{-1/2+\epsilon,0}}\\ &\quad+\|Q_N(\overline{g_N}w_1D_x^2\tilde{w}_2)\|_{X_I^{-1/2+\epsilon,0}}+\|Q_N(\overline{D_x^2g_N}w_1w_2)\|_{X_I^{-1/2+\epsilon,0}}\\ &\quad+\|Q_N(\overline{g_N}D_x^2g_Nw_2)\|_{X_I^{-1/2+\epsilon,0}} \qquad (53)\end{aligned}$$

with $(\tilde{w_1},\tilde{w_2})\in\{P_Nu,z\}$. The terms containing no derivative on g_N of the above right-hand side can be estimated thanks to (7), (9), (49) and (50) by

$$\begin{aligned}&\|g_N\|_{X_I^{1/2,0}}(\|P_Nu\|_{X_I^{1/2,2}}+\|z\|_{X_I^{1/2,2}}+\|g_N\|_{X_I^{1/2,0}})(\|P_Nu\|_{X_I^{1/2,0}}+\|z\|_{X_I^{1/2,0}}+\|g_N\|_{X_I^{1/2,0}})\\ &\quad\lesssim\varepsilon(N)(M_0^2N^2+M_0\|z\|_{X_I^{1/2,2}}+\varepsilon(N))\;. \qquad (54)\end{aligned}$$

For the terms that contains two derivatives on g_N we write

$$\begin{aligned}\|Q_N(D_x^2\overline{g_N}w_1w_2)\|_{X_I^{-1/2+\epsilon,0}} &\lesssim \|g_N\|_{H^2(\mathbb{T})}\|w_1\|_{L_I^\infty L^\infty(\mathbb{T})}\|w_2\|_{L_I^\infty L^\infty(\mathbb{T})}\\ &\lesssim \varepsilon(N)\|w_1\|^{3/4}_{L_I^\infty L^2(\mathbb{T})}\|w_2\|^{3/4}_{L_t^\infty L^2(\mathbb{T})}\|w_1\|^{1/4}_{L_I^\infty H^2(\mathbb{T})}\|w_2\|^{1/4}_{L_I^\infty H^2(\mathbb{T})}\\ &\quad\lesssim\varepsilon(N)(M_0^2N^2+\varepsilon(N)\|z\|_{X_I^{1/2+,2}}+\varepsilon(N))\;. \qquad (55)\end{aligned}$$

The terms of the form $\|Q_N(\overline{w}g_Nw_2)\|_{X_I^{-1/2+\epsilon,2}}$ can be treated exactly in the same way. It remains to consider the terms where g_N is not involved. From (7), (9) and (51) ,

$$\||z|^2z\|_{X_I^{-1/2+\epsilon,2}}\lesssim\|z\|^2_{X_I^{1/2,0}}\|z\|_{X_I^{1/2,2}}\lesssim\varepsilon(N)^2\|z\|_{X_I^{1/2,2}}, \qquad (56)$$

and

$$\begin{aligned}&\|P_Nu|z|^2\|_{X_I^{-1/2+\epsilon,2}}+\|\overline{P_Nu}z^2\|_{X_I^{-1/2+\epsilon,2}}\\ &\qquad\lesssim\|P_Nu\|_{X_I^{1/2,0}}\|z\|_{X_I^{1/2,0}}\|z\|_{X_I^{1/2,2}}+\|P_Nu\|_{X^{1/2,2}}\|z\|^2_{X^{1/2,0}}\\ &\qquad\lesssim\varepsilon(N)M_0\|z\|_{X_I^{1/2,2}}+N^2M_0\varepsilon(N)N^{-2}\|z\|_{X_I^{1/2,2}}\\ &\qquad\lesssim\varepsilon(N)M_0\|z\|_{X_I^{1/2,2}}\;. \qquad (57)\end{aligned}$$

To deal with $(P_Nu)^2\overline{z}$ we decompose it as

$$(P_Nu)^2\overline{z}=(Q_{N/2}P_Nu)(Q_{N/2}P_Nu+2P_{N/2}u)\overline{z}+(P_{N/2}u)^2\overline{z}:=A_1+A_2\;. \qquad (58)$$

Clearly, (34) yields

$$\|A_1\|_{X_I^{-1/2+\epsilon,2}}\lesssim\varepsilon(N/2)M_0\|z\|_{X_I^{1/2,2}} \qquad (59)$$

and using Lemma 5.3 it is easy to check that

$$\|A_2\|_{X^{-1/2+\epsilon,2}} \lesssim N^{-1/4} M_0^2 \|z\|_{X_I^{1/2,2}} \ . \tag{60}$$

Finally,

$$\begin{aligned}\||P_N u|^2 z\|_{X^{-1/2+\epsilon,2}} &\lesssim \|P_N u\|_{X_I^{1/2,0}} \|P_N u\|_{X_I^{1/2,2}} \|z\|_{X_I^{1/2,0}} + \|P_N u\|^2_{X_I^{1/2,0}} \|z\|_{X_I^{1/2,2}} \\ &\lesssim M_0^2 (N^2 N^{-2} + 1) \|z\|_{X_I^{1/2,2}} \lesssim M_0^2 \|z\|_{X_I^{1/2,2}} \end{aligned} \tag{61}$$

and

$$\||P_N u|^2 P_N u\|_{X_I^{-1/2+\epsilon,2}} \lesssim N^2 M_0^3 \ . \tag{62}$$

Gathering all the above estimates, (52) follows. ☐ From the above lemma and (5)-(6)we deduce that $z \in X^{1/2+,2}_{\text{loc}}$ and that , for any $t \geq 0$ and any $0 < \delta < 1$, it holds

$$\|z\|_{X^{1/2+,2}_{]t-\delta,t+\delta[}} \lesssim \|z(t)\|_{H^2(\mathbb{T})} + C\,\delta^{0+}\Big(C(N) + \|z\|_{X^{1/2+,2}_{]t-\delta,t+\delta[}}\Big) \ .$$

This ensures that for $\delta > 0$ small enough,

$$\|z\|_{X^{1/2+,2}_{]t-\delta,t+\delta[}} \lesssim \|z(t)\|_{H^2(\mathbb{T})} + C(N) \ . \tag{63}$$

We will proceed as in the preceding subsection. From now on we fix $0 < \delta < 1$ such that (63) holds. As in (37), this implies that

$$\inf_{\tau \in]t-\delta,t+\delta[} \|z(\tau)\|_{H^2(\mathbb{T})} \geq C\|z(t)\|_{H^2(\mathbb{T})} - C(N), \ \forall t \in \mathbb{R} \ . \tag{64}$$

On the other hand, taking the real part of the $H^2(\mathbb{T})$ hermitian-product of (48) with $2z$, we get

$$\frac{d}{dt}\|z\|^2_{H^2(\mathbb{T})} + 2\gamma\|z\|^2_{H^2(\mathbb{T})} = \Re\Big[-i2((Q_N(|v|^2 v), z))_{H^2}\Big] \ . \tag{65}$$

Integrating with respect to time this implies the following estimate for any $t \geq 0$,

$$\begin{aligned}\|z(t+\delta)\|^2_{H^2(\mathbb{T})} \leq\ & \|z(t)\|^2_{H^2(\mathbb{T})} e^{-\gamma\delta} - \gamma \int_t^{t+\delta} e^{-\gamma(t-s)} \|z(s)\|^2_{H^2(\mathbb{T})}\, ds \\ & +2\Big[\int_t^{t+\delta} e^{-\gamma(t-s)} \Im\Big[((Q_N(|v(s)|^2 v(s)), z(s)))_{H^2}\Big]\, ds \\ \leq\ & \|z(t)\|^2_{H^2(\mathbb{T})} e^{-\gamma\delta} + (1-e^{-\gamma\delta})\Big(C(N) - C\|z(t)\|^2_{H^2(\mathbb{T})}\Big) \\ & +2\Big|\Im\Big[\int_t^{t+\delta} ((|v(s)|^2 v(s), z(s)))_{H^2}\, ds\Big]\Big| \end{aligned} \tag{66}$$

To estimate the last term of the above right-hand side we decompose v as in Lemma 5.5. In view of (54)-(62) to get the following estimate :

$$\Big|\Im\Big[\int_t^{t+\delta}((|v(s)|^2v(s),z(s)))_{H^2}\,ds\Big]\Big|\lesssim C(N)+\varepsilon(N)\|z\|^2_{X^{1/2+,2}}\ , \tag{67}$$

we only have to care about

$$I:=\Big|\Im\int_t^{t+\delta}(|P_Nu(s)|^2z(s),z(s))_{H^2}\,ds\Big|\ .$$

To deal with this term we decompose it as

$$\begin{aligned}I\lesssim&\Big|\Im\int_t^{t+\delta}(|P_Nu|^2z,z)_{H^1}\,ds\Big|+\Big|\Im\int_t^{t+\delta}\int_{\mathbb{T}}\Big(2\partial_x(|P_Nu|^2)\partial_xz+\partial_x^2(|P_Nu|^2)z\Big)\partial_x^2\overline{z}\,dxds\Big|\\&+\Big|\Im\int_t^{t+\delta}\int_{\mathbb{T}}|P_Nu|^2|\partial_x^2z|^2\,dxds\Big|\end{aligned}\tag{68}$$

and notice that the last term of the right-hand side vanishes. We thus get

$$\begin{aligned}I\lesssim&\||P_Nu|^2z\|_{X^{-1/2,1}_{]t,t+\delta[}}\|z\|_{X^{1/2,1}_{]t,t+\delta[}}+\Big\|2\partial_x(|P_Nu|^2)\partial_xz+\partial_x^2(|P_Nu|^2)z\Big\|_{X^{-1/2,0}_{]t,t+\delta[}}\|z\|_{X^{1/2,2}_{]t,t+\delta[}}\\\lesssim&M_0^2\varepsilon(N)\|z\|_{X^{1/2,2}_{]t,t+\delta[}}+\Big(M_0^2N\varepsilon(N)^{1/2}\|z\|^{1/2}_{X^{1/2,2}_{]t,t+\delta[}}+M_0^2N^2\varepsilon(N)\Big)\|z\|_{X^{1/2,2}_{]t,t+\delta[}}\\\lesssim&\varepsilon(N)\|z\|^2_{X^{1/2,2}_{]t,t+\delta[}}+C(N)\ .\end{aligned}\tag{69}$$

Combining this last estimate with (54)-(61), (67) follows.

We thus infer that

$$\begin{aligned}\|z(t+\delta)\|^2_{H^2(\mathbb{T})}\le&\|z(t)\|^2_{H^2(\mathbb{T})}e^{-\gamma\delta}+C(N)\\&+C_1\Big(\varepsilon(N)-C_2(1-e^{-\gamma\delta})\Big)\|z(t)\|^2_{H^2(\mathbb{T})}\,ds\ .\end{aligned}\tag{70}$$

For N large enough the last term of the right-hand side is clearly negative and is bounded from above by

$$-\alpha\,\|z(t)\|^2_{H^2(\mathbb{T})}\ ,$$

for some small real number $\alpha>0$. This ensures that, taking $N>0$ large enough, there exists $C(N)>0$ such that

$$\|z(t)\|_{H^2(\mathbb{T})}\le C(N),\ \forall t\ge 0,\tag{71}$$

and thus on account of (46), (49), (63) and the definition of z, there exists $K(N)>0$ such that

$$\|v\|_{X^{1/2+,2}_{]t-1,t+1[}}+\|v(t)\|_{H^2(\mathbb{T})}\le K(N),\ \forall t\ge 0\ .\tag{72}$$

This completes the proof of Proposition 5.1. □

5.2 Compactness in $H^2(\mathbb{T})$

To prove the compactness in $H^2(\mathbb{T})$, it suffices to show that

$$\|z(t)\|_{H^2(\mathbb{T})} \le \varepsilon(N),\ \forall t \ge 0 . \tag{73}$$

Indeed, this will imply the same estimate on v and thus on any $a \in \mathcal{A}$ which will clearly prove the $H^2(\mathbb{T})$ compactness of $\mathcal{A}$. For proving (73), we revisit Lemma 5.5 with (28) in hand. Combining Theorem 3.1 with (28), the crucial new information is that there exists $C > 0$ such that

$$\|u\|_{X^{1/2+,2}_{]t-1,t+1[}} \le C,\ \forall t \in \mathbb{R} .$$

It is then easy to check that the terms involving g_N in Lemma (52) can now be estimated by

$$\varepsilon(N)\Big(\|z\|_{X_I^{1/2+,2}} + 1\Big) . \tag{74}$$

and that (see (68) above)

$$\Big|\Im \int_t^{t+\delta} \Big(|P_N u(s)|^2 z(s), z(s)\Big)_{H^2} ds\Big| \lesssim \Big(\varepsilon(N)(M_0^2+C^2)+M_0^2 C N^{-1}\Big)\|z\|^2_{X^{1/2,2}_{]t,t+\delta[}} . \tag{75}$$

To conclude we need the following estimate that we will prove hereafter.

$$\Big\|Q_N\Big(|P_N u|^2 P_N u\Big)\Big\|_{X^{-1/2,2}_{]t-1,t+1[}} \lesssim \varepsilon(N),\ \forall t \in \mathbb{R} . \tag{76}$$

Proceeding as in the obtention of (67) but with(74)-(76) in hand it is easy to see that it actually holds

$$\Big|\Im\Big[\int_t^{t+\delta} \Big((Q_N(|v(s)|^2 v(s)), z(s))\Big)_{H^2} ds\Big]\Big| \lesssim \varepsilon(N)(1 + \|z\|^2_{X^{1/2+,2}_{]t,t+\delta[}}) \tag{77}$$

and thus

$$\begin{aligned}\|z(t+\delta)\|_{H^2(\mathbb{T})} \le\ & \|z(t)\|_{H^2(\mathbb{T})} e^{-\gamma\delta} + \varepsilon(N) \\ & + C_1\Big(\varepsilon(N) - C_2(1-e^{-\gamma\delta})\Big)\|z(t)\|^2_{H^2(\mathbb{T})} .\end{aligned} \tag{78}$$

which ensures that (73) holds.

5.2.1 *Proof of Estimate (76).*

We take an extension θ of $P_N u$ such that $\theta = P_N\theta$, $\|\theta\|_{X^{1/2,2}} \le 2\|P_N u\|_{X^{1/2,2}_{]t-1,t+1[}}$ and $\|Q_{N/6}\theta\|_{X^{1/2,2}} \le 2\|Q_{N/6}P_N u\|_{X^{1/2,2}_{]t-1,t+1[}}$. By duality we have to prove that

$$\sup_{\|h\|_{X^{1/2,-2}}=1} \Big| \Big(h, Q_N(|\theta|^2\theta)\Big)_{L^2(\mathbb{R}\times\mathbb{T})} \Big| \lesssim \varepsilon(N)\|\theta\|^3_{X^{1/2,2}} \; .$$

It thus suffices to estimate

$$I = \int_{\mathbb{R}^3} \sum_{(k_1,k_2,k_3)\in A(N)} |\widehat{h}(\tau,k)||\widehat{\theta}(\tau_1,k_1)||\widehat{\theta}(\tau_2,k_2)||\widehat{\overline{\theta}}(\tau_3,k_3)|\, d\tau_1\, d\tau_2\, d\tau_3$$

where $\tau = \tau_1+\tau_2+\tau_3$, $k = k_1+k_2+k_3$ and

$$A(N) := \{(k_1,k_2,k_3)\in\mathbb{Z}^3,\ |k_i|\le N \text{ for } i\in\{1,2,3\},\ N < |k_1+k_2+k_3| \le 3N\} \; .$$

By symetry we can assume that $|k_1| \ge |k_2|$. Note that by the definition of $A(N)$, we must have $\max(|k_i|) \ge |k|/3 > N/3$. We divide $A(N)$ into two regions :

• The region $|k_1+k_3| \le \max(|k_1|,|k_3|)/2$. In this region it holds $\min(|k_1|,|k_3|) > N/6$ and it is easy to check using (7) and (34) that

$$\begin{aligned} I &\lesssim \|h\|_{X^{1/2,-2}}\|\theta\|_{X^{1/2,0}}\|Q_{N/6}\theta\|_{X^{1/2,0}}\|\theta\|_{X^{1/2,2}} \\ &\lesssim CM_0\,\varepsilon(N/6)\|h\|_{X^{1/2,-2}} \; . \end{aligned}$$

• The region $|k_1+k_3| > \max(|k_1|,|k_3|)/2.$. In this region we use the resonant relation (31). Notice that, in $A(N)$, $|k_i+k_j| \ge 1$ for $i\ne j$. (31) thus ensures that on $\mathbb{R}^3\times A(N)$,

$$\max(|\sigma|,|\sigma_1|,|\sigma_2|,|\tilde{\sigma}_3|) \gtrsim \max_{i=1,2,3}(|k_i|) \gtrsim 2|(k_1+k_3)(k_2+k_3)| \gtrsim |k_1| \gtrsim N \; ,$$

where $\sigma = \tau + k^2$, $\sigma_1 = \tau_1 + k_1^2$, $\sigma_2 = \tau_2 + k_2^2$ and $\tilde{\sigma}_3 = \tau_3 - k_3^2$. Therefore proceeding as in the proof of Lemma 5.3 we obtain

$$\begin{aligned} I &\lesssim N^{-1/8}\|h\|_{X^{1/2,-2}} \prod_{i=1}^{3} \|\theta\|^2_{X^{1/2,0}}\|\theta\|_{X^{1/2,2}} \\ &\lesssim CM_0^2 N^{-1/8}\|h\|_{X^{1/2,-2}} \; , \end{aligned}$$

which completes the proof of (76).

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