

# HYPERBOLIC RELAXATION MODEL FOR GRANULAR FLOW

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**ABSTRACT.** In this work we describe an efficient model for the simulation of a two-phase flow made of a gas and a granular solid. The starting point is the two-velocity two-pressure model of Baer-Nunziato [1]. The model is supplemented by a relaxation source term in order to take into account the pressure equilibrium between the two phases and the granular stress in the solid phase. We show that the relaxation process can be made thermodynamically coherent with an adequate choice of the granular stress. We then propose a numerical scheme based on splitting approach. Each step of the time marching algorithm is made of two stages. In the first stage, the homogeneous convection equations are solved by a standard finite volume Rusanov scheme. In the second stage, the volume fraction is updated in order to take into account the equilibrium source term. The whole procedure is entropy dissipative. For simplified pressure laws (stiffened gas laws) we are able to prove that the approximated volume fraction satisfies a maximum principle.

## INTRODUCTION

We are interested in the numerical modeling of a two-phase (granular-gas) flow with two velocities and two pressures  $p_1$  and  $p_2$ . In one space dimension, the model is made up of seven non-homogeneous partial differential equations: two mass balance laws, two momentum balance laws, two energy balance laws and one volume fraction evolution equation. It is similar to the initial model proposed by Baer-Nunziato [1]. The main feature of this model is that the left hand side of the equations is hyperbolic. This property is very important because it ensures the mathematical stability of the model.

However, in many industrial applications it is not realistic to admit two independent pressures. Generally, an algebraic relation between the two pressures is assumed. An example (among many others) of such a modeling in the framework of internal ballistics is given by Gough in [7]. For a general presentation of two-phase flow models, we refer to the book of Gidaspow [4]. The relation between the two pressures is classically of the form  $p_2 = p_1 + R$  where  $R$  is the granular stress. In the general case, the granular stress depends on all the thermodynamic variables of the two phases.

Because of the pressure relation, the system is now overdetermined. The volume fraction equation can be eliminated and a six-equation model is obtained. Unfortunately, the new model has a reduced hyperbolicity domain. The worst situation corresponds to a vanishing granular stress  $R = 0$ . In this case, the model is almost never hyperbolic.

In the case of a vanishing granular stress, several authors have proposed to relax the algebraic relation  $p_2 = p_1$  by adding an adequate source term to the volume fraction evolution [12], [3], [8], *etc.* An important parameter of the source term is the characteristic equilibrium time. When the equilibration time tends to zero, the

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six equation model is recovered. When the equilibrium time is positive the stability of the model is improved.

In this paper, we extend the relaxation approach to non-vanishing granular stress  $R > 0$ . With a positive granular stress, the hyperbolicity domain of the six-equation model is slightly extended. But it is generally not possible to remove all the elliptic regions. Therefore, we apply a relaxation source term to the volume fraction evolution equation of the seven-equation model. This source term takes into account the granular stress (see (20)). When the relaxation parameter  $\tau_p$  tends to 0, we recover the equilibrium six-equation model. When the relaxation parameter  $\tau_p > 0$ , the stability of the model is enhanced.

An important aspect of the model is that the granular stress cannot be chosen arbitrarily once the pressure law of the solid phase is fixed. Indeed, it has to satisfy some thermodynamical relations in order that an entropy dissipation equation can be established. We illustrate this fact when the equation of state of the solid phase is a stiffened gas law. We propose a very simple but useful expression (see (14)) for the granular stress, which is mathematically and physically coherent.

Then we propose a numerical method to solve the relaxed system. For that purpose, we use a splitting algorithm. Each time step of the algorithm consists in:

- evolving the seven equation model without the source term;
- solving the relaxed pressure equilibrium with granular stress;
- solving the other source terms.

In the second stage the relaxed pressure equilibrium implies to solve an update for the volume fraction, keeping constant the conserved variables. Under some monotony hypothesis on the granular stress, and when the pressure laws of the two phases are stiffened gas laws, we are able to prove the existence and uniqueness of the new volume fraction in the interval  $]0, 1[$ .

The relaxed equilibrium becomes an exact equilibrium when the relaxation parameter  $\tau_p = 0$ . Thus our method can also be used to solve the equilibrium six-equation model on coarse meshes, which is important in industrial applications.

Finally, we propose some numerical experiments. For academic test cases, we demonstrate some behavior of the relaxed approach in the case of a non-stable (elliptic) case. We then compare the results of our new approach with the standard Gough model [11] in the case of a simplified internal ballistics problem.

## 1. NOTATIONS AND MODEL

We consider a two-phase flow of a granular solid mixed with a compressible gas. The solid is denoted by the index (2) and the gas by the index (1). For more generality, the solid is supposed to be compressible. The unknowns are, for each phase  $k = 1, 2$ , the partial density  $\rho_k$ , the velocity  $u_k$ , the internal energy  $e_k$ . The volume fractions  $\alpha_k$  satisfy  $\alpha_1 + \alpha_2 = 1$ . The solid volume fraction  $\alpha_2$  is also called the *porosity* in the context of granular flows. The pressure of each phase is given by an equation of state of the form

$$(1) \quad p_k = p_k(\rho_k, e_k).$$

We note  $\alpha_k \rho_k = m_k$ .

The balance of mass, momentum and energy read

$$(2) \quad \begin{aligned} m_{k,t} + (m_k u_k)_x &= \pm M, \\ (m_k u_k)_t + (m_k u_k^2 + \alpha_k p_k)_x - p_I \alpha_{k,x} &= \pm Q, \\ (m_k E_k)_t + ((m_k E_k + \alpha_k p_k) u_k)_x + p_I \alpha_{k,t} &= \pm S, \\ \alpha_{k,t} + v_I \alpha_{k,x} &= \pm P, \end{aligned}$$

where

$$(3) \quad E_k = e_k + \frac{u_k^2}{2}.$$

The right hand side terms  $M$ ,  $Q$ ,  $P$ ,  $S$  are internal exchange source terms that will be discussed later. Here, the sign  $\pm = +$  if  $k = 1$  and  $\pm = -$  if  $k = 2$ . For the moment, we suppose that there are no external force and energy source (this explains the  $\pm$  signs in the source terms). The quantities  $p_I$  and  $v_I$  are respectively the interface pressure and the interface velocity. In this paper, we take the special choice of Baer-Nunziato

$$(4) \quad \begin{aligned} p_I &= p_1, \\ v_I &= u_2. \end{aligned}$$

which enjoys good properties (see [3], [9]):

- the left hand side of the system is hyperbolic (the proof is recalled in Section 7);
- this choice ensures that the non-conservative products are well defined. This is due to the fact that the volume fraction only jumps in linearly degenerated fields. In a linearly degenerated field, the jump relations are simply provided by the Riemann invariants of this field. See [3];
- in the applications, the Baer-Nunziato model is particularly adapted to granular flows.

Using the volume fraction equation, the time derivative  $\alpha_{k,t}$  can be replaced by space derivatives

$$(5) \quad \begin{aligned} m_{k,t} + (m_k u_k)_x &= \pm M, \\ (m_k u_k)_t + (m_k u_k^2)_x + \alpha_k p_{k,x} + (p_k - p_I) \alpha_{k,x} &= \pm Q, \\ (m_k E_k)_t + (m_k E_k u_k)_x + \alpha_k (p_k u_k)_x + (p_k u_k - p_I v_I) \alpha_{k,x} &= \pm S \mp p_I P, \\ \alpha_{k,t} + v_I \alpha_{k,x} &= \pm P. \end{aligned}$$

The equations can then be written under the form of a first order non-conservative and non-homogeneous system

$$(6) \quad W_t + F(W)_x + A(W)L(W)_x = S(W),$$

with

$$(7) \quad \begin{aligned} W &= (m_1, m_1 u_1, m_1 E_1, m_2, m_2 u_2, m_2 E_2, \alpha_1)^T, \\ F(W) &= (m_1 u_1, m_1 u_1^2, m_1 E_1 u_1, m_2 u_2, m_2 u_2^2, m_2 E_2 u_2, 0)^T, \\ L(W) &= (p_1, p_1 u_1, p_2, p_2 u_2, \alpha_1)^T, \\ A(W)L(W)_x &= (0, \alpha_1 p_{1,x}, \alpha_1 (p_1 u_1)_x + p_1 (u_1 - u_2) \alpha_{1,x}, \\ &\quad 0, \alpha_2 p_{2,x} + (p_2 - p_1) \alpha_{2,x}, \alpha_2 (p_2 u_2)_x + u_2 (p_2 - p_1) \alpha_{2,x}, v_2 \alpha_{1,x})^T, \\ S(W) &= (M, Q, S - p_1 P, -M, -Q, -S + p_1 P, P)^T. \end{aligned}$$

Let us note that this writing is not unique. We have chosen a formulation in which the non-conservative terms vanish when the pressures and velocities are constant, *i.e.*

$$(8) \quad p_1 = p_2 = p_0 = \text{Cst and } u_1 = u_2 = u_0 = \text{Cst}.$$

## 2. ENTROPY DISSIPATION

In this section, we establish an entropy dissipation equation. This equation is very important because it permits to select the source terms that are compatible with the second principle of thermodynamics.

For that purpose we first rewrite the system as follows

$$\begin{aligned}
 & m_k (u_{k,t} + u_k u_{k,x}) + (\alpha_k p_k)_x - p_1 \alpha_{k,x} = \pm Q \mp u_k M, \\
 (9) \quad & m_k (e_{k,t} + u_k e_{k,x}) + \frac{u_k}{2} [(m_k u_k)_t + (m_k u_k^2 + \alpha_k p_k)_x + (\alpha_k p_k)_x] \\
 & + \frac{1}{2} m_k u_k (u_{k,t} + u_k u_{k,x}) + \alpha_k p_k u_{k,x} + p_1 \alpha_{k,t} = \pm S \mp e_k M.
 \end{aligned}$$

The last equation also reads

$$\begin{aligned}
 (10) \quad & m_k (e_{k,t} + u_k e_{k,x}) + \frac{u_k}{2} [p_1 \alpha_{k,x} \pm Q + (\alpha_k p_k)_x] \\
 & + \frac{1}{2} u_k (- (\alpha_k p_k)_x + p_1 \alpha_{k,x} \pm Q \mp u_k M) + \alpha_k p_k u_{k,x} + p_1 \alpha_{k,t} = \pm S \mp e_k M.
 \end{aligned}$$

and

$$\begin{aligned}
 (11) \quad & m_k (e_{k,t} + u_k e_{k,x}) + \alpha_k p_k u_{k,x} + p_1 (u_k - u_2) \alpha_{k,x} = \\
 & \pm S \mp e_k M \mp u_k Q \pm \frac{1}{2} u_k^2 M \mp p_1 P
 \end{aligned}$$

Finally, we obtain

$$\begin{aligned}
 (12) \quad & \alpha_{k,t} + u_2 \alpha_{k,x} = \pm P, \\
 & \alpha_k (\rho_{k,t} + u_k \rho_{k,x}) + \rho_k (u_k - u_2) \alpha_{k,x} + m_k u_{k,x} = \pm M \mp \rho_k P, \\
 & m_k (u_{k,t} + u_k u_{k,x}) + (\alpha_k p_k)_x - p_1 \alpha_{k,x} = \pm Q \mp u_k M, \\
 & m_k (e_{k,t} + u_k e_{k,x}) + \alpha_k p_k u_{k,x} + p_1 (u_k - u_2) \alpha_{k,x} = \\
 & \pm S \mp e_k M \mp u_k Q \pm \frac{1}{2} u_k^2 M \mp p_1 P
 \end{aligned}$$

Now, we introduce entropies  $s_k$  for the two phases. The entropy of the phase  $k = 1, 2$  satisfies the following first order partial differential equation

$$\begin{aligned}
 (13) \quad & T_k ds_k = de_k - \frac{p_k}{\rho_k^2} d\rho_k - \frac{R_k}{m_k} d\alpha_k \\
 & = T_k \left( \frac{\partial s_k}{\partial e_k} \right) de_k + T_k \left( \frac{\partial s_k}{\partial \rho_k} \right) d\rho_k + T_k \left( \frac{\partial s_k}{\partial \alpha_k} \right) d\alpha_k.
 \end{aligned}$$

The temperature of phase  $k$  is denoted by  $T_k$ . The granular stress in the phase  $k$  is denoted by  $R_k$ . Physically, the term  $-R_k/m_k d\alpha_k$  represents the work of the granular stress due to a change of volume  $d\alpha_k$ . For sake of simplicity and in order to be more realistic, we assume that the "granular" stress  $R_1$  vanishes in the gas phase, thus  $R_1 = 0$ . Without ambiguity, we can also denote the granular stress in the solid phase by  $R = R_2$ . Generally, the granular stress  $R$  depends on  $(\rho_2, e_2, \alpha_2)$ . When the solid phase is modeled by a stiffened gas equation of state (with a polytropic parameter  $\gamma_2$ ), a very nice choice is

$$(14) \quad R = R(\rho_2, \alpha_2) = \lambda (\rho_2 \alpha_2)^{\gamma_2}.$$

This choice is discussed in Section 3.

The chemical potential of phase  $k$  is noted  $\mu_k$  and is defined by

$$(15) \quad \mu_k = e_k + \frac{p_k}{\rho_k} - T_k s_k$$

We multiply the last equation in (??) by  $1/T_k$ , the second by  $-p_k/\rho_k/T_k$ , the first by  $-R_k/T_k$  and take the sum

$$\begin{aligned}
 (16) \quad & m_k(s_{k,t} + u_k s_{k,x}) + \frac{p_1 - p_k}{T_k}(u_k - u_2)\alpha_{k,x} = \\
 & \frac{1}{T_k} \left( \pm P(-R_k + p_k - p_1) \pm M\left(\frac{u_k^2}{2} - \frac{p_k}{\rho_k} - e_k\right) \pm Q(-u_k) \pm S \right), \\
 & (m_k s_k)_t + (m_k u_k s_k)_x = \\
 & \frac{1}{T_k} \left( \pm P(p_k - R_k - p_1) \pm M(T_k s_k + \frac{u_k^2}{2} - \frac{p_k}{\rho_k} - e_k) \pm Q(-u_k) \pm S \right).
 \end{aligned}$$

Adding now the two entropy equations leads to the entropy dissipation partial differential equation that we sum up in the following proposition.

**Proposition 1.** *Consider a smooth solution of the system (2) and two entropy functions  $s_1$  and  $s_2$  satisfying (13). Then, the smooth solution satisfies the following entropy dissipation equation*

$$\begin{aligned}
 (17) \quad & (\sum m_k s_k)_t + (\sum m_k u_k s_k)_x = \frac{P}{T_2} (p_1 + R - p_2) + \\
 & M \left( \frac{u_1^2}{2T_1} - \frac{u_2^2}{2T_2} - \frac{\mu_1}{T_1} + \frac{\mu_2}{T_2} \right) + Q \left( \frac{u_2}{T_2} - \frac{u_1}{T_1} \right) + S \left( \frac{1}{T_1} - \frac{1}{T_2} \right).
 \end{aligned}$$

**Remark 1.** *According to the second principle of thermodynamics the right hand side of (17) has to be non-negative. But each term in the formula (17) has not a clear physical meaning. It is often more convenient to rewrite the source term in a different way. For example, we can set*

$$\begin{aligned}
 (18) \quad & Q = Q_0 + u_1 M, \\
 & S = S_0 + u_1 Q_0 + \frac{u_1^2}{2} M + \mu_1 M.
 \end{aligned}$$

In this way, the dissipation rate becomes

$$(19) \quad \frac{P}{T_2} (p_1 + R - p_2) + \frac{M}{T_2} (\mu_2 - \mu_1 - \frac{(u_2 - u_1)^2}{2}) + \frac{Q_0}{T_2} (u_2 - u_1) + \frac{S_0}{T_1 T_2} (T_2 - T_1).$$

It is  $> 0$  if each term in the sum is  $> 0$ . The source  $S_0$  can then be interpreted as the heat flux (it is  $> 0$  when  $T_2 > T_1$ , i.e. when the phase 2 heats the phase 1). The source  $Q_0$  is the drag force. Finally,  $M$  is the mass transfer due to chemical reaction. When  $u_1 = u_2$ , we recover that the chemical reaction tends to create the phase with the smallest chemical potential.

**Remark 2.** *Generally, the equations (13) satisfied by the entropies  $s_k$  have not a unique solution, once the pressure laws are given. For example, if  $s_k$  is a solution,  $-s_k$  is also a solution. A supplementary condition has thus to be given in order to fix the sign of the entropy dissipation rate. In the case of conservative systems the entropies are supposed to satisfy some convexity property. For a non-conservative system, it is not possible to apply the Godunov-Mock theorem and it is difficult to extend naturally the convexity approach. We propose here only to forbid the change  $s \rightarrow -s$  by imposing that the temperature remains  $> 0$ . It implies*

$$(20) \quad \frac{1}{T_k} = \frac{\partial s_k}{\partial e_k} > 0.$$

In this paper, we will concentrate on the pressure relaxation source term. We will assume the following form, which ensures a positive entropy dissipation

$$(21) \quad P = \frac{1}{\tau_p} \alpha_1 \alpha_2 (p_1 + R - p_2), \quad \tau_p > 0,$$

where  $\tau_p$  is the relaxation parameter. An instantaneous relaxation corresponds to the limit  $\tau_p \rightarrow 0$ .

**Remark 3.** *Let  $p_{\text{ref}}$  be a reference pressure. We can define a characteristic time for the pressure equilibrium by*

$$(22) \quad t_{\text{ref}} = \frac{\tau_p}{p_{\text{ref}}}.$$

*The knowledge of this characteristic time is important for a proper modeling.*

### 3. APPLICATION TO STIFFENED GAS LAWS: ADMISSIBLE GRANULAR STRESS

**3.1. Admissible granular stress.** In many papers, the granular stress is supposed to depend only on the solid volume fraction  $\alpha_2$ . This hypothesis is reasonable when the solid phase is incompressible. However, this choice is not compatible with the existence of an entropy satisfying (13) in the case of a compressible phase. The choice of the granular stress expression cannot be arbitrary. That is why in this section, we compute a simple expression of it when the pressure law of the solid phase is a stiffened gas equation of state.

Let us note

$$(23) \quad \Theta(1/\rho_2, e, \alpha_2) = \frac{R_2(1/\rho_2, e, \alpha_2)}{\alpha_2 \rho_2}.$$

We omit now the subscript  $k = 2$  because we concentrate only on the solid phase. We have to find an entropy  $s$ , a temperature  $T$  and a function  $\Theta$  (containing the granular modeling) such that

$$(24) \quad Tds = de - \frac{p}{\rho^2} d\rho - \Theta d\alpha.$$

We note

$$(25) \quad \begin{aligned} \tau &= 1/\rho, \\ T &= 1/\varphi. \end{aligned}$$

Then,  $\varphi = \varphi(\tau, e, \alpha)$  is an integrating factor for the form

$$(26) \quad de + p d\tau - \Theta d\alpha,$$

which reads

$$(27) \quad ds = \varphi de + \varphi p(\tau, e) d\tau - \varphi \Theta d\alpha.$$

In order to construct a practical and simple model, we suppose that the granular stress only depends on the density and the volume fraction of the solid phase. This leads to the choice  $\Theta = \Theta(\tau, \alpha)$ . The differential form is closed if

$$(28) \quad \begin{aligned} \varphi_\alpha &= -\Theta \varphi_e, \\ p \varphi_\alpha &= -\Theta \varphi_\tau - \varphi \Theta_\tau, \\ p \varphi_e + p_e \varphi &= \varphi_\tau. \end{aligned}$$

The general case corresponds to  $\varphi_e \neq 0$ ,  $\varphi_\alpha \neq 0$  and  $\varphi \neq 0$ . We then have necessarily

$$(29) \quad \frac{\Theta_\tau}{\Theta} = -p_e.$$

**3.2. Practical example.** Now we propose some computations when the pressure law is a stiffened gas EOS

$$(30) \quad p(\rho, e) = (\gamma - 1)\rho e - \gamma\pi.$$

The parameter  $\gamma$  must be  $> 1$ . The parameter  $\pi$  has the dimension of a pressure and can be arbitrary. But in practice, for a solid phase, it is positive and large compared to a characteristic pressure of the low. In the case of a stiffened gas equation, we thus find

$$(31) \quad \frac{\Theta_\tau}{\Theta} = -\frac{\gamma - 1}{\tau} \Rightarrow \Theta(1/\rho, \alpha) = \theta(\alpha)\rho^{\gamma-1},$$

which leads to

$$(32) \quad R(\tau, \alpha) = \alpha\theta(\alpha)\rho^\gamma.$$

In this paper, we will perform numerical experiments with a very simple particular choice

$$(33) \quad R(\tau, \alpha) = \lambda\rho^\gamma\alpha^\gamma.$$

The value of  $\lambda$  can be adjusted to experiments. With this choice, the parameter  $R_0 = \lambda\rho_2^{\gamma_2}$  has the dimension of a pressure. It represents the maximal stress corresponding to the maximal compaction  $\alpha_2 = 1$ . This model is not so different from classical approaches (as described for example in [6]): usually, the granular stress vanishes under some critical porosity  $\alpha_c$  (dilute case) and increases with  $\alpha$  when  $\alpha > \alpha_c$  (packed case). In our approach, the parameter  $\gamma$  allows to ensure that the granular stress is small when  $\alpha$  is small. Actually, the higher  $\gamma$  is, the faster the granular stress tends to zero when  $\alpha$  tends to zero.

Of course, it would be also possible to consider the most general case where the granular stress also depends on the internal energy

$$(34) \quad \Theta = \Theta(\alpha, \tau, e).$$

However, we will see that the choice (33) is very interesting for the modeling and the numerics because it ensures a maximum principle on the volume fraction during the pressure equilibrium resolution.

Is also possible to compute the whole thermodynamic underlying model. The expressions for the associated entropy and temperature are given in Section 8.

#### 4. FINITE VOLUME APPROACH

For the numerical implementation, we consider a finite volume discretisation and a splitting approach. The convection step is solved by a standard Rusanov scheme already described in many works as [11]. In the second stage, the source terms are applied. We concentrate on the pressure relaxation source term in the next section.

For the finite volume scheme, we consider a step  $h$ , a time step  $\Delta t$ . The cells are intervals  $]x_{i-1/2}, x_{i+1/2}[$ . For simplicity, we consider a regular mesh  $x_i = ih$  (but this of course is not mandatory). The vector  $W$  is approximated in each cell at time  $t_n$  by

$$(35) \quad W_i^n \simeq W(x_i, t_n).$$

The numerical scheme for the convective term is a standard Rusanov scheme for non-conservative systems, which reads

$$(36) \quad h(W_i^{n+1/2} - W_i^n) + \Delta t(F_{i+1/2}^n - F_{i-1/2}^n) + \Delta t A(W_i^n) \frac{L_{i+1}^n - L_{i-1}^n}{2} = 0.$$

The conservative numerical flux is given by

$$(37) \quad \begin{aligned} F(W_L, W_R) &= \frac{F(W_L) + F(W_R)}{2} - \zeta \frac{W_R - W_L}{2}, \\ \zeta &= \max(\rho(B(Y_L)), \rho(B(Y_R))). \end{aligned}$$

where  $\rho(B)$  denotes the spectral radius of the matrix  $B$ . The convection matrix  $B(Y)$  in the primitive variables  $Y$  is given in Section 7.

Our particular choice of the non-conservative terms  $L$  ensures that constant velocity-pressure states will be maintained by the Rusanov scheme.

## 5. RELAXATION ALGORITHM

In this section, we address now the numerical approximation of the pressure relaxation source term. As usual, we use a fractional step method in order to separate the convection step, the pressure equilibrium step and the step involving the remaining source terms. The standard source terms step is solved by a simple explicit first-order Euler method.

Thus, we concentrate only on the description of the pressure equilibrium step, which can be formally written

$$(38) \quad \begin{aligned} \alpha_{k,t} &= \pm P, \\ m_{k,t} &= u_{k,t} = 0, \\ (m_k e_k)_t + p_1 \alpha_{k,t} &= 0. \end{aligned}$$

In order to simplify the notations, we denote now by a 0 superscript the physical values in a given cell  $i$  at the end of the advection step. These values are computed from the vector  $W_i^{n+1/2}$  given by the Rusanov scheme (36). The updated values at time  $n+1$  are noted without any superscript.

Owing to mass and momentum conservation we have  $m_k = m_k^0$  and  $u_k = u_k^0$ . We have now to compute  $(\alpha_1, p_1, p_2)$  in order to pursue the computation. The system is

$$(39) \quad \begin{aligned} p_2 - R - p_1 &= \tau_p \alpha_{2,t}, \\ m_1 e_1 + m_2 e_2 &= m_1^0 e_1^0 + m_2^0 e_2^0, \\ (m_1 e_1 - m_1^0 e_1^0) + p_1 (\alpha_1 - \alpha_1^0) &= 0. \end{aligned}$$

We assume now that the two phases obey stiffened gas equations of state

$$(40) \quad p_k(\rho_k, e_k) = (\gamma_k - 1)\rho_k e_k - \gamma_k \pi_k, \quad \gamma_k > 1, \quad k = 1, 2.$$

It is also physically reasonable to suppose that

$$(41) \quad \pi_2 > \pi_1,$$

because phase (2) is the solid phase. We have also to state some assumptions for the granular stress. We suppose that

$$(42) \quad R = R(\rho_2, \alpha_2) = \rho_2^{\gamma_2} \alpha_2 \theta(\alpha_2)$$

(as already discussed in Section 3). Because of the stiffened gas law, we have

$$(43) \quad m_k e_k = \alpha_k \frac{p_k + \gamma_k \pi_k}{\gamma_k - 1}.$$

Thus we have to solve for  $(\alpha_1, p_1, p_2)$  the following three-equation system, at each time step and in each cell

$$\begin{aligned}
 (44) \quad & p_2 - \alpha_2 \rho_2^{\gamma_2} \theta(\alpha_2) - p_1 = \frac{\tau_p}{\alpha_2(1 - \alpha_2)} \alpha_{2,t}, \\
 & \alpha_2 \frac{p_2 + \pi_2}{\gamma_2 - 1} - \alpha_2^0 \frac{p_2^0 + \pi_2}{\gamma_2 - 1} + (p_1 + \pi_2)(\alpha_2 - \alpha_2^0) = 0, \\
 & \alpha_1 \frac{p_1 + \pi_1}{\gamma_1 - 1} - \alpha_1^0 \frac{p_1^0 + \pi_1}{\gamma_1 - 1} + (p_1 + \pi_1)(\alpha_1 - \alpha_1^0) = 0.
 \end{aligned}$$

We have

$$\begin{aligned}
 (45) \quad & p_2 - \alpha_2 \rho_2^{\gamma_2} \theta(\alpha_2) - p_1 = \frac{\tau_p}{\alpha_2(1 - \alpha_2)} \alpha_{2,t}, \\
 & (\alpha_2 + (\gamma_2 - 1)(\alpha_2 - \alpha_2^0))(p_2 + \pi_2) - \alpha_2^0(p_2^0 + \pi_2) - \\
 & (\gamma_2 - 1)(\alpha_2 \rho_2^{\gamma_2} \theta(\alpha_2) + \tau_p \alpha_{2,t})(\alpha_2 - \alpha_2^0) = 0, \\
 & (\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^0))(p_1 + \pi_1) - \alpha_1^0(p_1^0 + \pi_1) = 0.
 \end{aligned}$$

We then note

$$\begin{aligned}
 (46) \quad & A_1 = \alpha_1^0(p_1^0 + \pi_1), \\
 & A_2 = \alpha_2^0(p_2^0 + \pi_2).
 \end{aligned}$$

For a stiffened gas law, the sound speed  $c$  is given by the formula

$$(47) \quad c = \sqrt{\frac{\gamma(p + \pi)}{\rho}}.$$

It implies that the two quantities  $A_1$  and  $A_2$  are  $> 0$ . We suppose the following conditions on the granular constraint

$$\begin{aligned}
 (48) \quad & \theta \text{ continuous on } [0, 1], \\
 & \theta(\alpha) \geq 0, \\
 & \theta(\alpha) = o(\alpha^{\gamma_2 - 2}) \text{ when } \alpha \rightarrow 0.
 \end{aligned}$$

Let us remark that these conditions are fulfilled by the granular stress that we proposed in (33) where

$$(49) \quad \theta(\alpha) = \lambda \alpha^{\gamma_2 - 1}.$$

Of course, we also suppose that the initial volume fraction  $0 < \alpha_2^0 < 1$ .

After the elimination of  $p_1$  and  $p_2$ , the system can be rewritten

$$\begin{aligned}
 (50) \quad & G(\alpha_2) = (\pi_2 - \pi_1)(\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^0))(\alpha_2 + (\gamma_2 - 1)(\alpha_2 - \alpha_2^0)) \\
 & + (\alpha_2^{2 - \gamma_2} m_2^{\gamma_2} \theta(\alpha_2) + \frac{\tau_p}{(1 - \alpha_2)} \alpha_{2,t} - A_2)(\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^0)) \\
 & + A_1(\alpha_2 + (\gamma_2 - 1)(\alpha_2 - \alpha_2^0)) = 0.
 \end{aligned}$$

We use an implicit first-order time discretisation of  $\alpha_{2,t}$

$$(51) \quad \alpha_{2,t} = \frac{\alpha_2 - \alpha_2^0}{\Delta t}.$$

(recall that  $\Delta t$  is the time step in the convection step (36)). The implicit approach is natural because  $\tau_p$  may be arbitrary small and thus the source term (21) may be stiff.

We first compute  $G$  at the left point of the interval  $[0, 1]$

$$(52) \quad \begin{aligned} G(0) &= -(\pi_2 - \pi_1)(\gamma_2 - 1)\alpha_2^0(1 + (\gamma_1 - 1)(1 - \alpha_1^0)) \\ &\quad - (A_2 + \frac{\tau_p \alpha_2^0}{\Delta t})(1 + (\gamma_1 - 1)(1 - \alpha_1^0)) \\ &\quad - A_1(\gamma_2 - 1)\alpha_2^0 < 0, \end{aligned}$$

because of the hypothesis (48) and  $A_1, A_2 > 0$ . For the computation at the right point, we introduce

$$(53) \quad \beta_1 = \frac{\gamma_1 - 1}{\gamma_1} \alpha_1^0.$$

We have

$$(54) \quad \alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^0) > 0 \Leftrightarrow \alpha_1 > \beta_1 \Leftrightarrow \alpha_2 < 1 - \beta_1.$$

We compute

$$(55) \quad \begin{aligned} G(1 - \beta_1) &= A_1(1 - \beta_1 + (\gamma_2 - 1)(\alpha_1^0 - \beta_1)) \\ &= \frac{A_1}{\gamma_1 - 1} ((\gamma_1 - 1)(1 - \beta_1) + (\gamma_2 - 1)\beta_1) > 0. \end{aligned}$$

Thus we have existence of a solution  $\alpha_2$  to  $G(\alpha_2) = 0$  in the interval  $]0, 1 - \beta_1[$ .

We have to check that this solution leads to a correct pressures  $p_1$  and  $p_2$  *i.e.* that  $p_k + \pi_k > 0$ . But we have

$$(56) \quad p_1 + \pi_1 = \frac{A_1}{\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^0)}.$$

This quantity is  $> 0$  if the solution satisfies  $\alpha_2 < 1 - \beta_1$ . Finally we also have

$$(57) \quad p_2 + \pi_2 = p_1 + \pi_1 + \alpha_2 \rho_2^{\gamma_2} \theta(\alpha_2) + \tau_p \frac{\alpha_2 - \alpha_2^0}{\tau \alpha_2 (1 - \alpha_2)} + \pi_2 - \pi_1.$$

When  $\alpha_2 - \alpha_2^0 \geq 0$  the previous quantity is obviously  $> 0$ . In the case  $\alpha_2 - \alpha_2^0 < 0$ , we use the second equation of (44) and we find

$$(58) \quad \alpha_2 \frac{p_2 + \pi_2}{\gamma_2 - 1} = \alpha_2^0 \frac{p_2^0 + \pi_2}{\gamma_2 - 1} - (p_1 + \pi_1 + \pi_2 - \pi_1)(\alpha_2 - \alpha_2^0) > 0.$$

And the algorithm can continue.

**Remark 4.** Formula (56) shows that we have to discard any solution that is not in the interval  $]0, 1 - \beta_1[$ . Lets us notice that in many cases, we can find another solution to  $G(\alpha_2) = 0$  in  $]1 - \beta_1, 1[$ .

Now, we will give a sufficient condition on the granular stress in such a way that the solution  $\alpha_2$  is unique in the interval  $]0, 1 - \beta_1[$ . From now on we set

$$(59) \quad f(\alpha) = \alpha^{2-\gamma_2} \theta(\alpha) + \frac{\tau_p}{\Delta t m_2^{\gamma_2}} \frac{\alpha - \alpha_2^0}{1 - \alpha}.$$

and we assume that

$$(60) \quad f \text{ is convex.}$$

We then have

$$(61) \quad G''(\alpha_2) = -2\gamma_1\gamma_2(\pi_2 - \pi_1) - 2\gamma_1 m_2^{\gamma_2} f'(\alpha_2) + (\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^0)) m_2^{\gamma_2} f''(\alpha_2).$$

The function  $f$  is convex and satisfies  $f(\alpha) > f(0)$  for  $0 < \alpha < 1$ . It implies that  $f$  is also increasing. Then the two first terms in (61) are  $< 0$  and the last is  $> 0$ . On the other hand, it is sufficient that  $G$  is concave to prove the uniqueness. But

$$(62) \quad \begin{aligned} G''(\alpha_2) &\leq -2\gamma_1\gamma_2(\pi_2 - \pi_1) - \\ &2\gamma_1m_2^{\gamma_2}f'(\alpha_2) + \gamma_1m_2^{\gamma_2}(1 - \alpha_2)f''(\alpha_2) \leq \\ &2\gamma_1m_2^{\gamma_2} \left[ \frac{1}{2}(1 - \alpha_2)f''(\alpha_2) - f'(\alpha_2) - \frac{\gamma_2}{m_2^{\gamma_2}}(\pi_2 - \pi_1) \right] \end{aligned}$$

A sufficient condition to obtain uniqueness is then

$$(63) \quad \frac{1}{2}(1 - \alpha_2)f''(\alpha_2) - f'(\alpha_2) - \frac{\gamma_2}{m_2^{\gamma_2}}(\pi_2 - \pi_1) \leq 0.$$

**Example 1.** As suggested in (33), we take

$$(64) \quad \theta(\alpha_2) = \lambda\alpha_2^{\gamma_2-1}, \quad \lambda > 0.$$

Thus we have

$$(65) \quad f(\alpha_2) = \lambda\alpha_2 + \frac{\tau_p}{\Delta t m_2^{\gamma_2}} \frac{\alpha_2 - \alpha_2^0}{1 - \alpha_2},$$

which is indeed convex. The above inequality (63) becomes

$$(66) \quad -\lambda - \frac{\gamma_2}{m_2^{\gamma_2}}(\pi_2 - \pi_1) \leq 0$$

and it is obviously satisfied, independantly of  $\tau_p$ .

We sum up the previous computations in the following proposition, which is useful for the implementation of the algorithm.

**Proposition 2.** Let the granular stress be defined by (64) and (32). Let

$$(67) \quad \begin{aligned} 0 &< \alpha_1^0 < 1, \\ p_k^0 + \pi_k &> 0, \quad k = 1, 2, \end{aligned}$$

then, the algebraic system (39), (51) admits a unique solution  $(\alpha_1, p_1, p_2)$  that complies with

$$(68) \quad \begin{aligned} 0 &< \alpha_1 < 1, \\ p_k + \pi_k &> 0, \quad k = 1, 2. \end{aligned}$$

Moreover we also have

$$(69) \quad \alpha_1 > \frac{\gamma_1 - 1}{\gamma_1} \alpha_1^0.$$

Finally, the solution can be computed by the Newton's method by solving equation (50) for  $\alpha_2$ . A safe choice for the initialisation of the Newton's method is  $\alpha_2 = 0$  (since  $G$  is concave).

## 6. NUMERICAL RESULTS

**6.1. Academic test cases.** First, we consider two one-dimensional test cases in order to evaluate the influence of the granular stress on the system stability. We take  $\tau_p = 0$ , which corresponds to instantaneous pressure equilibrium. It is known that generally, the equilibrium system is not hyperbolic (the computations are recalled in Section 7). The numerical parameters are taken from [2] (and also studied in [8]). We consider a simple Riemann problem in the interval  $[-1/2, 1/2]$ . The two phases are supposed to satisfy perfect gas equations of state, with  $\gamma_1 = 1.0924$  and  $\gamma_2 = 1.0182$ . The initial condition is made of two constant states jumping at  $x = 0$ .

We plot the solution at time  $t = 0.0008$ . The CFL number is fixed to 0.9. The initial data are

$$(70) \quad \begin{array}{ccc} & (L) & (R) \\ \rho_1 & 76.45430093 & 57.34072568 \\ u_1 & 0 & 0 \\ p_1 & 200 \times 10^5 & 150 \times 10^5 \\ \rho_2 & 836.1239718 & 358.8982226 \\ u_2 & 0 & 0 \\ p_2 & 200 \times 10^5 & 150 \times 10^5 \\ \alpha_1 & 0.25 & 0.25 \end{array}$$

We perform our algorithm with a granular stress  $R = 0$ . With a mesh of 1,000 cells we observe that the solution is rather smooth but a small oscillation starts to develop in the center of the computational domain. The volume fraction  $\alpha_1$ , the velocities and pressures are plotted on Figures 1, 2 and 3.

The same computation is made with 10,000 cells. We observe on Figures 4, 5 and 6 that instabilities arise, due to the non-hyperbolic behavior of the model.

We have also performed a computation on a 100,000 cells mesh. The oscillations clearly increase as can be seen on Figure 7 for the volume fraction.

We perform then another computation on the finer mesh with a granular stress given by (33). For the numerics, we have chosen  $\lambda = 500$ . We observe a very slight damping of the oscillations on Figure 8 (to be carefully compared with Figure 4). We have also plotted the two pressures on Figure 10 in order to show the difference of pressures imposed by the granular stress.

It clearly arises that the magnitude of the granular stress is not sufficient to recover an hyperbolic regime. This is confirmed on Figure 11 where we compare

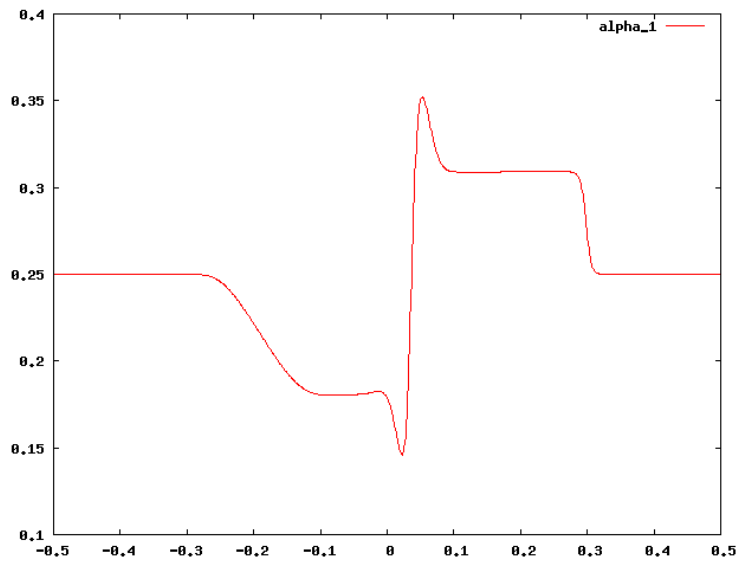
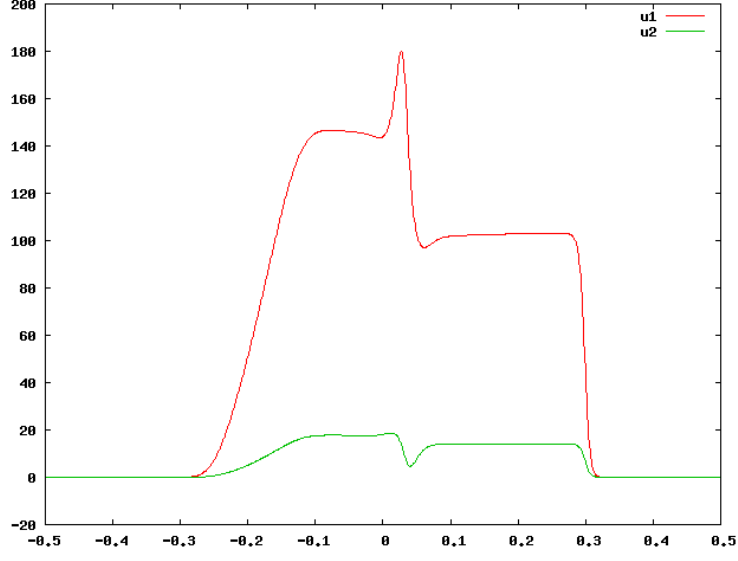
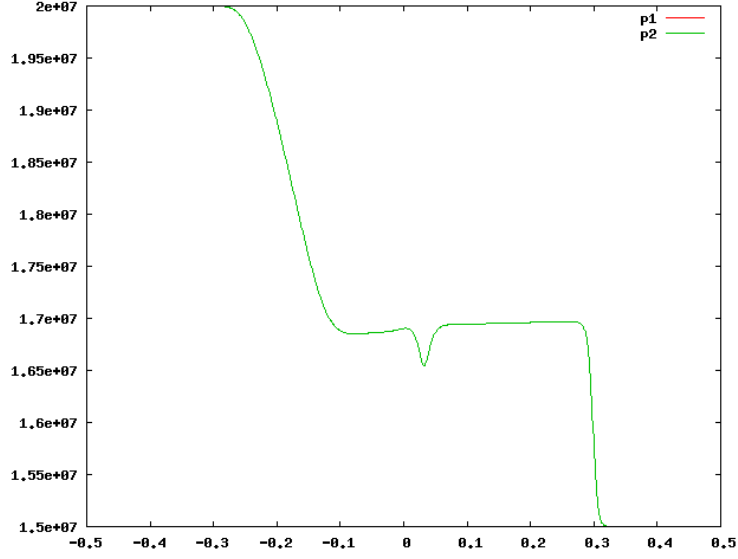
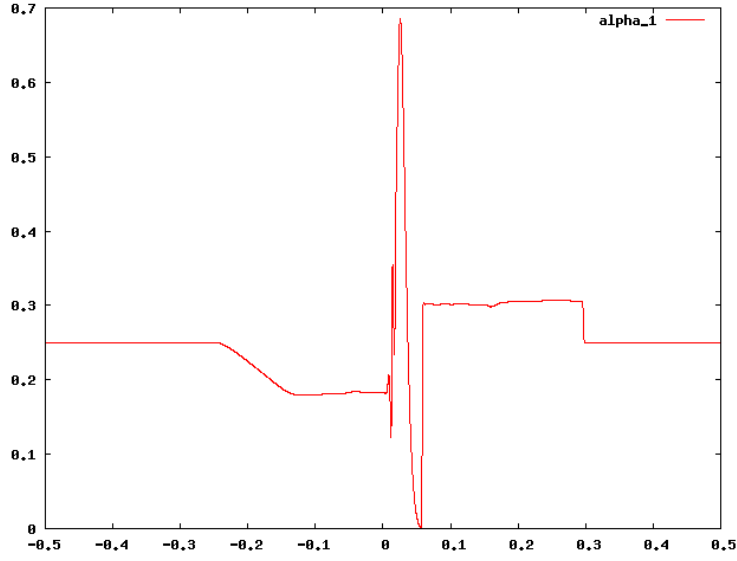
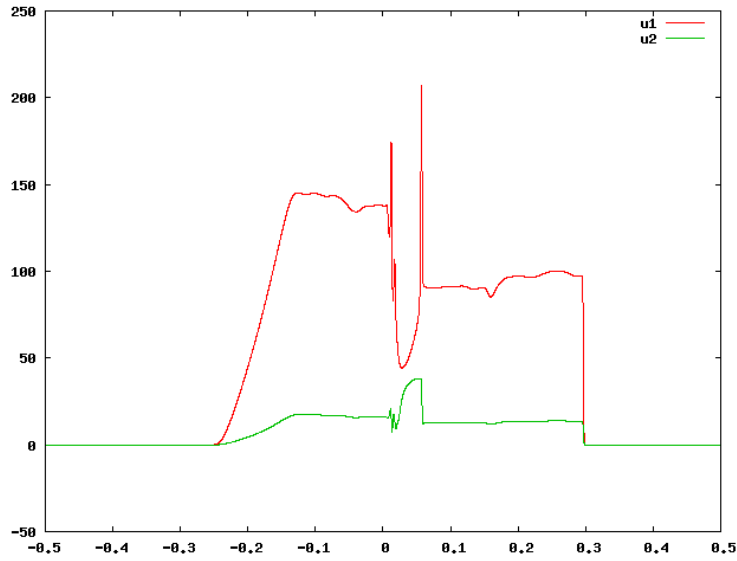


FIGURE 1. Void fraction, 1,000 cells,  $\tau_p = 0$ ,  $R = 0$ .

FIGURE 2. Velocities  $u_1$  and  $u_2$ , 1,000 cells,  $\tau_p = 0$ ,  $R = 0$ .FIGURE 3. Pressures, 1,000 cells,  $\tau_p = 0$ ,  $R = 0$ .

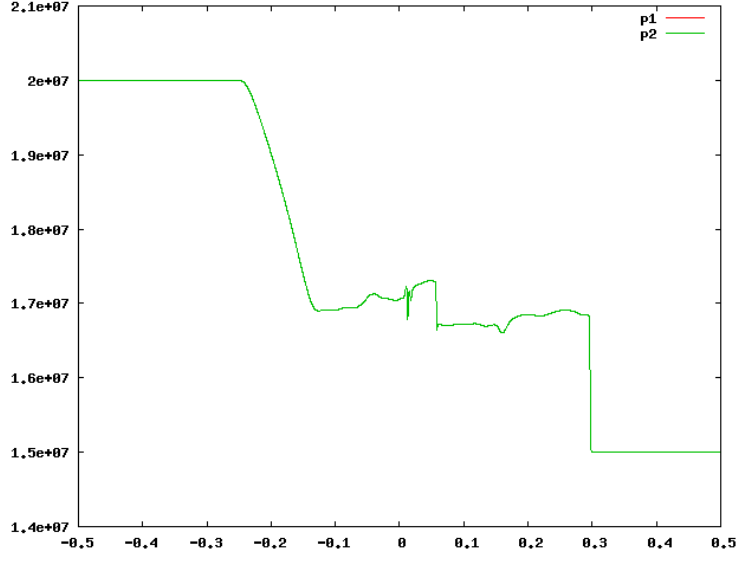
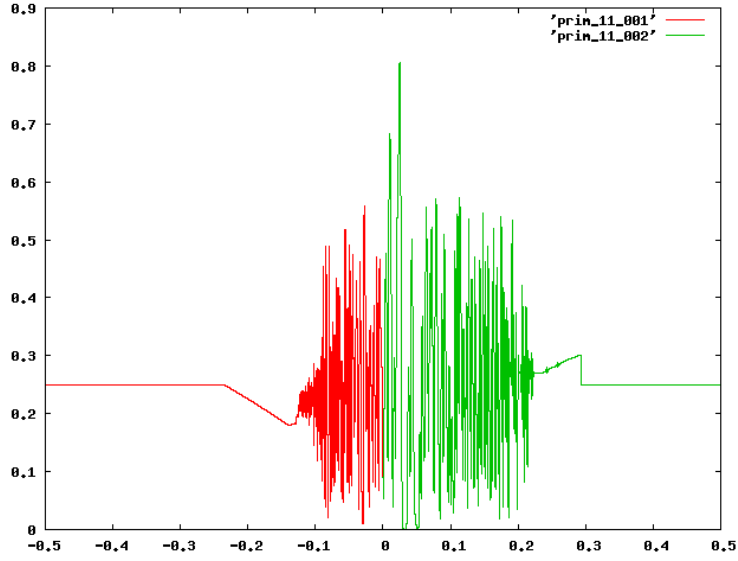
a  $L^2$  norm of the imaginary parts of the eigenvalues with  $R = 0$  or  $R > 0$ . The computation of the convection matrix of the equilibrium system in the case  $\tau_p \rightarrow 0$  is given in Section 7. The eigenvalues are evaluated numerically. The quantity that has been plotted is

$$(71) \quad I = \sqrt{\sum_{i=1}^6 \text{Im}(\lambda_i)^2}.$$

FIGURE 4. Void fraction, 10,000 cells,  $\tau_p = 0$ ,  $R = 0$ .FIGURE 5. Velocities  $u_1$  and  $u_2$ , 10,000 cells,  $\tau_p = 0$ ,  $R = 0$ .

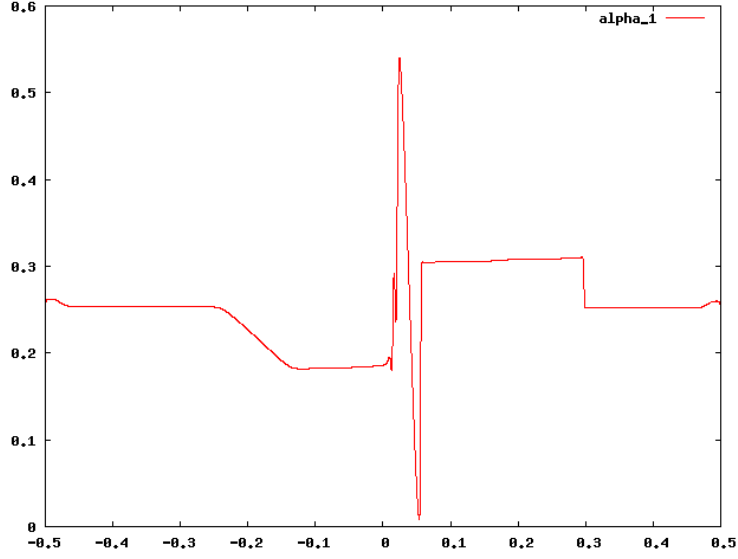
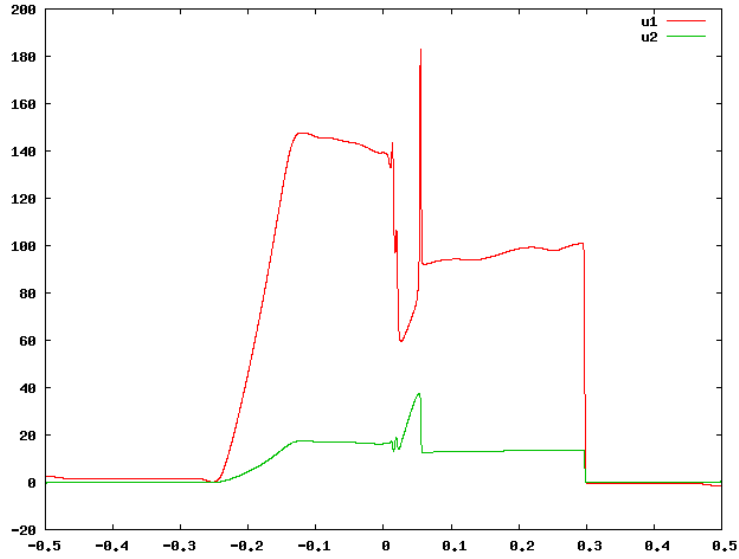
We observe on Figure 11 that the imaginary part slightly decreases, owing to the introduction of the granular stress; however, it does not vanish.

We now evaluate the influence of the positive time scale parameter  $\tau_p$ , and on its ability to stabilize the model. While setting  $r_{ref} = 10^7$ , we compare in Figure 12 the pressures obtained with different relaxation coefficients (corresponding with true time scales  $\frac{\tau_p}{p_{ref}} = 0, 10^{-7}, 10^{-6}, 10^{-5}$  and  $10^{-4}$ ). On this rather coarse mesh of 1,000 cells, we observe that the pressures are very similar for small enough relaxation time scales. When focusing on a finer mesh of 50,000 cells, the stability

FIGURE 6. Pressures, 10,000 cells,  $\tau_p = 0$ ,  $R = 0$ .FIGURE 7. Void fraction, 100,000 cells,  $\tau_p = 0$ ,  $R = 0$  (the two colors only correspond to the two processors used in the MPI computation).

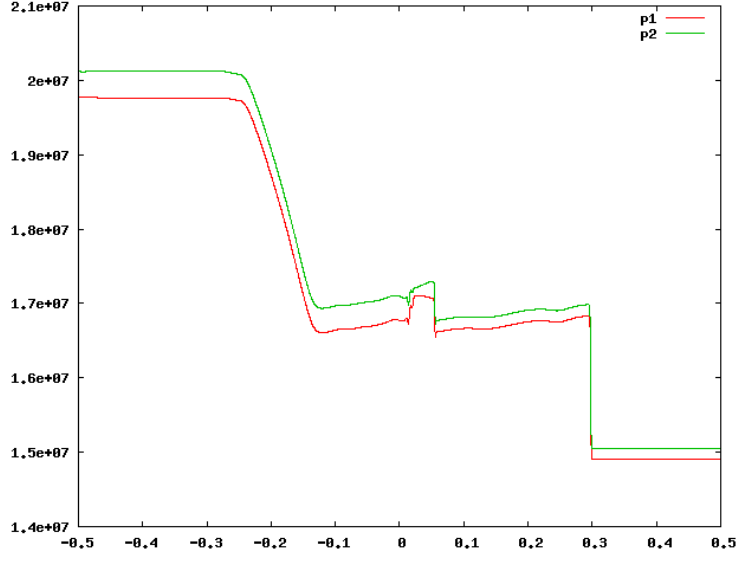
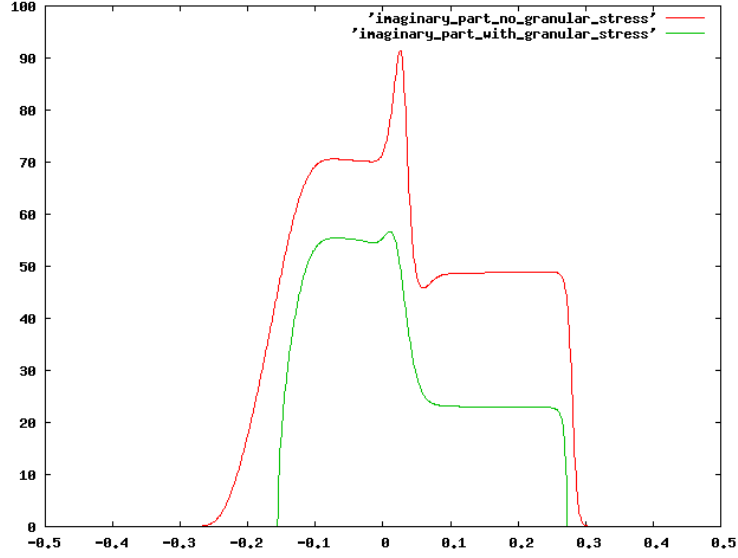
of the approximations increases when the relaxation time scales are larger (Figure 13). We also observe on this test cases that even with high values of  $\tau_p$ , the difference between both pressures  $p_1$  and  $p_2$  is indeed rather small.

**6.2. Simplified combustion chamber.** We consider now a more realistic case taken from [10]. We are interested in the modeling of a simplified gun. The gun is modeled by a one-dimensional tube filled with a solid phase (the powder grains)

FIGURE 8. Void fraction, 10,000 cells,  $\tau_p = 0$ ,  $R = 500m_2^{\gamma_2}$ .FIGURE 9. Velocities  $u_1$  and  $u_2$ , 10000 cells, with granular stress .

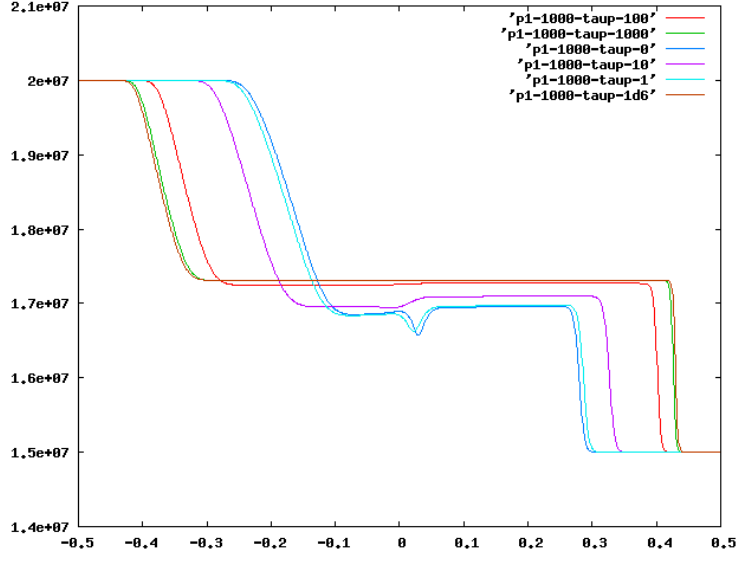
and a gas phase (the combustion gases). The breech is on the left and the shot base at the right boundary. The shot base moves according to Newton's law because the bullet is accelerated by the increase of pressure due to the combustion of the powder. For this test case, we have adapted and simplified the physical parameters given in [11]. The mass transfer term is defined by the simplified relations

$$\begin{aligned}
 (72) \quad M &= \alpha_2 \rho_2 \frac{3\dot{r}}{r} \\
 \dot{r} &= 5 \times 10^{-3} m/s \text{ (combustion velocity of the grains)} \\
 r &= 10^{-3} m \text{ (radius of the grains)}
 \end{aligned}$$

FIGURE 10. Pressures, 10,000 cells,  $\tau_p = 0$ ,  $R = 500m_2^{\gamma_2}$ .FIGURE 11. Imaginary parts, 1,000 cells,  $\tau_p = 0$ ,  $R = 500m_2^{\gamma_2}$  or  $R = 0$ . stress.

The momentum source term is given by

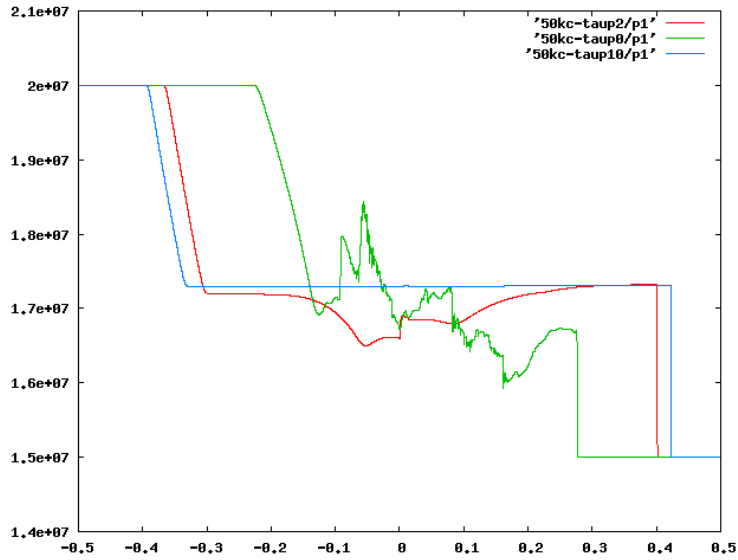
$$\begin{aligned}
 Q &= Mu_2 - D \\
 D &= C\alpha_1\alpha_2\rho_2(u_1 - u_2)|u_1 - u_2| \text{ (drag force)} \\
 C &= \frac{3}{4r} \text{ (simplified shape factor)}
 \end{aligned}
 \tag{73}$$

FIGURE 12. Pressure  $p_1$ , 1,000 cells, with  $0 \leq \tau_p \leq 1000$ .

The energy source terms are

$$\begin{aligned}
 S_1 &= -u_2 D + M Q_{ex} \\
 S_2 &= u_2 D \\
 Q_{ex} &= 37.3839 \times 10^6 \text{ J/kg (chemical combustion energy)}
 \end{aligned}
 \tag{74}$$

Let us remark that they do not cancel when summed up. This is only apparently a violation of the total energy conservation. Actually, we can rewrite the model in order to have opposite source terms. The rewriting is based on a translation of the internal energy in the pressure laws.

FIGURE 13. Pressure, 50,000 cells,  $\tau_p = 0, 2, 10$ ,  $R = 0$ .

**Remark 5.** *If we set*

$$(75) \quad e_k = e'_k - e_k^0,$$

where  $e'_k$  is the translated internal energy of phase  $k$ , and  $e_k^0$  is a reference energy for phase  $k$ , and if we define the translated total energies

$$(76) \quad E'_k = e'_k + e_k^0 + \frac{u_k^2}{2}.$$

the energy balance equations can be rewritten

$$(77) \quad (m_k E'_k)_t + ((m_k E'_k + \alpha_k p_k) u_k)_x + p_I \alpha_{k,t} = \pm S \mp M e_k^0.$$

Now the total translated energy  $m_1 E'_1 + m_2 E'_2$  is no longer conserved since the  $\pm$  terms do not cancel. The term  $M(e_2^0 - e_1^0)$  can be identified to the chemical reaction heat.

The other parameters of the computations are

$$(78) \quad \begin{aligned} \gamma_1 &= 1.4 \\ \gamma_2 &= 3 \\ \pi_2 &= 2.1333 \times 10^9 \text{ Pa} \\ \rho_2 &= 1600 \text{ kg/m}^3 \text{ (initial solid density)} \\ m_p &= 30 \text{ kg (projectile mass)} \\ p_r &= 10^8 \text{ Pa (resistive pressure)} \\ p_0 &= 10^5 \text{ Pa (initial pressure)} \\ T_0 &= 294 \text{ K (initial temperature)} \\ \rho_0 &= 0.8713 \text{ kg/m}^3 \text{ (initial gas density)} \\ \alpha_{2,0} &= 0.5709 \text{ (initial porosity)} \\ \text{diam} &= 132 \text{ mm (diameter of the gun)} \\ \text{length} &= 762 \text{ mm (length of the tube)} \\ \text{mpow} &= 9.5255 \text{ kg (powder mass)} \\ \lambda &= 0.03 \text{ (granular parameter)} \\ \text{mmol} &= 21.3 \text{ g/mol (molar mass of the gas)} \end{aligned}$$

When the gas pressure  $p_1$  at the shot base is greater than the resistive pressure  $p_r$ , the acceleration of the projectile is given by

$$m_p \frac{dv}{dt} = (p_1 - p_r) \frac{\pi \text{diam}^2}{4}.$$

The algorithm to move the right boundary is based on an Arbitrary Lagrangian Eulerian (ALE) approach described in [10]. As the domain enlarges, the number of computational cells increases.

We compare our new compressible model with the classical Gough model described for example in [11].

We obtain the following results for the projectile velocity at the exit time

|                | Gough model | Relax. no granular stress | Relax. with granular stress |
|----------------|-------------|---------------------------|-----------------------------|
| velocity (m/s) | 425         | 414                       | 414                         |
| exit time (ms) | 2.9         | 3.07                      | 3.07                        |

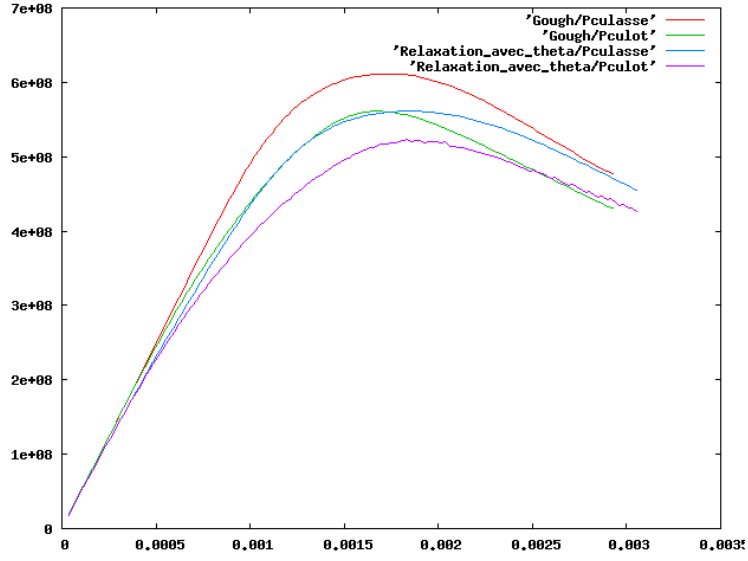


FIGURE 14. Pressure evolution at the breech and the shot base wrt time. Comparison between the Gough and the relaxation model.

On Figure 14, we compare the pressure evolution at the breech and the shot base of the projectile. We observe a good qualitative agreement between the Gough model and the relaxation model.

Finally, we plot some quantities in the tube at the final time. The porosity, the velocities and the pressures are given in Figures 15, 16, 17. We also plot on Figure 18 the density  $\rho_2$  of the solid phase at the final time in order to check that the variations of the powder density are small when compared with the initial density  $\rho_2 = 1600 \text{ kg/m}^3$ .

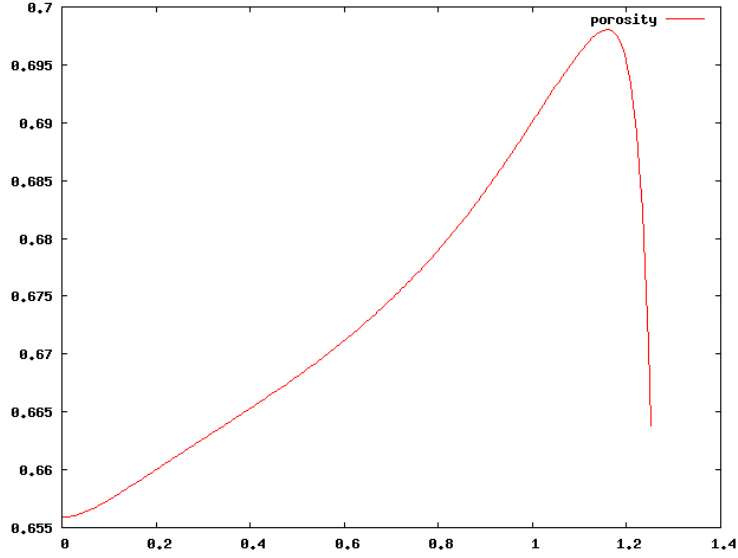


FIGURE 15. Porosity at the final time. Relaxation model with granular stress.

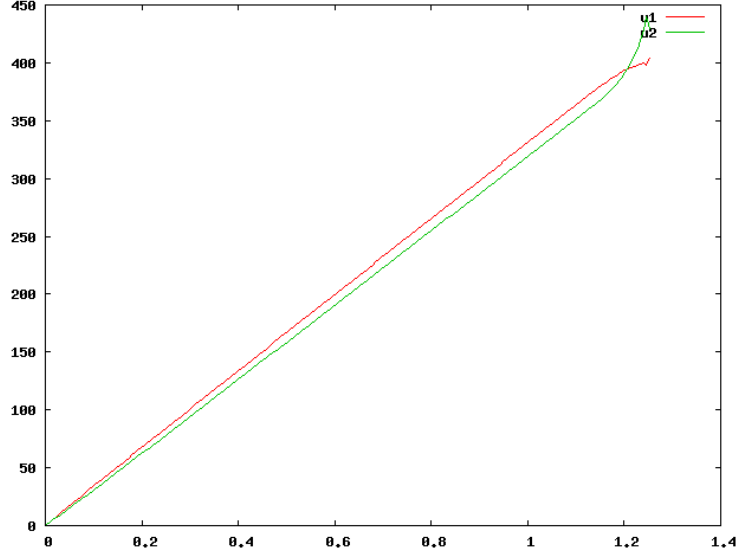


FIGURE 16. Velocities at the final time. Relaxation model with granular stress.

### CONCLUSION

In this paper, we have adapted the pressure relaxation method described in [12] and [8] to the case of a non-vanishing granular stress.

Starting from the two-velocity, two-pressure multiphase model of Baer-Nunziato, we have proposed a relaxation source term in the governing equation of the void

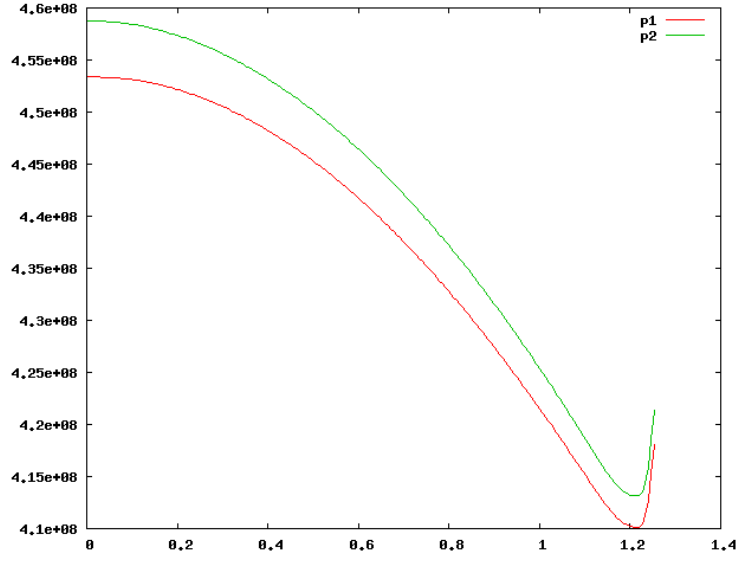


FIGURE 17. Pressures at the final time. Relaxation model with granular stress.

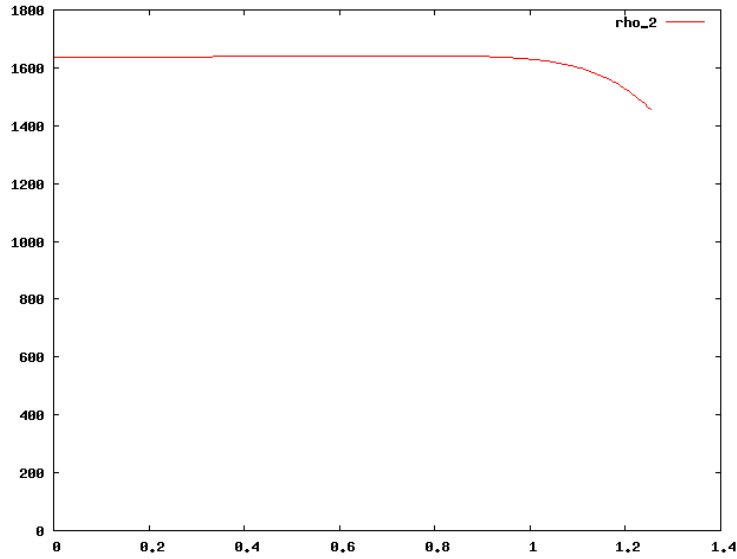


FIGURE 18. Density of the solid phase at the final time. Relaxation model with granular stress.

fraction that is compatible with the second principle of thermodynamics. In this study, we have shown that

- the source term increases the entropy of the phase mixture;
- the granular stress cannot have an arbitrary form. It is related to the fact that the differential form satisfied by the entropy is closed.

When the relaxation time tends to zero, we have then proposed a numerical method based on the underlying two-pressure model to approximate the single-pressure model. In the pressure relaxation step, the void fraction is updated in order to equilibrate the jump of pressures with the granular stress. We have proved

existence and uniqueness of the equilibrium void fraction under some hypothesis on the granular stress. Those hypothesis are satisfied by physically reasonable models.

Eventually, we have proposed some numerical experiments in order to validate our approach. In an ideal test case, we have checked that when the mesh is refined, the instability of the one-pressure model is (fortunately) not suppressed. We also checked that the introduction of the granular stress slightly improves the whole stability. We finally performed more realistic simulations where we were able to reproduce correct quantitative features of a simplified gun.

The whole approach is thus very promising and must now be extended to more sophisticated granular pressure laws, equations of state and geometries.

## 7. APPENDIX I: HYPERBOLICITY

**7.1. Relaxed system.** For the sake of completeness, we recall the proof of hyperbolicity of the convection part of the equations. It is convenient to study it in the variables

$$(79) \quad Y = (\alpha_1, \rho_1, u_1, s_1, \rho_2, u_2, s_2)^T.$$

In this set of variables the system becomes

$$(80) \quad Y_t + B(Y)Y_x = 0,$$

with

$$(81) \quad c_k = \frac{\partial p(\rho_k, s_k)}{\partial \rho_k}, \quad k = 1, 2$$

$$B(Y) = \begin{bmatrix} u_2 & & & & & & \\ \frac{\rho_1(u_1 - u_2)}{\alpha_1} & u_1 & \rho_1 & & & & \\ & \frac{c_1^2}{\rho_1} & u_1 & \frac{p_{1,s_1}}{\rho_1} & & & \\ & & & u_1 & & & \\ & & & & u_2 & \rho_2 & \\ \frac{p_1 - p_2}{m_2} & & & & \frac{c_2^2}{\rho_2} & u_2 & \frac{p_{2,s_2}}{\rho_2} \\ & & & & & & u_2 \end{bmatrix}$$

The characteristic polynomial is

$$(82) \quad P(\lambda) = (u_2 - \lambda)^2(u_1 - \lambda)(u_1 - c_1 - \lambda)(u_1 + c_1 - \lambda)(u_2 - c_2 - \lambda)(u_2 + c_2 - \lambda)$$

We can then state the following proposition

**Proposition 3.** *If  $|u_1 - u_2| \neq c_k$ ,  $k = 1, 2$  then, the system (2) is hyperbolic. If  $|u_1 - u_2| = c_k$  for  $k = 1$  or  $2$  then the system is resonant.*

**7.2. Equilibrium system.** We also study the hyperbolicity behavior of the equilibrium system, which correspond to the limit  $\tau_p = 0$ . When the granular stress vanishes, the results are given in several papers. We thus only detail the case  $R > 0$  with a granular stress satisfying (32). The computations given below have been used to draw Figure 11. We note, for any quantity  $z$ ,

$$(83) \quad D_k z = z_t + u_k z_x.$$

At equilibrium, we can remove the transport equations in  $\alpha_k$  and replace them by the pressure relation

$$(84) \quad p_2 = p_1 + \alpha_2 \rho_2^{\gamma_2} \theta(\alpha_2) = p_1 + \rho_2^{\gamma_2} g(\alpha_2).$$

We note

$$(85) \quad h = g^{-1}.$$

Of course, when the granular stress vanishes, the inverse of  $g$  is not defined and the computations must be carried out in another way. At equilibrium, we have

$$\begin{aligned}
 \alpha_2 &= h \left( \frac{p_2 - p_1}{\rho_2^{\gamma_2}} \right) \\
 (86) \quad &\Rightarrow d\alpha_2 = \delta \left( \left( c_2^2 - \gamma_2 \frac{p_2 - p_1}{\rho_2} \right) d\rho_2 + p_{2,s_2} ds_2 - c_1^2 d\rho_1 - p_{1,s_1} ds_1 \right) \\
 &\text{with } \delta = \frac{h' \left( \frac{p_2 - p_1}{\rho_2^{\gamma_2}} \right)}{\rho_2^{\gamma_2}} > 0.
 \end{aligned}$$

**Example 2.** *We can consider*

$$(87) \quad \theta(\alpha) = \lambda \alpha^{\gamma_2 - 1}.$$

*We then have*

$$(88) \quad \delta = \frac{\alpha_2^{1-1/\gamma_2}}{\lambda \gamma_2 \rho_2^{\gamma_2}}.$$

It is natural to introduce

$$(89) \quad a_2^2 = \frac{\gamma_2 \rho_1}{\gamma_1 \rho_2} c_1^2 + \gamma_2 \frac{\pi_2 - \pi_1}{\rho_2} > 0,$$

In such a way that we have also

$$(90) \quad d\alpha_2 = \delta (a_2^2 d\rho_2 + p_{2,s_2} ds_2 - c_1^2 d\rho_1 - p_{1,s_1} ds_1)$$

It gives another expression of the source term  $P$  at equilibrium

$$(91) \quad P = -\delta (a_2^2 D_2 \rho_2 + p_{2,s_2} D_2 s_2 - c_1^2 D_2 \rho_1 - p_{1,s_1} D_2 s_1)$$

We then rewrite the equilibrium system in the variables

$$(92) \quad Z = (\rho_1, u_1, s_1, \rho_2, u_2, s_2)^T,$$

In this variables, the system is

$$(93) \quad Z_t + C(Z)Z_x = 0.$$

For the sake of completeness, we give some details of the computations

$$\begin{aligned}
 (94) \quad &\rho_{1,t} + u_1 \rho_{1,x} - \frac{\rho_1}{\alpha_1} (u_1 - u_2) \delta (a_2^2 \rho_{2,x} + p_{2,s_2} s_{2,x} - c_1^2 \rho_{1,x} - p_{1,s_1} s_{1,x}) + \rho_1 u_{1,x} \\
 &- \frac{\rho_1}{\alpha_1} \delta (a_2^2 D_2 \rho_2 + p_{2,s_2} D_2 s_2 - c_1^2 D_2 \rho_1 - p_{1,s_1} D_2 s_1) = 0, \\
 &\rho_{2,t} + u_2 \rho_{2,x} + \frac{\rho_2}{\alpha_2} \delta (a_2^2 D_2 \rho_2 + p_{2,s_2} D_2 s_2 - c_1^2 D_2 \rho_1 - p_{1,s_1} D_2 s_1) + \rho_2 u_{2,x} = 0
 \end{aligned}$$

$$\begin{aligned}
 (95) \quad &\rho_{1,t} + u_1 \rho_{1,x} - \frac{\rho_1}{\alpha_1} u_1 \delta (a_2^2 \rho_{2,x} + p_{2,s_2} s_{2,x} - c_1^2 \rho_{1,x} - p_{1,s_1} s_{1,x}) + \rho_1 u_{1,x} \\
 &- \frac{\rho_1}{\alpha_1} \delta (a_2^2 \rho_{2,t} + p_{2,s_2} s_{2,t} - c_1^2 \rho_{1,t} - p_{1,s_1} s_{1,t}) = 0,
 \end{aligned}$$

$$\begin{aligned}
 (96) \quad &u_{1,t} + u_1 u_{1,x} + \frac{1}{\rho_1} p_{1,x} = 0, \\
 &u_{2,t} + u_2 u_{2,x} + \frac{1}{\rho_2} p_{2,x} + \frac{p_2 - p_1}{m_2} \delta (a_2^2 \rho_{2,x} + p_{2,s_2} s_{2,x} - c_1^2 \rho_{1,x} - p_{1,s_1} s_{1,x}) = 0
 \end{aligned}$$

$$(97) \quad s_{k,t} + u_k s_{k,x} = 0$$

$$\begin{aligned}
(98) \quad & \left(1 + \frac{\rho_1 c_1^2 \delta}{\alpha_1}\right) \rho_{1,t} - \frac{\rho_1 a_2^2 \delta}{\alpha_1} \rho_{2,t} + u_1 \rho_{1,x} \\
& + \frac{\rho_1}{\alpha_1} \delta \left(-a_2^2 u_1 \rho_{2,x} + c_1^2 u_1 \rho_{1,x} + (u_2 - u_1) p_{2,s_2} s_{2,x}\right) + \rho_1 u_{1,x} = 0 \\
& \left(1 + \frac{\rho_2 a_2^2 \delta}{\alpha_2}\right) \rho_{2,t} - \frac{\rho_2 c_1^2 \delta}{\alpha_2} \rho_{1,t} + u_2 \rho_{2,x} \\
& + \frac{\rho_2}{\alpha_2} \delta \left(a_2^2 u_2 \rho_{2,x} - c_1^2 u_2 \rho_{1,x} - p_{1,s_1} (u_2 - u_1) s_{1,x}\right) + \rho_2 u_{2,x} = 0
\end{aligned}$$

$$\begin{aligned}
(99) \quad & \left(1 + \frac{\rho_1 c_1^2 \delta}{\alpha_1}\right) \rho_{1,t} - \frac{\rho_1 a_2^2 \delta}{\alpha_1} \rho_{2,t} + \left(1 + \frac{\rho_1 c_1^2 \delta}{\alpha_1}\right) u_1 \rho_{1,x} \\
& + \frac{\rho_1}{\alpha_1} \delta \left(-a_2^2 u_1 \rho_{2,x} + p_{2,s_2} (u_2 - u_1) s_{2,x}\right) + \rho_1 u_{1,x} = 0 \\
& \left(1 + \frac{\rho_2 a_2^2 \delta}{\alpha_2}\right) \rho_{2,t} - \frac{\rho_2 c_1^2 \delta}{\alpha_2} \rho_{1,t} + \left(1 + \frac{\rho_2 a_2^2 \delta}{\alpha_2}\right) u_2 \rho_{2,x} \\
& + \frac{\rho_2}{\alpha_2} \delta \left(-c_1^2 u_2 \rho_{1,x} - p_{1,s_1} (u_2 - u_1) s_{1,x}\right) + \rho_2 u_{2,x} = 0
\end{aligned}$$

$$\begin{aligned}
(100) \quad & u_{1,t} + u_1 u_{1,x} + \frac{c_1^2}{\rho_1} \rho_{1,x} + \frac{p_{1,s_1}}{\rho_1} s_{1,x} = 0, \\
& u_{2,t} + u_2 u_{2,x} + \frac{1}{\rho_2} \left(c_2^2 + \frac{p_2 - p_1}{\alpha_2} \delta a_2^2\right) \rho_{2,x} + \frac{p_{2,s_2}}{\rho_2} \left(1 + \frac{p_2 - p_1}{\alpha_2} \delta\right) s_{2,x} \\
& + \frac{p_2 - p_1}{m_2} \delta \left(-c_1^2 \rho_{1,x} - p_{1,s_1} s_{1,x}\right) = 0
\end{aligned}$$

Finally, setting

$$(101) \quad \Delta = \alpha_1 \alpha_2 + \delta (\alpha_1 \rho_2 a_2^2 + \alpha_2 \rho_1 c_1^2),$$

we find

$$C(Z) = \begin{bmatrix} u_1 + \frac{\rho_1 \rho_2 c_1^2 a_2^2 \delta^2 (u_1 - u_2)}{\Delta} & \frac{\alpha_1 \rho_1 (\alpha_2 + \rho_2 a_2^2 \delta)}{\Delta} & \frac{\rho_1 \rho_2 a_2^2 \delta^2 (u_1 - u_2) p_{1,s_1}}{\Delta} & \frac{\rho_1 a_2^2 \delta (\alpha_2 + \rho_2 a_2^2 \delta) (u_2 - u_1)}{\Delta} & \frac{\alpha_2 \rho_1 \rho_2 a_2^2 \delta}{\Delta} & \frac{\rho_1 \delta (\alpha_2 + \rho_2 a_2^2 \delta) (u_2 - u_1) p_{2,s_2}}{\Delta} \\ \frac{c_1^2}{\rho_1} & u_1 & \frac{p_{1,s_1}}{\rho_1} & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 & 0 \\ \frac{\rho_2 c_1^2 \delta (\alpha_1 + \rho_1 c_1^2 \delta) (u_1 - u_2)}{\Delta} & \frac{\alpha_1 \rho_1 \rho_2 c_1^2 \delta}{\Delta} & \frac{\rho_2 \delta (\alpha_1 + \rho_1 c_1^2 \delta) (u_1 - u_2) p_{1,s_1}}{\Delta} & u_2 + \frac{\rho_1 \rho_2 c_1^2 a_2^2 \delta^2 (u_2 - u_1)}{\Delta} & \frac{\alpha_2 \rho_2 (\alpha_1 + \rho_1 c_1^2 \delta)}{\Delta} & \frac{\rho_1 \rho_2 c_1^2 \delta^2 (u_2 - u_1) p_{2,s_2}}{\Delta} \\ \frac{(p_1 - p_2) \delta c_1^2}{\alpha_2 \rho_2} & 0 & \frac{(p_1 - p_2) \delta p_{1,s_1}}{\alpha_2 \rho_2} & \frac{\delta (p_2 - p_1) + \alpha_2 c_2^2}{\alpha_2 \rho_2} & u_2 & \frac{(\alpha_2 + \delta (p_2 - p_1)) p_{2,s_2}}{\alpha_2 \rho_2} \\ 0 & 0 & 0 & 0 & 0 & u_2 \end{bmatrix}$$

It is not easy to compute the eigenvalues analytically. It is also difficult to give a practical sufficient condition on all the parameters in order to prove that the eigenvalues are all real. In the case  $\delta = 0$ , corresponding to an infinite granular stress, the characteristic polynomial is

$$(102) \quad P(\lambda) = (u_2 - \lambda)(u_1 - \lambda)(u_1 - c_1 - \lambda)(u_1 + c_1 - \lambda)(u_2 - c_2 - \lambda)(u_2 + c_2 - \lambda)$$

We recover the same eigenvalues as in (82).

With a small  $\lambda$ , which corresponds to a big  $\delta$  as can be seen by formula (88), we observe numerically that the system is elliptic when  $u_1 \neq u_2$ . When  $\lambda$  increases,  $\delta$  decreases and we recover an hyperbolic behavior.

**Numerical application:** we take

$$(103) \quad \begin{aligned} \gamma_1 &= 1.0924 \\ \gamma_2 &= 1.0182 \\ \pi_1 &= \pi_2 = 0 \\ \alpha_1 &= 0.25 \\ p_1 &= 0.2 \times 10^8 \\ \lambda = 0.01 &\Rightarrow p_2 = 0.20000007050881 \times 10^8 \\ u_2 &= -u_1 = 50 \\ \rho_1 &= 76.45430093 \\ \rho_2 &= 836.1239718 \end{aligned}$$

The eigenvalues are

$$(104) \quad \begin{aligned} &-310.79 \\ &-50.00 \\ &48.96 - 9.36i \\ &48.96 + 9.36i \\ &50 \\ &212.86 \end{aligned}$$

We modify  $\lambda$  to  $\lambda = 500$ , the pressure  $p_2$  is now  $p_2 = 0.20352544 \times 10^8$ . The eigenvalues become real

$$(105) \quad \begin{aligned} &-312.54 \\ &-50.000 \\ &30.507 \\ &50.000 \\ &67.438 \\ &214.59 \end{aligned}$$

## 8. APPENDIX II: ASSOCIATED ENTROPY

It is also possible to compute an entropy associated to the choice (33). For this, we postulate the following form of the entropy of the solid phase (as in Section 3, we omit the subscript)

$$(106) \quad s = K(\alpha)U((e - \pi\tau)\tau^{\gamma-1}).$$

This choice is justified by the fact that when  $K$  is constant, then we recover the general entropy of a stiffened gas. We can verify that our entropy and our stiffened gas law are compatible. Without the subscripts, the equation (13) reads

$$(107) \quad T ds = de + p d\tau - \Theta d\alpha.$$

The temperature is given by

$$(108) \quad s_e = \frac{1}{T} = K(\alpha) \tau^{\gamma-1} U'((e - \pi\tau) \tau^{\gamma-1})$$

In a similar way, we can deduce a relationship between  $p$  and  $T$

$$(109) \quad s_\tau = \frac{p}{T} = p s_e.$$

This relation enables to compute the pressure

$$(110) \quad \begin{aligned} p &= \frac{s_\tau}{s_e} \\ &= \frac{K(\alpha) ((\gamma - 1) e \tau^{\gamma-2} - \gamma \pi \tau^{\gamma-1}) U'((e - \pi\tau) \tau^{\gamma-1})}{K(\alpha) \tau^{\gamma-1} U'((e - \pi\tau) \tau^{\gamma-1})} \\ &= (\gamma - 1) \frac{e}{\tau} - \gamma \pi \end{aligned}$$

and we indeed recover the stiffened gas equation of state.

We try now to find an expression for the function  $U(x)$ . From (107) we can write

$$(111) \quad s_\alpha = -\frac{\Theta}{T}$$

and thus

$$(112) \quad \Theta = -\frac{s_\alpha}{s_e} = -\frac{K'(\alpha) U((e - \pi\tau) \tau^{\gamma-1})}{K(\alpha) U'((e - \pi\tau) \tau^{\gamma-1})} \rho^{\gamma-1}$$

But  $\Theta$  has also to be of the form (32). It implies that

$$(113) \quad \begin{aligned} K(\alpha) &= B \exp\left(\int_0^\alpha \theta(u) du\right), \\ U(x) &= A \exp(-Bx). \end{aligned}$$

We choose now the sign of the constants  $A$  and  $B$  in such way that the temperature is positive and that the function  $U$  is concave. It implies that  $A$  and  $B$  are  $> 0$ . The positivity of  $T$  and the concavity of  $U$  are required for obtaining real sound speeds in the pure phases (see [5]).

## REFERENCES

- [1] M. R. Baer and J. W. Nunziato. A two phase mixture theory for the deflagration to detonation transition (ddt) in reactive granular materials. *Int. J. for Multiphase Flow*, 16(6):861–889, 1986.
- [2] F. Coquel, K. El Amine, E. Godlewski, B. Perthame, and P. Rascle. A numerical method using upwind schemes for the resolution of two-phase flows. *J. Comput. Phys.*, 136(2):272–288, 1997.
- [3] Thierry Gallouët, Jean-Marc Hérard, and Nicolas Seguin. Numerical modeling of two-phase flows using the two-fluid two-pressure approach. *Math. Models Methods Appl. Sci.*, 14(5):663–700, 2004.
- [4] Dimitri Gidaspow. *Multiphase flow and fluidization*. Academic Press Inc., Boston, MA, 1994. Continuum and kinetic theory descriptions.
- [5] E. Godlewski and P.-A. Raviart. *Numerical approximation of hyperbolic systems of conservation laws*, volume 118 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996.
- [6] A. Goldshtein, M. Shapiro, and C. Gutfinger. Mechanics of collisional motion of granular materials. part 3: Self similar shock wave propagation. *Journal of FLuid Mechanics*, 316:29–51, 1996.
- [7] P. S. Gough. Modeling of two-phase flows in guns. *AIAA*, 66:176–196, 1979.
- [8] O. Hurisse. *Couplage interfacial instationnaire de modèles diphasiques*. PhD thesis, Université Aix-Marseille I, octobre 2006.
- [9] A. K. Kapila, R. Menikoff, J. B. Bdzil, S. F. Son, and D. S. Stewart. Two-phase modeling of deflagration-to-detonation transition in granular materials: reduced equations. *Physics of Fluids*, 13(10):3002–3024, 2001.

- [10] J. Nussbaum. *Modélisation et simulation numérique d'un écoulement diphasique de la balistique intérieure*. PhD thesis, Université de Strasbourg, December 2007.
- [11] Julien Nussbaum, Philippe Helluy, Jean-Marc Hérard, and Alain Carrière. Numerical simulations of gas-particle flows with combustion. *Flow, Turbulence and Combustion*, 76(4):403–417, 2006.
- [12] R. Saurel and R. Abgrall. A multiphase Godunov method for compressible multifluid and multiphase flows. *Journal of Computational Physics*, 150(2):425–467, 1999.

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