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A double obstacle problem arising in differential game theory

Pierre Cardaliaguet*

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Abstract : A Hamilton-Jacobi equation involving a double obstacle problem is investigated. The link between this equation and the notion of dual solutions—introduced in [1, 2, 3] in the framework of differential games with lack of information—is established. As an application we characterize the convex hull of a function in the simplex as the unique solution of some nonlinear obstacle problem.

Keywords : Viscosity solutions of Hamilton-Jacobi equations, differential games, lack of information, convex hull.

1 Introduction

The paper investigates the notion of viscosity solution of Hamilton-Jacobi equations of the form

$$\min \left\{ \max \left\{ w + H(x, Dw, D^2w, p, q); -\lambda_{\min} \left(\frac{\partial^2 w}{\partial p^2} \right) \right\}; -\lambda_{\max} \left(\frac{\partial^2 w}{\partial q^2} \right) \right\} = 0 \quad (1.1)$$

In the above equation, the unknown $w = w(x, p, q)$ depends on the variables $x \in \mathbb{R}^N$, $p \in \Delta(I)$ and $q \in \Delta(J)$, where $\Delta(I)$ and $\Delta(J)$ are the simplexes of \mathbb{R}^I and \mathbb{R}^J . We have denoted by Dw and D^2w the first and second order derivatives of the map w with respect to x , and by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the minimal and maximal eigenvalue of a symmetric matrix A . The map H is a standard hamiltonian, non increasing with respect to the matrix D^2w . In particular, (1.1) is a degenerate elliptic equation. We note that, because of the min-max Theorem, (1.1) can also be written as

$$\max \left\{ \min \left\{ w_t + H(t, x, Dw, D^2w, p, q); -\lambda_{\max} \left(\frac{\partial^2 w}{\partial q^2} \right) \right\}; -\lambda_{\min} \left(\frac{\partial^2 w}{\partial p^2} \right) \right\} = 0$$

Heuristically a map w is a solution w of (1.1) if:

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- w is convex with respect to p (since $-\lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right) \leq 0$),
- concave with respect to q (since $-\lambda_{\max}\left(\frac{\partial^2 w}{\partial q^2}\right) \geq 0$),
- and, at points where it is strictly convex in p and strictly concave in q , w satisfies the Hamilton-Jacobi equation

$$w + H(x, Dw, D^2w, p, q) = 0 \quad (1.2)$$

In particular (1.1) appears as a double obstacle problem for equation (1.2), where the second order obstacles are $\lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right) \geq 0$ and $\lambda_{\max}\left(\frac{\partial^2 w}{\partial q^2}\right) \leq 0$.

Example : The following equation appears in [6] and [9]:

$$\max \left\{ w - g(p); -\lambda_{\min} \left(\frac{\partial^2 w}{\partial p^2} \right) \right\} = 0. \quad (1.3)$$

where $g : \mathbb{R}^I \rightarrow \mathbb{R}$. In [9] it is proved that the convex envelope of the function g is a viscosity solution of (1.3). Note that this equation is exactly of the form (1.1) for $H = H(p) = -g(p)$ (there is no dependance in x and q here).

Our main motivation for studying equation (1.1) comes from the theory of stochastic zero-sum differential games with lack of information [1, 2, 3]. In these games, the value function u depends on the usual space variable $x \in \mathbb{R}^N$ (or time-space variables $(t, x) \in (0, T) \times \mathbb{R}^N$ for evolution problems) and on some parameters $(p, q) \in \Delta(I) \times \Delta(J)$. These parameters are interpreted as probability measures on the sets $\{1, \dots, I\}$ and $\{1, \dots, J\}$ respectively. Due to the specific structure of the game, u has to be convex with respect to p and concave with respect to q . In particular u *cannot* be a viscosity solution of the Hamilton-Jacobi equation (1.2) for every value of the parameters p and q , since otherwise the convexity/concavity constraints would not be fulfilled. However u satisfies equation (1.2) in a very specific sense, called “dual sense” [1, 2, 3]. By this we mean that the convex Fenchel conjugate $u^* = u(x, \hat{p}, q)$ of u with respect to p satisfies in the viscosity sense

$$u + H(x, -Du^*, -D^2u^*, \frac{\partial u^*}{\partial \hat{p}}, q) \geq 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^I \times \Delta(J),$$

while its concave conjugate $u^\sharp = u^\sharp(x, p, \hat{q})$ with respect to q satisfies

$$u + H(x, -Du^\sharp, -D^2u^\sharp, p, \frac{\partial u^\sharp}{\partial \hat{q}}) \leq 0 \quad \text{in } \mathbb{R}^N \times \Delta(I) \times \mathbb{R}^J.$$

The main result of this paper amounts to show that a map w is a dual solution of (1.2) if and only if it is a viscosity solution of (1.1). This equivalence is a little surprising because in the double obstacle problem p and q are considered as plain variables, while in the dual sense interpretation of (1.2) they are merely parameters.

The advantage of equation (1.1) on the notion of dual solutions of (1.2) is clear: the dual formulation involves a non local operator (the Fenchel conjugate of the solution), while equation (1.1) is a purely local one. Hence handling equation (1.1) is much easier than handling dual solutions.

As an application we first show that the restriction of solutions of (1.1) to any face of $\Delta(I)$ and of $\Delta(J)$ is still a solution of (1.1). In particular, this means that equation (1.2) has to be satisfied at the extreme points of $\Delta(I)$ and of $\Delta(J)$. Such a result is reminiscent of a similar one given in [3] for dual solutions.

We also prove a comparison principle for (1.1), which implies in particular that (1.1) has at most one solution. For instance this shows that equation (1.3) characterizes the convex envelope of a Lipschitz continuous map g defined on $\Delta(I)$ (in [9] this characterization was not established, and, in fact, did not hold since the author was working in the full space \mathbb{R}^I). The proof of this comparison principle is somewhat surprising because it involves an induction on the space.

Let us recall that a comparison result was already given in [3] for dual solutions of evolution equations (see also [1] for the simpler case where (1.2) is of first order). However the proof of this comparison principle was really involved, requiring a very intricate extension of the classical maximum principle. Our introduction of the double obstacle problem has largely been motivated by the desire to simplify this proof. In that respect our goal is achieved since our new proof no longer requires the use of this intricate maximum principle, but only of the standard one.

As a last byproduct of our equivalence theorem, we also get a new characterization of the value function of the stochastic differential games studied in [1, 2, 3]. This point shall not be discussed here and will be the aim of the forthcoming paper [4]. In [4] we show in particular that the intriguing terms $\lambda_{\min} \left(\frac{\partial^2 w}{\partial p^2} \right)$ and $\lambda_{\max} \left(\frac{\partial^2 w}{\partial q^2} \right)$ have a natural interpretation: they appear as second order term generated by fictitious martingales in p and q controlled by the players.

The paper is organized in the following way: in section 2, we collect the main notations used in the paper and recall the definition of dual solutions for (1.2). Section 3 is devoted to the proof of the equivalence between dual solutions and solutions of (1.1). In section 4

we show the restriction property and in section 5 the comparison principle for solutions of (1.1). Finally section 5 is devoted to examples: in particular we characterize the convex hull of a function g as the unique solution of (1.3) and extend this result to the Φ -operator of Mertens-Zamir [8].

2 Definitions and notations

Notations : if x and y belong to \mathbb{R}^K (for some $K \geq 1$), we denote by $x.y$ or by $\langle x, y \rangle$ their scalar product, by $|x|$ the euclidean norm of x and by $B_r(x)$ the ball centered at x and of radius r . We denote by \mathcal{S}_K the set of symmetric matrices of size K . For $I \in \mathbb{N}^*$, $\Delta(I)$ is the set of all $p = (p_1, \dots, p_I) \in [0, 1]^I$ that satisfy $\sum_{i=1}^I p_i = 1$. By abuse of notation, we denote by $\text{Int}(\Delta(I))$ the set of $p = (p_i) \in \Delta(I)$ such that $p_i > 0$ for any $i \in \{1, \dots, I\}$. If C is a closed convex subset of normed vectoriel space and $x \in C$, $T_C(x)$ is the tangent cone to C at x , namely

$$T_C(x) = \text{closure} \left(\bigcup_{\epsilon > 0} \frac{K - p}{\epsilon} \right) .$$

In particular we often use the cone

$$T_{\Delta(I)}(p) = \left\{ v \in \mathbb{R}^I \mid \sum_i v_i = 0, v_i \geq 0 \text{ if } p_i = 0 \right\} \quad \forall p \in \Delta(I) .$$

For $n \in \mathbb{N}^*$ and $w : \mathbb{R}^n \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$, we define the Fenchel conjugate w^* of w with respect to p by

$$w^*(x, \hat{p}, q) = \sup_{p \in \Delta(I)} \{ \langle \hat{p}, p \rangle - w(x, p, q) \}, \quad (x, \hat{p}, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^I \times \Delta(J) .$$

We note that this actually corresponds to the Fenchel conjugate of the map w extended to \mathbb{R}^I by setting $w = +\infty$ on $\mathbb{R}^I \setminus \Delta(I)$. For w defined on the dual space $\mathbb{R}^n \times \mathbb{R}^I \times \Delta(J)$, we also set

$$w^*(x, p, q) = \sup_{\hat{p} \in \mathbb{R}^I} \{ \langle \hat{p}, p \rangle - w(x, \hat{p}, q) \}, \quad (x, p, q) \in [0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) .$$

It is well known that, if w is convex in p on $\Delta(I)$, we have $(w^*)^* = w$. We also have to introduce the concave conjugate with respect to q of a map $w : \mathbb{R}^n \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$:

$$w^\sharp(x, p, \hat{q}) = \inf_{q \in \Delta(J)} \{ \langle \hat{q}, q \rangle - w(x, p, q) \}, \quad (x, p, \hat{q}) \in \mathbb{R}^n \times \Delta(I) \times \mathbb{R}^J .$$

We use the following notations for the sub- and superdifferentials with respect to \hat{p} and \hat{q} respectively: if $w : \mathbb{R}^n \times \mathbb{R}^I \times \Delta(J) \rightarrow \mathbb{R}$, we set

$$\partial_{\hat{p}}^- w(x, \hat{p}, q) = \{p \in \mathbb{R}^I, w(x, \hat{p}, q) + \langle p, \hat{p}' - \hat{p} \rangle \leq w(x, \hat{p}', q) \forall \hat{p}' \in \mathbb{R}^I\}$$

and if $w : \mathbb{R}^n \times \Delta(I) \times \mathbb{R}^J \rightarrow \mathbb{R}$

$$\partial_{\hat{q}}^+ w(x, p, \hat{q}) = \{q \in \mathbb{R}^J, w(x, p, \hat{q}) + \langle q, \hat{q}' - \hat{q} \rangle \geq w(x, p, \hat{q}') \forall \hat{q}' \in \mathbb{R}^J\}.$$

Let $H : \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S}_N \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ be a continuous map.

Definition 2.1 (Dual solutions) *We say that a map $w : \mathbb{R}^N \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ is a supersolution in the dual sense of equation (1.2) if $w = w(x, p, q)$ is lower semicontinuous, concave with respect to q and if, for any $\mathcal{C}^2(\mathbb{R}^N)$ function ϕ such that $x \rightarrow w^*(x, \hat{p}, \bar{q}) - \phi(x)$ has a maximum at some point \bar{x} for some $(\hat{p}, \bar{q}) \in \mathbb{R}^I \times \Delta(J)$ at which $\frac{\partial w^*}{\partial \hat{p}}$ exists, we have*

$$w(\bar{x}, \bar{p}, \bar{q}) + H(\bar{x}, -D\phi(\bar{x}), -D^2\phi(\bar{x}), \bar{p}, \bar{q}) \geq 0 \quad \text{where } \bar{p} = \frac{\partial w^*}{\partial \hat{p}}(\bar{x}, \hat{p}, \bar{q}).$$

We say that w is a subsolution of (1.2) in the dual sense if w is upper semicontinuous, convex with respect to p and if, for any $\mathcal{C}^2(\mathbb{R}^N)$ function ϕ such that $x \rightarrow w^\sharp(x, \bar{p}, \hat{q}) - \phi(x)$ has a minimum at some point \bar{x} for some $(\bar{p}, \hat{q}) \in \Delta(I) \times \mathbb{R}^J$ at which $\frac{\partial w^\sharp}{\partial \hat{q}}$ exists, we have

$$w(\bar{x}, \bar{p}, \bar{q}) + H(\bar{x}, -D\phi(\bar{x}), -D^2\phi(\bar{x}), \bar{p}, \bar{q}) \leq 0 \quad \text{where } \bar{q} = \frac{\partial w^\sharp}{\partial \hat{q}}(\bar{x}, \bar{p}, \hat{q}).$$

A solution of (1.2) in the dual sense is a map which is sub- and supersolution in the dual sense.

3 Equivalence between dual solutions and solutions of the double obstacle problem

In this section we explain that the notion of dual solution of (1.2) can be recasted in terms of viscosity solution of the Hamilton-Jacobi equation (1.1).

Definition 3.1 (Viscosity solution of (1.1)) *We say that w is a subsolution of (1.1) if, for any smooth test function $\phi : \mathbb{R}^N \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ such that $w - \phi$ has a local maximum at some point $(x, p, q) \in \mathbb{R}^N \times \text{Int}(\Delta(I)) \times \Delta(J)$, one has*

$$\min \left\{ \max \left\{ w + H(x, D\phi, D^2\phi); -\lambda_{\min} \left(p, \frac{\partial^2 \phi}{\partial p^2} \right) \right\}; -\lambda_{\max} \left(q, \frac{\partial^2 \phi}{\partial q^2} \right) \right\} \leq 0$$

at (x, p, q) , where, for any $(p, A) \in \Delta(I) \times \mathcal{S}_I$,

$$\lambda_{\min}(p, A) = \min_{z \in T_{\Delta(I)}(p) \setminus \{0\}} \langle Az, z \rangle / |z|^2$$

and, for any $(q, A) \in \Delta(J) \times \mathcal{S}_J$

$$\lambda_{\max}(p, A) = \max_{z \in T_{\Delta(J)}(q) \setminus \{0\}} \langle Az, z \rangle / |z|^2$$

We say that w is a supersolution of (1.1) if, for any test function $\phi : \mathbb{R}^N \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ such that $w - \phi$ has a local minimum at some point $(x, p, q) \in \mathbb{R}^N \times \Delta(I) \times \text{Int}(\Delta(J))$, one has

$$\min \left\{ \max \left\{ w + H(x, D\phi, D^2\phi) ; -\lambda_{\min} \left(p, \frac{\partial^2 \phi}{\partial p^2} \right) \right\} ; -\lambda_{\max} \left(q, \frac{\partial^2 \phi}{\partial q^2} \right) \right\} \geq 0$$

at (x, p, q) .

Remarks :

1. In the definition we have to require that the solution satisfies the equation up to the boundary of $\Delta(I)$ or $\Delta(J)$. Indeed, if for instance the function $u = u(x)$ does not depend on p and q , then it automatically satisfies equation (1.1) in $\text{Int}(\Delta(I)) \times \text{Int}(\Delta(J))$, because in these sets,

$$\lambda_{\min} \left(p, \frac{\partial^2 w}{\partial p^2} \right) = \lambda_{\max} \left(q, \frac{\partial^2 w}{\partial q^2} \right) = 0$$

(in the viscosity sense), and

$$\max \{ \min \{ z, 0 \} ; 0 \} = 0 \quad \forall z \in \mathbb{R} .$$

This shows that relevant informations have to be carried out by the boundary of $\Delta(I)$ and $\Delta(J)$.

2. Following [5], the definition could be formulated in terms of super- and subjets.

Our main result is the following:

Theorem 3.1 *Let us assume that $H : \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S}_N \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ is continuous. Let $w : \mathbb{R}^N \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ be bounded and continuous, convex with respect to p and concave with respect to q .*

Then w is a supersolution (resp. subsolution, solution) of (1.2) in the dual sense if and only if it is a supersolution (resp. subsolution, solution) of (1.1).

Remarks :

1. The requirement that w is convex in p and concave in q is necessary: without this condition, a dual supersolution of (1.2) need not be a supersolution of (1.1) (the reverse always holds, as can be seen in the proof).

Here is a counter-example for equation (1.3). Let $g : [0, 1] \rightarrow \mathbb{R}$ be Lipschitz continuous and such that $g > 0$ in $(0, 1)$, $g(0) = g(1) = 0$. Let u be of class \mathcal{C}^2 such that $0 < u < g$ in $(0, 1)$, $u(0) = u(1) = 0$ and such that there exists $\bar{p} \in (0, 1)$ with $u''(\bar{p}) > 0$. Then u is a dual supersolution of equation $w - g = 0$ but not a supersolution of (1.1) at \bar{p} .

2. A symmetric result holds for dual solutions of the evolution equation of the form

$$w_t + H(t, x, Dw, D^2w, p, q) = 0$$

in $(0, T) \times \mathbb{R}^N \times \Delta(I) \times \Delta(J)$ (see [3] for the definition). In this case the double obstacle problem reads

$$\min \left\{ \max \left\{ w_t + H(t, x, Dw, D^2w); -\lambda_{\min} \left(\frac{\partial^2 w}{\partial p^2} \right) \right\}; -\lambda_{\max} \left(\frac{\partial^2 w}{\partial q^2} \right) \right\} = 0$$

in $(0, T) \times \mathbb{R}^N \times \Delta(I) \times \Delta(J)$. Generalizations to problems stated in bounded domains (for the x variable) are also straightforward, the arguments being only local in space.

Proof of Theorem 3.1 : Let us first assume that w is a supersolution of (1.2) in the dual sense. Let ϕ be smooth and such that $w \geq \phi$ on $\mathbb{R}^N \times \Delta(I) \times \Delta(J)$ with an equality at $(\bar{x}, \bar{p}, \bar{q})$ where $\bar{q} \in \text{Int}(\Delta(J))$. Then, since w is concave with respect to q and $\bar{q} \in \text{Int}(\Delta(J))$, one readily gets that $\lambda_{\max} \left(\bar{q}, \frac{\partial^2 \phi}{\partial q^2}(\bar{x}, \bar{p}, \bar{q}) \right) \leq 0$.

Let us now assume that

$$\lambda_{\min} \left(\bar{p}, \frac{\partial^2 \phi}{\partial p^2}(\bar{x}, \bar{p}, \bar{q}) \right) > 0. \quad (3.4)$$

Then we have to prove that

$$w + H(\bar{x}, D\phi, D^2\phi, \bar{p}, \bar{q}) \geq 0 \quad \text{at } (\bar{x}, \bar{p}, \bar{q}). \quad (3.5)$$

We claim that there are some $\delta, \epsilon > 0$ such that

$$w(x, p, \bar{q}) \geq \phi(x, \bar{p}, \bar{q}) + \left\langle \frac{\partial \phi}{\partial p}(x, \bar{p}, \bar{q}), p - \bar{p} \right\rangle + \delta |p - \bar{p}|^2 \quad \forall x \in B_\epsilon(\bar{x}), \forall p \in \Delta(I). \quad (3.6)$$

Proof of (3.6) : From (3.4), there are $\eta > 0$ and $\delta > 0$ such that

$$\left\langle \frac{\partial^2 \phi}{\partial p^2}(x, p, \bar{q})z, z \right\rangle \geq 4\delta |z|^2 \quad \forall (x, p) \in B_\eta(\bar{x}, \bar{p}), z \in T_{\Delta(I)}(\bar{p}). \quad (3.7)$$

Hence for $(x, p) \in B_\eta(\bar{x}, \bar{p})$ we have

$$w(x, p, \bar{q}) \geq \phi(x, p, \bar{q}) \geq \phi(x, \bar{p}, \bar{q}) + \left\langle \frac{\partial \phi}{\partial p}(x, \bar{p}, \bar{q}), p - \bar{p} \right\rangle + 2\delta|p - \bar{p}|^2. \quad (3.8)$$

In particular

$$\hat{p} := \frac{\partial \phi}{\partial p}(\bar{x}, \bar{p}, \bar{q}) \in \partial_p^- w(\bar{x}, \bar{p}, \bar{q}).$$

We also note that, for any $p \in \Delta(I) \setminus B_\eta(\bar{p})$, we have

$$w(\bar{x}, p, \bar{q}) \geq \phi(\bar{x}, \bar{p}, \bar{q}) + \left\langle \frac{\partial \phi}{\partial p}(\bar{x}, \bar{p}, \bar{q}), p - \bar{p} \right\rangle + 2\delta\eta^2. \quad (3.9)$$

Indeed let $p \in \Delta(I) \setminus B_\eta(\bar{p})$ and let us set $p_1 = \bar{p} + \frac{p - \bar{p}}{|p - \bar{p}|}\eta \in \Delta(I)$ and let $\hat{p}_1 \in \partial_p^- w(\bar{x}, p_1, \bar{q})$. Then, since $|p_1 - p| = \eta$, we have

$$\begin{aligned} w(\bar{x}, p, \bar{q}) &\geq w(\bar{x}, p_1, \bar{q}) + \langle \hat{p}_1, p - p_1 \rangle \\ &\geq \phi(\bar{x}, \bar{p}, \bar{q}) + \left\langle \frac{\partial \phi}{\partial p}(\bar{x}, \bar{p}, \bar{q}), p_1 - \bar{p} \right\rangle + 2\delta\eta^2 + \langle \hat{p}_1, p - p_1 \rangle \\ &\geq \phi(\bar{x}, \bar{p}, \bar{q}) + \left\langle \frac{\partial \phi}{\partial p}(\bar{x}, \bar{p}, \bar{q}), p - \bar{p} \right\rangle + 2\delta\eta^2 + \langle \hat{p}_1 - \frac{\partial \phi}{\partial p}(\bar{x}, \bar{p}, \bar{q}), p - p_1 \rangle \end{aligned}$$

where

$$\langle \hat{p}_1 - \frac{\partial \phi}{\partial p}(\bar{x}, \bar{p}, \bar{q}), p - p_1 \rangle \geq 0$$

because w is convex, $\hat{p}_1 \in \partial_p^- w(\bar{x}, p_1, \bar{q})$, $\frac{\partial w}{\partial p}(x, \bar{p}, \bar{q}) \in \partial_p^- w(\bar{x}, \bar{p}, \bar{q})$ and $p - p_1 = \gamma(p_1 - \bar{p})$ for some $\gamma > 0$. This proves (3.9). Let us now argue by contradiction and assume that our claim (3.6) is false. Then there are $(t_n, x_n) \rightarrow \bar{x}$ and $p_n \rightarrow p \in \Delta(I)$ such that

$$w(t_n, x_n, p_n, \bar{q}) < \phi(t_n, x_n, \bar{p}, \bar{q}) + \left\langle \frac{\partial \phi}{\partial p}(t_n, x_n, \bar{p}, \bar{q}), p_n - \bar{p} \right\rangle + \delta|p_n - \bar{p}|^2$$

Note that $p_n \notin B_\eta(\bar{p})$ because of (3.8). Letting $n \rightarrow +\infty$, we get $p \in \Delta(I) \setminus B_\eta(\bar{p})$ and

$$w(\bar{x}, p, \bar{q}) \leq \phi(\bar{x}, p, \bar{q}) + \left\langle \frac{\partial \phi}{\partial p}(\bar{x}, \bar{p}, \bar{q}), p - \bar{p} \right\rangle + \delta\eta^2$$

which contradicts (3.9). So (3.6) holds true for some $\epsilon > 0$ sufficiently small.

Using (3.6) we have, for any $\hat{p}' \in \mathbb{R}^I$ and any $x \in B_\epsilon(\bar{x})$,

$$\begin{aligned} w^*(x, \hat{p}', \bar{q}) &= \max_{p \in \Delta(I)} \{p \cdot \hat{p}' - w(x, p, \bar{q})\} \\ &\leq -\phi(x, \bar{p}, \bar{q}) + \max_{p \in \mathbb{R}^I} \left\{ p \cdot \hat{p}' - \left\langle \frac{\partial \phi}{\partial p}(x, \bar{p}, \bar{q}), p - \bar{p} \right\rangle - \delta|p - \bar{p}|^2 \right\} \\ &\leq -\phi(x, \bar{p}, \bar{q}) + \bar{p} \cdot \hat{p}' + \frac{1}{4\delta} \left| \hat{p}' - \frac{\partial \phi}{\partial p}(x, \bar{p}, \bar{q}) \right|^2 \end{aligned}$$

Recalling that $\hat{p} := \frac{\partial \phi}{\partial p}(\bar{x}, \bar{p}, \bar{q})$, the above inequality shows that $\frac{\partial w^*}{\partial \hat{p}'}(\bar{x}, \hat{p}, \bar{q})$ exists and is equal to \bar{p} and, since w is a dual supersolution of (1.2), that (3.5) holds.

Let us now assume that w is a supersolution of (1.1). Let $\phi = \phi(x)$ be smooth and such that $\phi(x) \geq w^*(x, \hat{p}, \bar{q})$ for any x , with an equality only at \bar{x} and assume that $\bar{p} := \frac{\partial w^*}{\partial \hat{p}'}(\bar{x}, \hat{p}, \bar{q})$ exists. Without loss of generality we also assume that ϕ is coercive, i.e., $\phi(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$. For $\epsilon > 0$ small let $(x_\epsilon, \hat{p}_\epsilon, q_\epsilon)$ be a maximum point of the map

$$(x, \hat{p}', q) \rightarrow w^*(x, \hat{p}', q) - \phi(x) - \frac{1}{2\epsilon}|\hat{p}' - \hat{p}|^2 + \epsilon\sigma(q) - \frac{1}{2\epsilon}|q - \bar{q}|^2,$$

where $\sigma(q) = \sum_j \ln(q_j(1 - q_j))$. We note that such a maximum point exists because w^* has at most a linear growth with respect to \hat{p}' and ϕ is coercive. From standard arguments, we have that $q_\epsilon \in \text{Int}(\Delta(J))$ and that $(x_\epsilon, \hat{p}_\epsilon, q_\epsilon) \rightarrow (\bar{x}, \hat{p}, \bar{q})$ as $\epsilon \rightarrow 0$. From the definition of $(x_\epsilon, \hat{p}_\epsilon, q_\epsilon)$ we have

$$\begin{aligned} w^*(x, \hat{p}', q) &\leq \phi(x) + \frac{1}{2\epsilon}|\hat{p}' - \hat{p}|^2 - \epsilon\sigma(q) + \frac{1}{2\epsilon}|q - \bar{q}|^2 \\ &\quad + w^*(x_\epsilon, \hat{p}_\epsilon, q_\epsilon) - \phi(x_\epsilon) - \frac{1}{2\epsilon}|\hat{p}_\epsilon - \hat{p}|^2 + \epsilon\sigma(q_\epsilon) - \frac{1}{2\epsilon}|q_\epsilon - \bar{q}|^2 \end{aligned} \quad (3.10)$$

Hence, for any (x, p, q) we have

$$\begin{aligned} w(x, p, q) &\geq -\phi(x) + \max_{\hat{p}'} \left\{ p \cdot \hat{p}' - \frac{1}{2\epsilon}|\hat{p}' - \hat{p}|^2 \right\} + \epsilon\sigma(q) - \frac{1}{2\epsilon}|q - \bar{q}|^2 \\ &\quad - w^*(x_\epsilon, \hat{p}_\epsilon, q_\epsilon) + \phi(x_\epsilon) + \frac{1}{2\epsilon}|\hat{p}_\epsilon - \hat{p}|^2 - \epsilon\sigma(q_\epsilon) + \frac{1}{2\epsilon}|q_\epsilon - \bar{q}|^2 \\ &\geq -\phi(x) + p \cdot \hat{p} + \frac{\epsilon}{2}|p|^2 + \epsilon\sigma(q) - w^*(x_\epsilon, \hat{p}_\epsilon, q_\epsilon) - \frac{1}{2\epsilon}|q - \bar{q}|^2 \\ &\quad + \phi(x_\epsilon) + \frac{1}{2\epsilon}|\hat{p}_\epsilon - \hat{p}|^2 - \epsilon\sigma(q_\epsilon) + \frac{1}{2\epsilon}|q_\epsilon - \bar{q}|^2 \end{aligned} \quad (3.11)$$

Let $p_\epsilon = \frac{1}{\epsilon}(\hat{p}_\epsilon - \hat{p})$. From (3.10) we have that

$$p_\epsilon \in \partial_{\hat{p}}^- w^*(x_\epsilon, \hat{p}_\epsilon, q_\epsilon) \subset \Delta(I). \quad (3.12)$$

We note that $p_\epsilon \rightarrow \bar{p}$ as $\epsilon \rightarrow 0$ because $(x_\epsilon, \hat{p}_\epsilon, q_\epsilon) \rightarrow (\bar{x}, \hat{p}, \bar{q})$ and $\partial_{\hat{p}}^- w^*(\bar{x}, \hat{p}, \bar{q}) = \{\bar{p}\}$. Moreover (3.12) also implies that inequality (3.11) is an equality at $(x_\epsilon, p_\epsilon, q_\epsilon)$. Since w is a supersolution of (1.1) and since the right-hand side of (3.11) is strictly convex in p and $q_\epsilon \in \text{Int}(\Delta(J))$, we get

$$w + H(x_\epsilon, -D\phi, -D^2\phi, p_\epsilon, q_\epsilon) \geq 0 \quad \text{at } (x_\epsilon, p_\epsilon, q_\epsilon).$$

Letting $\epsilon \rightarrow 0$ gives the desired result. \square

4 Restriction of solutions

In this section, we show that the restriction of a solution of (1.1) to any face of $\Delta(I)$ or $\Delta(J)$ is still a solution. For this, we use the following conventions. If $I' \subset \{1, \dots, I\}$, we

denote by $\Delta(I')$ the set

$$\Delta(I') = \{p \in \Delta(I) \mid p_i = 0 \text{ if } i \notin I'\}.$$

We identify $\Delta(I')$ with $\Delta(|I'|)$ (where $|I'|$ is the cardinal of I').

Proposition 4.1 *Let w be a subsolution (resp. supersolution, solution) of (1.1). Let I' and J' be a subsets of $\{1, \dots, I\}$ and $\{1, \dots, J\}$. Then the restriction of w to $\Delta(I')$ and $\Delta(J')$ is still a subsolution (resp. supersolution, solution) of (1.1) in $\mathbb{R}^N \times \Delta(I') \times \Delta(J')$.*

Remarks :

1. In particular, if w is a solution of (1.1), then, for any $i \in I$ and $j \in J$, the map $x \rightarrow w(x, e_i, e'_j)$ is a supersolution of the usual Hamilton-Jacobi equation

$$w + H(x, Dw, D^2w) = 0$$

where (e_i) and (e'_j) are the standard basis of \mathbb{R}^I and \mathbb{R}^J .

2. The result extends with the same proof to evolution equations and to equations stated in bounded domains in the x variable.

Proof : Let w' be the restriction of w and assume that $(x, p', q') \rightarrow w'(x, p', q') - \phi(x, p', q')$ has a maximum at some point $(\bar{x}, \bar{p}', \bar{q}') \in \text{Int}(\Delta(I')) \times \Delta(J')$ for some smooth function $\phi : \mathbb{R}^N \times \Delta(I') \times \Delta(J')$. Without loss of generality we can assume that

$$w'(x, p', q') < \phi(x, p', q') \quad \forall (x, p', q') \in \Delta(I') \times \Delta(J')$$

with an equality at $(\bar{x}, \bar{p}', \bar{q}')$. We also suppose that there is some $\alpha > 0$ such that

$$\lambda_{\max} \left(\bar{q}', \frac{\partial^2 \phi}{\partial q^2}(\bar{x}, \bar{p}', \bar{q}') \right) \leq -2\alpha. \quad (4.13)$$

Then we have to prove that

$$w' + H(x, D\phi, D^2\phi) \leq 0 \quad \text{and} \quad \lambda_{\min} \left(\bar{p}', \frac{\partial^2 \phi}{\partial p^2} \right) \geq 0$$

at $(\bar{x}, \bar{p}', \bar{q}')$. Let $\mu \in \mathbb{R}^I$ and $\nu \in \mathbb{R}^J$ be such that $\mu_i = 0$ if $i \in I'$, $\mu_i = 1$ otherwise and $\nu_j = 0$ if $j \in J'$, $\nu_j = 1$ otherwise. For $(p, q) \in \Delta(I) \times \Delta(J)$ we denote by Π_1 and Π_2 the projections of p and q onto $\Delta(I')$ and $\Delta(J')$ respectively. Note that

$$\Pi_1(p)_i = \begin{cases} p_i + (\sum_{j \notin I'} p_j)/|I'| & \text{if } i \in I' \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \Pi_2(q)_i = \begin{cases} q_i + (\sum_{j \notin J'} q_j)/|J'| & \text{if } j \in J' \\ 0 & \text{otherwise} \end{cases}$$

where $|I'|$ and $|J'|$ denote the cardinal of I' and J' . Since w is k -Lipschitz continuous with respect to p and q uniformly with respect to x , we have

$$w(x, p, q) \leq w'(x, \Pi_1(p), \Pi_2(q)) + (k+1)(|\Pi_1(p) - p| + |\Pi_2(q) - q|)$$

with equality only at $(\bar{x}, \bar{p}', \bar{q}')$. So, for any $(p, q) \in \Delta(I) \times \Delta(J)$, we get

$$w(x, p, q) \leq \phi(x, \Pi_1(p), \Pi_2(q)) + 2(k+1)(\langle \nu, p \rangle + \langle \mu, q \rangle)$$

(with equality only at $(\bar{x}, \bar{p}', \bar{q}')$) because

$$|\Pi_1(p) - p| \leq \sum_{i \in I'} |\Pi_1(p)_i - p_i| + \sum_{j \notin I'} p_j = (1 + 1/|I'|) \sum_{j \notin I'} p_j \leq 2\langle \nu, p \rangle.$$

Let now $\epsilon > 0$ be small and let us look at the problem

$$\begin{aligned} & \max_{x \in \mathbb{R}^N,} w(x, p, q) - \{\phi(x, \Pi_1(p), \Pi_2(q)) + 2(k+1)(\langle \nu, p \rangle + \langle \mu, q \rangle) - \epsilon\sigma(p)\} \\ & (p, q) \in \Delta(I) \times \Delta(J) \end{aligned}$$

where

$$\sigma(p) = \sum_{i \in I \setminus I'} \ln(p_i(1 - p_i)).$$

For ϵ sufficiently small, this problem has a maximum $(x_\epsilon, p_\epsilon, q_\epsilon)$ which converges to $(\bar{x}, \bar{p}', \bar{q}')$ as $\epsilon \rightarrow 0$. Moreover, from the definition of σ and the fact that $\bar{p}' \in \text{Int}(\Delta(I'))$, we have $p_\epsilon \in \text{Int}(\Delta(I))$. Since w is a subsolution, we get

$$\max \left\{ \min \left\{ w + H(x, D\tilde{\phi}, D^2\tilde{\phi}); -\lambda_{\max} \left(q_\epsilon, \frac{\partial^2 \tilde{\phi}}{\partial q^2} \right) \right\}; -\lambda_{\min} \left(p_\epsilon, \frac{\partial^2 \tilde{\phi}}{\partial p^2} \right) \right\} \leq 0 \quad (4.14)$$

at $(x_\epsilon, p_\epsilon, q_\epsilon)$, where

$$\tilde{\phi}(x, p, q) = \phi(x, \Pi_1(p), \Pi_2(q)) + 2(k+1)(\langle \nu, p \rangle + \langle \mu, q \rangle) - \epsilon\sigma(p).$$

Note that Π_1 is affine in $\Delta(I)$ and that $\Pi_1(p') = p'$ on $\Delta(I')$. Since $\sigma(p)$ does not depend on p_i for $i \in I'$, we get

$$\begin{aligned} & \liminf_{\epsilon \rightarrow 0} \lambda_{\min} \left(p_\epsilon, \frac{\partial^2 \tilde{\phi}}{\partial p^2}(x_\epsilon, p_\epsilon, q_\epsilon) \right) \\ & \leq \liminf_{\epsilon \rightarrow 0} \min_{z \in T_{\Delta(I')}(p') \setminus \{0\}} \left\langle \frac{\partial^2 \tilde{\phi}}{\partial p^2}(x_\epsilon, p_\epsilon, q_\epsilon)z, z \right\rangle / |z|^2 \\ & = \liminf_{\epsilon \rightarrow 0} \min_{z \in T_{\Delta(I')}(p') \setminus \{0\}} \left\langle \frac{\partial^2 \tilde{\phi}}{\partial p^2}(x_\epsilon, \Pi_1(p_\epsilon), \Pi_2(q_\epsilon))z, z \right\rangle / |z|^2 \\ & = \lambda_{\min} \left(\bar{p}', \frac{\partial^2 \phi}{\partial p^2}(\bar{x}, \bar{p}', \bar{q}') \right) \end{aligned}$$

Since from (4.14),

$$\lambda_{\min} \left(p_\epsilon, \frac{\partial^2 \tilde{\phi}}{\partial p^2}(x_\epsilon, p_\epsilon, q_\epsilon) \right) \geq 0,$$

we get

$$\lambda_{\min} \left(\bar{p}', \frac{\partial^2 \phi}{\partial p^2}(\bar{x}, \bar{p}', \bar{q}') \right) \geq 0.$$

Moreover, since Π_2 is affine in $\Delta(J)$ and $\Pi_2(q) \in \Delta(J')$ for any $q \in \Delta(J)$ and since (4.13) holds, we have, for any $z \in T_{\Delta(J)}(q_\epsilon) \setminus \{0\}$ and at $(x_\epsilon, p_\epsilon, q_\epsilon)$

$$\left\langle \frac{\partial^2 \tilde{\phi}}{\partial q^2} z, z \right\rangle = \left\langle \frac{\partial^2 \phi}{\partial q^2} d\Pi_2(q_\epsilon)(z), d\Pi_2(q_\epsilon)(z) \right\rangle \leq -\alpha |d\Pi_2(q_\epsilon)(z)|^2$$

as soon as ϵ is small enough. We note that $z \in T_{\Delta(J)}(q_\epsilon) \setminus \{0\}$ implies that $\sum_i z_i = 0$, so that $d\Pi_2(q_\epsilon)(z)$ cannot vanish unless $z = 0$. So

$$\lambda_{\max} \left(q_\epsilon, \frac{\partial^2 \tilde{\phi}}{\partial q^2}(x_\epsilon, p_\epsilon, q_\epsilon) \right) < 0.$$

Then (4.14) implies that

$$w + H(x, D\tilde{\phi}, D^2\tilde{\phi}) \leq 0 \quad \text{at } (x_\epsilon, p_\epsilon, q_\epsilon).$$

We get the desired inequality by letting $\epsilon \rightarrow 0$. □

5 A comparison principle

Let $H : \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S}_N \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ be continuous and satisfy

$$\begin{aligned} H(y, \xi_2, X_2, p, q) - H(x, \xi_1, X_1, p, q) \leq \\ \omega(|\xi_1 - \xi_2| + a|x - y|^2 + b + |x - y|(1 + |\xi_1| + |\xi_2|)) \end{aligned} \quad (5.15)$$

where ω is continuous and non decreasing with $\omega(0) = 0$, for any $a, b \geq 0$, $(p, q) \in \Delta(I) \times \Delta(J)$, $x, y, \xi_1, \xi_2 \in \mathbb{R}^N$ and $X_1, X_2 \in \mathcal{S}_N$ such that

$$\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq a \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + bI$$

In this section we give a proof of following result:

Theorem 5.1 *Let us assume that H satisfies the structure condition (5.15). Let w_1 be a bounded, uniformly continuous subsolution of (1.1) which is uniformly Lipschitz continuous w.r. to p and q and w_2 be a bounded, uniformly continuous supersolution of (1.1) which is uniformly Lipschitz continuous w.r. to p and q . Then*

$$w_1(x, p, q) \leq w_2(x, p, q) \quad \forall (x, p, q) \in \mathbb{R}^N \times \Delta(I) \times \Delta(J) .$$

Proof of Theorem 5.1 : Quite surprisingly, we have to do the proof by induction on $|I| + |J|$. For $|I| + |J| = 2$, i.e., when (1.1) reduces to the ordinary Hamilton-Jacobi equation

$$w_t + H(x, Dw, D^2w) = 0 \quad \text{in } \mathbb{R}^N ,$$

then the result holds from standard theory of viscosity solutions (see [5]). We now assume that the result holds true whenever $|I| + |J| \leq n$, for some $n \geq 2$. Let I, J be such that $|I| + |J| = n + 1$, and let w_1 and w_2 be as in the Theorem. From Proposition 4.1 the restriction of w_1 and w_2 to any face of $\Delta(I)$ and $\Delta(J)$ is still a subsolution (resp. supersolution) of (1.1). From the recurrence condition, this implies that

$$w_1 \leq w_2 \quad \text{on } \mathbb{R}^N \times [(\partial\Delta(I) \times \Delta(J)) \cup (\Delta(I) \times \partial\Delta(J))] . \quad (5.16)$$

We now start the proof of the inequality $w_1 \leq w_2$ in $\mathbb{R}^N \times \Delta(I) \times \Delta(J)$ in the usual way, by assuming that

$$\sup_{x,p,q} (w_1 - w_2) > 0 .$$

Since w_1 and w_2 are uniformly continuous and bounded, classical arguments show that, for $\epsilon, \alpha > 0$,

$$M_{\epsilon,\alpha} := \sup_{x,y,p,q} \left\{ w_1(x, p, q) - w_2(y, p, q) - \left(\frac{|x-y|^2}{2\epsilon} + \frac{\alpha}{2}(|x|^2 + |y|^2) \right) \right\} \quad (5.17)$$

is finite and achieved at a point $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$. One can also show that

$$\lim_{\epsilon, \alpha \rightarrow 0^+} M_{\epsilon,\alpha} = \sup_{x,p,q} (w_1 - w_2) \quad (5.18)$$

and that

$$\frac{|\bar{x} - \bar{y}|^2}{\epsilon^2} , \alpha|\bar{x}|^2 , \alpha|\bar{y}|^2 \leq 2M_\infty \quad (5.19)$$

where $M_\infty = |w_1|_\infty + |w_2|_\infty$. We also note that, because of (5.16), for any maximum point $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ of (5.17), one has $(\bar{p}, \bar{q}) \in \text{Int}(\Delta(I) \times \Delta(J))$ as soon as $M_{\epsilon,\alpha} > 0$ (i.e., for ϵ and α small enough).

We now introduce a new penalization: For $\beta > 0$ and $\delta > 0$ small, the problem

$$M_{\epsilon, \alpha, \delta} := \max_{\substack{x, y \in \mathbb{R}^N \\ (p, q), (p', q') \in \Delta(I) \times \Delta(J)}} w_1(x, p, q) - w_2(y, p', q') - \frac{|x - y|^2}{2\epsilon} \\ - \frac{\alpha}{2}(|x|^2 + |y|^2) - \frac{|(p, q) - (p', q')|^2}{2\delta} + \frac{\beta}{2}(|p|^2 + |p'|^2 + |q|^2 + |q'|^2) \quad (5.20)$$

has a maximum $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{p}', \tilde{q}')$. Since as $\beta, \delta \rightarrow 0$, $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{p}', \tilde{q}')$ converges (up to subsequences) to some $(\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{p}', \bar{q}')$ where $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ is a maximum point of (5.17), one has $(\bar{p}, \bar{q}), (\bar{p}', \bar{q}') \in \text{Int}(\Delta(I)) \times \text{Int}(\Delta(J))$ for β and δ sufficiently small.

From the usual maximum principle (see [5]), for any $\sigma \in (0, 1)$ there are $X_1, X_2 \in \mathcal{S}_N$, $(P_1, Q_1), (P_2, Q_2) \in \mathcal{S}_I \times \mathcal{S}_J$ such that (denoting by P'_i and Q'_i the restrictions of P_i and Q_i to the spaces $T_I = \{z \in \mathbb{R}^I \mid \sum_i z_i = 0\}$ and $T_J = \{z \in \mathbb{R}^J \mid \sum_i z_i = 0\}$):

$$\left(\frac{(\tilde{x} - \tilde{y})}{\epsilon} + \alpha \tilde{x}, \frac{(\tilde{p} - \tilde{p}')}{\delta} - \beta \tilde{p}, \frac{(\tilde{q} - \tilde{q}')}{\delta} - \beta \tilde{q}, X_1, P'_1, Q'_1 \right) \in \overline{D}^{2,-} w_1(\tilde{x}, \tilde{p}, \tilde{q}), \\ \left(-\frac{(\tilde{y} - \tilde{x})}{\epsilon} - \alpha \tilde{y}, -\frac{(\tilde{p}' - \tilde{p})}{\delta} + \beta \tilde{p}', -\frac{(\tilde{q}' - \tilde{q})}{\delta} + \beta \tilde{q}', X_2, P'_2, Q'_2 \right) \in \overline{D}^{2,+} w_2(\tilde{y}, \tilde{p}', \tilde{q}')$$

and

$$\text{diag} \left(\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix}, \begin{pmatrix} P'_1 & 0 \\ 0 & -P'_2 \end{pmatrix}, \begin{pmatrix} Q'_1 & 0 \\ 0 & -Q'_2 \end{pmatrix} \right) \leq A + \sigma A^2$$

on $\mathbb{R}^{2N} \times (T_I)^2 \times (T_J)^2$, where

$$A = \text{diag} \left(\frac{1}{\epsilon} \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix} + \alpha I_{2N}, \frac{1}{\delta} \begin{pmatrix} I_I & -I_I \\ -I_I & I_I \end{pmatrix} - \beta I_{2I}, \frac{1}{\delta} \begin{pmatrix} I_J & -I_J \\ -I_J & I_J \end{pmatrix} - \beta I_{2J} \right)$$

Hence

$$\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq \left(\frac{1}{\epsilon} + 2\frac{\sigma}{\epsilon^2} + \frac{2\alpha\sigma}{\epsilon} \right) \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix} + (\alpha + \alpha^2\sigma) I_{2N} \quad (5.21)$$

and

$$P'_1 - P'_2 \leq 2(-\beta + \sigma\beta^2) I_I \quad \text{and} \quad Q'_1 - Q'_2 \leq 2(-\beta + \sigma\beta^2) I_J \quad (5.22)$$

on T_I and T_J . Since w_1 is a subsolution of (1.1) we have

$$\max \left\{ \min \left\{ w_1 + H \left(\tilde{x}, \frac{\tilde{x} - \tilde{y}}{\epsilon} + \alpha \tilde{x}, X_1, \tilde{p}, \tilde{q} \right); -\lambda_{\max}(\tilde{q}, Q_1) \right\}; -\lambda_{\min}(\tilde{p}, P_1) \right\} \leq 0. \quad (5.23)$$

In particular $\lambda_{\min}(\tilde{p}, P_1) \geq 0$ and, since $\tilde{p} \in \text{Int}(\Delta(I))$, we get from (5.22):

$$\lambda_{\min}(\tilde{p}', P_2) > 0. \quad (5.24)$$

In the same way, since w_2 is a supersolution of (1.2) in the dual sense, we have

$$\max \left\{ \min \left\{ w_2 + H \left(\tilde{y}, -\frac{(\tilde{y} - \tilde{x})}{\epsilon} - \alpha \tilde{y}, X_2, \tilde{p}', \tilde{q}' \right); -\lambda_{\max}(\tilde{q}', Q_2) \right\}; -\lambda_{\min}(\tilde{p}', P_2) \right\} \geq 0.$$

which, thanks to (5.24), entails that

$$w_2 + H \left(\tilde{y}, -\frac{(\tilde{y} - \tilde{x})}{\epsilon} - \alpha \tilde{y}, X_2, \tilde{p}', \tilde{q}' \right) \geq 0$$

and

$$\lambda_{\max}(\tilde{q}', Q_2) \leq 0.$$

Since $\tilde{q}' \in \text{Int}(\Delta(J))$, putting this inequality into (5.22) gives

$$\lambda_{\max}(\tilde{q}, Q_1) < 0,$$

which, combined with (5.23) yields to

$$w_1 + H \left(\tilde{x}, \frac{\tilde{x} - \tilde{y}}{\epsilon} + \alpha \tilde{x}, X_1, \tilde{p}, \tilde{q} \right) \leq 0.$$

We now let $\delta \rightarrow 0$ and $\beta \rightarrow 0$ to obtain

$$w_1(\bar{x}, \bar{p}, \bar{q}) + H \left(\bar{x}, \frac{\bar{x} - \bar{y}}{\epsilon} + \alpha \bar{x}, X_1, \bar{p}, \bar{q} \right) \leq 0 \quad (5.25)$$

and

$$w_2(\bar{y}, \bar{p}, \bar{q}) + H \left(\bar{y}, -\frac{(\bar{y} - \bar{x})}{\epsilon} - \alpha \bar{y}, X_2, \bar{p}, \bar{q} \right) \geq 0 \quad (5.26)$$

for some maximum point $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ of (5.17). Using the structure condition (5.15) on H , and plugging estimates (5.18), (5.19) and (5.21) into (5.25) and (5.26) yields to a contradiction for ϵ and α sufficiently small as in [5]. \square

6 Examples

6.1 Convex hull of a Lipschitz map

In [9] it is proved that, if $g : \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous, then the convex hull $u = \text{Vex}(g)$ of g is a solution of

$$\max \left\{ u - g; -\lambda_{\min} \left(\frac{\partial^2 u}{\partial p^2} \right) \right\} = 0. \quad (6.27)$$

As a consequence of Theorem 5.1 we get

Proposition 6.1 *Let $g : \Delta(I) \rightarrow \mathbb{R}$ be Lipschitz continuous. Then $u = \text{Vex}_p(g)$ is the unique Lipschitz continuous viscosity solution of (6.27).*

Remark : In \mathbb{R}^N the solution of equation (6.27) is *never* unique. Indeed, any affine map below g satisfies (6.27).

Proof : Following [9] we know that u is a solution of (6.27). Moreover it is Lipschitz continuous since g is Lipschitz continuous and we are working on the simplex $\Delta(I)$. This this equation has at most one Lipschitz continuous solution this proves the characterization.

□

6.2 The Φ operator of Mertens-Zamir

The above arguments extend to the Φ operator of Mertens-Zamir [8]. It is a mapping from the set $Lip(\Delta(I) \times \Delta(J))$ of Lipschitz continuous function on $\Delta(I) \times \Delta(J)$ into itself defined by: for any $g \in Lip(\Delta(I) \times \Delta(J))$, $u = \Phi(g)$ is the unique solution to

$$u = \text{Vex}_p(\max\{u, g\}) = \text{Cav}_q(\min\{u, g\}) ,$$

where $\text{Vex}_p(\phi)$ and $\text{Cav}_q(\phi)$ denote the convex and the concave envelopes of ϕ with respect to p and q respectively.

Proposition 6.2 *For any $g \in Lip(\Delta(I) \times \Delta(J))$, $u = \Phi(g)$ is the unique Lipschitz continuous solution of the equation*

$$\min \left\{ \max \left\{ w - g ; -\lambda_{\min} \left(\frac{\partial^2 w}{\partial p^2} \right) \right\} ; -\lambda_{\max} \left(\frac{\partial^2 w}{\partial q^2} \right) \right\} = 0 \quad (6.28)$$

Proof : From Theorem 5.1 equation (6.28) has at most one Lipschitz continuous solution. Hence it is enough to show that $u = \Phi(g)$ is a viscosity solution. We only prove that it is a supersolution: the fact that it is a subsolution can be established in the same way. Let ϕ be a smooth test function such that $u \geq \phi$ in $\Delta(I) \times \Delta(J)$ with an equality at $(\bar{p}, \bar{q}) \in \Delta(I) \times \text{Int}(\Delta(J))$. Since u is concave and $\bar{q} \in \text{Int}(\Delta(J))$ we get $\lambda_{\max} \left(\bar{q}, \frac{\partial^2 \phi}{\partial q^2}(\bar{p}, \bar{q}) \right) \leq 0$. Let us assume that $\lambda_{\min} \left(\bar{p}, \frac{\partial^2 \phi}{\partial p^2}(\bar{p}, \bar{q}) \right) > 0$ and prove that $(u - g)(\bar{p}, \bar{q}) \geq 0$. Indeed, since $u = \text{Vex}_p(\max\{u, g\})$ is strictly convex at (\bar{p}, \bar{q}) , one necessarily get that $u(\bar{p}, \bar{q}) = \max\{u(\bar{p}, \bar{q}), g(\bar{p}, \bar{q})\}$. Hence $u(\bar{p}, \bar{q}) \geq g(\bar{p}, \bar{q})$. □

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