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THE EXTREMAL SOLUTION OF A BOUNDARY REACTION PROBLEM

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ABSTRACT. We consider

$$\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = \lambda f(u) \text{ on } \Gamma_1, \quad u = 0 \text{ on } \Gamma_2$$

where $\lambda > 0$, $f(u) = e^u$ or $f(u) = (1 + u)^p$ and Γ_1, Γ_2 is a partition of $\partial\Omega$ and $\Omega \subset \mathbb{R}^N$. We determine sharp conditions on the dimension N and $p > 1$ such that the extremal solution is bounded, where the extremal solution refers to the one associated to the largest λ for which a solution exists.

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1. INTRODUCTION

We study the semilinear boundary value problem

$$(1) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \lambda f(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases}$$

where $\lambda > 0$ is a parameter, $f(u)$ is a nonlinear smooth function of u , $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface. We will assume that

$$(2) \quad f \text{ is smooth, nondecreasing, convex and } f(0) > 0,$$

$$(3) \quad \lim_{u \rightarrow \infty} \frac{f(u)}{u} = \infty, \quad \text{and}$$

$$(4) \quad \liminf_{t \rightarrow +\infty} \frac{f'(t)t}{f(t)} > 1.$$

Assumption (4) is not essential, but it simplifies some of the arguments and holds for the examples $f(u) = e^u$, $f(u) = (1+u)^p$, $p > 1$.

We say that u is a weak solution of (1) if $u \in W^{1,1}(\Omega)$, $f(u) \in L^1(\Gamma_1)$ and

$$\int_{\Omega} u(-\Delta\varphi) = \int_{\Gamma_1} \lambda f(u)\varphi \quad \text{for all } \varphi \in C^2(\bar{\Omega}) \text{ such that } \varphi|_{\Gamma_2} \equiv 0 \text{ and } \frac{\partial\varphi}{\partial\nu}|_{\Gamma_1} \equiv 0.$$

Problem (1) shares many properties with the following generalization of the so-called Gelfand's problem

$$(5) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

which has been widely considered [3, 4, 10, 11, 19, 20]. In particular, the following result can be proved as in [3].

Proposition 1.1. *There exists $\lambda^* \in (0, \infty)$ such that*

- (1) has a smooth solution for $0 \leq \lambda < \lambda^*$,
- (1) has a weak solution for $\lambda = \lambda^*$,
- (1) has no solution for $\lambda > \lambda^*$ (even in the weak sense).

Moreover, for $0 \leq \lambda < \lambda^*$ there exists a minimal solution u_λ which is bounded, positive and stable, in the sense that the linearized operator at u_λ is positive, i.e.

$$(6) \quad \inf_{\varphi \in C^1(\bar{\Omega}), \varphi=0 \text{ on } \Gamma_2} \frac{\int_{\Omega} |\nabla\varphi|^2 dx - \lambda \int_{\Gamma_1} f'(u_\lambda)\varphi^2 ds}{\int_{\Gamma_1} \varphi^2 ds} > 0.$$

The monotone limit $u^* := \lim_{\lambda \nearrow \lambda^*} u_\lambda$ is a weak solution for $\lambda = \lambda^*$ and satisfies

$$(7) \quad \lambda^* \int_{\Gamma_1} f'(u^*)\varphi^2 \leq \int_{\Omega} |\nabla\varphi|^2 dx, \quad \forall \varphi \in C^1(\bar{\Omega}), \varphi = 0 \text{ on } \Gamma_2.$$

We call u^* the extremal solution of (1).

Remark 1.2. *Under assumption (4) we have $u^* \in H^1(\Omega)$. The proof is analogous to the argument for (5) in [4], so we skip it.*

Proposition 1.1 suggests the following natural question : is u^* a bounded solution?

In the context of (5), no complete answer has been given yet. For the case $f(u) = e^u$, that is the original Gelfand problem, it was shown by Joseph and Lundgren [19] that if Ω is a ball, then u^* is bounded if and only if $N < 10$. Crandall and Rabinowitz [11] showed that if $f(u) = e^u$ and $N < 10$ then for *any* smooth and bounded domain, u^* is bounded. Brezis and Vázquez [4] provided a different proof of the unboundedness of u^* in the case $\Omega = B_1$ and $N \geq 10$: they established in particular that a singular solution which is stable must be the extremal one. In applying this criterion in dimension $N \geq 10$ they use Hardy's inequality valid for $N \geq 3$:

$$(8) \quad \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N)$$

Other explicit nonlinearities, for instance $f(u) = (1+u)^p$ with $p > 1$, are considered in these references, but in the general case, little is known. In this direction, we mention the result of Nedev [21], which asserts that for any function f satisfying (2) and (3), and any smooth bounded domain in \mathbb{R}^N , $N \leq 3$, u^* is bounded. This

result has been extended by Cabré to the case $N = 4$ and Ω strictly convex [5]. Finally, Cabré and Capella [6] showed that if Ω is a ball and $N \leq 9$ then for any nonlinearity f satisfying (2),(3) the extremal solution is bounded.

Proving that u^* is unbounded seems to be much more difficult. Besides the radial case Dávila and Dupaigne [13] have shown that in domains that are small perturbations of a ball and for the nonlinearities e^u and $(1+u)^p$ the extremal solution is singular for large dimensions ($N \geq 11$ and $N > 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$ respectively).

Returning to (1), we are interested in determining whether the extremal solution u^* is bounded or singular in the cases $f(u) = e^u$ and $f(u) = (1+u)^p$, $p > 1$.

Theorem 1.3. *Let $f(u) = e^u$. In any dimension $N \geq 10$ there exists a domain $\Omega \subset \mathbb{R}^N$ and a partition in smooth sets Γ_1, Γ_2 of $\partial\Omega$ such that $u^* \notin L^\infty(\Omega)$.*

The proof is an adaptation of the argument of Brezis and Vázquez [4], using a stability criterion. In our case the singular solution has the form $u_0(x) = -\log|x|$ for $x \in \partial\mathbb{R}_+^N$ and its linearized stability in dimension $N \geq 10$ is obtained thanks to :

$$(9) \quad \int_{\mathbb{R}_+^N} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}_+^N} \frac{\varphi^2}{|x|}, \quad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}_+^N}),$$

which holds for $N \geq 3$ and where the best constant

$$(10) \quad H_N := \inf \left\{ \frac{\int_{\mathbb{R}_+^N} |\nabla\varphi|^2}{\int_{\partial\mathbb{R}_+^N} \frac{\varphi^2}{|x|}} : \varphi \in H^1(\mathbb{R}_+^N), \varphi|_{\partial\mathbb{R}_+^N} \not\equiv 0 \right\}$$

is given by

$$(11) \quad H_N = 2 \frac{\Gamma(\frac{N}{4})^2}{\Gamma(\frac{N-2}{4})^2} \quad \forall N \geq 3,$$

where Γ is the Gamma function. Inequality (9) is known as Kato's inequality and a proof of it was given by Herbst [18].

We will give here a different proof of this result which offers a sharper version, analogous to improvements of (8) found by Brezis and Vázquez [4] or Vázquez and Zuazua [22] (see also [2, 4, 12, 17, 22] for other improved versions of the Hardy inequality (8)) :

Theorem 1.4. *Let $B = B_1(0)$ be the unit ball in \mathbb{R}^N , $N \geq 3$. Then for any $1 \leq q < 2$ there exists $c = c(N, q) > 0$ such that*

$$\int_{\mathbb{R}_+^N \cap B} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}_+^N \cap B} \frac{\varphi^2}{|x|} + c \|\varphi\|_{W^{1,q}(\mathbb{R}_+^N \cap B)}^2, \quad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}_+^N} \cap B),$$

As a converse to Theorem 1.3 we prove :

Theorem 1.5. *Let $f(u) = e^u$, $N \leq 9$ and suppose $\Omega \subset \mathbb{R}_+^N$ is open, bounded and satisfies:*

- $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}_+^N$ and $\Gamma_2 \subset \mathbb{R}_+^N$
- Ω is symmetric with respect to the hyperplanes $x_1 = 0, \dots, x_{N-1} = 0$, and
- Ω is convex with respect to all directions x_1, \dots, x_{N-1} .

Then the extremal solution u^* of (1) belongs to $L^\infty(\Omega)$.

Remark 1.6. *In order to prove Theorem 1.5, one is at first tempted to imitate the classical argument of Crandall and Rabinowitz [11]: roughly speaking, one uses the stability inequality (7) and the equation (1) with test functions of the form $\varphi = e^{ju}$, $j \geq 1$. This does not lead to the optimal dimension $N = 9$ but applies to general domains (see Proposition 1.7 below). We use instead test functions φ , which are not functions of u , but which have the expected behavior of e^{ju} near a singular point, assuming it exists.*

Proposition 1.7. *Let $f(u) = e^u$ and assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}_+^N$ and $\Gamma_2 \subset \mathbb{R}_+^N$. Assume further that $N < 6$. Then the extremal solution u^* of (1) belongs to $L^\infty(\Omega)$.*

This raises the following question

Open Problem 1. *Does Theorem 1.5 hold in any smooth bounded domain $\Omega \subset \mathbb{R}_+^N$ such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}_+^N$ and $\Gamma_2 \subset \mathbb{R}_+^N$?*

Next we look at (1) in the case $f(u) = (1+u)^p$, $p > 1$. Given $0 < \alpha < N-1$ define

$$(12) \quad w_\alpha(x) = \int_{\partial\mathbb{R}_+^N} K(x,y)|y|^{-\alpha} dy \quad \text{for } x \in \mathbb{R}_+^N,$$

where $K(x,y) = \frac{2x_N}{N\omega_N}|x-y|^{-N}$ is the Green's function for the Dirichlet problem in \mathbb{R}_+^N (see e.g. [16]). Clearly, $w_\alpha > 0$ in \mathbb{R}_+^N . Moreover w_α is harmonic in \mathbb{R}_+^N and w_α extends to a function belonging to $C^\infty(\mathbb{R}_+^N \setminus \{0\})$ with

$$(13) \quad w_\alpha(x) = |x|^{-\alpha} \quad \text{for all } x \in \partial\mathbb{R}_+^N \setminus \{0\}.$$

It is not difficult to verify that for some constant $C(N, \alpha)$ we have

$$\frac{\partial w_\alpha}{\partial \nu}(x) = C(N, \alpha)|x|^{-\alpha-1} \quad \forall x \in \partial\mathbb{R}_+^N \setminus \{0\}.$$

In Section 2 we shall show that

$$(14) \quad C(N, \alpha) = 2 \frac{\Gamma(\frac{\alpha}{2} + \frac{1}{2})\Gamma(\frac{N-1}{2} - \frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{N-2}{2} - \frac{\alpha}{2})}.$$

A heuristic calculation shows that for (1) with nonlinearity $f(u) = (1+u)^p$, the expected behavior of a solution u which is singular at $0 \in \partial\Omega$ should be $u(x) \sim |x|^{\frac{1}{p-1}}$. The boundedness of u^* is then related to the value of $C(N, \frac{1}{p-1})$. Observe that $C(N, \frac{1}{p-1})$ is defined for $p > \frac{N}{N-1}$. In the sequel, when writing $C(N, \frac{1}{p-1})$ we will implicitly assume that this condition holds.

Theorem 1.8. *Consider (1) with $f(u) = (1+u)^p$. Assume $\Omega \subset \mathbb{R}_+^N$ is a bounded domain such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}_+^N$ and $\Gamma_2 \subset \mathbb{R}_+^N$ and such that the following condition holds*

- Ω is convex with respect to x' and
- $\Pi_N(\Omega) = \Gamma_1$, where Π_N is the projection on $\partial\mathbb{R}_+^N$.

If $pC(N, \frac{1}{p-1}) > H_N$ or $1 < p < \frac{N}{N-2}$ then u^ is bounded.*

In the above, Ω is said to be convex with respect to x' if $(tx', x_N) + ((1-t)y', x_N) \in \Omega$ whenever $t \in [0, 1]$, $x = (x', x_N) \in \Omega$ and $y = (y', x_N) \in \Omega$. Π_N is defined by $\Pi_N(x', x_N) = x_N$ for all $x = (x', x_N) \in \mathbb{R}_+^N$.

Remark 1.9. *It is not difficult to verify that the same result holds if*

- Ω is convex with respect to all directions x_1, \dots, x_{N-1} and
- Ω is symmetric with respect to the hyperplanes $x_1 = 0, \dots, x_{N-1} = 0$.

Theorem 1.10. *Consider (1) with $f(u) = (1 + u)^p$. If $pC(N, \frac{1}{p-1}) \leq H_N$ and $p \geq \frac{N}{N-2}$ there exists a domain Ω such that u^* is singular.*

Remark 1.11. *The condition $pC(N, \frac{1}{p-1}) \leq H_N$ is not enough to guarantee that the extremal solution is singular for some domain. Actually this condition can hold for some values of p in the range $\frac{N}{N-1} < p < \frac{N}{N-2}$. In this case a singular solution exists in some domains, but it does not correspond to the extremal one. See Theorem 6.2 in [4] for a similar phenomenon.*

The organization of the paper is as follows. In Section 2 we derive formula (14) and we prove Theorem 1.4 in Section 3. In Section 4 we analyze the exponential case and give a proof of Theorems 1.3 and 1.5. The proofs of Theorems 1.8 and 1.10 are given in Section 5.

2. COMPUTATION OF $C(N, \alpha)$

We write $x = (x', x_N) \in \mathbb{R}_+^N$ with $x' \in \mathbb{R}^{N-1}$, $x_N > 0$. It follows from (12) and a simple change of variables that

$$w_\alpha(x', x_N) = w_\alpha(e(x'), x_N) \quad \text{for all rotations } e \in \mathcal{O}(N-1).$$

and similarly

$$(15) \quad w_\alpha(Rx', Rx_N) = R^{-\alpha} w_\alpha(x', x_N).$$

Differentiating with respect to x_N yields

$$\frac{\partial w_\alpha}{\partial x_N}(Rx', Rx_N) = R^{-\alpha-1} \frac{\partial w_\alpha}{\partial x_N}(x', x_N).$$

Let $x \in \partial\mathbb{R}_+^N$, $x = (x', 0)$ and plug $R = \frac{1}{|x|} = \frac{1}{|x'|}$ in the previous formula to find

$$\frac{\partial w_\alpha}{\partial \nu}(x) = -\frac{\partial w_\alpha}{\partial x_N}(x', 0) = |x|^{-\alpha-1} \left(-\frac{\partial w_\alpha}{\partial x_N} \left(\frac{x'}{|x'|}, 0 \right) \right).$$

Define

$$(16) \quad C(N, \alpha) = -\frac{\partial w_\alpha}{\partial x_N} \left(\frac{x'}{|x'|}, 0 \right)$$

and observe that it is independent of $x' \in \mathbb{R}^{N-1}$.

Using (15) and the radial symmetry of w in the variables x' , there exists a function $v : [0, \infty) \rightarrow \mathbb{R}$ such that

$$(17) \quad w_\alpha(x', x_N) = |x'|^{-\alpha} w_\alpha \left(\frac{x'}{|x'|}, \frac{x_N}{|x'|} \right) = |x'|^{-\alpha} v \left(\frac{x_N}{|x'|} \right).$$

Writing $r = |x'|$, $t = \frac{x_N}{|x'|}$, we have

$$r^{-\alpha} v(t) = w_\alpha(x', rt), \quad \forall x' \in \mathbb{R}^{N-1}, |x'| = r.$$

The equation $\Delta w = 0$ is equivalent to

$$(18) \quad (1 + t^2)v''(t) + (2\alpha + 4 - N)tv'(t) + \alpha(\alpha - N + 3)v(t) = 0, \quad t > 0,$$

while (13) implies

$$v(0) = 1.$$

The initial condition for v' is related to (16)

$$v'(0) = -C(N, \alpha).$$

In addition to these initial conditions we remark that w_α is a smooth function in \mathbb{R}_+^N and this together with (17) implies that

$$(19) \quad \lim_{t \rightarrow \infty} v(t)t^\alpha \text{ exists.}$$

Using the change of variables $z = it$ with i the imaginary unit and defining the new unknown $h(z) := v(-iz)$ equation (18) becomes

$$(20) \quad (1 - z^2)h''(z) - (2\alpha + 4 - N)zh'(z) - \alpha(\alpha - N + 3)h(z) = 0,$$

with initial conditions

$$(21) \quad \lim_{t > 0, t \rightarrow 0} h(it) = 1, \quad \lim_{t > 0, t \rightarrow 0} h'(it) = iC(N, \alpha).$$

On the other hand (19) implies

$$(22) \quad \lim_{t \in \mathbb{R}, t \rightarrow \infty} h(it)t^\alpha \text{ exists.}$$

The substitution

$$(23) \quad g(z) = (1 - z^2)^{\frac{\alpha}{2} + \frac{1}{2} - \frac{N}{4}} h(z)$$

transforms equation (20) into

$$(24) \quad (1 - z^2)g''(z) - 2zg'(z) + \left(\nu(\nu + 1) - \frac{\mu^2}{1 - z^2}\right)g(z) = 0,$$

with

$$(25) \quad \mu = \alpha + \frac{2 - N}{2}, \quad \nu = \frac{N - 4}{2}.$$

The general solution to (24) is well known. Indeed, equation (24) belongs to the class of Legendre's equations. Following [1], two linearly independent solutions of (24) are given by the Legendre functions $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$, which are defined in $\mathbb{C} \setminus \{-1, 1\}$ and analytic in $\mathbb{C} \setminus (-\infty, 1]$ (see [1, Formulas 8.1.2 – 8.1.6]). Moreover the limits of $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ on both sides of $(-1, 1)$ exist and we shall use the notation

$$(26) \quad \begin{aligned} P_\nu^\mu(x + i0) &= \lim_{z \rightarrow x, \operatorname{Re}(z) > 0} P_\nu^\mu(z), & -1 < x < 1, \\ P_\nu^\mu(x - i0) &= \lim_{z \rightarrow x, \operatorname{Re}(z) < 0} P_\nu^\mu(z), & -1 < x < 1, \end{aligned}$$

and a similar notation for Q_ν^μ .

The solution g of (24) is therefore given by

$$g(z) = c_1 P_\nu^\mu(z) + c_2 Q_\nu^\mu(z),$$

for appropriate constants c_1 , c_2 . These constants are determined by the initial conditions (21), which imply:

$$(27) \quad c_1 P_\nu^\mu(0 + i0) + c_2 Q_\nu^\mu(0 + i0) = 1,$$

$$(28) \quad c_1 \frac{d}{dz} P_\nu^\mu(0 + i0) + c_2 \frac{d}{dz} Q_\nu^\mu(0 + i0) = iC(N, \alpha).$$

In order to evaluate $C(N, \alpha)$, we use also condition (22), which is equivalent to

$$(29) \quad \lim_{t \rightarrow \infty, t \in \mathbb{R}} (c_1 P_\nu^\mu(it) + c_2 Q_\nu^\mu(it)) t^{\frac{N}{2}-1} \text{ exists.}$$

But according to [1, Formulas 8.1.3, 8.1.5]

$$\begin{aligned} P_\nu^\mu(z) &\sim z^\nu \text{ as } |z| \rightarrow \infty \\ Q_\nu^\mu(z) &\sim z^{-\nu-1} \text{ as } |z| \rightarrow \infty \end{aligned}$$

This and (23),(29) imply that $c_1 = 0$ and we obtain from (27),(28)

$$(30) \quad C(N, \alpha) = -i \frac{\frac{d}{dz} Q_\nu^\mu(0 + i0)}{Q_\nu^\mu(0 + i0)}$$

From the properties and formulas in [1] the following values can be deduced:

$$(31) \quad Q_\nu^0(0 + i0) = -i 2^{\mu-1} \pi^{\frac{1}{2}} e^{i\mu\pi - i\nu\frac{\pi}{2}} \frac{\Gamma(\frac{\nu}{2} + \frac{\mu}{2} + \frac{1}{2})}{\Gamma(\frac{\nu}{2} - \frac{\mu}{2} + 1)}$$

$$(32) \quad \frac{d}{dz} Q_\nu^0(0 + i0) = 2^\mu \pi^{\frac{1}{2}} e^{i\mu\pi - i\nu\frac{\pi}{2}} \frac{\Gamma(\frac{\nu}{2} + \frac{\mu}{2} + 1)}{\Gamma(\frac{\nu}{2} - \frac{\mu}{2} + \frac{1}{2})}$$

The relations (30),(31),(32) and the values (25) yield formula (14). □

3. IMPROVED KATO INEQUALITY

We begin with some remarks on (9).

Remark 3.1. *a) The singular weight $\frac{1}{|x|}$ in the right-hand side of (9) is optimal, in the sense that it may not be replaced by $\frac{1}{|x|^\alpha}$ with $\alpha > 1$. This can be easily seen by choosing $\varphi \in H^1(\mathbb{R}_+^N)$ such that $\varphi(x) = |x|^{-\frac{N-2}{2} + \frac{\alpha-1}{2}}$ in a neighborhood of the origin.*

Moreover, the infimum in (10) is not achieved.

b) In dimension $N = 2$ the infimum (10) is zero, see [14]. Nonetheless, if the test-functions φ are required to vanish on the half line $x_1 > 0$ then the infimum has been computed in [14] :

$$(33) \quad \inf \left\{ \frac{\int_{\mathbb{R}_+^2} |\nabla \varphi|^2}{\int_{\partial \mathbb{R}_+^2} \frac{\varphi^2}{|x|}} : \varphi \in H^1(\mathbb{R}_+^2), \varphi(x_1, 0) = 0 \text{ if } x_1 > 0, \varphi|_{\partial \mathbb{R}_+^2} \not\equiv 0 \right\} = \frac{1}{\pi}$$

c) Using Stirling's formula it is easy to see that

$$(34) \quad H_N = \frac{N-3}{2} + O\left(\frac{1}{N}\right) \quad \text{as } N \rightarrow \infty.$$

d) The estimates

$$(35) \quad \frac{N-3}{2} \leq H_N \leq \frac{\sqrt{(N-3)^2 + 1}}{2}$$

can be obtained in a more straightforward way using particular test functions. We give a proof of this at the end of Section 3. Also observe that (34) could be deduced from (35).

Let us explain first informally the idea behind the proof of Theorem 1.4, assuming for a moment that a minimizer $\bar{w} \in H^1(\mathbb{R}_+^N)$ of (10) exists. \bar{w} then satisfies the associated Euler-Lagrange equation:

$$(36) \quad \begin{cases} \Delta \bar{w} = 0 & \text{in } \mathbb{R}_+^N, \\ \frac{\partial \bar{w}}{\partial \nu} = H_N \frac{\bar{w}}{|x|} & \text{on } \partial \mathbb{R}_+^N. \end{cases}$$

Elementary changes of variables show that given $R > 0$ and a rotation $e \in \mathcal{O}(N-1)$, $\bar{w}_R := R^{\frac{2-N}{2}} \bar{w}(Rx)$ and $\bar{w}_e := \bar{w}(e(x'), x_N)$ are also minimizers of (10). Thus it is natural to assume $\bar{w} = \bar{w}_R = \bar{w}_e$ for all $R > 0$ and $e \in \mathcal{O}(N-1)$. In particular a constant multiple of \bar{w} solves

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}_+^N, \\ w = |x|^{-\frac{N-2}{2}} & \text{on } \partial \mathbb{R}_+^N. \end{cases}$$

Unfortunately, such a function w does not belong to $H^1(\mathbb{R}_+^N)$. Let $w = w_\alpha$ with $\alpha = \frac{N-2}{2}$ as defined in (12). Observe that $C(N, \frac{N-2}{2}) = H_N$ by (16) and hence w is indeed a solution of (36).

Following an idea of Brezis and Vázquez (equation (4.6) on page 453 of [4]), we restate (9) in terms of the new variable $v = \varphi/w$.

Proof of Theorem 1.4. When $N \geq 3$, $C_0^\infty(\mathbb{R}_+^N \setminus \{0\})$ is dense in $H^1(\mathbb{R}_+^N)$. So it suffices to prove (9) for $\varphi \in C_0^\infty(\mathbb{R}_+^N \setminus \{0\})$. Fix such a $\varphi \neq 0$ and let w be the function defined by (12). Notice that, on $\text{supp } \varphi$, w is smooth and bounded from above and from below by some positive constants. Hence $v := \frac{\varphi}{w} \in C_0^\infty(\mathbb{R}_+^N)$ is well defined. Now, $\varphi = vw$, $\nabla \varphi = v \nabla w + w \nabla v$ and

$$|\nabla \varphi|^2 = v^2 |\nabla w|^2 + w^2 |\nabla v|^2 + 2vw \nabla v \nabla w.$$

Integrating

$$\int_{\mathbb{R}_+^N} |\nabla \varphi|^2 = \int_{\mathbb{R}_+^N} v^2 |\nabla w|^2 + \int_{\mathbb{R}_+^N} w^2 |\nabla v|^2 + 2 \int_{\mathbb{R}_+^N} vw \nabla v \nabla w$$

and by Green's formula

$$\begin{aligned} \int_{\mathbb{R}_+^N} v^2 |\nabla w|^2 &= \int_{\partial \mathbb{R}_+^N} v^2 w \frac{\partial w}{\partial \nu} - \int_{\mathbb{R}_+^N} w \nabla(v^2 \nabla w) \\ &= \int_{\partial \mathbb{R}_+^N} v^2 w \frac{\partial w}{\partial \nu} - 2 \int_{\mathbb{R}_+^N} vw \nabla w \nabla v, \end{aligned}$$

since w is harmonic in \mathbb{R}_+^N . Thus,

$$(37) \quad \int_{\mathbb{R}_+^N} |\nabla \varphi|^2 = \int_{\mathbb{R}_+^N} w^2 |\nabla v|^2 + \int_{\partial \mathbb{R}_+^N} v^2 w \frac{\partial w}{\partial \nu} = \int_{\mathbb{R}_+^N} w^2 |\nabla v|^2 + \int_{\partial \mathbb{R}_+^N} \frac{\varphi^2}{w} \frac{\partial w}{\partial \nu}.$$

But by (16) $\frac{\partial w}{\partial \nu}(x) = \frac{H_N}{|x|}$ for $x \in \partial \mathbb{R}_+^N$ and hence,

$$(38) \quad \int_{\mathbb{R}_+^N} |\nabla \varphi|^2 \geq H_N \int_{\partial \mathbb{R}_+^N} \frac{\varphi^2}{|x|} + \int_{\mathbb{R}_+^N} w^2 |\nabla v|^2 \quad \forall \varphi \in H^1(\mathbb{R}_+^N).$$

The second term in the right hand side of the above inequality yields the improvement of Kato's inequality when φ has support in the unit ball.

Now we assume $\varphi \in C_0^\infty(\overline{\mathbb{R}_+^N} \setminus \{0\} \cap B)$ and, as before, set $v = \frac{\varphi}{w}$. Our aim is to prove that given $1 \leq q < 2$ there exists $C > 0$ such that

$$(39) \quad I := \int_{\mathbb{R}_+^N} w^2 |\nabla v|^2 \geq \frac{1}{C} \|\varphi\|_{W^{1,q}}.$$

In spherical coordinates

$$I = \int_0^1 r^{N-1} \int_{S_1^+} w^2(r\theta) |\nabla v(r\theta)|^2 d\theta dr$$

where $S_1^+ = S_1 \cap \mathbb{R}_+^N$ and $S_1 = \{x \in \mathbb{R}^N / |x| = 1\}$ is the sphere of radius 1. From (15) we have $w(x) \geq \frac{1}{C} |x|^{-\frac{N-2}{2}}$ for some $C > 0$ and all $x \in B \cap \mathbb{R}_+^N$. Hence

$$I \geq \frac{1}{C} \int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta dr.$$

Let us compute the Sobolev norm of φ :

$$\begin{aligned} \|\varphi\|_{W^{1,q}}^q &= \int_{\mathbb{R}_+^N \cap B} |\nabla \varphi|^q dx = \int_0^1 r^{N-1} \int_{S_1^+} |\nabla \varphi(r\theta)|^q d\theta dr \\ &= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta) w(r\theta) + \nabla w(r\theta) v(r\theta)|^q d\theta dr \\ &\leq C_q \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)|^q |w(r\theta)|^q + |\nabla w(r\theta)|^q |v(r\theta)|^q d\theta dr. \end{aligned}$$

Define

$$\begin{aligned} I_1 &:= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)|^q |w(r\theta)|^q d\theta dr \\ I_2 &:= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla w(r\theta)|^q |v(r\theta)|^q d\theta dr. \end{aligned}$$

Since $w(x) \leq C|x|^{-\frac{N-2}{2}}$ we have by Hölder's inequality

$$(40) \quad \begin{aligned} I_1 &\leq C \int_0^1 r^{N-1-\frac{(N-2)q}{2}} \int_{S_1^+} |\nabla v(r\theta)|^q d\theta dr \\ &\leq C \left[\int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta dr \right]^{\frac{q}{2}} \left[\int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} dr \right]^{\frac{2-q}{2}} = CI^{\frac{q}{2}}, \end{aligned}$$

since $q < 2$.

Using $|\nabla w(x)| \leq C|x|^{-\frac{N}{2}}$ we estimate I_2 :

$$I_2 \leq C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q dr d\theta.$$

From the classical Hardy inequality

$$\int_0^1 r^\gamma |f(r)|^p dr \leq \left(\frac{p}{\gamma+1} \right)^p \int_0^1 r^{\gamma+p} |f'(r)|^p dr$$

($p \geq 1$, $\gamma > -1$, $f \in C_0^\infty(0,1)$) we deduce

$$\int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q dr \leq C \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q dr$$

and therefore

$$I_2 \leq C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q dr d\theta.$$

Hölder's inequality yields

(41)

$$I_2 \leq C \left[\int_{S_1^+} \int_0^1 r |\nabla v(r\theta)|^2 dr d\theta \right]^{\frac{q}{2}} \left[\int_{S_1^+} \int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} dr d\theta \right]^{1-\frac{q}{2}} = CI^{\frac{q}{2}},$$

where we have used $q < 2$. Gathering (40) and (41) we conclude that (39) holds. \square

Now we pass to the proof of item (d) of Remark 3.1.

Proof of (35). We shall first show the inequality

$$\frac{N-3}{2} \leq H_N, \quad \forall N \geq 4.$$

One may assume that $u = u(r, t)$ where $r = |(x_1, \dots, x_{N-1})|$ and $t = x_N$. Then

$$\int_{\partial\mathbb{R}_+^N} \frac{u^2}{|x|} = (N-1)\omega_{N-1} \int_0^\infty u(r, 0)^2 r^{N-3} dr,$$

where ω_{N-1} is the volume of the unit ball in \mathbb{R}^{N-1} . But

$$u(r, 0) = -2 \int_0^\infty u(r, t) \frac{\partial u}{\partial t}(r, t) dt.$$

So,

$$\begin{aligned} \int_{\partial\mathbb{R}_+^N} \frac{u^2}{|x|} &= -2(N-1)\omega_{N-1} \int_0^\infty \int_0^\infty u(r, t) \frac{\partial u}{\partial t}(r, t) r^{N-3} dr dt \\ &\leq 2(N-1)\omega_{N-1} \int_0^\infty \left(\int_0^\infty u(r, t)^2 r^{N-4} dr \right)^{1/2} \left(\int_0^\infty \left(\frac{\partial u}{\partial t}(r, t) \right)^2 r^{N-2} dr \right)^{1/2} dt. \end{aligned}$$

We use now the inequality

$$\int_0^\infty u(r, t)^2 r^{N-4} dr \leq \frac{4}{(N-3)^2} \int_0^\infty \left(\frac{\partial u}{\partial r}(r, t) \right)^2 r^{N-2} dr,$$

which is one of the classical version of Hardy's inequality (in dimension $N-1$). We obtain

$$\begin{aligned} &\int_{\partial\mathbb{R}_+^N} \frac{u^2}{|x|} \\ &\leq \frac{4}{N-3} (N-1)\omega_{N-1} \int_0^\infty \left[\int_0^\infty \left(\frac{\partial u}{\partial r}(r, t) \right)^2 r^{N-2} dr \right]^{\frac{1}{2}} \left[\int_0^\infty \left(\frac{\partial u}{\partial t}(r, t) \right)^2 r^{N-2} dr \right]^{\frac{1}{2}} dt \\ &\leq \frac{2}{N-3} (N-1)\omega_{N-1} \int_0^\infty \int_0^\infty \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial r} \right)^2 \right] r^{N-2} dr dt \\ &= \frac{2}{N-3} \int_{\mathbb{R}_+^N} |\nabla u|^2. \end{aligned}$$

To prove

$$(42) \quad H_N \leq \frac{\sqrt{(N-3)^2 + 1}}{2},$$

we consider, for fixed $a > 0$ and $\varepsilon \downarrow 0$, the function

$$\tilde{\varphi}(r, x_N) = \begin{cases} r^{\frac{2-N}{2}} e^{-ax_N/r} & \text{if } r > \varepsilon \\ \varepsilon^{\frac{2-N}{2}} e^{-ax_N/\varepsilon} & \text{if } r \leq \varepsilon, \end{cases}$$

where $x = (x', x_N) \in \mathbb{R}_+^{N-1} \times \mathbb{R}_+$, $r = |x'|$. With the test function

$$\varphi = \eta \tilde{\varphi}$$

where $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_1(0)$ and $\eta \equiv 0$ outside of $B_2(0)$ and a suitable choice of a one obtains (42). We omit the details.

4. THE EXPONENTIAL CASE

We need the following result that characterizes extremal singular solutions belonging to $H^1(\Omega)$, see [4, Theorem 3.1]. The proof is an adaptation of the one in this reference.

Lemma 4.1. *Assume that $v \in H^1(\Omega)$ is an unbounded solution of (1) for some $\lambda > 0$. Assume furthermore the stability condition*

$$(43) \quad \lambda \int_{\Gamma_1} f'(v) \varphi^2 \leq \int_{\Omega} |\nabla \varphi|^2 \quad \forall \varphi \in C^1(\bar{\Omega}), \varphi = 0 \text{ on } \Gamma_2.$$

Then $\lambda = \lambda^*$ and $v = u^*$.

Remark 4.2. *We have not shown that there is a unique weak solution of (1) when $\lambda = \lambda^*$. A result of Martel [20] guarantees that this is indeed the case for problem (5) and this was used by Brezis and Vázquez in the proof of [4, Theorem 3.1]. In our context, we take $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ as the definition of the extremal solution. Knowing that $u^* \in H^1(\Omega)$, the proof of [4, Theorem 3.1] shows that $\lambda = \lambda^*$ and $v = u^*$.*

To prove Theorems 1.3 and 1.5 it will be convenient to study the function u_0 defined by

$$(44) \quad u_0(x) = \int_{\partial \mathbb{R}_+^N} K(x, y) \log \frac{1}{|y|} dy \quad \text{for } x \in \mathbb{R}_+^N,$$

where as before $K(x, y) = \frac{2x_N}{N\omega_N} |x - y|^{-N}$. Then u_0 is harmonic in \mathbb{R}_+^N and

$$u_0(x) = \log \frac{1}{|x|} \quad \text{for } x \in \partial \mathbb{R}_+^N, x \neq 0.$$

Note that

$$u_0(Rx) = u_0(x) + \log \frac{1}{R}.$$

Let $r = |x'|$. Then

$$(45) \quad u_0(x', x_N) = v\left(\frac{x_N}{r}\right) + \log \frac{1}{r},$$

for some $v : [0, \infty) \rightarrow \mathbb{R}$ such that $v(0) = 0$. We see that

$$\frac{\partial u_0}{\partial \nu} = -\frac{\partial u_0}{\partial x_N} \Big|_{x_N=0} = -\frac{1}{r} v'(0)$$

so

$$\frac{\partial u_0}{\partial \nu} = \lambda_{0,N} e^{u_0} \quad \text{on } \partial \mathbb{R}_+^N,$$

where we let

$$\lambda_{0,N} = -v'(0).$$

Let

$$\begin{aligned}\Omega_0 &= \{x \in \mathbb{R}_+^N : u_0(x) > 0\} \\ \Gamma_1 &= \partial\Omega \cap \partial\mathbb{R}_+^N \quad \Gamma_2 = \partial\Omega \setminus \partial\mathbb{R}_+^N.\end{aligned}$$

The boundary $\partial\Omega_0$ is not smooth itself but Γ_1, Γ_2 are, and it can be checked that Proposition 1.1 still holds in this case.

It can be verified that Ω_0 can be written as $\Omega_0 = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+ : |x'| < e^{v(x_N/|x'|)}\}$.

Lemma 4.3. *We have*

$$\lambda_{0,N} = \begin{cases} (N-3) \frac{\sqrt{\pi}\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)} & \text{if } N \geq 4, \\ 1 & \text{if } N = 3. \end{cases}$$

Proof. We give details for $N \geq 4$, the case $N = 3$ being similar. We need to compute $v'(0)$. Calculating Δu_0 in terms of v (see (45)) we obtain that v satisfies

$$(1+t^2)v''(t) + (4-N)tv'(t) + 3-N = 0$$

and thus v' is given by

$$v'(t) = (N-3)(1+t^2)^{\frac{N-4}{2}} \int_0^t (1+s^2)^{\frac{2-N}{2}} ds + (1+t^2)^{\frac{N-4}{2}} v'(0).$$

Integrating and using $v(0) = 0$ yields

(46)

$$v(t) = (N-3) \int_0^t (1+\tau^2)^{\frac{N-4}{2}} \int_0^\tau (1+s^2)^{\frac{2-N}{2}} ds d\tau + v'(0) \int_0^t (1+\tau^2)^{\frac{N-4}{2}} d\tau.$$

We look at the asymptotics of the two integrals above, as $t \rightarrow \infty$. For the second integral, we have

$$\lim_{t \rightarrow \infty} \frac{\int_0^t (1+\tau^2)^{\frac{N-4}{2}} d\tau}{t^{N-3}} = \frac{(1+t^2)^{\frac{N-4}{2}}}{(N-3)t^{N-4}} = \frac{1}{N-3}.$$

And for the first integral,

$$\begin{aligned}\lim_{t \rightarrow \infty} \frac{\int_0^t (1+\tau^2)^{\frac{N-4}{2}} \int_0^\tau (1+s^2)^{\frac{2-N}{2}} ds d\tau}{t^{N-3}} &= \lim_{t \rightarrow \infty} \frac{(1+t^2)^{\frac{N-4}{2}} \int_0^t (1+s^2)^{\frac{2-N}{2}} ds}{(N-3)t^{N-4}} \\ &= \frac{1}{N-3} \int_0^\infty (1+s^2)^{\frac{2-N}{2}} ds \\ &= \frac{1}{N-3} \frac{\sqrt{\pi}\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)}.\end{aligned}$$

Going back to (46), we obtain that

$$v(t) = \left(\frac{\sqrt{\pi}\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)} + \frac{v'(0)}{N-3} \right) t^{N-3} + o(t^{N-3}).$$

Now, recall that for $x_N > 0$, $\lim_{r \rightarrow 0} v(x_N/r) + \log \frac{1}{r} = u_0(0, x_N) \in \mathbb{R}$ exists and is finite. Hence, we must have

$$v'(0) = -(N-3) \frac{\sqrt{\pi} \Gamma(\frac{N}{2} - \frac{3}{2})}{2\Gamma(\frac{N}{2} - 1)}.$$

□

Proof of Theorem 1.3. We have shown that u_0 defined in (44) is a solution to (1) with $\Omega = \Omega_0$ and $\lambda = \lambda_{0,N}$. This solution satisfies the stability condition (43) if and only if (by scaling)

$$\lambda_{0,N} \int_{\partial \mathbb{R}_+^N} \frac{\varphi^2}{|x|} \leq \int_{\mathbb{R}_+^N} |\nabla \varphi|^2, \quad \forall \varphi \in C_0^1(\overline{\mathbb{R}_+^N} \setminus \{0\}).$$

In the Appendix we prove that

$$(47) \quad H_N \geq \lambda_{0,N} \text{ if and only if } N \geq 10$$

and this completes the proof of the theorem. □

Proof of Theorem 1.5.

We prove the theorem by contradiction, assuming that u^* is unbounded. We use an idea of Crandall and Rabinowitz [11], but with different test functions.

Let $\phi(x) = \int_{\partial \mathbb{R}_+^N} K(x, y) |y|^{2-N+\varepsilon} dy$ and $\psi(x) = \int_{\partial \mathbb{R}_+^N} K(x, y) |y|^{\frac{2-N+\varepsilon}{2}} dy$. Then,

$$(48) \quad \frac{\partial \phi}{\partial \nu} = K_\phi |x|^{1-N+\varepsilon} \quad \frac{\partial \psi}{\partial \nu} = K_\psi |x|^{\frac{-N+\varepsilon}{2}},$$

where the constants K_ϕ, K_ψ are given by

$$K_\phi = \lambda_{0,N} \varepsilon + O(\varepsilon^2) \quad \text{and} \quad K_\psi = H_N + O(\varepsilon).$$

Indeed, since u_0 and ϕ are harmonic in Ω ,

$$\int_{\partial \Omega} u_0 \frac{\partial \phi}{\partial \nu} = \int_{\partial \Omega} \phi \frac{\partial u_0}{\partial \nu}.$$

Clearly, $\int_{\Gamma_2} |\phi \frac{\partial u_0}{\partial \nu}| \leq C$, for some constant C independent of ε . So

$$K_\phi \int_0^1 \ln\left(\frac{1}{r}\right) \frac{1}{r} r^{2-N+\varepsilon} r^{N-2} dr = \lambda_{0,N} \int_0^1 \frac{1}{r} r^{2-N+\varepsilon} r^{N-2} dr + O(1) = \frac{\lambda_{0,N}}{\varepsilon} + O(1).$$

Now, $\int_0^1 \ln \frac{1}{r} r^{-1+\varepsilon} dr = \frac{1}{\varepsilon^2}$ so we end up with

$$K_\phi = \lambda_{0,N} \varepsilon + O(\varepsilon^2).$$

Similarly, since ψ and w (defined in (12)) are harmonic in Ω , we have

$$\int_{\partial \Omega} w \frac{\partial \psi}{\partial \nu} = \int_{\partial \Omega} \psi \frac{\partial w}{\partial \nu}.$$

As before the boundary terms on Γ_2 are bounded independently of ε so

$$K_\psi \int_0^1 r^{-1+\varepsilon} dr = H_N \int_0^1 r^{-1+\varepsilon} dr + O(1).$$

Hence,

$$K_\psi = H_N + O(\varepsilon).$$

Multiplying the equation (1) by ϕ and integrating by parts twice yields

$$(49) \quad \int_{\partial\Omega} u^* \frac{\partial\phi}{\partial\nu} = \lambda^* \int_{\partial\Omega} \phi e^{u^*}.$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $\eta \equiv 1$ in $B_R(0)$ where $R > 0$ is small and fixed, and $\eta = 0$ on Γ_2 . Using the stability condition (7) with $\eta\psi$ yields

$$(50) \quad \begin{aligned} \lambda^* \int_{\Gamma_1 \cap B_R(0)} e^{u^*} \psi^2 &\leq \int_{\Omega} |\nabla(\eta\psi)|^2 = \int_{\partial\Omega} \frac{\partial}{\partial\nu}(\eta\psi)(\eta\psi) - \int_{\Omega} (\eta\psi)\Delta(\eta\psi) \\ &\leq \int_{\Gamma_1 \cap B_R(0)} \frac{\partial\psi}{\partial\nu} \psi + C \end{aligned}$$

where the constant C does not depend on ε . Since $\psi^2 = \phi$ on $\partial\mathbb{R}_+^N$ combining (49) and (50) we obtain

$$\int_{\partial\Omega} u^* \frac{\partial\phi}{\partial\nu} \leq \int_{\Gamma_1 \cap B_R(0)} \frac{\partial\psi}{\partial\nu} \psi + C.$$

Using (48) we arrive at

$$K_\phi \int_{\Gamma_1 \cap B_R(0)} u^* |x|^{1-N+\varepsilon} \leq K_\psi \int_{\Gamma_1 \cap B_R(0)} |x|^{1-N+\varepsilon} + C$$

and thus

$$(51) \quad \int_{\Gamma_1 \cap B_R(0)} u^* |x|^{1-N+\varepsilon} \leq \omega_{N-1} \frac{H_N}{\lambda_{0,N}} \frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right),$$

where ω_{N-1} is the area of the $N-1$ dimensional sphere.

Next we claim that for any given $0 < \sigma < 1$ there exists $r(\sigma) > 0$ such that

$$(52) \quad u^*(x) \geq (1-\sigma) \log \frac{1}{|x|} \quad \forall x \in \Gamma_1, |x| \leq r(\sigma).$$

Observe first that for all $0 < \lambda < \lambda^*$ the minimal solution u_λ is symmetric in the variables x_1, \dots, x_{N-1} by uniqueness of the minimal solution and it achieves its maximum at the origin by the moving plane method (see Proposition 5.2 in [7]).

Assume by contradiction that (52) is false. Then there exists $\sigma > 0$ and a sequence $x_k \in \Gamma_1$ with $x_k \rightarrow 0$ such that

$$(53) \quad u^*(x_k) < (1-\sigma) \log \frac{1}{|x_k|}.$$

Let $s_k = |x_k|$ and choose $0 < \lambda_k < \lambda^*$ such that

$$(54) \quad \max_{\bar{\Omega}} u_{\lambda_k} = u_{\lambda_k}(0) = \log \frac{1}{s_k}.$$

Note that $\lambda_k \rightarrow \lambda^*$, otherwise u_{λ_k} would remain bounded. Let

$$v_k(x) = \frac{u_{\lambda_k}(s_k x)}{\log \frac{1}{s_k}} \quad x \in \Omega_k \equiv \frac{1}{s_k} \Omega.$$

Then $0 \leq v_k \leq 1$, $v_k(0) = 1$, $\Delta v_k = 0$ in Ω_k and

$$\begin{aligned} \frac{\partial v_k}{\partial \nu}(x) &= \frac{1}{\log \frac{1}{s_k}} s_k \lambda_k \exp(u_{\lambda_k}(s_k x)) \\ &\leq \frac{\lambda_k}{\log \frac{1}{s_k}} \rightarrow 0. \end{aligned}$$

by (54). By elliptic regularity $v_k \rightarrow v$ uniformly on compact sets of $\overline{\mathbb{R}_+^N}$ to a function v satisfying $0 \leq v \leq 1$, $v(0) = 1$, $\Delta v = 0$ in \mathbb{R}_+^N , $\frac{\partial v}{\partial \nu} = 0$ on $\partial \mathbb{R}_+^N$. Extending v evenly to \mathbb{R}^N we deduce that $v \equiv 1$. Since $|x_k| = s_k$ we deduce that

$$\frac{u_{\lambda_k}(x_k)}{\log \frac{1}{s_k}} \rightarrow 1,$$

which contradicts (53).

Going back to (51) and using (52) we find

$$(55) \quad (1 - \sigma) \int_0^{r(\sigma)} \log \frac{1}{r} r^{\varepsilon-1} dr \leq \frac{K_\psi}{K_\phi} \frac{1}{\varepsilon} + C = \frac{H_N}{\lambda_{0,N}} \frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right).$$

Integrating

$$(1 - \sigma) \left(\frac{1}{\varepsilon^2} r(\sigma)^\varepsilon + \frac{1}{\varepsilon} r(\sigma)^\varepsilon \log \frac{1}{r(\sigma)} \right) \leq \frac{H_N}{\lambda_{0,N}} \frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right).$$

Letting $\varepsilon \rightarrow 0$ yields

$$(1 - \sigma) \leq \frac{H_N}{\lambda_{0,N}}.$$

As σ is arbitrarily small we deduce $\frac{H_N}{\lambda_{0,N}} \geq 1$ which by (47) forces $N \geq 10$, a contradiction. \square

Proof of Proposition 1.7. Let indeed $u = u_\lambda$ be the minimal solution of (1). Working as in [11] we take $\varphi = e^{ju} - 1$, $j > 0$ in (6) and multiply (1) by $\psi = e^{2ju} - 1$. We obtain

$$\frac{\lambda}{j^2} \int_{\Gamma_1} e^u (e^{ju} - 1)^2 ds \leq \frac{\lambda}{2j} \int_{\Gamma_1} e^u (e^{2ju} - 1) ds.$$

It follows that

$$\begin{aligned} \left(\frac{1}{j} - \frac{1}{2} \right) \int_{\Gamma_1} e^{(2j+1)u} ds &\leq \frac{2}{j} \int_{\Gamma_1} e^{(j+1)u} ds \\ &\leq \frac{2}{j} \int_{\Gamma_1 \cap A} e^{(j+1)u} ds + \frac{2}{j} \int_{\Gamma_1 \cap B} e^{(j+1)u} ds, \end{aligned}$$

where $A = [(1/j - 1/2)e^{(2j+1)u} < \frac{4}{j}e^{(j+1)u}]$ and $B = [(1/j - 1/2)e^{(2j+1)u} \geq \frac{4}{j}e^{(j+1)u}]$. Given $j \in (0, 2)$, we see that u remains uniformly bounded on A , while

$$\frac{2}{j} \int_{\Gamma_1 \cap B} e^{(j+1)u} ds \leq \frac{1}{2} \left(\frac{1}{j} - \frac{1}{2} \right) \int_{\Gamma_1} e^{(j+1)u} ds.$$

We conclude that e^u is bounded in $L^{2j+1}(\partial\Omega)$ independently of λ . If $2j+1 > N-1$ we obtain by elliptic estimates a bound for u in $C^\alpha(\overline{\Omega})$, for some $\alpha \in (0, 1)$. Thus if $N < 6$ we can choose $j \in (0, 2)$ such that $N-1 < 2j+1 < 5$ and obtain a bound for u in $C^\alpha(\overline{\Omega})$ independent of λ . However, for $N = 6, 7, 8$ or 9 , this argument does not prove that $u^* \in L^\infty(\Omega)$. \square

5. THE POWER CASE

Proof of Theorem 1.8. We shall give here the proof of the case $pC(N, \frac{1}{p-1}) > H_N$. If $p < \frac{N}{N-2}$, the boundedness of u^* follows from standard techniques, using the Sobolev trace embedding theorem $H^1(\Omega) \rightarrow L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$.

Let $v = C(N, \frac{1}{p-1})^{\frac{1}{p-1}} w_{\frac{1}{p-1}}$. Then v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}_+^N \\ \frac{\partial v}{\partial \nu} = v^p & \text{on } \partial\mathbb{R}_+^N. \end{cases}$$

Observe that $pv^{p-1} = \frac{pC(N, \frac{1}{p-1})}{|x|} > \frac{H_N}{|x|}$ on $\partial\mathbb{R}_+^N \setminus \{0\}$ and hence

$$(56) \quad \inf \frac{\int_{\partial\mathbb{R}_+^N} |\nabla\varphi|^2 - p \int_{\partial\mathbb{R}_+^N} v^{p-1}\varphi^2}{\int_{\partial\mathbb{R}_+^N} \varphi^2} = -\infty$$

where the infimum is taken over the functions $\varphi \in C_0^\infty(\overline{\mathbb{R}_+^N})$ that do not vanish identically on $\partial\mathbb{R}_+^N$.

Assume that u^* is singular. For $R > 0$ and $0 < \lambda < \lambda^*$ let

$$u_R(x) = \lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} u_\lambda(Rx + x_\lambda),$$

where x_λ denotes a point of maximum of u_λ . Observe that since u_λ is positive and harmonic in Ω , $x_\lambda \in \Gamma_1$.

For $0 < \lambda < \lambda^*$, we choose R such that $u_R(0) = 1$ i.e. such that $\lambda^{\frac{1}{1-p}} R^{\frac{1}{p-1}} u_\lambda(x_\lambda) = 1$. Since $u_\lambda(x_\lambda) \rightarrow \infty$ as $\lambda \uparrow \lambda^*$ we have $R \rightarrow 0$ as $\lambda \uparrow \lambda^*$.

Then u_R verifies

$$\begin{cases} \Delta u_R = 0 & \text{in } \Omega_R \\ \frac{\partial u_R}{\partial \nu} = (\lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} + u_R)^p & \text{on } \Gamma_1^R \\ u_R = 0 & \text{on } \Gamma_2^R, \end{cases}$$

where

$$\Omega_R = (\Omega - x_\lambda)/R, \quad \Gamma_1^R = (\Gamma_1 - x_\lambda)/R, \quad \Gamma_2^R = (\Gamma_2 - x_\lambda)/R.$$

Furthermore u_R satisfies the stability condition

$$\int_{\Omega_R} |\nabla\varphi|^2 \geq p \int_{\Gamma_1^R} (\lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} + u_R)^{p-1} \varphi^2 \quad \forall \varphi \in C_0^\infty(\Omega_R \cup \Gamma_1^R).$$

Let

$$\phi(R) = \sup\{r > 0 / B_r \cap \partial\mathbb{R}_+^N \subset \Gamma_1^R\}.$$

The moving plane method implies that the distance of the point $x_\lambda \in \Gamma_1$ to $\Gamma_1 \cap \Gamma_2$ stays bounded away from zero, see [7] for this method in the context of non-linear Neumann condition. Thus implies that

$$(57) \quad \phi(R) \rightarrow +\infty \quad \text{as } R \rightarrow 0.$$

Step 1. We have

$$(58) \quad u_R \leq v \quad \text{in } \Gamma_1^R \cap B_{\phi(R)}.$$

Proof. Suppose not. Define

$$r_0 = \sup\{r > 0 \mid r < \phi(R), u_R \leq v \text{ in } B_r \cap \Gamma_1^R\}.$$

Since v is singular at 0, $r_0 > 0$ and we have $u_R \leq v$ in $B_{r_0} \cap \Gamma_1^R$. Furthermore, there exists $x_0 \in \partial\mathbb{R}_+^N$ such that $|x_0| = r_0$ and $u_R(x_0, 0) = v(x_0, 0)$.

Let $x \in \Gamma_1^R$ be such that $|x| = r_0$. If $u_R(x, 0) = v(x, 0)$ then $\frac{\partial v}{\partial \nu}(x, 0) = v(x, 0)^p < (\lambda^{\frac{1}{1-p}} R^{\frac{1}{p-1}} + u_R(x, 0))^p = \frac{\partial u_R}{\partial \nu}(x, 0)$ and hence for some $\delta_x > 0$

$$(59) \quad \frac{\partial v}{\partial x_N}(y, t) > \frac{\partial u_R}{\partial x_N}(y, t) \quad |y - x|^2 + t^2 < \delta_x^2.$$

It follows that for some $m_x > 0$ (and decreasing if necessary δ_x)

$$(60) \quad u_R(y, t) < v(y, t), \quad \text{if } |y - x|^2 + t^2 < \delta_x^2, t > m(|y| - r_0), t > 0.$$

Indeed, because of (59) and $u_R(y, 0) \leq v(y, 0)$ for $|y| \leq r_0$ we immediately obtain

$$u_R(y, t) < v(y, t) \quad \text{for } |y - x|^2 + t^2 < \delta_x^2, |y| \leq r_0, t > 0.$$

If (60) is false, then there are sequences $y_k \rightarrow x$, $t_k \rightarrow 0$ with $|y_k| > r_0$ and $\frac{t_k}{|y_k - x|} \rightarrow \infty$ such that $v(y_k, t_k) \leq u_R(y_k, t_k)$. Then by the mean value theorem there exists a point ξ_k in the segment from (y_k, t_k) to $(x, 0)$ such that $\nabla(v(\xi_k) - u_R(\xi_k)) \cdot w_k \leq 0$ where w_k is the unit vector $w_k = \frac{(y_k - x, t_k)}{\|w_k\|}$. Taking the limit we obtain $\frac{\partial}{\partial x_N}(v(x, 0) - u_R(x, 0)) \leq 0$ which contradicts (59) (recall that $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial x_N}$). This proves (60).

Suppose $x \in \Gamma_1^R$ is such that $|x| = r_0$ and $u_R(x, 0) < v(x, 0)$. The by continuity there is $\delta_x > 0$ such that (60) still holds.

Then by compactness for some $\delta > 0$ and $m > 0$ we have

$$(61) \quad u_R(x, t) < v(x, t), \quad \text{if } (|x| - r_0)^2 + t^2 < \delta^2, t > m(|x| - r_0), t > 0.$$

Now consider

$$z(x, t) = \rho^\alpha (\sin(\alpha(\theta - \theta_0)) + b(\theta - \theta_0)^2),$$

where (ρ, θ) are polar coordinates around $(r_0, 0)$ i.e.

$$|x| = r_0 + \rho \cos(\theta), \quad t = \rho \sin(\theta).$$

We choose $\theta_0 \in (0, \frac{\pi}{2})$ is close enough to $\frac{\pi}{2}$ so that $\tan \theta_0 > m$. The parameters α , b are chosen later on.

We shall use the maximum principle to prove that for sufficiently small $\delta > 0$, $\varepsilon > 0$ we have

$$v(x, 0) - u_R(x, 0) \geq \varepsilon z(x, 0), \quad r_0 - \delta < |x| < r_0.$$

We have

$$(62) \quad \begin{aligned} \Delta z &= \frac{\rho^{\alpha-2}}{r_0 + \rho \cos(\theta)} \left[2br_0 + \alpha^2 br_0 (\theta - \theta_0)^2 + \rho \left(-(N-2)\alpha \sin(\theta) \cos(\alpha(\theta - \theta_0)) \right. \right. \\ &\quad \left. \left. - 2b(N-2) \sin(\theta)(\theta - \theta_0) + 2b \cos(\theta) + (N-2)\alpha \cos(\theta) \sin(\alpha(\theta - \theta_0)) \right. \right. \\ &\quad \left. \left. + b\alpha(\alpha + N - 2) \cos(\theta)(\theta - \theta_0)^2 \right) \right] \\ &= \rho^{\alpha-2} \left(2b + \alpha^2 b(\theta - \theta_0)^2 + O(\rho) \right), \quad \text{as } \rho \rightarrow 0 \end{aligned}$$

and, observing that for $\theta = \pi$ we are on $\partial\mathbb{R}_+^N$

$$\frac{\partial z}{\partial \nu} = \frac{1}{\rho} \frac{\partial z}{\partial \theta} \Big|_{\theta=\pi} = \rho^{\alpha-1} (\alpha \cos(\alpha(\pi - \theta_0)) + 2b(\pi - \theta_0)).$$

But $\pi - \theta_0 > \frac{\pi}{2}$. We fix $0 < \alpha < 1$ close enough to 1 such that $\cos(\alpha(\pi - \theta_0)) < 0$, and then take $b > 0$ small enough so that $\alpha \cos(\alpha(\pi - \theta_0)) + 2b(\pi - \theta_0) < 0$. Thus, setting

$$a \equiv -(\alpha \cos(\alpha(\pi - \theta_0)) + 2b(\pi - \theta_0)) > 0,$$

we have

$$\frac{\partial z}{\partial \nu} = -a\rho^{\alpha-1} \quad \text{for } \rho > 0 \text{ small.}$$

On the other hand, by (62)

$$(63) \quad \Delta z \geq \rho^{\alpha-2} (2b + O(\rho)) \quad \text{as } \rho \rightarrow 0$$

uniformly for $\theta_0 < \theta < \pi$. Now consider the region

$$D = \{ (x, t) \mid (|x| - r_0)^2 + t^2 < \delta^2, t > m(|x| - r_0), t > 0 \} = \{ (\rho, \theta) \mid 0 < \rho < \delta, \theta_0 < \theta < \pi \}$$

and write $\partial D = S_1 \cup S_2 \cup A$ where

$$S_1 = \{ (\rho, \theta) \mid 0 < \rho < \delta, \theta = \theta_0 \}, \quad S_2 = \{ (\rho, \theta) \mid 0 < \rho < \delta, \theta = \pi \}$$

and

$$A = \{ (\rho, \theta) \mid \rho = \delta, \theta_0 < \theta < \pi \}.$$

By (63) and choosing $\delta > 0$ smaller if necessary we achieve $\Delta z > 0$ in D . On S_1 we have $z = 0$ and $v - u_R > 0$. Now we seek $\varepsilon > 0$, $\delta > 0$ smaller than before such that

$$\inf_{S_2} \left(\frac{\partial v}{\partial \nu} - \frac{\partial u_R}{\partial \nu} \right) \geq \varepsilon \sup_{S_2} \frac{\partial z}{\partial \nu} = -a\varepsilon\delta^{\alpha-1}$$

and

$$\inf_A (v - u_R) > \varepsilon \sup_A z = \varepsilon\delta^\alpha C_1$$

where $C_1 = \sin(\alpha(\pi - \theta_0)) + b(\pi - \theta_0)^2$. Writing $K = -\inf_{S_2} (\frac{\partial v}{\partial \nu} - \frac{\partial u_R}{\partial \nu}) < \infty$ and $c_0 = \inf_A (v - u_R) > 0$ we first choose $\delta > 0$ small such that

$$\delta \frac{K}{a} < \frac{c_0}{C_1}$$

and then ε such that $\delta^{1-\alpha} \frac{K}{a} \leq \varepsilon < \frac{c_0}{C_1} \delta^{-\alpha}$.

The calculations above and the maximum principle then yield $v - u_R \geq \varepsilon z$ in D , which was the desired conclusion. Now this implies that $v - u_R$ is not differentiable at $(x_0, 0)$, a contradiction.

Step 2. We let $\lambda \uparrow \lambda^*$ and hence $R \rightarrow 0$. Since $0 \leq u_R \leq u_R(0) = 1$, $u_R \rightarrow u$ uniformly on compact sets of \mathbb{R}_+^N and u satisfies

$$(64) \quad \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^N \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\mathbb{R}_+^N \end{cases}$$

and

$$u(0) = 1.$$

Also, u satisfies

$$(65) \quad \int_{\mathbb{R}_+^N} |\nabla \varphi|^2 \geq p \int_{\partial \mathbb{R}_+^N} u^{p-1} \varphi^2 \quad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}_+^N}).$$

By (57) and the previous step we deduce

$$u \leq v \quad \text{in } \partial \mathbb{R}_+^N.$$

Let

$$(66) \quad \mu = \sup_{\partial \mathbb{R}_+^N} \frac{u}{v} \leq 1$$

We claim that $\mu = 1$. Let indeed

$$\tilde{u}(x) = c_1 \int_{\partial \mathbb{R}_+^N} \frac{u(y)^p}{|x-y|^{N-2}} dy \quad x \in \partial \mathbb{R}_+^N.$$

Then \tilde{u} is harmonic in \mathbb{R}_+^N and agrees with u on $\partial \mathbb{R}_+^N$. Since u is bounded by 1 and \tilde{u} is bounded, we see that $\tilde{u} - u$ must be a constant. But then, since $\tilde{u}(x, 0) \rightarrow 0$ and $u(x, 0) \rightarrow 0$ as $|x| \rightarrow \infty$ we see that $u \equiv \tilde{u}$.

Thus

$$u(x) = \int_{\partial \mathbb{R}_+^N} \frac{u(y)^p}{|x-y|^{N-2}} dy \leq c_1 \mu^p \int_{\partial \mathbb{R}_+^N} \frac{v(y)^p}{|x-y|^{N-2}} dy = \mu^p v(x) \quad x \in \partial \mathbb{R}_+^N.$$

This implies $\mu \leq \mu^p$ and since $\mu \neq 0$, we conclude $\mu = 1$.

Step 3. Observe that the supremum in (66) is not attained. Otherwise $v - u$ would achieve a minimum at a point $x \in \partial \mathbb{R}_+^N$, where the normal derivative would be zero. By Hopf's lemma, we would have $u \equiv v$, which is impossible since u is bounded and v is not. Let $x_k \in \partial \mathbb{R}_+^N$ be such that $|x_k| \rightarrow \infty$ and $\frac{u(x_k)}{v(x_k)} \rightarrow 1$. Let

$$u_k(x) = |x_k|^{\frac{1}{p-1}} u(|x_k|x).$$

Since v is invariant under the above transformation we have $u_k \leq v$ in $\partial \mathbb{R}_+^N$. Thus, for a subsequence we have $u_k \rightarrow u_0$ and u_0 solves (64). Since $u_k(\frac{x_k}{|x_k|}) \rightarrow v(y)$ where $y = \lim \frac{x_k}{|x_k|}$, again using Hopf's lemma we see that $u_0 \equiv v$. But u_0 satisfies the condition (65), contradicting (56).

Proof of Theorem 1.10. Set $u = w^{\frac{1}{p-1}} - 1$ so that

$$(67) \quad \Delta u = 0 \quad \text{in } \mathbb{R}_+^N$$

$$(68) \quad \frac{\partial u}{\partial \nu} = C(N, \frac{1}{p-1})(1+u)^p \quad \text{on } \partial \mathbb{R}_+^N$$

Let $\Omega = \{x \in \mathbb{R}_+^N \mid u(x) > 0\}$, $\Gamma_1 = \partial \Omega \cap \partial \mathbb{R}_+^N$, $\Gamma_2 = \partial \Omega \setminus \partial \mathbb{R}_+^N$. Then u is a singular solution to (67), (68) with

$$u = 0 \quad \text{on } \Gamma_2.$$

To apply Lemma 4.1 we need to verify that $u \in H^1(\Omega)$. We are assuming that $pC(N, \frac{1}{p-1}) \leq H_N$ and $p \geq \frac{N}{N-2}$. Actually we must have $p > \frac{N}{N-2}$. For this it is convenient to observe that :

$$(69) \quad C(N, \alpha) = C(N, N - 2 - \alpha) \quad \forall 0 < \alpha < N - 2$$

$$(70) \quad \alpha \mapsto C(N, \alpha) \quad \text{is increasing for } 0 < \alpha < \frac{N-2}{2}.$$

Property (69) is direct from (16) and we leave (70) to the appendix. From these properties we deduce that $p > \frac{N}{N-2}$ and therefore $u \in H^1(\Omega)$. This solution satisfies the stability condition (43) if and only if (by scaling)

$$pC(N, \frac{1}{p-1}) \int_{\partial \mathbb{R}_+^N} \frac{\varphi^2}{|x|} \leq \int_{\mathbb{R}_+^N} |\nabla \varphi|^2, \quad \forall \varphi \in C_0^1(\overline{\mathbb{R}_+^N} \setminus \{0\})$$

which is guaranteed by Kato's inequality (9). Thus we may apply Lemma 4.1 and conclude that u is the extremal solution. \square

APPENDIX

Proof of (47). We have $H_N = 2 \left[\frac{\Gamma(\frac{N}{4})}{\Gamma(\frac{N}{4} - \frac{1}{2})} \right]^2 = 2f(N/4)^2$ where $f(z) = \frac{\Gamma(z)}{\Gamma(z - \frac{1}{2})}$ and similarly we have $\lambda_{0,N} = \sqrt{\pi} \frac{N-3}{2} \frac{\Gamma(\frac{N}{2} - \frac{3}{2})}{\Gamma(\frac{N}{2} - 1)} = \sqrt{\pi} \frac{\Gamma(\frac{N}{2} - \frac{1}{2})}{\Gamma(\frac{N}{2} - 1)} = \sqrt{\pi} f\left(\frac{N}{2} - \frac{1}{2}\right)$. Then

$$\frac{H_N}{\lambda_{0,N}} = \frac{2f(N/4)^2}{\sqrt{\pi} f\left(\frac{N}{2} - \frac{1}{2}\right)}.$$

Since $H_{10} = \frac{9\pi}{8} > \lambda_{0,10} = \frac{35\pi}{32}$ it follows that

$$\frac{H_{10}}{\lambda_{0,10}} > 1.$$

On the other hand

$$\frac{H_9}{\lambda_{0,9}} = \frac{\left(\frac{5\Gamma(\frac{1}{4})^2}{12\pi}\right)^2}{\frac{16}{5}} \approx \frac{3.039}{3.333} < 1.$$

Let us compute

$$\begin{aligned} \frac{d}{dx} \frac{f(\frac{x}{4})^2}{f(\frac{x}{2} - \frac{1}{2})} &= \frac{\frac{1}{2} f(\frac{x}{4}) f'(\frac{x}{4}) f(\frac{x}{2} - \frac{1}{2}) - \frac{1}{2} f(\frac{x}{4})^2 f'(\frac{x}{2} - \frac{1}{2})}{f(\frac{x}{2} - \frac{1}{2})^2} \\ &= \frac{f(\frac{x}{4})^2}{2f(\frac{x}{2} - \frac{1}{2})} \left[\frac{f'(\frac{x}{4})}{f(\frac{x}{4})} - \frac{f'(\frac{x}{2} - \frac{1}{2})}{f(\frac{x}{2} - \frac{1}{2})} \right] \end{aligned}$$

Recall that

$$\Gamma'(z) = \psi_0(z)\Gamma(z), \quad \text{where } \psi_0(z) = - \left(\frac{1}{z} + \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right) \right).$$

In particular, ψ_0 is a positive increasing function on $]\Xi, \infty[$ where $\Xi \approx 1.4$ is the unique zero of ψ_0 in \mathbb{R}_+ , and

$$f'(z) = \left[\psi_0(z) - \psi_0\left(z - \frac{1}{2}\right) \right] \frac{\Gamma(z)}{\Gamma(z - \frac{1}{2})} > 0 \quad \text{for } z \in]1/2, \infty[.$$

Calculating

$$\frac{d}{dx} \frac{f'(x)}{f(x)} = \psi'_0(x) - \psi'_0\left(x - \frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} - \frac{1}{\left(n+x-\frac{1}{2}\right)^2} < 0 \quad \text{for } x > \frac{1}{2}.$$

So $\frac{f'(x)}{f(x)}$ is decreasing for $x > \frac{1}{2}$ and it follows that $\frac{f'(\frac{x}{4})}{f(\frac{x}{4})} - \frac{f'(\frac{x}{2}-\frac{1}{2})}{f(\frac{x}{2}-\frac{1}{2})} > 0$ for $x > 2$.

Hence

$$H_N > \lambda_{0,N} \quad \text{only for } N \geq 10.$$

□

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