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WELL-POSEDNESS OF THE SPATIALLY HOMOGENEOUS LANDAU EQUATION FOR SOFT POTENTIALS

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Abstract. We consider the spatially homogeneous Landau equation of kinetic theory, and provide a differential inequality for the Wasserstein distance with quadratic cost between two solutions. We deduce some well-posedness results. The main difficulty is that this equation presents a singularity for small relative velocities. Our uniqueness result is the first one in the important case of soft potentials. Furthermore, it is almost optimal for a class of moderately soft potentials, that is for a moderate singularity. Indeed, in such a case, our result applies for initial conditions with finite mass, energy, and entropy. For the other moderatley soft potentials, we assume additionnally some moment conditions on the initial data. For very soft potentials, we obtain only a local (in time) well-posedness result, under some integrability conditions. Our proof is probabilistic, and uses a stochastic version of the Landau equation, in the spirit of Tanaka [14].

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1. Introduction and results

1.1. **The Landau equation.** We consider the spatially homogeneous Landau equation in dimension 3 for soft potentials. This equation of kinetic physics, also called Fokker-Planck-Landau equation, has been derived from the Boltzmann equation when the grazing collisions prevail in the gas. It describes the density $f_t(v)$ of particles having the velocity $v \in \mathbb{R}^3$ at time $t \geq 0$:

$$\partial_t f_t(v) = \frac{1}{2} \sum_{i,j=1}^{3} \partial_i \left\{ \int_{\mathbb{R}^3} a_{ij}(v-v^*) \Big[f_t(v^*) \partial_j f_t(v) - f_t(v) \partial_j f_t(v^*) \Big] dv^* \right\}, \tag{1.1}$$

where $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial v_i}$ and $a(z)$ is a symmetric nonnegative matrix, depending on a parameter γ (we will deal here with soft potentials, that is $\gamma \in (-3, 0)$), defined by

$$a_{ij}(z) = |z|^\gamma (|z|^2 \delta_{ij} - z_i z_j). \tag{1.2}$$

The weak form of (1.1) writes, for any test function $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$

$$\frac{d}{dt} \int_{\mathbb{R}^3} \varphi(v) f_t(v) dv = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_t(dv) f_t(dv^*) L\varphi(v, v^*)$$

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where the operator L is defined by

$$L\varphi(v,v^*) = \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}(v-v^*)\partial^2_{ij}\varphi(v) + \sum_{i=1}^{3} b_i(v-v^*)\partial_i\varphi(v) \tag{1.3}$$

$$\text{with } b_i(z) = \sum_{j=1}^{3}\partial_j a_{ij}(z) = -2|z|^\gamma z_i, \text{ for } i=1,2,3. \tag{1.4}$$

We observe that the solutions to (1.1) conserve, at least formally, the mass, the momentum and the kinetic energy: for any $t \geq 0$,

$$\int_{\mathbb{R}^3} f_t(v)\varphi(v)dv = \int_{\mathbb{R}^3} f_0(v)\varphi(v)dv, \quad \text{for} \quad \varphi(v) = 1,\ v,\ |v|^2.$$

We classically may assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v)dv = 1$.
Another fundammental *a priori* estimate is the decay of entropy, that is solutions satisfy, at least formally, for all $t \geq 0$,

$$\int_{\mathbb{R}^3} f_t(v)\log f_t(v)dv \leq \int_{\mathbb{R}^3} f_0(v)\log f_0(v)dv.$$

We refer to Villani [17, 18] for many details on this equation, its physical meaning, its derivation from the Boltzmann equation, and on what is known about this equation.

1.2. **Existing results and goals.** One usually speaks of hard potentials for $\gamma > 0$, Maxwell molecules for $\gamma = 0$, soft potentials for $\gamma \in (-3, 0)$, and Coulomb potential for $\gamma = -3$.
The Landau equation is a *continuous* approximation of the Boltzmann equation: when there are infinitely many infinitesimally small collisions, the particle velocities become continuous in time, which can be modeled by equation (1.1). The most interesting case is that of Coulomb potential, since then the Boltzmann equation seems to be meaningless. Unfortunately, it is also the most difficult case to study. However, the Landau equation can be derived from the Boltzmann equation with *true very soft potentials*, that is $\gamma \in (-3, -1)$. The main idea is that the more γ is negative, the more the Landau equation is physically interesting. We refer again to [18] for a detailed survey about such considerations.
Another possible issue concerns numerics for the Boltzmann equation without cutoff: one can approximate grazing collisions by the Landau equation.

Let us mention that existence of weak solutions, under physically reasonnable assumptions on initial conditions, has been obtained by Villani [17] for all previously cited potentials.

We study here the question of uniqueness (and stability with respect to the initial condition). This question is of particular importance, since uniqueness is needed to justify the derivation of the equation, the convergence from the Boltzmann equation to the Landau equation, the convergence of some numerical schemes, ...
Villani [16] has obtained uniqueness for Maxwell molecules, and this was extended by Desvillettes-Villani [5] to the case of hard potentials. To our knowledge, there is no available result in the important case of soft potentials. We adapt in this paper the ideas of some recent works on the Boltzmann equation, see [7, 8] (see also Desvillettes-Mouhot [6] for other ideas). We will essentially prove here that uniqueness and stability hold in the following situations:

(a) for $\gamma \in (\gamma_0, 0)$, with $\gamma_0 = 1 - \sqrt{5} \simeq -1.236$, as soon as f_0 satifies the sole physical assumptions of finite mass, energy and entropy;

(b) for $\gamma \in (-2, \gamma_0]$, as soon f_0 has finite mass, energy, entropy, plus some finite moment of order $q(\gamma)$ large enough;
(c) for $\gamma \in (-3, -2]$, as soon f_0 has a finite mass, energy, and belongs to L^p, with $p > 3/(3+\gamma)$, and the result is local in time.

Observe that (a) is extremely satisfying, and (c) is quite disappointing.
Observe also that we obtain some much better results than for the Boltzmann equation [7, 8], where well-posedness is proved in the following cases: for $\gamma \in (-0.61, 0)$ and f_0 with finite mass, energy, entropy; for $\gamma \in (-1, -0.61)$ and f_0 with finite mass, energy, entropy, and a moment of order $q(\gamma)$ sufficiently large, and for $\gamma \in (-3, -1]$ and $f_0 \in L^p$ with finite mass and energy, with $p > 3/(3+\gamma)$, and the result being local in time in the latest case.

In [17], Villani proved several results on the convergence of the Boltzmann equation to the Landau equation. His results work *up to extraction of a subsequence*. Of course, our uniqueness result allows us to get a true convergence.

1.3. **Some notation.** Let us denote by C^2_b the set of C^2-functions $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ bounded with their derivatives of order 1 and 2. Let also $L^p(\mathbb{R}^3)$ be the space of measurable functions f with $||f||_{L^p} := (\int_{\mathbb{R}^3} f^p(v)dv)^{1/p} < \infty$, and let $\mathcal{P}(\mathbb{R}^3)$ be the space of probability measures on $\mathbb{R}^3$. For $k \geq 1$, we set

$$\mathcal{P}_k(\mathbb{R}^3) = \{f \in \mathcal{P}(\mathbb{R}^3) : m_k(f) < \infty\} \quad \text{with} \quad m_k(f) := \int_{\mathbb{R}^3} |v|^k f(dv).$$

For $\alpha \in (-3, 0)$, we introduce the space $\mathcal{J}_\alpha$ of probability measures f on $\mathbb{R}^3$ such that

$$J_\alpha(f) \quad := \quad \sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\alpha f(dv_*) < \infty, \tag{1.5}$$

We denote by $L^\infty\big([0,T], \mathcal{P}_k\big)$, $L^1\big([0,T], L^p\big)$, $L^1([0,T], \mathcal{J}_\alpha)$ the set of measurable families $(f_t)_{t\in[0,T]}$ of probability measures on $\mathbb{R}^3$ such that $\sup_{t\in[0,T]} m_k(f_t) < \infty$, $\int_0^T ||f_t||_{L^p} dt < \infty$, $\int_0^T J_\alpha(f_t)dt < \infty$ respectively. Observe that (see [8, (5.2)]):

Remark 1.1. *For $\alpha \in (-3, 0)$ and $p > 3/(3+\alpha)$, there exists a constant $C_{\alpha,p} > 0$ such that for all nonnegative measurable $f : \mathbb{R}^3 \mapsto \mathbb{R}$, any $v^* \in \mathbb{R}^d$*

$$J_\alpha(f) \leq ||f||_{L^1} + C_{\alpha,p}||f||_{L^p}.$$

For a nonnegative function $f \in L^1(\mathbb{R}^3)$, we denote its entropy by

$$H(f) = \int_{\mathbb{R}^3} f(v) \log(f(v))dv.$$

Finally we denote $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$ for $x, y \in \mathbb{R}_+$, and $z.\tilde{z}$ the scalar product of $z, \tilde{z} \in \mathbb{R}^3$ and $\mathcal{L}(X)$ the distribution of a random variable X. For some set A we write $\mathbf{1}_A$ the usual indicator function of A.

1.4. **Weak solutions.** We introduce now the notion of weak solution for the Landau equation. Some much more refined definitions were introduced by Villani [17] to allow solutions with only finite mass, energy, and entropy.

We observe here that for $\varphi \in C^2_b$, $|L\varphi(v, v_*)| \leq C_\varphi(|v - v_*|^{\gamma+1} + |v - v_*|^{\gamma+2})$. Thus if $\gamma \in [-1, 0]$, $|L\varphi(v, v_*)| \leq C_\varphi(1+|v|^2+|v_*|^2)$, while if $\gamma \in (-3, -1)$, $|L\varphi(v, v_*)| \leq C_\varphi(1+|v|^2+|v_*|^2+|v-v_*|^{\gamma+1})$. This guarantees that all the terms are well-defined in the definition below.

Definition 1.2. *Let $\gamma \in (-3,0)$. Consider a measurable family $f = (f_t)_{t\in[0,T]} \in L^\infty\big([0,T],\mathcal{P}_2\big)$. If $\gamma \in (-3,-1)$, assume additionnaly that $f \in L^1([0,T],\mathcal{J}_{\gamma+1})$. We say that f solves the Landau equation (1.1) if for any $\varphi \in \mathcal{C}^2_b(\mathbb{R}^3)$ and any $t \in [0,T]$*

$$\int_{\mathbb{R}^3} \varphi(v) f_t(dv) = \int_{\mathbb{R}^3} \varphi(v) f_0(dv) + \int_0^t ds \iint_{\mathbb{R}^3\times\mathbb{R}^3} f_t(dv) f_t(dv^*) L\varphi(v,v^*),$$

where the operator L is defined by (1.3).

1.5. **The Wasserstein distance and the Monge-Kantorovich problem.** Given $\pi \in \mathcal{P}_2(\mathbb{R}^3\times\mathbb{R}^3)$, we respectively denote by π_1 and π_2 its first and second marginals on $\mathbb{R}^3$. For two probability measures $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^3)$ and $\pi \in \mathcal{P}_2(\mathbb{R}^3\times\mathbb{R}^3)$, we write $\pi <^\mu_\nu$ if $\pi_1 = \mu$ and $\pi_2 = \nu$.
The Wassertein distance $\mathcal{W}_2$ on $\mathcal{P}_2(\mathbb{R}^3)$ is defined by

$$\begin{aligned}\mathcal{W}_2^2(\mu,\nu) &= \inf_{\pi<^\mu_\nu} \int_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^2 \pi(dx,dy)\\ &= \inf\{\mathbb{E}[|V-\tilde V|^2] : (V,\tilde V) \in \mathbb{R}^3\times\mathbb{R}^3 \text{ such that } \mathcal{L}(V)=\mu \text{ and } \mathcal{L}(\tilde V)=\nu\}.\end{aligned}$$

The set $(\mathcal{P}_2(\mathbb{R}^3),\mathcal{W}_2)$ is a Polish space, see e.g. Rachev and Rüschendorf [13]. The topology is stronger that the usual weak topology (for more details, see Villani, [19] Theorem 7.12).
It is well-known (see e.g. Villani [19] Chapter 1 for details) that the infimum is actually a minimum, and that if μ (or ν) has a density with respect to the Lebesgue measure on $\mathbb{R}^3$, then there is a unique $\pi <^\mu_\nu$ such that $\mathcal{W}_2^2(\mu,\nu) = \int_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^2\pi(dx,dy)$. Moreover, if we consider a family $(\mu_\lambda,\nu_\lambda)_{\lambda\in\Lambda}$ of $\mathcal{P}_2(\mathbb{R}^3)\times\mathcal{P}_2(\mathbb{R}^3)$ such that the map $\lambda \mapsto (\mu_\lambda,\nu_\lambda)$ is measurable, and if μ_λ has a density for all λ, then the function $\lambda \mapsto \pi_\lambda$ is measurable, see Fontbona-Guérin-Méléard [9].

1.6. **A general inequality.** Our main result reads as follows.

Theorem 1.3. *Let $T>0$ and $\gamma \in (-3,0)$. Consider two weak solutions f and $\tilde f$ to the Landau equation (1.1) such that $f,\tilde f \in L^\infty\big([0,T],\mathcal{P}_2\big) \cap L^1([0,T],\mathcal{J}_\gamma)$. We also assume that f_t (or $\tilde f_t$) has a density with respect the Lebesgue measure on $\mathbb{R}^3$ for each $t\in[0,T]$. Then there exists an explicit constant $C_\gamma > 0$, depending only on γ, such that for all $t \in [0,T]$,*

$$\mathcal{W}_2^2(f_t,\tilde f_t) \le \mathcal{W}_2^2(f_0,\tilde f_0) \exp\left(C_\gamma \int_0^t (J_\gamma(f_s)+J_\gamma(\tilde f_s))dt\right).$$

Observe that if $H(f_0)<\infty$, then $H(f_t)<\infty$ for all $t\ge 0$, so that f_t has a density for all times. Thus this result always applies for solutions with finite entropy.
This result asserts that uniqueness and stability hold in $L^\infty\big([0,T],\mathcal{P}_2\big)\cap L^1([0,T],\mathcal{J}_\gamma)$. Using Remark 1.1, we immediately deduce that uniqueness and stability also hold in $L^\infty\big([0,T],\mathcal{P}_2\big)\cap L^1([0,T],L^p)$, as soon as $p > 3/(3+\gamma)$.

1.7. **Applications.** We now show the existence of solutions in $L^\infty\big([0,T],\mathcal{P}_2\big)\cap L^1([0,T],L^p)$.

Corollary 1.4. *(i) Assume that $\gamma\in(-2,0)$. Let $q(\gamma) := \gamma^2/(2+\gamma)$. Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfy also $H(f_0)<\infty$ and $m_q(f_0)<\infty$, for some $q > q(\gamma)$. Consider $p \in (3/(3+\gamma),(3q-3\gamma)/(q-3\gamma)) \subset (3/(3+\gamma),3)$. Then the Landau equation (1.1) has an unique weak solution in $L^\infty\big([0,T],\mathcal{P}_2\big)\cap L^1([0,T],L^p)$.*
(ii) Assume that $\gamma\in(-3,0)$, and let $p > 3/(3+\gamma)$. Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$. Then there exists $T^*>0$ depending on $\gamma,p,||f_0||_p$ such that there is an unique weak solution in $L^\infty_{loc}([0,T_*),L^p)$.

Observe that if $\gamma \in (1-\sqrt{5}, 0) \simeq (-1.236, 0)$, then $q(\gamma) < 2$, and thus we obtain the well-posedness for the Landau equation under the physical assumptions of finite mass, energy, and entropy. This is of course extremely satisfying.

1.8. **Plan of the paper.** The paper is organized as follows: we first prove Theorem 1.3 in Section 2, by means of stochastic Landau processes. Next, we prove Corollary 1.4 in Section 3.

2. A general inequality

We give in this section the proof of Theorem 1.3. In the whole section, $T > 0$ and $\gamma \in (-3, 0)$ are fixed, and we consider two weak solutions $f = (f_t)_{t\in[0,T]}$ and $\tilde{f} = (\tilde{f}_t)_{t\in[0,T]}$ to (1.1) belonging to $L^\infty\big([0,T], \mathcal{P}_2\big) \cap L^1([0,T], \mathcal{J}_\gamma)$. We also assume that f_t has a density with respect to the Lebesgue measure on $\mathbb{R}^3$ for each $t \geq 0$, which implies the uniqueness of the minimizer in $\mathcal{W}_2(f_t, \tilde{f}_t)$.

We now introduce two coupled Landau stochastic processes, the first one associated with f, the second one associated with $\tilde{f}$, in such a way that they remain as close to each other as possible. Our approach is inspired by [11], which was itself inspired by the work of Tanaka [14] on the Boltzmann equation.

For any $s \in [0,T]$, we denote by R_s the unique solution of the Monge-Kantorovich transportation problem for the couple $(f_s, \tilde{f}_s)$. Recall that $R_s(dv, d\tilde{v})$ is a probability measure on $\mathbb{R}^3 \times \mathbb{R}^3$, with marginals f_s and $\tilde{f}_s$, and that $\mathcal{W}_2^2(f_s, \tilde{f}_s) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde{v}|^2 R_s(dv, d\tilde{v})$. Let us notice that the map $s \mapsto R_s$ is measurable thanks to Theorem 1.3 of [9].

On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider $W(dv, d\tilde{v}, ds) = (W_1, W_2, W_3)(dv, d\tilde{v}, ds)$ a three-dimensional space-time white noise on $\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T]$ with covariance measure $R_s(dv, d\tilde{v})ds$ (in the sense of Walsh [20]): W is an orthogonal martingale measure with covariance $R_s(dv, d\tilde{v})ds$. Let us now consider two random variables $V_0, \tilde{V}_0$ with valued in $\mathbb{R}^3$ with laws $f_0, \tilde{f}_0$, independant of W, such that $\mathcal{W}_2^2(f_0, \tilde{f}_0) = \mathbb{E}[|V_0 - \tilde{V}_0|^2]$. We finally consider the two following $\mathbb{R}^3$-valued stochastic differential equations.

$$V_t = V_0 + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \sigma(V_s - v) W(dv, d\tilde{v}, ds) + \int_0^t \int_{\mathbb{R}^3} b(V_s - v) f_s(dv) ds, \tag{2.1}$$

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \sigma(\tilde{V}_s - \tilde{v}) W(dv, d\tilde{v}, ds) + \int_0^t \int_{\mathbb{R}^3} b(\tilde{V}_s - \tilde{v}) f_s(d\tilde{v}) ds, \tag{2.2}$$

where σ is a square root matrix of a (recall (1.2)), that is $a(z) = \sigma(z)\sigma^t(z)$, defined by

$$\sigma(z) = |z|^{\frac{\gamma}{2}} \begin{bmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{bmatrix}. \tag{2.3}$$

We denote $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by W and $V_0, \tilde{V}_0$, that is $\mathcal{F}_t = \sigma\{W([0,s] \times A),\ s \in [0,t], A \in \mathcal{B}(\mathbb{R}^3 \times \mathbb{R}^3)\} \vee \mathcal{N}$, where $\mathcal{N}$ is the set of all $\mathbb{P}$-null subsets.

Proposition 2.1. *(i) There exists a unique pair of continuous $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $(V_t)_{t\in[0,T]}$, $(\tilde{V}_t)_{t\in[0,T]}$ solutions to (2.1) and (2.2), satisfying $\mathbb{E}[\sup_{[0,T]}(|V_t|^2 + |\tilde{V}_t|^2)] < \infty$.*
(ii) For $t \in [0,T]$, consider the distributions $g_t = \mathcal{L}(V_t)$ and $\tilde{g}_t = \mathcal{L}(\tilde{V}_t)$. Then $g_t = f_t$ and $\tilde{g}_t = \tilde{f}_t$ for all $t \in [0,T]$.

The processes $(V_t)_{t\in[0,T]}$ and $(\tilde{V}_t)_{t\in[0,T]}$ are called *Landau processes* associated with f and $\tilde{f}$ respectively. We start the proof with a simple remark.

Remark 2.2. *Let us consider the functions σ and b respectively defined by (2.3) and (1.4), and recall that $\gamma < 0$. There exists a constant $C_\gamma > 0$ such that for any $z, \tilde{z} \in \mathbb{R}^3$,*

$$|\sigma(z) - \sigma(\tilde{z})| \leq \quad C_\gamma |z - \tilde{z}| \left(|z|^{\frac{\gamma}{2}} + |\tilde{z}|^{\frac{\gamma}{2}} \right) \quad \text{and} \quad |b(z) - b(\tilde{z})| \leq C_\gamma |z - \tilde{z}| \left(|z|^\gamma + |\tilde{z}|^\gamma \right).$$

Proof. We have

$$\begin{aligned} |\sigma(z) - \sigma(\tilde{z})| &\leq \left| |z|^{\gamma/2} - |\tilde{z}|^{\gamma/2} \right| |z| + |z - \tilde{z}| \, |\tilde{z}|^{\gamma/2} \\ &\leq \frac{|\gamma|}{2} |z| \, |z - \tilde{z}| \max(|z|^{\frac{\gamma}{2}-1}, |\tilde{z}|^{\frac{\gamma}{2}-1}) + (|z|^{\frac{\gamma}{2}} + |\tilde{z}|^{\frac{\gamma}{2}}) |z - \tilde{z}|. \end{aligned}$$

By symmetry, we deduce that

$$\begin{aligned} |\sigma(z) - \sigma(\tilde{z})| &\leq |z - \tilde{z}| \left(\frac{|\gamma|}{2} \min(|z|, |\tilde{z}|) \max(|z|^{\frac{\gamma}{2}-1}, |\tilde{z}|^{\frac{\gamma}{2}-1}) + |\tilde{z}|^{\frac{\gamma}{2}} + |z|^{\frac{\gamma}{2}} \right) \\ &\leq \left(\frac{|\gamma|}{2} + 1 \right) |z - \tilde{z}| \left(|z|^{\frac{\gamma}{2}} + |\tilde{z}|^{\frac{\gamma}{2}} \right) \end{aligned}$$

The same computation works with b, starting with $|b(z) - b(\tilde{z})| \leq 2 \, ||z|^\gamma - |\tilde{z}|^\gamma| \, |z| + 2|z - \tilde{z}| \, |\tilde{z}|^\gamma$. □

Proof of Proposition 2.1. We just deal with (2.1), the study of (2.2) being of course the same.

Step 1. We start with the proof of (i), that is existence and uniqueness of a solution for (2.1). We consider the map Φ which associates to a continuous adapted process $X = (X_t)_{t\in[0,T]} \in \mathcal{C}([0,T], \mathbb{R}^3)$, with $\mathbb{E}[\sup_{0\leq s\leq T} |X_s|^2] < \infty$, the continuous adapted process $\Phi(X) \in \mathcal{C}([0,T], \mathbb{R}^3)$ defined by

$$\Phi(X)_t = V_0 + \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sigma(X_s - v) W(dv, d\tilde{v}, ds) + \int_0^t \int_{\mathbb{R}^3} b(X_s - v) f_s(dv) ds.$$

Step 1.1. Let us first prove that

$$\mathbb{E}\left[\sup_{[0,T]} |\Phi(X)_t|^2 \right] \leq C_T \left(1 + \mathbb{E}[|V_0|^2] + \mathbb{E}[\sup_{[0,T]} |X_t|^2] + \sup_{[0,T]} m_2(f_s) + \left(\int_0^T J_\gamma(f_s) ds \right)^2 \right),$$

which is finite thanks to the conditions imposed on V_0, f, and X. Using Doob's inequality, we easily get, for some constant C,

$$\begin{aligned} \mathbb{E}\left[\sup_{[0,T]} |\Phi(X)_t|^2 \right] &\leq C\mathbb{E}[|V_0|^2] + C \int_0^T \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbb{E}[|\sigma(X_s - v)|^2] R_s(dv, d\tilde{v}) ds \\ &\quad + C\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^3} |b(X_s - v)| f_s(dv) ds \right)^2 \right] =: C(\mathbb{E}[|V_0|^2] + A + B). \end{aligned}$$

Using that the first marginal of R_s is f_s, that $|\sigma(z)|^2 \leq |z|^{\gamma+2} \leq (1 + |z|^2 + |z|^\gamma)$, we observe that

$$\begin{aligned} A &\leq C \int_0^T \int_{\mathbb{R}^3} (1 + \mathbb{E}[|X_s|^2] + |v|^2 + \mathbb{E}[|X_s - v|^\gamma] f_s(dv) ds \\ &\leq C \int_0^T (1 + \mathbb{E}[|X_s|^2] + m_2(f_s) + J_\gamma(f_s)) ds, \end{aligned}$$

by definition of J_γ, see (1.5). Next, using that $|b(z)| = 2|z|^{\gamma+1} \le 2 + 2|z|^\gamma + 2|z|$,

$$\begin{aligned} B &\le C\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^3} (1 + |X_s| + |v| + |X_s - v|^\gamma) f_s(dv)ds\right)^2\right] \\ &\le C\left[T^2 + T^2\mathbb{E}[\sup_{[0,T]} |X_s|^2] + T\int_0^T m_2(f_s)ds + \left(\int_0^T J_\gamma(f_s)ds\right)^2\right]. \end{aligned}$$

Step 1.2. Let us now consider two adapted processes $X, Y \in \mathcal{C}([0,T], \mathbb{R}^3)$, and show that

$$\mathbb{E}\left[\sup_{[0,t]} |\Phi(X)_s - \Phi(Y)_s|^2\right] \le C_\gamma \left(1 + \int_0^t J_\gamma(f_s)ds\right) \int_0^t \mathbb{E}[|X_s - Y_s|^2] J_\gamma(f_s)ds.$$

Arguing as previously and using Remark 2.2, we deduce that

$$\begin{aligned} \mathbb{E}\big[\sup_{[0,t]} |\Phi(X)_s - \Phi(Y)_s|^2\big] &\le 2\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbb{E}\left[|\sigma(X_s - v) - \sigma(Y_s - v)|^2\right] R_s(dv, d\tilde{v})ds \\ &\quad + 2\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^3} |b(X_s - v) - b(Y_s - v)| f_s(dv)ds\right)^2\right] \\ &\le C_\gamma \int_0^t \int_{\mathbb{R}^3} \mathbb{E}\left[|X_s - Y_s|^2 (|X_s - v|^\gamma + |Y_s - v|^\gamma)\right] f_s(dv)ds \\ &\quad + C_\gamma \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^3} |X_s - Y_s| (|X_s - v|^\gamma + |Y_s - v|^\gamma) f_s(dv)ds\right)^2\right] \\ &\le C_\gamma \int_0^t \mathbb{E}[|X_s - Y_s|^2] J_\gamma(f_s)ds + C_\gamma \mathbb{E}\left[\left(\int_0^t |X_s - Y_s| J_\gamma(f_s)ds\right)^2\right]. \end{aligned}$$

We conclude by using the Cauchy-Schwarz inequality.

Step 1.3. The uniqueness of the solution to (2.1) immediately follows from Step 1.2. Indeed, consider two solutions V^1 and V^2. Then $V^1 = \Phi(V^1)$ and $V^2 = \Phi(V^2)$, so that Step 1.2. implies that $\mathbb{E}[\sup_{[0,t]} |V_s^1 - V_s^2|^2] \le C\int_0^t E[|V_s^1 - V_s^2|^2] J_\gamma(f_s)ds$ for some constant $C > 0$ depending on γ, f. Since $t \mapsto J_\gamma(f_t) \in L^1([0,T])$ by assumption, the Gronwall Lemma implies that $\mathbb{E}[\sup_{[0,t]} |V_t^1 - V_t^2|^2] = 0$, whence $V^1 = V^2$.

Step 1.4. Finally, one classically obtains the existence of a solution using a Picard iteraction: consider the process V^0 defined by $V_t^0 = V_0$, and then define by induction $V^{n+1} = \Phi(V^n)$ (this is well-defined thanks to Step 1.1.). Using Step 1.2 and classical arguments, one easily checks that there exists a continuous adapted process V such that $\mathbb{E}[\sup_{[0,T]} |V_t - V_t^n|^2]$ tends to 0. It is not difficult to pass to the limit in $V^{n+1} = \Phi(V^n)$, whence $V = \Phi(V)$, and thus V solves (2.1).

Step 2. We now prove (ii). Let V be the unique solution of (2.1) and $g_s = \mathcal{L}(V_s)$ for all $s \in [0,T]$.

Step 2.1. We first check that the family g solves the *linear Landau equation*: for any $\varphi \in \mathcal{C}_b^2(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \varphi(x) g_t(dx) = \int_{\mathbb{R}^3} \varphi(x) f_0(dx) + \int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} L\varphi(x,v) g_s(dx) f_s(dv)ds. \tag{2.4}$$

with L defined by (1.3). Applying the Itô formula, we immediately get

$$\varphi(V_t) = \varphi(V_0) + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \sum_{i,j} \partial_i\varphi(V_s)\sigma_{ij}(V_s - v)W_j(dv, d\tilde{v}, ds) + \int_0^t \int_{\mathbb{R}^3} \sum_i \partial_i\varphi(V_s)b_i(V_s - v)f_s(dv)ds + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \sum_{i,j,k} \partial_{ij}\varphi(V_s)\sigma_{ik}(V_s - v)\sigma_{jk}(V_s - v)R_s(dv, d\tilde{v})ds.$$

Taking expectations (which makes vanish the first integral), using that the first marginal of R_s is f_s and that $\mathcal{L}(V_0) = f_0$, we obtain

$$\int_{\mathbb{R}^3} \varphi(x)g_t(dx) = \int_{\mathbb{R}^3} \varphi(x)f_0(dx) + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \sum_i \partial_i\varphi(x)b_i(x - v)g_s(dx)f_s(dv)ds + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \sum_{i,j,k} \partial_{ij}\varphi(x)\sigma_{ik}(x - v)\sigma_{jk}(x - v)g_s(dx)f_s(dv)ds,$$

from which the conclusion follows, recalling (1.3), since $\sigma.\sigma^t = a$.

Step 2.2. One may apply the general uniqueness result of Bhatt-Karandikar [4, Theorem 5.2], which implies uniqueness for (2.4). We omit the proof here, but we refer to [8, Lemma 4.6] for a very similar proof concerning the Boltzmann equation. The main idea is that roughly, Bhatt-Karandikar have proved that existence and uniqueness for a stochastic differential equation implies uniqueness for the associated Kolmogorov equation, so that essentially, point (i) implies uniqueness for (2.4).

Step 2.3. But f, being a weak solution to (1.1), is also a weak solution to (2.4). We deduce that for all $t \in [0, T]$, $g_t = f_t$. □

To prove Proposition 2.1 there was no need to couple the stochastic processes V and $\tilde{V}$ with the same white noise. But to evaluate the Wasserstein distance between two Landau solutions f and $\tilde{f}$ using the stochastic processes, it is essential to connect them with the same white noise as we can see below.

Proposition 2.3. *Consider the unique solutions V and $\tilde{V}$ to (2.1) and (2.2) defined in Proposition 2.1. There exists a constant $C_\gamma > 0$ depending only on γ such that*

$$\mathbb{E}[|V_t - \tilde{V}_t|^2] \le \mathbb{E}[|V_0 - \tilde{V}_0|^2] + C_\gamma \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbb{E}\big[\big(|V_s - \tilde{V}_s|^2 + |v - \tilde{v}|^2\big)\big(|V_s - v|^\gamma + |\tilde{V}_s - \tilde{v}|^\gamma\big)\big] R_s(dv, d\tilde{v})ds.$$

Proof. First of all, we observe that since $R_s(dv, d\tilde{v})$ has the marginals $f_s(dv)$ and $\tilde{f}_s(d\tilde{v})$, we may rewrite equations (2.1) and (2.2) as

$$\begin{aligned} V_t &= V_0 + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \sigma(V_s - v)W(dv, d\tilde{v}, ds) + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} b(V_s - v)R_s(dv, d\tilde{v})ds, \\ \tilde{V}_t &= \tilde{V}_0 + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \sigma(\tilde{V}_s - \tilde{v})W(dv, d\tilde{v}, ds) + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} b(\tilde{V}_s - \tilde{v})R_s(dv, d\tilde{v})ds. \end{aligned}$$

Using the Itô formula and taking expectations, we obtain

$$\begin{aligned}\mathbb{E}[|V_t-\tilde V_t|^2] &= \mathbb{E}[|V_0-\tilde V_0|^2]+\sum_{i,l=1}^3\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathbb{E}\big[\big(\sigma_{il}(V_s-v)-\sigma_{il}(\tilde V_s-\tilde v)\big)^2\big]R_s(dv,d\tilde v)ds\\ &\quad+2\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathbb{E}\big[\big(b(V_s-v)-b(\tilde V_s-\tilde v)\big).(V_s-\tilde V_s)\big]R_s(dv,d\tilde v)ds.\end{aligned}$$

Using Remark 2.2, we deduce that for some constant C_γ,

$$\begin{aligned}\mathbb{E}[|V_t-\tilde V_t|^2] &\le \mathbb{E}[|V_0-\tilde V_0|^2]\\ &+ C_\gamma\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathbb{E}\big[|V_s-\tilde V_s-v+\tilde v|^2(|V_s-v|^\gamma+|\tilde V_s-\tilde v|^\gamma)\big]R_s(dv,d\tilde v)ds\\ &+ C_\gamma\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathbb{E}\big[|V_s-\tilde V_s-v+\tilde v||V_s-\tilde V_s|(|V_s-v|^\gamma+|\tilde V_s-\tilde v|^\gamma)\big]R_s(dv,d\tilde v)ds,\end{aligned}$$

from which the result immediately follows. □

We are finally in a position to conclude this section.

Proof of Theorem 1.3. Let us recall briefly the situation. We have two weak solutions f and $\tilde f$ to the Landau equation, belonging to $L^\infty\big([0,T],\mathcal{P}_2\big)\cap L^1([0,T],\mathcal{J}_\gamma)$. For each $s\in[0,T]$, R_s has marginals $f_s,\tilde f_s$, and satisfies $\mathcal{W}_2^2(f_s,\tilde f_s)=\iint_{\mathbb{R}^3\times\mathbb{R}^3}|v-\tilde v|^2R_s(dv,d\tilde v)$. Then we have introduced the solutions V and $\tilde V$ to (2.1-2.2), and we have shown that for each $t\in[0,T]$, $\mathcal{L}(V_t)=f_t$ and $\mathcal{L}(\tilde V_t)=\tilde f_t$. An immediate consequence of this is that $\mathcal{W}_2^2(f_t,\tilde f_t)\le\mathbb{E}[|V_t-\tilde V_t|^2]$.
Using Proposition 2.3, we get

$$\begin{aligned}\mathbb{E}[|V_t-\tilde V_t|^2] &\le \mathbb{E}[|V_0-\tilde V_0|^2]+C_\gamma\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathbb{E}[|V_s-\tilde V_s|^2]\big(|V_s-v|^\gamma+|\tilde V_s-\tilde v|^\gamma\big)\big]R_s(dv,d\tilde v)ds\\ &\quad+C_\gamma\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-\tilde v|^2\mathbb{E}\big(|V_s-v|^\gamma+|\tilde V_s-\tilde v|^\gamma\big)R_s(dv,d\tilde v)ds.\end{aligned}$$

But since $\mathcal{L}(V_s)=f_s$, we have $\mathbb{E}(|V_s-v|^\gamma)=\int_{\mathbb{R}^3}|x-v|^\gamma f_s(dx)\le J_\gamma(f_s)$. By the same way, $\mathbb{E}(|\tilde V_s-\tilde v|^\gamma)\le J_\gamma(\tilde f_s)$. On the other hand, since the first marginal of R_s is f_s, we deduce that $\int_{\mathbb{R}^3\times\mathbb{R}^3}|V_s-v|^\gamma R_s(dv,d\tilde v)=\int_{\mathbb{R}^3}|V_s-v|^\gamma f_s(dv)\le J_\gamma(f_s)$, and by the same way, $\int_{\mathbb{R}^3\times\mathbb{R}^3}|\tilde V_s-\tilde v|^\gamma R_s(dv,d\tilde v)\le J_\gamma(\tilde f_s)$. Thus, since $\mathbb{E}[|V_0-\tilde V_0|^2]=\mathcal{W}_2^2(f_0,\tilde f_0)$ and since $\mathcal{W}_2^2(f_t,\tilde f_t)\le\mathbb{E}[|V_t-\tilde V_t|^2]$,

$$\begin{aligned}\mathbb{E}[|V_t-\tilde V_t|^2] &\le \mathcal{W}_2^2(f_0,\tilde f_0)+C_\gamma\int_0^t(J_\gamma(f_s)+J_\gamma(\tilde f_s))\left[\mathbb{E}[|V_s-\tilde V_s|^2]+\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-\tilde v|^2R_s(dv,d\tilde v)\right]ds\\ &\le \mathcal{W}_2^2(f_0,\tilde f_0)+C_\gamma\int_0^t(J_\gamma(f_s)+J_\gamma(\tilde f_s))\Big[\mathbb{E}[|V_s-\tilde V_s|^2]+\mathcal{W}_2^2(f_s,\tilde f_s)\Big]ds\\ &\le \mathcal{W}_2^2(f_0,\tilde f_0)+C_\gamma\int_0^t(J_\gamma(f_s)+J_\gamma(\tilde f_s))\mathbb{E}[|V_s-\tilde V_s|^2]ds.\end{aligned}$$

Using finally the Gronwall Lemma, we get

$$\mathbb{E}[|V_t-\tilde V_t|^2]\le\mathcal{W}_2^2(f_0,\tilde f_0)\exp\left(C_\gamma\int_0^t(J_\gamma(f_s)+J_\gamma(\tilde f_s))ds\right),$$

which concludes the proof since $\mathcal{W}_2^2(f_t,\tilde{f}_t) \le \mathbb{E}[|V_t - \tilde{V}_t|^2]$. □

3. Applications

We now want to show that the uniqueness result proved in the previous section is relevant, in the sense that solutions in $L^\infty([0,T],\mathcal{P}_2) \cap L^1([0,T],\mathcal{J}_\gamma)$ indeed exist. We will use Remark 1.1, which says that $L^p \cap L^1 \subset \mathcal{J}_\gamma$ for $p > 3/(3+\gamma)$.
We first recall that moments of solutions propagate, we give some ellipticity estimate on the diffusion coefficient, and recall the chain rule for the Landau equation.
Then Corollary 1.4-(i) is obtained by using the dissipation of entropy. Finally, Corollary 1.4-(ii) is checked, using a direct computation.

3.1. **Moments.** We first recall the following result, which shows that moments of the solutions to the Landau equation propagate. We will use moments only in the case where $\gamma \in [-2,0)$, and the proposition below is quite easy. We refer to [18, Section 2.4. p 73]. When $\gamma \in (-3,-2]$, the situation is much more delicate, but Villani [15, Appendix B p 193] has also proved the propagation of moments.

Proposition 3.1. *Let $\gamma \in (-2,0)$. Let us consider a weak solution $(f_t)_{t\in[0,T]}$ to (1.1). Assume that for some $k \ge 2$, $m_k(f_0) \le M < \infty$. There is a constant C, depending only on k,γ,M,T such that $\sup_{[0,T]} m_k(f_s) \le C$.*

3.2. **Ellipticity.** We need a result as Desvillettes-Villani [5, Proposition 4] (who work with $\gamma \ge 0$) on the ellipticity of the matrix a.

Proposition 3.2. *Let $\gamma \in [-2,0)$. Let $E_0 > 0$ and $H_0 > 0$ be two constants, and consider a nonnegative function f such that $\int_{\mathbb{R}^3} f(v)dv = 1$, $m_2(f) \le E_0$ and $H(f) \le H_0$. There exists a constant $c > 0$ depending on γ, E_0, H_0 such that*

$$\forall \xi \in \mathbb{R}^3, \quad \sum_{i,j} \overline{a}^f_{ij}(v)\xi_i\xi_j \ge c(1+|v|)^\gamma |\xi|^2$$

where $\overline{a}^f(v) = \int_{\mathbb{R}^3} a(v-v_*)f(v_*)dv_*$.

Proof. Since $\gamma \in [-2,0]$, the proof of [5, Proposition 4] can be applied: following line by line their proof, one can check that they use only that $\gamma + 2 \ge 0$. □

One could easily extend this result to the case where $\gamma \in (-3,-2)$. However, we will not use it.

3.3. **Chain rule for the Landau equation.** As noted by Desvillettes-Villani [5, Section 6], we may write, for f a weak solution to the Landau equation and β is a C^1 function with $\beta(0) = 0$, at least formally,

$$\frac{d}{dt}\int_{\mathbb{R}^3} \beta(f_t(v))dv = -\int_{\mathbb{R}^3} \overline{a}(t,v)\nabla f_t(v)\nabla f_t(v)\beta''(f_t)(v)dv - \int_{\mathbb{R}^3} \overline{c}(t,v)\phi_\beta(f_t(v))dv \tag{3.1}$$

where

$$\begin{gathered}
\overline{a}(t,v) = \overline{a}^{f_t}(v) = \textstyle\int_{\mathbb{R}^3} a(|v-v_*|)f_t(v_*)dv_*, \\
\overline{a}(t,v)\nabla f_t(v)\nabla f_t(v) = \textstyle\sum_{i,j} \overline{a}_{ij}(t,v)(\nabla f_t(v))_i(\nabla f_t(v))_j, \\
\overline{c}(t,v) = -2(\gamma+3)\textstyle\int_{\mathbb{R}^3} |v-v_*|^\gamma f_t(v_*)dv_*, \\
\phi'_\beta(x) = x\beta''(x) \quad \text{and} \quad \phi_\beta(0) = 0.
\end{gathered}$$

3.4. **Moderately soft potentials.** Using the dissipation of entropy, we will deduce, at least for γ not too much negative, the L^p estimate we need. Such an idea was handled in the much more delicate case of the Boltzmann equation by Alexandre-Desvillettes-Villani-Wennberg [2].

Proposition 3.3. *We assume that $\gamma \in (-2,0)$. Let $\varepsilon \in (0,1)$ with $3-\varepsilon > 3/(3+\gamma)$. Consider a weak solution $(f_t)_{t\in[0,T]}$ to (1.1) starting from f_0 with $H(f_0) < \infty$, $m_2(f_0) < \infty$ and $m_q(f_0) < \infty$ with $q > 3|\gamma|(2-\varepsilon)/\varepsilon$. Then, at least formally, $f \in L^1([0,T], L^{3-\varepsilon})$.*

Before proving Proposition 3.3, we show how it allows us to conclude the well-posedness for the Landau equation when $\gamma \in (-2,0)$.

Proof of Corollary 1.4-(i). We only have to prove the existence, since the uniqueness immediately from Theorem 1.3 and Remark 1.1.
We thus assume that $\gamma \in (-2,0)$, and consider an initial condition $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$, with $H(f_0) < \infty$, and $m_2(f_0) < \infty$. Then Villani [17] has shown the existence of a weak solution $(f_t)_{t\in[0,T]} \in L^\infty([0,T],\mathcal{P}_2)$, with constant energy and nonnincreasing entropy, that is $m_2(f_t) = m_2(f_0)$ and $H(f_t) \le H(f_0)$ for all $t \in [0,T]$. We now use that $m_q(f_0) < \infty$, for some $q > q(\gamma) := \gamma^2/(2+\gamma)$, and we consider $p \in (3/(3+\gamma), (3q-3\gamma)/(q-3\gamma))$. Then one may check that for $\varepsilon > 0$ such that $p = 3-\varepsilon$, we have $3|\gamma|(2-\varepsilon)/\varepsilon < q$. Thus the *a priori* estimate proved in Proposition 3.3 implies that the solution $(f_t)_{t\in[0,T]}$ can be built in such a way that it lies in $L^1([0,T],L^p)$, which concludes the proof. □

Proof of Proposition 3.3. We thus consider a weak solution $(f_t)_{t\in[0,T]}$ to the Landau equation. Then this solution satisfies, at least formally, $H(f_t) \le H(f_0)$ and $m_2(f_t) = m_2(f_0)$, for all $t \in [0,T]$. As a consequence, the ellipticity estimate of Proposition 3.2 is uniform in time when applied to f_t. We now divide the proof into several steps.

Step 1. We apply (3.1) with the function $\beta(x) = (x+1)\ln(x+1)$. One easily checks that $\beta''(x) = \frac{1}{x+1}$ and $0 \le \phi_\beta(x) = x - \ln(x+1) \le x$. Since $H(f_0) < \infty$ by assumption, we easily see that $\int_{\mathbb{R}^3} \beta(f_0(v))dv < \infty$. Using Proposition 3.2, there exists a positive constant c (depending only on $\gamma, H(f_0), m_2(f_0)$) such that

$$\frac{d}{dt}\int_{\mathbb{R}^3}\beta(f_t(v))dv \le -c\int_{\mathbb{R}^3}(1+|v|)^\gamma \frac{|\nabla f_t(v)|^2}{1+f_t(v)}dv + 2(\gamma+3)\iint_{\mathbb{R}^3\times\mathbb{R}^3}|v-v_*|^\gamma f_t(v)f_t(v^*)dvdv^*. \tag{3.2}$$

First,

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^3}(1+|v|)^\gamma\frac{|\nabla f_t(v)|^2}{1+f_t(v)}dv = 4\int_{\mathbb{R}^3}(1+|v|)^\gamma|\nabla(\sqrt{1+f_t(v)}-1)|^2dv \\ &\ge 4(1+R)^\gamma||\nabla(\sqrt{1+f_t}-1)||^2_{L^2(B_R)}, \end{aligned}$$

for any $R > 0$, where $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$. By Sobolev embedding (see for example Adams [1]), we also know that there exists a constant $C > 0$ such that for any $R > 0$, any measurable $g : \mathbb{R}^3 \mapsto \mathbb{R}$

$$||g||_{L^6(B_R)} \le C||g||_{H^1(B_R)} = C\left(||\nabla g||_{L^2(B_R)} + ||g||_{L^2(B_R)}\right).$$

Consequently, there exists a constant $C > 0$ such that

$$\begin{aligned} ||\nabla(\sqrt{1+f_t}-1)||^2_{L^2(B_R)} &\ge C||\sqrt{1+f_t}-1||^2_{L^6(B_R)} - ||\sqrt{1+f_t}-1||^2_{L^2(B_R)} \\ &\ge C||\sqrt{f_t}\mathbf{1}_{\{f_t\ge1\}}||^2_{L^6(B_R)} - ||f_t||_{L^1(B_R)} \ge C||f_t\mathbf{1}_{\{f_t\ge1\}}||_{L^3(B_R)} - ||f_t||_{L^1(B_R)}. \end{aligned}$$

Finally, since $||f_t||_{L^1(B_R)} \leq ||f_t||_{L^1(\mathbb{R}^3)} = 1$, we get, for some $C > 0$, for all $R \geq 1$ (which implies $(1+R) \leq 2R$),

$$I_1 \geq 2^{2+\gamma} R^\gamma \left(C||f_t \mathbf{1}_{\{f_t \geq 1\}}||_{L^3(B_R)} - 1 \right). \tag{3.3}$$

Next we use Remark 1.1, the Hölder inequality and that $||f_t||_{L^1} = 1$, to get, for $p \in (3/(3+\gamma), 3-\varepsilon)$, for some $C = C_{\gamma,p}$,

$$\begin{aligned} I_2 &:= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v - v_*|^\gamma f_t(v) f_t(v^*) dv dv^* \leq ||f_t||_{L^1} J_\gamma(f_t) \\ &\leq C\left(1 + ||f_t||_{L^p}\right) = C\left(1 + \left(\int_{\mathbb{R}^3} f_t(v) f_t^{p-1}(v) dv\right)^{\frac{1}{p}}\right) \\ &\leq C\left(1 + \left(\int_{\mathbb{R}^3} f_t(v) f_t^{2-\varepsilon}(v) dv\right)^{\frac{(p-1)}{(2-\varepsilon)p}}\right) = C\left(1 + ||f_t||_{L^{3-\varepsilon}}^{\frac{(p-1)(3-\varepsilon)}{(2-\varepsilon)p}}\right). \end{aligned} \tag{3.4}$$

Set $\delta = \frac{(p-1)(3-\varepsilon)}{(2-\varepsilon)p}$. Using (3.2-3.3-3.4), we deduce that there is $C > 0$ such that for any $R \geq 1$,

$$\begin{aligned} R^\gamma \int_0^T ||f_s \mathbf{1}_{\{f_s \geq 1\}}||_{L^3(B_R)} ds &\leq C\left(\int_{\mathbb{R}^3} \beta(f_0(v)) dv - \int_{\mathbb{R}^3} \beta(f_T(v)) dv + \int_0^T \left(1 + ||f_s||_{L^{3-\varepsilon}}^\delta\right) ds\right) \\ &\leq C\left(1 + \int_0^T ||f_s||_{L^{3-\varepsilon}}^\delta ds\right). \end{aligned}$$

Step 2. For $\alpha > 0$, we define $g_t(v) = f_t(v)(1+|v|)^{\gamma-\alpha} \mathbf{1}_{\{f_t \geq 1\}}$. We have, using Step 1,

$$\begin{aligned} \int_0^T ||g_s||_{L^3} ds &\leq \int_0^T \sum_{k \geq 0} ||g_s||_{L^3(\{2^k - 1 \leq |v| \leq 2^{k+1} - 1\})} ds \leq \int_0^T \sum_{k \geq 0} 2^{k(\gamma-\alpha)} ||f_s \mathbf{1}_{\{f_s \geq 1\}}||_{L^3(B_{2^{k+1}})} ds \\ &\leq C 2^{-\gamma} \sum_{k \geq 0} 2^{-\alpha k} \left(1 + \int_0^T ||f_s||_{L^{3-\varepsilon}}^\delta ds\right) \leq C\left(1 + \int_0^T ||f_s||_{L^{3-\varepsilon}}^\delta ds\right). \end{aligned}$$

Step 3. We now prove if $\alpha > 0$ is small enough, for some constant C,

$$||f_t \mathbf{1}_{\{f_t \geq 1\}}||_{L^{3-\varepsilon}} \leq C\left(1 + ||g_t||_{L^3}\right). \tag{3.5}$$

We consider a nonnegative function h with $\int_{\mathbb{R}^3} h(v) dv \leq 1$. By Hölder's inequality, for $\varepsilon \in (0,1)$

$$\begin{aligned} ||h||_{L^{3-\varepsilon}} &= \left(\int_{\mathbb{R}^3} h^{2-\varepsilon}(v) \left(\frac{(1+|v|)^{(\gamma-\alpha)}}{(1+|v|)^{(\gamma-\alpha)}}\right)^{3(2-\varepsilon)/2} h(v) dv\right)^{1/(3-\varepsilon)} \\ &\leq \left(\int_{\mathbb{R}^3} \left[(1+|v|)^{\gamma-\alpha} h(v)\right]^3 dv\right)^{\frac{2-\varepsilon}{2(3-\varepsilon)}} \left(\int_{\mathbb{R}^3} (1+|v|)^{3(\alpha-\gamma)(2-\varepsilon)/\varepsilon} h(v) dv\right)^{\frac{\varepsilon}{2(3-\varepsilon)}} \\ &\leq \left(\int_{\mathbb{R}^3} \left[(1+|v|)^{\gamma-\alpha} h(v)\right]^3 dv\right)^{\frac{2-\varepsilon}{2(3-\varepsilon)}} \left(1 + \int_{\mathbb{R}^3} |v|^{3(\alpha-\gamma)(2-\varepsilon)/\varepsilon} h(v) dv\right). \end{aligned}$$

Then, for $h = f_t \mathbf{1}_{\{f_t \geq 1\}}$, setting $r = 3(\alpha-\gamma)(2-\varepsilon)/\varepsilon$, and recalling that $g_t(v) = f_t(v)(1+|v|)^{\gamma-\alpha}\mathbf{1}_{\{f_t\geq 1\}}$

$$||f_t \mathbf{1}_{\{f_t\geq 1\}}||_{L^{3-\varepsilon}} \leq (1+m_r(f_t))\,||g_t||_{L^3}^{\frac{3(2-\varepsilon)}{2(3-\varepsilon)}} \leq (1+m_r(f_t))(1+||g_t||_{L^3}).$$

But by assumption, $m_q(f_0) < \infty$ for some $q > 3|\gamma|(2-\varepsilon)/\varepsilon$, whence, by Proposition 3.1, $\sup_{[0,T]} m_q(f_t) < \infty$. Choosing $\alpha > 0$ small enough, in order that $q \geq r$, we deduce (3.5).

Step 4. Using that $\int_{\mathbb{R}^3} f_s(v)dv = 1$, Steps 2 and 3, we obtain, for some constant C (depending in particular on T),

$$\int_0^T ||f_s||_{L^{3-\varepsilon}} ds \leq C + \int_0^T ||f_s \mathbf{1}_{\{f_s\geq 1\}}||_{L^{3-\varepsilon}} ds \leq C + C\int_0^T ||f_s||_{L^{3-\varepsilon}}^{\delta} ds \leq C + C\left(\int_0^T ||f_s||_{L^{3-\varepsilon}}\right)^{\delta}.$$

But one may choose $p \in (3/(3+\gamma), 3-\varepsilon)$ (recall Step 1) such that $\delta = \frac{(p-1)(3-\varepsilon)}{p(2-\varepsilon)} \in (0,1)$ (choose p very close to $3/(3+\gamma)$ and use that by assumption, $3/(3+\gamma) < 3-\varepsilon$, whence $\varepsilon < \frac{6+3\gamma}{3+\gamma}$). As a consequence, $\int_0^T ||f_s||_{L^{3-\varepsilon}} ds \leq x_0$, where x_0 is the largest solution of $x = C + Cx^{\delta}$. Following carefully the proof above, one may check that C, and thus x_0, depends only on $T, f_0, q, \gamma, \varepsilon$. □

3.5. **Soft potentials.** We now would like to obtain a result which includes the case of very soft potentials, that is $\gamma \in (-3, -2]$.

Proposition 3.4. *Let $\gamma \in (-3,0)$ and $p \in (\frac{3}{3+\gamma}, +\infty)$. Let f be a weak solution of the Landau equation with $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$. Then there exists a time $T^* > 0$ depending on γ, p and $||f_0||_{L^p}$ such that for any $T \in [0, T^*)$, at least formally, $\sup_{[0,T]} ||f_t||_{L^p} < \infty$.*

Proof. Let us consider the function $\beta(x) = x^p$. Since $p > 1$, we have $\beta'' \geq 0$ and $\phi_\beta(x) = (p-1)x^p$. Using (3.1), neglecting all nonnegative terms, and using Remark 1.1, since $p > \frac{3}{3+\gamma}$, there exists a constant $C_{\gamma,p} > 0$ such that

$$\begin{aligned}\frac{d}{dt}||f_t||_{L^p}^p &\leq 2(\gamma+3)(p-1)\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v-v_*|^\gamma f_t(v_*) f_t^p(v) dv dv_* \\ &\leq ||f_t||_{L^p}^p J_\gamma(f_t) \leq C_{\gamma,p}(1+||f_t||_{L^p})||f_t||_{L^p}^p \leq C_{\gamma,p}(1+||f_t||_{L^p}^{2p}).\end{aligned}$$

Thus for all $0 \leq t < T^* := (\frac{\pi}{2} - \arctan ||f_0||_{L^p}^p)/C_{\gamma,p}$, we have $||f_t||_{L^p}^p \leq \tan(\arctan ||f_0||_{L^p}^p + C_{\gamma,p} t)$, which concludes the proof. □

Proof of Corollary 1.4-(ii). We only have to prove the existence, since the uniqueness follows immediately follows from Theorem 1.3 and Remark 1.1.
We thus assume that $\gamma \in (-3,0)$, $p > 3/(3+\gamma)$, and consider an initial condition $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ (which implies that $H(f_0) < \infty$). Then Villani [17] has shown the existence of a weak solution $(f_t)_{t\in[0,T]} \in L^\infty([0,T], \mathcal{P}_2)$, for arbitrary T.
But using the *a priori* estimate of Proposition 3.4, we deduce that this solution can be built in such a way that it belongs to $L^\infty_{loc}([0,T_*), L^p)$. This concludes the proof. □

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