# Renewal series and square-root boundaries for Bessel processes

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# RENEWAL SERIES AND SQUARE-ROOT BOUNDARIES FOR BESSEL PROCESSES

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**Abstract.** We show how a description of Brownian exponential functionals as a renewal series gives access to the law of the hitting time of a square-root boundary by a Bessel process. This extends classical results by Breiman and Shepp, concerning Brownian motion, and recovers by different means, extensions for Bessel processes, obtained independently by Delong and Yor.

Let $B_t$ be the standard real valued Brownian motion and for $\nu > 0$, introduce the geometric Brownian motion $\mathcal{E}_t^{(-\nu)}$ and its exponential functional $\mathcal{A}_t^{(-\nu)}$

$$\mathcal{E}_t^{(-\nu)} := exp(B_t - \nu t)$$

$$\mathcal{A}_t^{(-\nu)} := \int_0^t (\mathcal{E}_s^{(-\nu)})^2 ds.$$

Lamperti's representation theorem [5] applied to $\mathcal{E}_t^{(-\nu)}$ states

$$(0.1) \qquad \mathcal{E}_t^{(-\nu)} = R_{\mathcal{A}_t^{(-\nu)}}^{(-\nu)}$$

where $(R_u^{(-\nu)}, u \leq T_0(R^{(-\nu)}))$ denotes the Bessel process of index $(-\nu)$ (equivalently of dimension $\delta = 2(1 - \nu)$), starting at 1, which is an $\mathbb{R}_+$-valued diffusion with infinitesimal generator $\mathcal{L}^{(-\nu)}$ given by

$$\mathcal{L}^{(-\nu)} f(x) = \frac{1}{2} f''(x) + \frac{1 - 2\nu}{2x} f'(x), \quad f \in C_b^2(\mathbb{R}_+^\star).$$

Let us remark that, in the special case $\nu = 1/2$, equation (0.1) is nothing else but the Dubins-Schwarz representation of the exponential martingale $\mathcal{E}_t^{(-1/2)}$ as Brownian motion time changed with $\mathcal{A}_t^{(-1/2)}$.

For a short summary of relations between Bessel processes and exponentials of Brownian motion, see e.g. Yor [10].

Let us consider now the following random variable $Z$, which is often called a perpetuity in the mathematical finance literature:

$$Z := \mathcal{A}_\infty^{(-\nu)} = \int_0^\infty (\mathcal{E}_s^{(-\nu)})^2 ds$$

We deduce directly from (0.1) that

$$\mathcal{A}_\infty^{(-\nu)} = T_0(R^{(-\nu)})$$

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where $T_0 := \inf\{u : X_u = 0\}$, and it is well-known (see [4], [11]), that

$$(0.2) \qquad \qquad \mathcal{A}_\infty^{(-\nu)} \overset{(law)}{=} \frac{1}{2\gamma_\nu}$$

where $\gamma_\nu$ is a gamma variable with parameter $\nu$ (i.e. with density $\frac{1}{\Gamma(\nu)} x^{\nu-1} e^{-x} \mathbf{1}_{\mathbb{R}_+}$).

Our main result characterizes the law of the hitting time of a parabolic boundary by $R_u^{(-\nu)}$ which corresponds to a Bessel process of dimension $d < 2$.

**Theorem 1.** *Let* $0 < b < c$, *and* $\sigma := \inf\{u : (R_u^{(-\nu)})^2 = \frac{1}{c}(b + u)\}$ *with* $R_0^{(-\nu)} = 1$.

$$(0.3) \qquad E[(b+\sigma)^{-s}] = c^{-s} \frac{E[(1 + 2b\gamma_{\nu+s})^{-s}]}{E[(1 + 2c\gamma_{\nu+s})^{-s}]}, \quad \textit{for any } s \geq 0$$

Proof: using the strong Markov property and the stationarity of the increments of Brownian motion, we obtain that for any stopping time $\tau$ of the Brownian motion

$$\mathcal{A}_\infty^{(-\nu)} =: Z = \mathcal{A}_\tau^{(-\nu)} + (\mathcal{E}_\tau^{(-\nu)})^2 Z'$$

where $Z'$ is independent of $(\mathcal{A}_\tau^{(-\nu)}, \mathcal{E}_\tau^{(-\nu)})$ and $Z \overset{(law)}{=} Z'$.

This implies, by (0.1), that $Z$ satisfies the following affine equation (see [8] for a survey about these equations)

$$(0.4) \qquad \mathcal{A}_\infty^{(-\nu)} =: Z = \mathcal{A}_\tau^{(-\nu)} + (R_{\mathcal{A}_\tau^{(-\nu)}}^{(-\nu)})^2 Z'$$

where $Z'$ is independent of $(\mathcal{A}_\tau^{(-\nu)}, R_{\mathcal{A}_\tau^{(-\nu)}}^{(-\nu)})$ and $Z \overset{(law)}{=} Z'$.

Obviously, $\sigma < T_0(R^{(-\nu)})$. Taking now :

$$\tau = \inf\{t : (R_{\mathcal{A}_t^{(-\nu)}}^{(-\nu)})^2 = \frac{1}{c}(b + \mathcal{A}_t^{(-\nu)})\}$$

we get $\mathcal{A}_\tau^{(-\nu)} = \sigma$, and the identity in law

$$(0.5) \qquad \qquad b + Z \overset{(law)}{=} (b+\sigma)(1 + \frac{Z}{c})$$

where the variables $\sigma$ and $Z$ on the right-hand side are independent.

As a result, we obtain the Mellin-Stieltjes transform of $\sigma$:

$$E[(b+\sigma)^{-s}] = c^{-s} \frac{E[(b+Z)^{-s}]}{E[(c+Z)^{-s}]}$$

But, from (0.2)

$$E[(b+\sigma)^{-s}] = c^{-s} \frac{E[(2\gamma_\nu)^s \frac{1}{(1+2b\gamma_\nu)^s}]}{E[(2\gamma_\nu)^s \frac{1}{(1+2c\gamma_\nu)^s}]}$$

which gives the result.                                                            $\square$

One can now use the duality between the laws of Bessel processes of dimension $d$ and $4 - d$ to get the analogous result of Theorem 1, and recover the result of Delong [2], [3], and Yor [9] which deals with the case $d \geq 2$.

**Theorem 2.** *Let* $0 < b < c$, *and* $\sigma := \inf\{u : (R_u^{(\nu)})^2 = \frac{1}{c}(b + u)\}$ *with* $R_0^{(\nu)} = 1$.

$$(0.6) \qquad E[(b+\sigma)^{-s}] = c^{-s} \frac{E[(1 + 2b\gamma_s)^{-s+\nu}]}{E[(1 + 2c\gamma_s)^{-s+\nu}]}, \quad \textit{for any } s \geq 0.$$

Proof : it is based on the following classical relation between the laws of the Bessel processes with indices $\nu$ and $-\nu$:

$$(0.7) \qquad \mathcal{P}_x^{(\nu)}{}_{|\mathcal{F}_t} = \frac{(X_{t \wedge T_0})^{2\nu}}{x^{2\nu}} . \mathcal{P}_x^{(-\nu)}{}_{|\mathcal{F}_t}$$

which implies that

$$E_1^{(\nu)}[(b+\sigma)^{-s}] = E_1^{(-\nu)}[X_\sigma^{2\nu}(b+\sigma)^{-s}] = \frac{1}{c^\nu} E_1^{(-\nu)}[(b+\sigma)^{-s+\nu}]$$

Theorem 1 gives the result. $\qquad\square$

Finally, it is easily shown, thanks to the classical representations of the Whittaker functions (see Lebedev [6]), that the right-hand sides of (0.3) and (0.6) are expressed in terms of ratios of Whittaker functions.

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