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► **To cite this version:**

Florent Benaych-Georges, Thierry Cabanal-Duvillard. A matrix interpolation between classical and free max operations: I. The univariate case. 2008. hal-00289807v1

**HAL Id: hal-00289807**

**<https://hal.science/hal-00289807v1>**

Preprint submitted on 23 Jun 2008 (v1), last revised 23 Feb 2009 (v3)

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**A MATRIX INTERPOLATION BETWEEN CLASSICAL AND FREE MAX  
OPERATIONS.  
I. THE UNIVARIATE CASE**

FLORENT BENAYCH-GEORGES AND THIERRY CABANAL-DUVILLARD

ABSTRACT. Recently, Ben Arous and Voiculescu considered taking the maximum of two free random variables and brought to light a deep analogy with the operation of taking the maximum of two independent random variables. We present here a new insight on this analogy, based on random matrices giving an interpolation between classical and free settings.

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1. INTRODUCTION

The free probability theory has been a very active field in mathematics over the last two decades, constructed in a deep analogy with classical probability theory. Nowadays, there is a unofficial dictionary of concepts in both theories: many fundamental notions or results of classical probability theory, such as

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*Date:* June 23, 2008.

*MSC 2000 subject classifications.* 15A52, 46L54

*Key words.* random matrices, free probability, max-stable laws

Law of Large Numbers, Central Limit Theorem, Gaussian distribution, convolution, cumulants, infinite divisibility have their precise counterpart in free probability theory. Recently, Ben Arous and Voiculescu have added a new item in this dictionary, bringing to light the fact that the operation which associates the distribution of the supremum  $X \vee Y$  to the distributions of two independent random variables  $X$  and  $Y$ , has an analogue in free probability theory (cf [2]).

This last analogy goes deep. A first evidence is given by the correspondence between max-stable laws and free max-stable laws established in [2], which are associated to each other by their domains of attraction. This is formally very close to the Bercovici-Pata bijection between classical and free infinitely divisible laws (w.r.t. additive convolution). A second evidence is the fact that there is a morphism (denoted by  $\Lambda^\vee$  in this text) between the set of probability measures on the real line endowed with the classical upper extremal convolution<sup>1</sup> and the same set endowed with the free upper extremal convolution. We shall improve on this analogy also with a rather natural random matricial model.

Since the publication of a fundamental paper by Voiculescu (cf [11]), random matrices are known to provide models for free random variables, and many free-probabilistic phenomena have a matricial interpretation. For instance, Gaussian Hermitian matrices perform asymptotically free semi-circular random variables. For more general free infinitely divisible laws, we have constructed infinitely divisible random matrices in a rather canonic way (cf [3] and [5]).

It occurs that our strategy can be adapted easily to the present case. For any probability measure  $\mu$  on the real line, and for any dimension, we will associate a random matrix whose spectral empirical law converges to the free corresponding law of  $\mu$ . Such a model produces a new clue with regards to the relevance of the analogy between the two max-operations, and it may be used to provide more intuitive proofs of some of Ben Arous and Voiculescu's results.

In a forthcoming paper, we will show how this approach is also appropriate to study the multivariate counterpart of the free upper extreme convolution.

The paper is organized as follows. First, we shall focus on the upper extremal convolution for classical, free and Hermitian random variables. Then, we shall introduce our models and set the main theorems. The last part will be dedicated to the proofs.

## 2. THE MAXIMUM OPERATION

**2.1. for independent real random variables.** Let  $X$  and  $Y$  be real independent random variables, with cumulative distribution functions  $F_X$  and  $F_Y$ , and let us denote  $X \vee Y = \max(X, Y)$ . The cumulative distribution function<sup>2</sup> of  $X \vee Y$  is then equal to  $F_{X \vee Y} = F_X F_Y$ .

**2.2. for free random variables.** Let  $(M, \tau)$  be a tracial  $W^*$ -probability space, that is a von Neumann algebra  $M$  endowed with an ultraweakly continuous faithful trace-state  $\tau$ . The maximum  $X \vee Y$  of two self-adjoints elements  $X$  and  $Y$  in  $M$  is defined with respect to the spectral order introduced by Olson in 1971 (cf [9]; see also [1]). In 2006, Ben Arous and Voiculescu have determined the cumulative distribution function of  $X \vee Y$  whenever  $X$  is free with  $Y$  (cf [2]):

$$(1) \quad \forall t \in \mathbb{R}, \quad F_{X \vee Y}(t) = \max(0, F_X(t) + F_Y(t) - 1) =: F_X \boxminus F_Y(t).$$

It is easy to note that the function  $\Lambda^\vee(u) = \max(0, 1 + \log u)$  is a kind of morphism between classical and free upper extremal convolutions:

$$\Lambda^\vee(F_X F_Y) = \Lambda^\vee(F_X) \boxminus \Lambda^\vee(F_Y).$$

### Notations:

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<sup>1</sup>The classical (resp. free) *upper extremal convolution* of two probability measures  $\mu, \nu$  is the distribution of  $X \vee Y$ , for  $X, Y$  independent (resp. free) random variables with laws  $\mu, \nu$ .

<sup>2</sup>The *cumulative distribution function* of a random variable  $X$  with distribution  $\mu$  is the function, denoted either by  $F_X$  or  $F_\mu$ , is defined by  $F_\mu(x) = \mu((-\infty, x])$ , for all  $x \in \mathbb{R}$ .

- In order to avoid the use of too many notations, we shall define the binary operation  $\boxplus$  on the set of probability measures on the real line with the same symbol as the operation  $\boxplus$  on the set of cumulative distribution functions. For all pair  $\mu, \nu$  of probability measures on the real line,  $\mu \boxplus \nu$  is defined by:

$$F_{\mu \boxplus \nu} = F_{\mu} \boxplus F_{\nu} \quad ( = \max(0, F_{\mu}(t) + F_{\nu}(t) - 1) ).$$

- We shall also define the operator  $\Lambda^{\vee}$  on the set of probability measures on the real line with the same symbol as the function  $\Lambda^{\vee}(x) = \max(0, 1 + \log x)$  on  $[0, +\infty)$ : for all probability measure  $\mu$  on the real line,  $\Lambda^{\vee}(\mu)$  is the probability measure with cumulative distribution function:

$$F_{\Lambda^{\vee}(\mu)}(t) = \Lambda^{\vee}(F_{\mu}(t)) \quad ( = \max(0, 1 + \log F_{\mu}(t)) ).$$

**2.3. for independent Hermitian random variables.** Let  $d \in \mathbb{N}^*$ . The spectral order introduced by Olson is also defined on the space of  $d \times d$  Hermitian matrices:

**Definition (cf [9]):**

- If  $P$  and  $Q$  are two Hermitian projectors, then  $P \vee Q$  is the Hermitian projector on  $\mathfrak{S}(P) + \mathfrak{S}(Q)$ .
- If  $A$  and  $B$  are two Hermitian matrices, with resolutions of identity  $E_A$  and  $E_B$ , then  $A \vee B$  is the Hermitian matrix with resolution of identity given by  $E_A \vee E_B$ :  $A \vee B$  is the only Hermitian matrix  $H$  such that for all real number  $t$ ,

$$\chi_{(t, \infty)}(H) = \chi_{(t, \infty)}(A) \vee \chi_{(t, \infty)}(B),$$

where  $\chi_{(t, \infty)}$  denotes the indicator function of the interval  $(t, \infty)$  and is applied to Hermitian matrices *via* the functional calculus.

**Remark:** concretely,  $A \vee B$  can be defined in the following way: let  $\{\lambda_1 > \lambda_2 > \dots > \lambda_p\}$  be the union of the spectrums of  $A$  and of  $B$ . For  $i = 1, \dots, p$ , let us define  $E_i = \ker(A - \lambda_i I) + \ker(B - \lambda_i I)$ . Then  $A \vee B$  is the Hermitian matrix with:

- largest eigenvalue  $\lambda_1$  with corresponding eigenspace  $E_1$ ,
- second largest eigenvalue  $\lambda_2$  with corresponding eigenspace  $E_2 \cap (E_1^{\perp})$  (which may be reduced to  $\{0\}$ ),
- third largest eigenvalue  $\lambda_3$  with corresponding eigenspace  $E_3 \cap ((E_1 + E_2)^{\perp})$ ,
- ⋮
- $i$ th largest eigenvalue  $\lambda_i$  with corresponding eigenspace  $E_i \cap ((E_1 + \dots + E_{i-1})^{\perp})$ ,
- ⋮

with the convention that the process ends at the first  $i$  such that  $(E_1 + \dots + E_i)^{\perp} = \{0\}$ .

The following proposition resumes the characteristics of the operation  $\vee$  on unitarily invariant random matrices. Recall that the *empirical spectral law* of a matrix is the uniform measure on the multiset of its eigenvalues counted by multiplicity.

**Proposition 2.1.** *If  $A$  and  $B$  are two  $d \times d$  Hermitian random independent matrices, whose laws are invariant by conjugation with any unitary matrix, then*

- $A \vee B$  is an Hermitian random matrix, whose law is invariant under the conjugation with one unitary matrix;
- the  $d$  eigenvalues of  $A \vee B$  are the  $d$  largest of the  $2d$  eigenvalues of  $A$  and  $B$  (counted with multiplicity);
- the cumulative distribution functions  $F_A, F_B$  and  $F_{A \vee B}$  of the respective empirical spectral laws of  $A, B$  and  $A \vee B$  are almost surely linked by the relation

$$(2) \quad F_{A \vee B} = \max(0, F_A + F_B - 1).$$

**Remark 2.2.** *It is noteworthy that formulae (1) and (2) are identical, showing that the matricial and free upper extremal convolutions are somehow the same operation. As free random variables can be approximated by independent unitarily invariant random matrices, the rather trivial result (2) provides an intuitive proof of formula (1).*

### 3. THE MATRICIAL INTERPOLATION

The aim of this section is to introduce random matrices giving quite a natural interpretation of the mapping  $\Lambda^\vee$ . We simply follow the same approach as we did for the Bercovici-Pata bijection in [3] and [5].

Let  $F_\mu$  denote its cumulative distribution function for  $\mu$  probability measure on the real line. For all integer  $k \geq 1$ ,  $F_\mu^{\frac{1}{k}}$  is the cumulative distribution function of a probability measure on the real line, which will be denoted by  $\mu^{\frac{1}{k}}$ . It is clear that if  $X_1, \dots, X_k$  are independent random variables distributed according to  $\mu^{\frac{1}{k}}$ ,  $\max\{X_1, \dots, X_k\}$  should be distributed according to  $\mu$ .

Let us consider  $d, k \geq 1$ . Let  $X_1, \dots, X_d$  be independent random variables distributed according to  $\mu^{\frac{1}{k}}$  and  $U$  an unitary, Haar distributed,  $d \times d$  random matrix independent of  $X_1, \dots, X_d$ . Define the random matrix  $M$  by

$$(3) \quad M = U \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_d \end{pmatrix} U^*$$

Let  $M_1, \dots, M_k$  be independent replicas of  $M$ .

**Theorem 3.1.** *As  $k$  tends to infinity, the distribution of  $M_1 \vee \dots \vee M_k$  converges to a probability measure  $\Lambda_d^\vee(\mu)$  on the space of  $d \times d$  Hermitian matrices. This measure is invariant under the action of the unitary group by conjugation and in the case where  $\mu$  has no atom, under  $\Lambda_d^\vee(\mu)$ , the joint distribution of the ranked eigenvalues is absolutely continuous with respect to  $\mu^{\otimes d}$ , with density*

$$(4) \quad d^d 1_{t_1 \geq \dots \geq t_d} \prod_{i=1}^d \frac{F_\mu(t_d)}{F_\mu(t_i)}$$

at any point  $(t_1, \dots, t_d) \in \mathbb{R}^d$  such that for all  $i = 1, \dots, d$ ,  $F_\mu(t_i) \neq 0$  (the density being set to zero anywhere else).

Moreover, if  $\nu$  is another probability measure on the real line and  $A, B$  are independent random matrices respectively distributed according to  $\Lambda_d^\vee(\mu)$ ,  $\Lambda_d^\vee(\nu)$ , then  $A \vee B$  is distributed according to  $\Lambda_d^\vee(\rho)$ , where  $\rho$  is the probability measure on the real line such that  $F_\rho = F_\mu F_\nu$ .

**Corollary 3.2.** *With the preceding notations:*

- (i)  $\Lambda_d^\vee(\mu)$  is max-infinitely divisible.
- (ii) If  $\mu$  is max-stable, then so is  $\Lambda_d^\vee(\mu)$ .

In the following theorem, we are interested in the limit of the spectral measure of a  $\Lambda_d^\vee(\mu)$ -distributed random matrix, when its dimension  $d$  tends to infinity.

**Theorem 3.3.** *Let  $\mu$  be a probability measure on the real line. For all  $d \geq 1$ , let  $M_d$  a random matrix with law  $\Lambda_d^\vee(\mu)$ ,  $\lambda_1, \dots, \lambda_d$  its eigenvalues and  $\hat{\mu}_d = \frac{1}{d} \sum_{i=1}^d \delta_{\lambda_i}$  its empirical spectral law. Then, when  $d$  goes to infinity,  $\hat{\mu}_d$  converges weakly almost surely to  $\Lambda^\vee(\mu)$ .*

In [2], Ben Arous and Voiculescu establish that  $\Lambda^\vee$  provides a remarkable correspondence between classical max-stable laws and free max-stable laws. Moreover, the corresponding laws are characterized by their same domain of stability.

The preceding theorems give a kind of intuitive matricial explanation on this result. As an example, the fact that  $\Lambda^\vee$  maps any classical max-stable law to a free one is a direct consequence of (ii) of Corollary 3.2 and of formulae (1) and (2). An intuitive explanation of the results of Ben Arous and Voiculescu concerning the domains of stability is detailed now. Let us consider a probability measure  $\mu$  on the real line, and for each  $d \geq 1$  and for each  $k \geq 1$ , a family  $M_1(d, k), \dots, M_k(d, k)$  of independent replicas of the random matrix  $M$  of (3).

On the one hand, according to theorem 3.1, we have the following convergence in law as  $k$  tends to infinity,  $d$  being fixed:

$$(5) \quad M_1(d, k) \vee \dots \vee M_k(d, k) \xrightarrow[k \rightarrow \infty]{} \Lambda_d^\vee(\mu).$$

Letting now  $d$  tend to infinity, we have also the following almost sure convergence (cf theorem 3.3):

$$(6) \quad \text{Empirical Spectral Law}(M_d) \xrightarrow[d \rightarrow \infty]{} \Lambda^\vee(\mu).$$

where  $M_d$  is a  $\Lambda_d^\vee(\mu)$ -distributed random matrix.

On the other hand, according to the law of large numbers, the empirical spectral law of  $M_i(d, k)$  converges almost surely to  $\mu^{\frac{1}{k}}$  for each  $i = 1, \dots, k$ , as  $d$  tends first to infinity,  $k$  being fixed. Therefore, the equality between formulae (1) and (2) implies the following almost sure convergence:

$$(7) \quad \text{Empirical Spectral Law}(M_1(d, k) \vee \dots \vee M_k(d, k)) \xrightarrow[d \rightarrow \infty]{} \mu^{\frac{1}{k}} \boxtimes \dots \boxtimes \mu^{\frac{1}{k}}.$$

Joining (5), (6), (7), we get the following diagram

$$\begin{array}{ccc} M_1(d, k) \vee \dots \vee M_k(d, k) & \xrightarrow[k \rightarrow \infty]{} & \Lambda_d^\vee(\mu) \\ \downarrow d \rightarrow \infty & & \downarrow d \rightarrow \infty \\ \text{Empirical Spectral Law:} & & \text{Empirical Spectral Law:} \\ \mu^{\frac{1}{k}} \boxtimes \dots \boxtimes \mu^{\frac{1}{k}} & & \Lambda^\vee(\mu) \end{array}$$

The fact that  $\mu^{\frac{1}{k}}$ , in the  $\vee$ -domain of attraction of  $\mu$ , also belongs to the  $\boxtimes$ -domain of attraction of  $\Lambda^\vee(\mu)$ , only means that one can add the edge  $\xrightarrow[k \rightarrow \infty]$  between both bottom vertices of the diagram, i.e. that the operations  $k \rightarrow \infty$  and  $d \rightarrow \infty$  are commutative, which is quite expected.

## 4. PROOFS

**4.1. Preliminary results.** Both of the theorems will be proved first for measures which are absolutely continuous with respect to the Lebesgue measure, and then extended to all measures by approximation. In this context, the appropriate approximation tool is given by the following lemma.

**Lemma 4.1.** *Let us define, for  $\mu$  probability measure on the real line, the function  $F_\mu^{<-1>}$  on  $(0, 1)$  by  $F_\mu^{<-1>}(u) = \min\{x \in \mathbb{R}; F_\mu(x) \geq u\}$ .*

(i) *For  $\mu, \nu$  probability measures on the real line, we have the following equalities (between quantities which can be infinite)*

$$\inf\{\varepsilon > 0; F_\nu(\bullet - \varepsilon) \leq F_\mu(\bullet) \leq F_\nu(\bullet + \varepsilon)\} = \|F_\mu^{<-1>} - F_\nu^{<-1>}\|_\infty = \inf \|X - Y\|_\infty,$$

where the infimum in the third term is taken on pairs  $(X, Y)$  of random variables defined on a same probability space with respective distributions  $\mu, \nu$ .

(ii) *For any probability measure  $\mu$  on the real line, for any  $\varepsilon > 0$ , there exists a probability measure  $\mu_\varepsilon$  on the real line such that  $F_{\mu_\varepsilon}$  is smooth and*

$$F_{\mu_\varepsilon}(\bullet - \varepsilon) \leq F_\mu(\bullet) \leq F_{\mu_\varepsilon}(\bullet + \varepsilon).$$

**Proof.** (i) We prove these equalities by cyclic majorizations.

- Let us prove first

$$\inf\{\varepsilon > 0; \forall x \in \mathbb{R}, F_\nu(x - \varepsilon) \leq F_\mu(x) \leq F_\nu(x + \varepsilon)\} \geq \|F_\mu^{<-1>} - F_\nu^{<-1>}\|_\infty.$$

Let us consider  $\varepsilon > 0$  such that  $F_\nu(\bullet - \varepsilon) \leq F_\mu(\bullet) \leq F_\nu(\bullet + \varepsilon)$ . Since this is equivalent to  $F_\mu(\bullet - \varepsilon) \leq F_\nu(\bullet) \leq F_\mu(\bullet + \varepsilon)$ , this inequation is symmetric in  $\mu$  and  $\nu$ . Hence, it suffices to prove that  $F_\nu^{<-1>}(u) - F_\mu^{<-1>}(u) \leq \varepsilon$  for all  $u \in (0, 1)$ . We have:

$$F_\nu(F_\mu^{<-1>}(u) + \varepsilon) \geq F_\mu(F_\mu^{<-1>}(u)) \geq u,$$

hence,  $F_\nu^{<-1>}(u) \leq F_\mu^{<-1>}(u) + \varepsilon$ .

- The inequality

$$\|F_\mu^{<-1>} - F_\nu^{<-1>}\|_\infty \geq \inf \|X - Y\|_\infty$$

is due to the fact that for  $U$  random variable with uniform distribution on  $(0, 1)$ ,  $X := F_\mu^{<-1>}(U)$ ,  $Y := F_\nu^{<-1>}(U)$  are respectively distributed according to  $\mu, \nu$ .

- To conclude, it suffices to prove

$$\inf \|X - Y\|_\infty \geq \inf\{\varepsilon > 0; F_\nu(\bullet - \varepsilon) \leq F_\mu(\bullet) \leq F_\nu(\bullet + \varepsilon)\}.$$

So let us consider a pair  $(X, Y)$  of random variables defined on the same probability space with respective distributions  $\mu, \nu$ . Consider  $\varepsilon > 0$  such that  $|X - Y| \leq \varepsilon$  uniformly on the probability space. Then, for all real number  $x$ ,

$$\mathbb{P}(Y \leq x - \varepsilon) \leq \mathbb{P}(X \leq x) \leq \mathbb{P}(Y \leq x + \varepsilon).$$

(ii) Let  $\varepsilon > 0$ , and  $F$  a smooth non decreasing function on the real line such that for all  $k \in \mathbb{Z}$ ,  $F(k\varepsilon) = F_\mu(k\varepsilon)$ . Such a function exists obviously, as an example, one can consider  $F(x) = \int_{-\infty}^x f(t) dt$ , for

$$f(t) = \sum_{k \in \mathbb{Z}} [F(k\varepsilon) - F((k-1)\varepsilon)] \varphi(t - k\varepsilon)$$

with  $\varphi$  smooth non negative function on the real line, with support contained in  $[-\varepsilon, 0]$  satisfying  $\int_{-\varepsilon}^0 \varphi(t) dt = 1$ . Hence  $F$  is a càdlàg non decreasing function on the real line such that  $F(x)$  tends to zero (resp. one) as  $x$  tends to  $-\infty$  (resp.  $+\infty$ ). It follows that  $F = F_{\mu_\varepsilon}$  for a certain probability measure  $\mu_\varepsilon$  on the real line. Moreover, for all real number  $x$ , if  $k \in \mathbb{Z}$  is such that  $k\varepsilon \leq x < (k+1)\varepsilon$ , then, one has:

$$F_{\mu_\varepsilon}(x - \varepsilon) \leq F_{\mu_\varepsilon}(k\varepsilon) = F_\mu(k\varepsilon) \leq F_\mu(x) \leq F_\mu((k+1)\varepsilon) = F_{\mu_\varepsilon}((k+1)\varepsilon) \leq F_{\mu_\varepsilon}(x + \varepsilon).$$

□

For any positive integer  $d$ , let us define  $\max_d$  to be the function from  $\cup_{n \geq d} \mathbb{R}^n$  to  $\mathbb{R}^d$  which maps any vector  $x$  of  $\mathbb{R}^n$ ,  $n \geq d$ , to the vector of the  $d$ th largest coordinates of  $x$  ranked in decreasing order. The following property is a basic result, see for instance [7].

**Property 4.2.** *Let  $F$  be the cumulative distribution function of a probability measure  $\mu$  with no atom on the real line. Let  $X_1, \dots, X_n$  be independent random variables with law  $\mu$ . The distribution of  $\max_d(X_1, \dots, X_n)$  has density*

$$(t_1, \dots, t_d) \mapsto \frac{n!}{(n-d)!} 1_{t_1 \geq \dots \geq t_d} F(t_d)^{n-d}$$

with respect to  $\mu^{\otimes d}$ .

Before stating the next proposition, we shall recall that for  $\mu$  be a probability measure on the real line and  $k$  positive integer,  $\mu^{\frac{1}{k}}$  is the probability measure on the real line with cumulative distribution function  $F_\mu^{\frac{1}{k}}$ , i.e. the law  $m$  such that for  $X_1, \dots, X_k$  independent random variables with law  $m$ ,  $\max(X_1, \dots, X_k)$  has law  $\mu$ .

**Proposition 4.3.** (i) Let  $\mu$  be a probability measure on the real line. For all integer  $d \geq 1$ , the push forward of the probability measure  $(\mu^{\frac{1}{k}})^{\otimes kd}$  on  $\mathbb{R}^{kd}$  by the function  $\max_d$  converges weakly, as the integer  $k$  tends to infinity, to a probability measure on  $\mathbb{R}^d$  denoted by  $\mu_d$ .

(ii) When  $\mu$  has no atom,  $\mu_d$  is absolutely continuous with respect to  $\mu^{\otimes d}$ , with density

$$d^d 1_{t_1 \geq \dots \geq t_d} \prod_{i=1}^d \frac{F_\mu(t_d)}{F_\mu(t_i)}$$

at any point  $(t_1, \dots, t_d) \in \mathbb{R}^d$  such that for all  $i = 1, \dots, d$ ,  $F_\mu(t_i) \neq 0$ .

(iii) Let us endow  $\mathbb{R}^d$  with the norm  $\|x\| = \max_i |x_i|$ . Then, for any probability measures  $\mu, \nu$  on the real line,

$$\inf\{\|V - W\|_\infty; V, W \text{ random vectors defined on the same space with respective laws } \mu_d, \nu_d\} \\ \leq \inf\{\|X - Y\|_\infty; X, Y \text{ random variables defined on the same space with respective laws } \mu, \nu\}.$$

(iv) For any pair  $\mu, \nu$  of probability measures on the real line, if  $\rho$  is the probability measure on the real line such that  $F_\rho = F_\mu F_\nu$ , for all  $d \geq 1$ ,  $\rho_d$  is the push-forward, by the function  $\max_d$ , of the probability measure  $\mu_d \otimes \nu_d$  on  $\mathbb{R}^{2d}$ .

**Proof.** Let  $k, d \in \mathbb{N}^*$ , and  $\mu$  a probability measure with no atom. From property 4.2, we infer first that  $\mu^{\frac{1}{k}} = \frac{1}{k} F_\mu^{\frac{1}{k}-1} d\mu$ , and then that the distribution  $\max_d((\mu^{\frac{1}{k}})^{\otimes kd})$  has a density with respect to  $\mu^{\otimes d}$  equal to

$$(t_1, \dots, t_d) \mapsto \frac{(dk)!}{k^d (dk-d)!} 1_{t_1 \geq \dots \geq t_d} \prod_{i=1}^d \frac{F_\mu(t_d)^{1-\frac{1}{k}}}{F_\mu(t_i)^{1-\frac{1}{k}}}$$

at any point  $(t_1, \dots, t_d) \in \mathbb{R}^d$  such that for all  $i = 1, \dots, d$ ,  $F_\mu(t_i) \neq 0$  (the density can obviously be set to zero anywhere else). As  $k$  tends to infinity, this density stays uniformly bounded and converges pointwise to

$$d^d 1_{t_1 \geq \dots \geq t_d} \prod_{i=1}^d \frac{F_\mu(t_d)}{F_\mu(t_i)},$$

hence (i) for probability measures with no atom and (ii) are proved.

Now, let us complete the proof of (i). Let us consider a probability measure  $\mu$  and, for all positive integer  $k$ , a family  $X(k, 1), \dots, X(k, kd)$  of independent random variables with law  $\mu^{\frac{1}{k}}$ . To prove (i), by Theorem 1.12.4 of [10], it suffices to prove that for any real bounded Lipschitz function  $f$  on  $\mathbb{R}^d$ , the sequence  $\mathbb{E}(f(\max_d(X(k, 1), \dots, X(k, kd))))$  converges as  $k$  tends to infinity. So let us fix such a function. We shall prove that the previous sequence is a Cauchy sequence. Let us fix  $\varepsilon > 0$ . Let us consider  $\mu_\varepsilon$  as in (ii) of lemma 4.1. Note that for all positive integers  $k$ , we also have:

$$F_{\mu_\varepsilon}^{\frac{1}{k}}(\bullet - \varepsilon) \leq F_\mu^{\frac{1}{k}}(\bullet) \leq F_{\mu_\varepsilon}^{\frac{1}{k}}(\bullet + \varepsilon).$$

So, based on (i) of lemma 4.1, we may suppose that for all  $k \geq 1$ , on the probability space where the  $X(k, i)$ 's are defined, there is a family

$$Y(k, 1), \dots, Y(k, kd)$$

of random variables such that

- (a) the  $Y(k, i)$ 's are independent and distributed according to  $\mu_\varepsilon^{\frac{1}{k}}$ ,
- (b) for all  $i = 1, \dots, kd$ ,  $|X(k, i) - Y(k, i)| \leq 2\varepsilon$  almost surely.

Note that  $\mu_\varepsilon$  has no atom, hence by (a) and what we just proved, the sequence

$$\mathbb{E}(f(\max_d(Y(k, 1), \dots, Y(k, kd))))$$



is a Cauchy sequence. Note also that by (b), if  $C$  is a Lipschitz constant for  $f$  with respect to the norm  $\|x\| = \max_i |x_i|$ , then for all  $k$ ,

$$|\mathbb{E}(f(\max_d(X(k, 1), \dots, X(k, kd)))) - \mathbb{E}(f(\max_d(Y(k, 1), \dots, Y(k, kd))))| \leq 2C\varepsilon.$$

Hence there is  $k_0 \geq 1$  such that for all  $k, k' \geq k_0$ ,

$$|\mathbb{E}(f(\max_d(X(k, 1), \dots, X(k, kd)))) - \mathbb{E}(f(\max_d(X(k', 1), \dots, X(k', k'd))))| \leq (4C + 1)\varepsilon.$$

Thus we have proved (i) for any probability measure  $\mu$ .

Now, let us prove (iii). Consider  $\mu, \nu$  probability measures on the real line. Note that by part (i) of lemma 4.1, it suffices to prove that for all positive  $\varepsilon$  such that

$$F_\nu(\bullet - \varepsilon) \leq F_\mu(\bullet) \leq F_\nu(\bullet + \varepsilon),$$

for all  $\alpha > \varepsilon$ , there exists a pair  $V, W$  of random vectors defined on the same space with respective laws  $\mu_d, \nu_d$  such that  $\|V - W\|_\infty \leq \alpha$ . Let us consider such a positive  $\varepsilon$  and  $\alpha > \varepsilon$ . For all  $k \geq 1$ , we also have:

$$F_\nu^{\frac{1}{k}}(\bullet - \varepsilon) \leq F_\mu^{\frac{1}{k}}(\bullet) \leq F_\nu^{\frac{1}{k}}(\bullet + \varepsilon).$$

So, according to part (i) of lemma 4.1, we shall consider, for all  $k \geq 1$ , a family

$$X(k, 1), \dots, X(k, kd), Y(k, 1), \dots, Y(k, kd)$$

of random variables defined on the same space such that

- (a) the  $X(k, i)$ 's are independent and distributed according to  $\mu^{\frac{1}{k}}$ ,
- (b) the  $Y(k, i)$ 's are independent and distributed according to  $\nu^{\frac{1}{k}}$ ,
- (c) for all  $i = 1, \dots, kd$ ,  $|X(k, i) - Y(k, i)| \leq \alpha$  almost surely.

Let, for all  $k \geq 1$ ,  $\tau_k$  be the joint law, on  $\mathbb{R}^{2d}$ , of the random vector

$$(\max_d(X(k, 1), \dots, X(k, kd)), \max_d(Y(k, 1), \dots, Y(k, kd))).$$

The law of  $\max_d(X(k, 1), \dots, X(k, kd))$  (resp. of  $\max_d(Y(k, 1), \dots, Y(k, kd))$ ) converges weakly to  $\mu_d$  (resp. to  $\nu_d$ ) as  $k$  tends to infinity. It follows that the sequence  $\tau_k$  is tight, i.e. relatively compact for the topology of weak convergence (Theorem 6.1 of [4]). Let  $\tau$  be the weak limit of a subsequence of  $\tau_k$ . Let  $(V, W)$  be a  $\tau$ -distributed random vector of  $\mathbb{R}^d \times \mathbb{R}^d$ . Then the law of  $V$  (resp. of  $W$ ) is  $\mu_d$  (resp.  $\nu_d$ ). Moreover, it is easy to notice that for all  $k$ ,  $\tau_k$  is supported by  $\{(x, y); x \in \mathbb{R}^d, y \in \mathbb{R}^d, \|x - y\| \leq \alpha\}$ . Indeed, let  $\sigma, \tau \in S_n$  such that  $X(k, \sigma(1)) \geq \dots \geq X(k, \sigma(kd))$  and  $Y(k, \tau(1)) \geq \dots \geq Y(k, \tau(kd))$ ; then for any  $i = 1, \dots, kd$ ,

$$X(k, \sigma(i)) = \max_{V \subset \{1, \dots, kd\}, \#V=i} \min(X(k, j), j \in V) \geq \max_{V \subset \{1, \dots, kd\}, \#V=i} \min(Y(k, j) - \alpha, j \in V) = Y(k, \tau(i)) - \alpha \text{ a.s.}$$

By symmetry, this proves  $|X(k, \sigma(i)) - Y(k, \tau(i))| \leq \alpha$  a.s., and this implies

$$\|\max_d(X(k, 1), \dots, X(k, kd)) - \max_d(Y(k, 1), \dots, Y(k, kd))\| = \max_{i=1, \dots, d} |X(k, \sigma(i)) - Y(k, \tau(i))| \leq \alpha \text{ a.s.}$$

Letting  $k$  go to infinity, this establishes the inequality  $\|V - W\| \leq \alpha$  almost surely.

To prove (iv), consider for any  $k \geq 1$ ,

$$X(k, 1), \dots, X(k, kd), Y(k, 1), \dots, Y(k, kd)$$

is a family of independent random variables such that for all  $i = 1, \dots, kd$ ,  $X(k, i)$  (resp.  $Y(k, i)$ ) is distributed according to  $\mu^{\frac{1}{k}}$  (resp.  $\nu^{\frac{1}{k}}$ ). Then  $X(k, 1) \vee Y(k, 1), \dots, X(k, kd) \vee Y(k, kd)$  are i.i.d. with law  $\rho^{\frac{1}{k}}$ , and consequently  $\mu_d, \nu_d$  and  $\rho_d$  are the respective weak limit distributions, as  $k$  tends to infinity, of

$$\max_d[X(k, 1), \dots, X(k, kd)], \quad \max_d[Y(k, 1), \dots, Y(k, kd)]$$

$$(8) \quad \text{and} \quad \max_d[].$$

Hence, by continuity and commutativity of the maximum operations,  $\max_d(\mu_d \otimes \nu_d)$  is the weak limit, as  $k$  tends to infinity, of the distribution of

$$(9) \quad \max_d[X(k, 1), \dots, X(k, kd), Y(k, 1), \dots, Y(k, kd)].$$

To conclude, we shall establish that the distributions of the vectors of (8) and (9) have the same weak limits as  $k$  tends to infinity. Obviously, it reduces to prove that the probability of the event

$$(10) \quad \{\max_d[X(k, 1) \vee Y(k, 1), \dots, X(k, kd) \vee Y(k, kd)] \neq \max_d[X(k, 1), \dots, X(k, kd), Y(k, 1), \dots, Y(k, kd)]\}$$

tends to zero as  $k$  tends to infinity. Note that this event is equivalent to the fact that there is  $i \in \{1, \dots, kd\}$  such that  $X(k, i)$  and  $Y(k, i)$  are both coordinates of

$$\max_d[X(k, 1), \dots, X(k, kd), Y(k, 1), \dots, Y(k, kd)]$$

and are not equal to any of the others  $X(k, j)$ 's and  $Y(k, l)$ 's. Hence, if one denotes by  $I(k)$  (resp.  $J(k)$ ) the set of  $i$ 's in  $\{1, \dots, kd\}$  such that  $X(k, i)$  (resp.  $Y(k, i)$ ) is one of the coordinates of

$$\max_d[X(k, 1), \dots, X(k, kd)] \quad (\text{resp. of } \max_d[Y(k, 1), \dots, Y(k, kd)]),$$

the event of (10) is contained in the event  $I(k) \cap J(k) \neq \emptyset$ , which probability is equal to  $\binom{kd}{d}^{-1}$ , since we may choose for  $I(k)$  and  $J(k)$  to be independent and to have uniform distribution on the set of subsets of  $\{1, \dots, kd\}$  with cardinality  $d$ . Thus, (iv) is proved.  $\square$

**4.2. Proof of proposition 2.1.** The first statement is immediate. The third one follows immediately from the second one. To establish the second statement, it suffices to prove that almost surely, for any  $t \in \mathbb{R}$ ,  $\text{Im}(\chi_{(t, \infty)}(A)) + \text{Im}(\chi_{(t, \infty)}(B))$  has the maximal dimension conditionally to the dimensions of  $\text{Im}(\chi_{(t, \infty)}(A))$  and  $\text{Im}(\chi_{(t, \infty)}(B))$ , i.e. that

$$(11) \quad \dim(\text{Im}(\chi_{(t, \infty)}(A)) + \text{Im}(\chi_{(t, \infty)}(B))) = \min(d, \dim(\text{Im}(\chi_{(t, \infty)}(A))) + \dim(\text{Im}(\chi_{(t, \infty)}(B)))) \text{ a.s.}$$

Let  $p = \dim(\text{Im}(\chi_{(t, \infty)}(A)))$ ,  $q = \dim(\text{Im}(\chi_{(t, \infty)}(B)))$ . Due to the unitarily invariance of  $A$  (resp.  $B$ ), the law of  $\text{Im}(\chi_{(t, \infty)}(A))$  (resp.  $\text{Im}(\chi_{(t, \infty)}(B))$ ) is also invariant under the action of any unitary matrix, hence is uniform on the set of all subspaces of dimension  $p$  (resp.  $q$ ). Moreover,  $\text{Im}(\chi_{(t, \infty)}(A))$  and  $\text{Im}(\chi_{(t, \infty)}(B))$  are independent.

Let us consider a family  $(g_{i,j}, i \in \{1, \dots, d\}, j \geq 1)$  of independent complex standard Gaussian random variables. The matrix  $(g_{i,j}, i, j \in \{1, \dots, d\})$  is distributed w.r.t. the Circular Unitary Ensemble. Hence, it is unitarily invariant and a.s. invertible. Therefore, the subspace generated by vectors  $(g_{i,j}, i \in \{1, \dots, d\})_{j=1}^p$  (resp.  $(g_{i,j}, i \in \{1, \dots, d\})_{j=p+1}^{p+q}$ ,  $(g_{i,j}, i \in \{1, \dots, d\})_{j=1}^{p+q}$ ) has dimension  $p$  (resp.  $q$ ,  $\min(d, p+q)$ ) a.s., and its law is uniform on the set of all subspaces of same dimension. This proves (11).

**Remark 4.4.** Note that the proof of the second statement could also have been deduced from Theorem 2.2 of [6].

**4.3. Proof of theorem 3.1.** Following proposition 2.1, if we denote  $M^{(k)} = M_1 \vee \dots \vee M_k$ , then  $M^{(k)}$  is an Hermitian random matrix, whose law is invariant under conjugation with any unitary matrix and whose eigenvalues  $\lambda_1^{(k)} \geq \dots \geq \lambda_d^{(k)}$  are equal to the  $d$  largest of the  $dk$  eigenvalues of  $M_1, \dots, M_k$  which are all independent with same distribution  $\mu^{1/k}$ . Therefore, by theorem 4.3.5 of [8], there exists  $U$  unitary, Haar distributed and independent of the vector  $(\lambda_1^{(k)}, \dots, \lambda_d^{(k)})$ , such that:

$$M^{(k)} = U \begin{pmatrix} \lambda_1^{(k)} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_d^{(k)} \end{pmatrix} U^*.$$

Note that the distribution of the vector  $(\lambda_1^{(k)}, \dots, \lambda_d^{(k)})$  is the push forward of the probability measure  $(\mu^{\frac{1}{k}})^{\otimes kd}$  on  $\mathbb{R}^{kd}$  by the function  $\max_d$ . Hence, the first part of the theorem follows from (i) and (ii) of proposition 4.3, and the second part follows from (iv) of the same proposition.

**4.4. Proof of corollary 3.2.** Both assertions are easy consequences of the last part of theorem 3.1.

- (i) Let  $p \geq 1$ . By definition,  $F_\mu = \left(F_{\mu^{\frac{1}{p}}}\right)^p$ . Therefore  $\Lambda_d^\vee(F_\mu)$  is the distribution of  $H_1 \vee \dots \vee H_p$ , with  $H_1, \dots, H_p$  i.i.d. with law  $\Lambda_d^\vee(F_{\mu^{\frac{1}{p}}})$ . This means that  $\Lambda_d^\vee(F_\mu)$  is max-infinitely divisible.
- (ii) Let  $p \geq 1$ . Since  $\mu$  is max-stable, there exists  $a_p > 0$ ,  $b_p \in \mathbb{R}$  such that  $\mu$  is the distribution of  $(a_p X_1 + b_p) \vee \dots \vee (a_p X_p + b_p)$ , where  $X_1, \dots, X_p$  are i.i.d. with law  $\mu$ . Let  $H_1, \dots, H_p$  be i.i.d. with law  $\Lambda_d^\vee(\mu)$ . It is quite immediate from the construction that the image of the distribution of  $a_p X_i + b_p$  is the law of  $a_p H_i + b_p I_d$  where  $I_d$  denotes the identity matrix. Therefore,  $\Lambda_d^\vee(\mu)$  is the distribution of  $(a_p H_1 + b_p I_d) \vee \dots \vee (a_p H_p + b_p I_d)$ . This proves that  $\Lambda_d^\vee(\mu)$  is max-stable.

**4.5. Proof of theorem 3.3.** Suppose first that  $F_\mu$  is smooth on the real line. It implies that  $\mu$  is absolutely continuous with respect to the Lebesgue measure and that  $d\mu(x) = F_\mu'(x) dx$ . Since  $F_\mu$  is continuous on the real line, the almost sure weak convergence of the sequence of random probability measures  $\hat{\mu}_d$  to  $\Lambda^\vee(\mu)$  is equivalent to the fact the for all rational number  $t$ ,  $\hat{\mu}_d((-\infty, t])$  converges almost surely to  $\max(0, 1 + \log F_\mu(t))$ . We shall prove that for all  $t \in \mathbb{R}$ ,  $\hat{\mu}_d((-\infty, t])$  converges almost surely to  $\max(0, 1 + \log F_\mu(t))$ . Let us fix a real number  $t$ .

The first step is to compute  $\mathbb{E}[e^{\lambda \hat{\mu}_d((-\infty, t])}]$ : with the notations  $\lambda_0 = +\infty$ ,  $\lambda_{d+1} = -\infty$ , we have

$$\begin{aligned} \mathbb{E}[e^{\lambda \hat{\mu}_d((-\infty, t])}] &= \mathbb{E}\left[\sum_{p=0}^d e^{\frac{\lambda(d-p)}{d}} 1_{\lambda_{p+1} \leq t < \lambda_p}\right] \\ &= \sum_{p=0}^d e^{\frac{\lambda(d-p)}{d}} \mathbb{P}(\lambda_{p+1} \leq t < \lambda_p) \end{aligned}$$

Now, note that for all real number  $T$  such that  $F(T) > 0$ , for all integer non negative  $m$ ,

$$\int_{(T, \infty)} (\log F_\mu(t))^m \frac{d\mu(t)}{F_\mu(t)} = -\frac{(\log F_\mu(T))^{m+1}}{m+1} \quad \text{and} \quad \int_{(-\infty, T]} F_\mu(t)^m d\mu(t) = \frac{F_\mu(T)^{m+1}}{m+1}.$$

Hence for  $p = 0, \dots, d-1$ ,

$$\mathbb{P}(\lambda_{p+1} \leq t < \lambda_p) = \int_{t \in \mathbb{R}^d} d^d 1_{t_{p+1} \geq \dots \geq t_d} 1_{t < t_p \leq \dots \leq t_1} \prod_{i=1}^d \frac{F_\mu(t_d)}{F_\mu(t_i)} d\mu(t_i) = \frac{d^p}{p!} F_\mu(t)^d (-\log F_\mu(t))^p.$$

And for  $p = d$ :

$$\mathbb{P}(t < \lambda_d) = \frac{d^d}{(d-1)!} \int_t^{+\infty} F_\mu(t_d)^d (-\log F_\mu(t_d))^{d-1} d\mu(t_d) = \int_0^{-\log F_\mu(t)} \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du.$$

Thus, using the fact that  $\sum_{p=0}^{d-1} \frac{x^p}{p!} = e^x \int_x^{+\infty} \frac{1}{(d-1)!} u^{d-1} e^{-u} du$ , we get:

$$\begin{aligned}
& \mathbb{E} \left[ e^{\lambda \hat{\mu}_d((-\infty, t])} \right] \\
&= e^{\lambda F_\mu(t)^d} \sum_{p=0}^{d-1} \frac{(-de^{-\frac{\lambda}{d}} \log F_\mu(t))^p}{p!} + \int_0^{-\log F_\mu(t)} \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du \\
&= e^{\lambda F_\mu(t)^d} e^{-de^{-\frac{\lambda}{d}} \log F_\mu(t)} \int_{-de^{-\frac{\lambda}{d}} \log F_\mu(t)}^{+\infty} \frac{1}{(d-1)!} u^{d-1} e^{-u} du + \int_0^{-\log F_\mu(t)} \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du \\
&= e^{\lambda + d \left(1 - e^{-\frac{\lambda}{d}}\right) \log F_\mu(t)} \int_{-e^{-\frac{\lambda}{d}} \log F_\mu(t)}^{+\infty} \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du + \int_0^{-\log F_\mu(t)} \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du.
\end{aligned}$$

The following inequalities will be useful, and they can be established from the Chernov inequality or proved directly:

(12)

$$\forall x \in [0, 1), \int_0^x \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du \leq x^d e^{d(1-x)}, \quad \forall x > 1, \int_x^{+\infty} \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du \leq x^d e^{d(1-x)}.$$

We have three cases to consider:

**Case 1:**  $-\log F_\mu(t) > 1$ . If moreover  $\lambda \in [0, d \log(-\log F_\mu(t))]$ , then  $-e^{-\frac{\lambda}{d}} \log F_\mu(t) > 1$ , and by (12),

$$\begin{aligned}
& e^{\lambda + d \left(1 - e^{-\frac{\lambda}{d}}\right) \log F_\mu(t)} \int_{-e^{-\frac{\lambda}{d}} \log F_\mu(t)}^{+\infty} \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du \\
&\leq e^{\lambda + d \left(1 - e^{-\frac{\lambda}{d}}\right) \log F_\mu(t)} \left(-e^{-\frac{\lambda}{d}} \log F_\mu(t)\right)^d e^{d \left(1 + e^{-\frac{\lambda}{d}} \log F_\mu(t)\right)} \\
&= e^{d(1 + \log F_\mu(t) + \log(-\log F_\mu(t)))} < 1
\end{aligned}$$

since  $1 + \log F_\mu(t) + \log(-\log F_\mu(t)) < 0$ . Hence for all  $d \geq 1$ ,  $\mathbb{E} [e^{\lambda \hat{\mu}_d((-\infty, t])}] \leq 2$ , and from the Chernov inequality, we get then: for all  $\varepsilon > 0$ ,

$$\mathbb{P} \{ \hat{\mu}_d((-\infty, t]) > \varepsilon \} \leq \inf_{\lambda \in [0, d \log(-\log F_\mu(t))]} 2e^{-\varepsilon \lambda} = 2e^{-\varepsilon d \log(-\log F_\mu(t))}$$

This is enough, with Borel-Cantelli lemma, to prove the almost-sure convergence of  $\hat{\mu}_d((-\infty, t])$  to  $0 = \max(0, 1 + \log F(t))$ .

**Case 2:**  $-\log F_\mu(t) < 1$ . Then by (12),

$$\int_0^{-\log F_\mu(t)} \frac{d^d}{(d-1)!} u^{d-1} e^{-du} du \leq (-\log F_\mu(t))^d e^{d(1 + \log F_\mu(t))} = e^{d\delta_t}$$

with  $\delta_t = 1 + \log F_\mu(t) + \log(-\log F_\mu(t)) < 0$ .

• Moreover, we suppose  $\lambda \geq 0$ . Then

$$\begin{aligned}
\mathbb{E} \left[ e^{\lambda [\hat{\mu}_d((-\infty, t]) - (1 + \log F_\mu(t))]} \right] &\leq \left( e^{\lambda F_\mu(t)^d} e^{-de^{-\frac{\lambda}{d}} \log F_\mu(t)} + e^{d\delta_t} \right) e^{-\lambda(1 + \log F_\mu(t))} \\
&= e^{-d \log F_\mu(t) \left( \frac{\lambda}{d} - (1 - e^{-\frac{\lambda}{d}}) \right)} + e^{d(\delta_t - \frac{\lambda}{d}(1 + \log F_\mu(t)))}
\end{aligned}$$

From the Chernov inequality, we infer then: for all  $\varepsilon > 0$

$$\begin{aligned}
\mathbb{P} \{ \hat{\mu}_d((-\infty, t]) - (1 + \log F_\mu(t)) > \varepsilon \} &\leq \inf_{\lambda \geq 0} \left( e^{-d \log F_\mu(t) \left( \frac{\lambda}{d} - (1 - e^{-\frac{\lambda}{d}}) \right)} + e^{d(\delta_t - \frac{\lambda}{d}(1 + \log F_\mu(t)))} \right) e^{-\lambda \varepsilon} \\
&= \inf_{\lambda \geq 0} \left( e^{-d(\log F_\mu(t)(\lambda - (1 - e^{-\lambda})) + \lambda \varepsilon)} + e^{d(\delta_t - \lambda(1 + \log F_\mu(t) + \varepsilon))} \right) \\
&\leq 2e^{-dC}
\end{aligned}$$

for some constant  $C > 0$ . This gives, using Borel-Cantelli lemma:

$$\limsup_{d \rightarrow +\infty} \hat{\mu}_d((-\infty, t]) \leq 1 + \log F_\mu(t) \text{ a.s.}$$

• We suppose now  $\lambda \leq 0$ . Using the same trick, we get

$$\begin{aligned} \mathbb{P}\{\hat{\mu}_d((-\infty, t]) - (1 + \log F_\mu(t)) < -\varepsilon\} &\leq \inf_{\lambda \leq 0} \left( e^{-d(\log F_\mu(t)(\lambda - (1 - e^{-\lambda})) - \lambda\varepsilon)} + e^{d(\delta_t - \lambda(1 + \log F_\mu(t) - \varepsilon))} \right) \\ &\leq 2e^{-dC'} \end{aligned}$$

for some constant  $C' > 0$ . This gives:

$$\liminf_{d \rightarrow +\infty} \hat{\mu}_d((-\infty, t]) \geq 1 + \log F_\mu(t) \text{ a.s.}$$

And the result follows:

$$\lim_{d \rightarrow +\infty} \hat{\mu}_d((-\infty, t]) = 1 + \log F_\mu(t) = \max(0, 1 + \log F_\mu(t)) \text{ a.s.}$$

**Case 3:**  $\log F_\mu(t) = -1$ . We know that  $F_\mu(u)$  tends to 0 as  $u$  tends to  $-\infty$ . Hence, there is  $u < t$  such that  $\log F_\mu(u) < -1$ . For all  $d$ ,  $\hat{\mu}_d((-\infty, t]) \geq \hat{\mu}_d((-\infty, u])$ , hence,

$$\liminf_{d \rightarrow +\infty} \hat{\mu}_d((-\infty, t]) \geq \liminf_{d \rightarrow +\infty} \hat{\mu}_d((-\infty, u]) = 0 = 1 + \log F_\mu(t) \text{ a.s.}$$

Moreover, for any positive  $\varepsilon$ , by continuity of  $F_\mu$  and by the fact that  $F_\mu(u)$  tends to 1 as  $u$  tends to  $+\infty$ , there is  $u > t$  such that  $1 + \log F_\mu(t) < 1 + \log F_\mu(u) < 1 + \log F_\mu(t) + \varepsilon$ . Hence,

$$\limsup_{d \rightarrow +\infty} \hat{\mu}_d((-\infty, t]) \leq \limsup_{d \rightarrow +\infty} \hat{\mu}_d((-\infty, u]) = 1 + \log F_\mu(u) < 1 + \log F_\mu(t) + \varepsilon \text{ a.s.}$$

Therefore,  $\lim_{d \rightarrow +\infty} \hat{\mu}_d((-\infty, t]) = 1 + \log F_\mu(t)$  almost surely.

The theorem is proved in the case where  $F_\mu$  is smooth. Now, we consider a probability measure  $\mu$  on the real line without making any assumption about  $F_\mu$ . By Theorem 1.12.4 of [10], the distance

$$d(m, m') := \sup \left| \int f dm - \int f dm' \right|,$$

where the sup is taken on the set  $BL_1$  bounded Lipschitz functions  $f$  on the real line with Lipschitz constant  $\leq 1$  and such that  $\|f\|_\infty \leq 1$ , is a distance which defines the weak topology on the set of probability measures on the real line.

Thus we have to prove that almost surely, as  $d$  tends to infinity,

$$\lim d(\hat{\mu}_d, \Lambda^\vee(\mu)) = 0,$$

i.e. that for all  $\varepsilon > 0$ , almost surely, for  $d$  large enough,

$$(13) \quad d(\hat{\mu}_d, \Lambda^\vee(\mu)) \leq \varepsilon.$$

So let us fix  $\varepsilon > 0$ . By part (ii) of lemma 4.1, there exists a probability measure  $\nu$  on the real line such that  $F_\nu$  is smooth and

$$F_\nu(\bullet - \varepsilon/6) \leq F_\mu(\bullet) \leq F_\nu(\bullet + \varepsilon/6).$$

Note that the same obviously holds if one replaces  $\mu$  by  $\Lambda^\vee(\mu)$  and  $\nu$  by  $\Lambda^\vee(\nu)$ , since for all probability measure  $m$ ,  $F_{\Lambda^\vee(m)} = \max(0, 1 + \log F_m)$ . Hence, according to part (i) of lemma 4.1,

$$(14) \quad d(\Lambda^\vee(\mu), \Lambda^\vee(\nu)) \leq \varepsilon/6.$$

Note also that by definition of  $\Lambda_d^\vee(\mu)$ , the vector  $(X_1 \geq \dots \geq X_d)$  of ranked eigenvalues of  $M_d$  has law  $\mu_d$ . Following part (iii) of proposition 4.3, one can suppose that for all  $d$ , on the same space as  $M_d$ , there is a random matrix  $N_d$  with law  $\Lambda_d^\vee(\nu)$  with ranked eigenvalues  $(Y_1 \geq \dots \geq Y_k)$  such that for all  $i = 1, \dots, d$ ,  $|X_i - Y_i| \leq \varepsilon/3$  almost surely. If one denotes the spectral law of  $N_d$  by  $\hat{\nu}_d$ , it implies that

$$(15) \quad \forall d \geq 1, d(\hat{\mu}_d, \hat{\nu}_d) \leq \varepsilon/3 \quad \text{almost surely.}$$

Please note finally that by the first part of the proof,

$$(16) \quad \text{almost surely, for } d \text{ large enough, } d(\hat{\nu}_d, \Lambda^\vee(\nu)) \leq \varepsilon/3.$$

Equations (14), (15), (16), together, imply (13). Thus, the theorem is proved.

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