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► **To cite this version:**

Haïm Brezis, Louis Dupaigne, Alberto Tesei. On a semilinear elliptic equation with inverse square potential. 2008. hal-00288567

**HAL Id: hal-00288567**

**<https://hal.science/hal-00288567>**

Preprint submitted on 17 Jun 2008

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# On a semilinear elliptic equation with inverse-square potential

Haïm Brezis <sup>\*†</sup>, Louis Dupaigne <sup>\*</sup> and Alberto Tesei <sup>‡</sup>

In this paper we study existence and nonexistence of solutions  $u \geq 0$  of the equation :

$$-\Delta u = \frac{c}{r^2}u + u^p \quad (1)$$

in a ball  $B(0, R)$  of  $\mathbb{R}^N$ ,  $N \geq 3$ . Here  $r = |x|$ ,  $p > 1$  and the coefficient  $c$  satisfies the inequality  $0 < c \leq c_0$ , where  $c_0 = (N - 2)^2/4$  is the best constant in the Hardy inequality.

In this study an important role is played by the roots

$$\alpha = \alpha^\pm := (N - 2)/2 \pm \sqrt{c_0 - c}$$

of the equation

$$\alpha^2 - (N - 2)\alpha + c = 0. \quad (2)$$

Observe that  $\alpha^+ > \alpha^- > 0$ .

Our main result asserts that nontrivial solutions of equation (1) exist if and only if  $p < p^+$  where

$$p^+ = 1 + 2/\alpha^-.$$

**Theorem 1.** *Let  $0 \leq c \leq c_0$ . For any  $p \in (1, p^+)$ , there exists a nontrivial solution to equation (1) with  $u^p$  and  $u/r^2$  belonging to  $L^1(B_R)$  and (1) holds in  $\mathcal{D}'(B_R)$ .*

The proof of Theorem 1 is straightforward and elementary, except for the limiting value  $c = c_0$ . The conclusion of Theorem 1 was known in many—but presumably not all—cases (see e.g. [12]). Concerning nonexistence we have

**Theorem 2.** *Let  $0 < c \leq c_0$ ,  $p \geq p^+$ . Assume  $u \in L^p_{loc}(B_R \setminus \{0\})$ ,  $u \geq 0$  satisfies*

$$-\Delta u - \frac{c}{r^2}u \geq u^p$$

*in  $\mathcal{D}'(B_R \setminus \{0\})$ . Then  $u \equiv 0$ .*

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Theorem 2 is reminiscent of the nonexistence results of Brezis-Cabré [2] concerning the so-called very weak solutions to the inequality

$$-\Delta u \geq \frac{u^p}{r^2}, \quad u \geq 0, \quad u \in L_{loc}^p(B_R \setminus \{0\}),$$

for any  $p > 1$ . The nonexistence aspect in (1) when  $p \geq p^+$  was first investigated by Pohozaev-Tesei [11]. However the concept of solution used there was stronger; our concept is the weakest possible.

We also observe that Theorem 2 seems (formally) to contradict the Implicit Function Theorem since there is no solution of  $-\Delta u = (c/|x|^2)u + u^p + t$ , even when  $t > 0$  is small. As observed in [1], this is due to the lack of an appropriate functional space in which to apply the IFT.

### Proof of Theorem 1.

Set  $p^- = 1 + 2/\alpha^+$  and observe that

$$1 < p^- < \frac{N+2}{N-2} < p^+ \quad \text{for any } 0 < c < c_0 \text{ and}$$

$$\lim_{c \rightarrow 0} p^- = \frac{N}{N-2}, \quad \lim_{c \rightarrow c_0} p^- = \frac{N+2}{N-2},$$

$$\lim_{c \rightarrow 0} p^+ = +\infty, \quad \lim_{c \rightarrow c_0} p^+ = \frac{N+2}{N-2}.$$

We distinguish three cases :

**Case 1 :**  $0 \leq c < c_0$  and  $p < \frac{N+2}{N-2}$ .

Here the existence of a positive solution  $u \in H_0^1(B_R)$  of (1) is a standard and straightforward consequence of the Mountain Pass Theorem. In fact, one can find a radial solution by working in the class of radial functions.

**Case 2 :**  $0 \leq c < c_0$  and  $p^- < p < p^+$ .

Here we have an explicit solution of (1) of the form  $u = A/r^\beta$  with  $\beta = 2/(p-1)$ ,  $A > 0$  given by

$$A^{p-1} = -\beta^2 + (N-2)\beta - c > 0,$$

because  $\alpha^-, \alpha^+$  are the roots of (2) and the restriction  $\alpha^- < \beta < \alpha^+$  is equivalent to the condition  $p^- < p < p^+$ . Since  $\beta < N-2$ ,  $u$  satisfies (1) in the sense of  $\mathcal{D}'(B_R)$ .

**Case 3 :**  $c = c_0$  and  $1 < p < p^+ = \frac{N+2}{N-2}$ .

This case is a little more delicate : here we need the improved Hardy inequality which asserts that

$$\int_{B_R} |\nabla u|^2 \geq c_0 \int_{B_R} \frac{u^2}{r^2} + c_q \|u\|_{L^q(B_R)}^2,$$

for any  $1 \leq q < \frac{2N}{N-2}$  and  $u \in C_0^\infty(B_R)$ . See [5]. Let  $H$  be the Hilbert space obtained by completing  $C_0^\infty(B_R)$  with respect to the scalar product

$$a(u, v) = \int_{B_R} \nabla u \cdot \nabla v - c_0 \int_{B_R} uv/r^2.$$

Clearly  $H$  is contained in every  $L^q(B_R)$  with  $1 \leq q < \frac{2N}{N-2}$  with continuous injection. Moreover the injection is compact. This fact is due to H. Brezis and the proof is presented in Lemmas 3.2, 3.3 of [7]. We may then use the Mountain Pass Theorem in  $H$  and the (PS) condition is satisfied.

## Proof of Theorem 2.

We will use the following lemma :

**Lemma 1.** *Let  $\Sigma \subset\subset \Omega$  be a closed set of zero (newtonian) capacity and assume that  $u, f \in L^1_{loc}(\Omega \setminus \Sigma)$  are two nonnegative functions such that*

$$-\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma).$$

*Then  $u, f \in L^1_{loc}(\Omega)$  and*

$$-\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

*Furthermore given any smooth subdomain  $\Omega' \subset\subset \Omega$ , if  $v \in L^1(\Omega')$  is the solution of*

$$\begin{cases} -\Delta v = f & \text{in } \Omega' \\ v = 0 & \text{on } \partial\Omega', \end{cases}$$

*in the sense that*

$$\int v(-\Delta\phi) = \int f\phi \quad \forall \phi \in C^2(\bar{\Omega}') \quad \text{such that} \quad \phi|_{\partial\Omega'} \equiv 0,$$

*then*

$$u \geq v \quad \text{a.e. in } \Omega'. \tag{3}$$

### Proof of Lemma 1

This lemma can be seen as a fairly easy consequence of Theorem 7.7 in [9]. It is also closely related to a result in [4]. We provide a proof for completeness. Let  $u_k = \min(u, k)$ ,  $k > 0$ , which by Kato's Lemma (see [10]) satisfies

$$-\Delta u_k \geq f_k \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma), \quad (4)$$

where  $f_k := f\chi_{\{u < k\}}$ . Since  $-\Delta u_k$  is a nonnegative distribution on  $\Omega \setminus \Sigma$ , it extends to a nonnegative measure on  $\Omega \setminus \Sigma$ . Since  $u_k$  is bounded, it follows from a Gagliardo-Nirenberg-type inequality that  $u_k \in H_{loc}^1(\Omega \setminus \Sigma)$ . We show next that in fact  $u_k \in H_{loc}^1(\Omega)$ . We first take a nonnegative function  $\phi \in C_0^\infty(\Omega)$  and a sequence  $\phi_n \in C_0^\infty(\Omega \setminus \Sigma)$  converging to  $\phi$  in  $H^1(\Omega)$ . This is always possible since  $\text{cap}_\Omega(\Sigma) = 0$  (take e.g.  $\phi_n = \phi(1 - \chi_n)$  where  $\chi_n = 1$  near  $\Sigma$  and  $\|\chi_n\|_{H^1} \rightarrow 0 = \text{cap}_\Omega(\Sigma)$ ). We then have, with  $C_k = e^k$ ,

$$\begin{aligned} \int |\nabla u_k|^2 \phi_n^2 &\leq C_k \int e^{-u_k} |\nabla u_k|^2 \phi_n^2 = -C_k \int \phi_n^2 \nabla(e^{-u_k}) \cdot \nabla u_k \\ &= C_k \left( 2 \int e^{-u_k} \phi_n \nabla \phi_n \cdot \nabla u_k + \int e^{-u_k} \Delta u_k \phi_n^2 \right) \\ &\leq 2C_k^2 \int e^{-u_k} |\nabla \phi_n|^2 + \frac{1}{2} \int e^{-u_k} |\nabla u_k|^2 \phi_n^2, \end{aligned}$$

so that

$$\int |\nabla(u_k \phi_n)|^2 \leq C'_k \int |\nabla \phi_n|^2.$$

Passing to the limit as  $n \rightarrow \infty$  in the above inequality implies that  $u_k \in H_{loc}^1(\Omega)$ .

We next show that

$$-\Delta u_k \geq f_k \quad \text{in } \mathcal{D}'(\Omega). \quad (5)$$

Take  $\phi$  and  $\phi_n$  as above so that by (4),

$$\int u_k(-\Delta \phi_n) \geq \int f_k \phi_n. \quad (6)$$

Now, as  $n \rightarrow \infty$ ,

$$\int u_k(-\Delta \phi_n) = \int \nabla u_k \nabla \phi_n \rightarrow \int \nabla u_k \nabla \phi = - \int u_k \Delta \phi.$$

Passing to the limit in (6) as  $n \rightarrow \infty$ , we thus obtain (5).

In particular  $u_k$  is superharmonic in  $\Omega$  and given almost any  $x \in \Omega$  and any ball  $B \subset \Omega$  centered at  $x$ , we have

$$u_k(x) \geq \frac{1}{|B|} \int_B u_k(y) dy. \quad (7)$$

Now, since  $u \in L_{loc}^1(\Omega \setminus \Sigma)$  (and  $|\Sigma| = 0$ ),  $u_k \rightarrow u$  a.e. in  $\Omega$  as  $k \rightarrow \infty$  and  $u$  is finite almost everywhere. By Fatou's Lemma we then conclude from (7) that for almost every ball  $B$ ,

$$\int_B u < \infty,$$

which means that  $u \in L^1_{loc}(\Omega)$ . Using this information, we can now easily pass to the limit in (5) and conclude that  $f \in L^1_{loc}(\Omega)$  and that

$$-\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

It only remains to prove (3). We let  $\rho_n$  be a standard smooth mollifier and let  $u_n = u * \rho_n$ ,  $f_n = f * \rho_n$  so that for  $n$  large enough  $-\Delta u_n \geq f_n$  and  $u_n \geq 0$  in  $\Omega'$ . By the Maximum Principle

$$u_n \geq v_n \quad \text{in } \Omega',$$

where  $v_n$  solves

$$\begin{cases} -\Delta v_n = f_n & \text{in } \Omega' \\ v_n = 0 & \text{on } \partial\Omega'. \end{cases}$$

As  $n \rightarrow \infty$ ,  $u_n \rightarrow u$  in  $L^1(\Omega')$ ,  $f_n \rightarrow f$  in  $L^1(\Omega')$  and (by Lemma 4 in [3])  $v_n \rightarrow v$  in  $L^1(\Omega')$ , which yields the desired conclusion.  $\square$

## Proof of Theorem 2

We argue by contradiction and assume that  $u \not\equiv 0$ . By Lemma 1,  $u \in L^p_{loc}(B_R)$ ,  $u/r^2 \in L^1_{loc}(B_R)$  and by the mean-value formula for superharmonic functions, given  $R' \in (0, R)$ , there exists  $\epsilon > 0$  such that  $u \geq \epsilon$  a.e. in  $B_{R'}$ . Let  $\lambda := \epsilon^{q-1}/2 > 0$  and  $v_0$  be the solution of

$$\begin{cases} -\Delta v_0 = \lambda & \text{in } B_{R'} \\ v_0 = 0 & \text{on } \partial B_{R'}. \end{cases}$$

Once more by Lemma 1, we have

$$0 \leq v_0 \leq u. \tag{8}$$

Next, for  $n \geq 1$ , define inductively  $v_n$  by

$$\begin{cases} -\Delta v_n = \frac{c}{|x|^2} v_{n-1} + \frac{1}{2} v_{n-1}^p + \lambda & \text{in } B_{R'} \\ v_n = 0 & \text{on } \partial B_{R'}. \end{cases}$$

In order to have a well-defined solution  $v_n$  (in the sense of Lemma 4 in [3]) it suffices to prove that  $f := \frac{c}{|x|^2} v_{n-1} + \frac{1}{2} v_{n-1}^p \in L^1(B_{R'})$ . When  $n = 0$ , this follows from (8) and Lemma 1 which implies that  $\frac{c}{|x|^2} u + \frac{1}{2} u^p \in L^1(B_{R'})$ . Assume now that  $v_{n-1} \in L^1(B_{R'})$  is well-defined. Using the Maximum Principle, it is easy to see that

$$0 \leq v_0 \leq v_1 \leq \dots \leq v_{n-1} \leq u,$$

whence  $f \in L^1(B_{R'})$  and by the Maximum Principle again  $0 \leq v_{n-1} \leq v_n \leq u$ .

By monotone convergence, letting  $v := \lim_{n \rightarrow \infty} v_n$ , we have that

$$\begin{cases} -\Delta v = \frac{c}{|x|^2} v + \frac{1}{2} v^p + \lambda & \text{in } B_{R'} \\ v = 0 & \text{on } \partial B_{R'}, \end{cases}$$

in the sense that given any  $\phi \in C^2(\bar{B}_{R'})$  such that  $\phi|_{\partial B_{R'}} \equiv 0$ ,

$$\int v(-\Delta \phi) = \int \frac{c}{|x|^2} v \phi + \frac{1}{2} \int v^p \phi + \lambda \int \phi.$$

This contradicts Theorem 1 of [6].

□

**Remark 1.** Theorems 1 and 2 extend to more general situations— for example, when  $u^p$  is replaced by  $|x|^{-\beta} u^q$ . Assume  $0 < c \leq c_0$  and set  $q^+ = 1 + \frac{2-\beta}{\alpha^-}$ , where  $\alpha^-$  is as above. The conclusions of Theorems 1 and 2 remain valid with  $p^+$  replaced by  $q^+$ .

**Remark 2.** The argument presented in the proof of Theorem 2 may be used to provide a slightly simpler proof of Theorem 1 in [2].

**Remark 3.** Theorem 2 can be extended to problems of the type

$$-\Delta u = \frac{c}{\text{dist}(x, \Sigma)^2} u + u^p,$$

where  $c > 0$  is a small constant,  $\Sigma$  is a smooth compact manifold of codimension  $k \geq 3$  and  $p$  is larger than some critical exponent, which can be computed explicitly in terms of  $k$  and  $c$ . The argument is the same as in the proof of Theorem 2 except that the result of [6] is replaced by a result from [8].

**Acknowledgments.** The three authors are partially supported by an EC Grant through the RTN Program "Fronts-Singularities", HPRN-CT-2002-00274. The first author (H.B.) is also a member of the Institut Universitaire de France.

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