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Semilinear elliptic PDEs with a singular potential

Louis Dupaigne, Gueorgui Nedev

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Louis Dupaigne, Gueorgui Nedev. Semilinear elliptic PDEs with a singular potential. 2001. hal-00288543

HAL Id: hal-00288543

<https://hal.science/hal-00288543>

Preprint submitted on 17 Jun 2008

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Table of Contents

1. Semilinear elliptic PDE's with a singular potential (by L. Dupaigne and G. Nedev, Adv. Differential Equations 7 (2002), no. 8, 973–1002)	1
1.1. Introduction	1
1.1.1. Statement of the problem	1
1.1.2. Main results	6
1.2. Preliminary : linear theory	8
1.3. Existence vs. complete blow-up	13
1.4. Proof of Theorem 2	19
1.5. Application 1 : $a(x) = c/ x ^2$, $g(u) = u^p$	26
1.6. Application 2 : $a(x) = c/\delta(x)^2$, $\Omega = B_1$	28
1.7. Application 3 : $a(x) = c/dist(x, \Sigma)^2$, $g(u) = u^p$	29
References	34

Chapter 1

Semilinear elliptic PDE's with a singular potential

1.1 Introduction

1.1.1 Statement of the problem

This section focusses on the following equation :

$$\left\{ \begin{array}{ll} -\Delta u - \frac{c}{|x|^2}u = f(u) + \lambda b(x) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \right. \quad (P_\lambda)$$

Here, Ω is a smooth bounded domain of \mathbb{R}^n , $\lambda > 0$ is a (small) constant and a, b, f are non-negative functions, satisfying a number of conditions listed later on. At this point, we would like to look at an example treated in [D], which motivates the study of (P_λ) and clarifies the issues at stake : take $a(x) = c/|x|^2$ where $c \in (0, (n-2)^2/4)$, $f(u) = u^p$, with $p > 1$ and $b(x) \equiv 1$. (P_λ) becomes :

$$\left\{ \begin{array}{ll} -\Delta u - \frac{c}{|x|^2}u = u^p + \lambda & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \right.$$

It turns out that if $0 \in \Omega$ and $n \geq 3$, there exists a critical exponent $p_0 = p_0(c, n)$ such that the above equation has no solution for any pair (p, λ) satisfying $p \geq p_0$ and $\lambda > 0$, whereas solutions exist for $p < p_0$, provided $\lambda > 0$ is chosen small enough (while no solution exist if $p < p_0$ and λ is large). It should be noted that whenever they exist, the solutions are always singular at the origin. In this work, we show that this result can be extended to a greater class of potentials, examples of which can be taken to

have singularities on curves or higher dimensional submanifolds of Ω (see Section 6) : if $a(x) = c/\text{dist}(x, \Sigma)^2$, where Σ is a submanifold of codimension $k \geq 3$ of Ω , there is again a critical exponent $p_0 = p_0(c, n, k)$, which somewhat surprisingly decreases with k .

Roughly speaking, there is a better chance that (P_λ) has a solution when the potential is singular on a 'larger' set. In fact, when $\Sigma = \partial\Omega$, any power (or any nonlinearity f) is allowed. Also, this critical exponent phenomenon is just a specific case of a dichotomy between nonlinearities f that allow for existence of solutions and those that don't. We derive for this matter a sharp abstract criterium on f , in the spirit of [KV] and [BC], which is nevertheless easy to check in applications.

Even in the case of the inverse-square potential $a(x) = c/|x|^2$, this will lead us to new results complementing those of [D].

We now turn back to (P_λ) and to make all of our statements precise, list the assumptions on our data :

- $a \in L^1_{loc}(\Omega)$, $a(x) \geq 0$ a.e. and, for some > 0 ,

$$\int_{\Omega} |\nabla u|^2 - \int_{\Omega} a(x)u^2 \geq \int_{\Omega} u^2 \quad \text{for all } u \in C_c^\infty(\Omega) \quad (0.1)$$

(0.1) states that the first eigenvalue of the operator $L = -\Delta - \frac{c}{|x|^2}$ is positive. When $c \leq c_0 := (n-2)^2/4$, $n \geq 3$ and $a(x) = c/|x|^2$, (0.1) is just the celebrated Hardy inequality (see [BV] for its proof). However, if $c > c_0$ (and $0 \in \Omega$), (0.1) fails and in fact there are no nonnegative $u \not\equiv 0$ such that $-\Delta u - \frac{c}{|x|^2}u \geq 0$, hence no solution of (P_λ) (see [BG] or [CM]). Hence (0.1) is crucial.

It also follows from (0.1) that

$$\|u\|_H^2 := \int_{\Omega} |\nabla u|^2 - \int_{\Omega} a(x)u^2$$

is (the square of) a norm on $C_c^\infty(\Omega)$. Completing $C_c^\infty(\Omega)$ with respect to this norm, we obtain a Hilbert space H . Using Lax-Milgram lemma, we then define a unique $\zeta_0 \in H$ solving

$$\begin{cases} -\Delta \zeta_0 - \frac{c}{|x|^2} \zeta_0 = 1 & \text{in } \Omega \\ \zeta_0 = 0 & \text{on } \partial\Omega \end{cases} \quad (0.2)$$

in the sense that

$$(\zeta_0|\phi)_H = \langle 1, \phi \rangle_{H^*, H} \quad \text{for all } \phi \in H. \quad (0.3)$$

Observe that given any $\epsilon > 0$, if a satisfies (0.1) then the space H associated with $a_\epsilon := (1 - \epsilon)a$ coincides with $H_0^1(\Omega)$. So that in the generic case, our definition of ζ_0 reduces to the standard one. However, it was proved in [VZ] that if $a(x) = (n - 2)^2/(4|x|^2)$ (this potential corresponds to the limiting case of the Hardy inequality), the associated space H contains $H_0^1(\Omega)$ as a proper subset.

- $b \in L_\delta^1(\Omega) := L^1(\Omega, \text{dist}(x, \partial\Omega)dx)$, $b \not\equiv 0$, $b(x) \geq 0$ a.e. and

$$\int_\Omega b\zeta_0 < \infty \quad (0.4)$$

where $0 \leq \zeta_0 \in H$ is the solution of (0.3). For simplicity, the reader may think of b as a smooth and bounded function. As we shall see (in Lemma 1.2), what (0.4) ensures is that there exists $\zeta_1 \geq 0$ solving (in a certain sense to be defined later on)

$$\begin{cases} -\Delta\zeta_1 - \frac{c}{|x|^2}\zeta_1 = b & \text{in } \Omega \\ \zeta_1 = 0 & \text{on } \partial\Omega \end{cases} \quad (0.5)$$

which is a minimum requirement, if one wants to solve (P_λ) .

The following set of conditions on f , though technical, is satisfied by a wide class of nonlinearities.

- $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^1 , convex function with $f(0) = f'(0) = 0$ satisfying the two following growth conditions :

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{t} = +\infty \quad (0.6)$$

$$\int_1^\infty g(s)ds < \infty \quad \text{and} \quad sg(s) < 1 \text{ for } s > 1 \quad (0.7)$$

where we set, for $s \geq 1$,

$$g(s) = \sup_{t>0} f(t)/f(st) \quad (0.8)$$

Clearly, g is a decreasing, nonnegative function. Moreover, $s \rightarrow sg(s)$ is nonincreasing since, by convexity, $t \rightarrow f(t)/t$ is increasing and $f(0) = 0$.

We may also assume that g is continuous. If not, since g is used in our proofs solely for comparison arguments, it suffices to replace g with a continuous function $\tilde{g} \geq g$, satisfying (0.7), such that $t \rightarrow t\tilde{g}(t)$ is nonincreasing and

$$\int_1^\infty \tilde{g}(s)ds - \int_1^\infty g(s)ds \quad \text{is arbitrarily small}$$

We construct such a function in Lemma 2.3.

We also observe that since $g(s) \geq f(1)/f(s)$, (0.7) implies the following weaker condition, which often appears in the litterature :

$$\int_1^\infty \frac{1}{f(s)} ds < \infty.$$

In particular, our proofs yield no result for functions like $f(t) = t(\ln t)_+^\beta$, $\beta > 0$ for which $g(s) = 1/s$.

Examples of nonlinearities f which do satisfy our assumptions are : $f(u) = u^p$ for $p > 1$, $f(u) = e^u - u - 1$, $f(u) = u^2 - 1 + \cos(u)$, ... Next, we clarify the notion of solution used in this section. We need to do so because even linear problems of the form (0.5) may not be well posed in the usual distributional or Sobolev space settings. This is shown in [D] for the potential $a(x) = c/|x|^2$. • Following [BCMR], we shall say that $u \in L^1(\Omega)$ is a **weak solution** of (P_λ) if $u \geq 0$ a.e. and if it satisfies the two following conditions :

$$\begin{cases} \int_\Omega (a(x)u + f(u)) \delta(x) < \infty & \text{where } \delta(x) = \text{dist}(x, \partial\Omega) \\ \int_\Omega u \left(-\Delta\phi - \frac{c}{|x|^2}\phi \right) = \int_\Omega (f(u) + \lambda b)\phi & \text{for } \phi \in C^2(\bar{\Omega}), \phi|_{\partial\Omega} = 0 \end{cases}$$

Observe that the first condition merely ensures that the integrals in the second equation make sense. Similarly, a weak solution $u \in L^1(\Omega)$ of (0.5) with $b \in L^1_\delta := L^1(\Omega, \delta(x)dx)$, is one that satisfies the equation $\int_\Omega u \left(-\Delta\phi - \frac{c}{|x|^2}\phi \right) = \int_\Omega b\phi$ (for all $\phi \in C^2(\bar{\Omega})$, $\phi|_{\partial\Omega} = 0$) with the integrability condition $\int_\Omega (a(x) + 1)|u|\delta(x) < \infty$. We will also refer to inequalities holding in the weak sense or talk about (weak) supersolutions. This means that we integrate the equation with nonnegative test functions. For example, $-\Delta u - \frac{c}{|x|^2}u \geq f$ holds **in the weak sense**, given $f \in L^1_\delta$, if $u \in L^1(\Omega)$, $a(x)u \in L^1_\delta(\Omega)$

and if

$$\int_{\Omega} u \left(-\Delta \phi - \frac{c}{|x|^2} \phi \right) \geq \int_{\Omega} f \phi \quad \text{for all } \phi \in C^2(\bar{\Omega}) \text{ with } \phi \geq 0 \text{ and } \phi|_{\partial\Omega} = 0$$

These definitions are motivated by the following lemma (proved in [BCMR]) :

Lemma 0.1. *Let $f \in L^1_{\delta}(\Omega) := L^1(\Omega, \delta(x)dx)$. There exists a unique (weak) solution $u \in L^1(\Omega)$ of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense that

$$\int_{\Omega} u (-\Delta \phi) = \int_{\Omega} f \phi \quad \text{for all } \phi \in C^2(\bar{\Omega}), \phi|_{\partial\Omega} = 0$$

Furthermore, there exists a constant $C = C(\Omega) > 0$ such that

$$\|u\|_{L^1(\Omega)} \leq C \|f\|_{L^1_{\delta}(\Omega)}$$

and

$$f \geq 0 \quad \text{a.e.} \quad \implies \quad u \geq 0 \quad \text{a.e.}$$

Lemma 0.2. *Let $a(x) \in L^1_{loc}(\Omega)$, $b \in L^1_{\delta}(\Omega)$ and $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ be nonnegative functions. Let $\lambda > 0$. Suppose there exists a (weak) supersolution $w \geq 0$ of (P_{λ}) (respectively (0.2),(0.5)). Then there exists a unique weak solution $u \geq 0$ of (P_{λ}) (respectively (0.2),(0.5)) such that*

$$0 \leq u \leq \tilde{w}$$

for any (weak) supersolution $\tilde{w} \geq 0$ of (P_{λ}) (respectively (0.2),(0.5)).

u is then called **the** minimal nonnegative weak solution of (P_{λ}) (respectively (0.2),(0.5)).

Remark. The function $\zeta_0 \in H$ solving (0.3) also solves (0.2) in the weak sense. In fact, it is the minimal nonnegative weak solution of (0.2), so that no confusion may arise (see the remark following Lemma 1.1).

Proof. The proof is identical for all three equations $(P_{\lambda}), (0.2), (0.5)$ so we restrict to the case where w is a supersolution of (P_{λ}) . First if $u_1 \geq 0$ and $u_2 \geq 0$ are two weak solutions such that

$$0 \leq u_i \leq \tilde{w} \quad i = 1, 2$$

for all supersolutions $\tilde{w} \geq 0$, then we must have $u_1 \leq u_2$ and $u_2 \leq u_1$, hence $u_1 = u_2$ so that the minimal solution – if it exists – is unique. Next, let $w \geq 0$ be a weak supersolution of (P_λ) and let $u_0 \in L^1(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta u_0 = \lambda b & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of Lemma 0.1. It follows easily from Lemma 0.1 that $0 \leq u_0 \leq w$. Next, we show by induction that there exists a unique $u_n \in L^1(\Omega)$ for $n = 1, 2, \dots$ solving

$$\begin{cases} -\Delta u_n = a(x)u_{n-1} + f(u_{n-1}) + \lambda b & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of Lemma 0.1, and such that $0 \leq u_n \leq w$. Indeed, since $0 \leq u_0 \leq w$ and w is a weak supersolution,

$$0 \leq a(x)u_0 + f(u_0) \leq a(x)w + f(w) \in L^1_\delta(\Omega).$$

So that u_1 is well defined (by Lemma 0.1) and $0 \leq u_0 \leq u_1 \leq w$ (applying Lemma 0.1 again). The same argument can be applied inductively to show that u_n is well defined (provided $0 \leq u_{n-1} \leq w$) and that

$$0 \leq u_{n-1} \leq u_n \leq w.$$

Hence $\{u_n\}_n$ is a nondecreasing sequence of nonnegative functions dominated by w . By monotone convergence, its (pointwise) limit u solves (P_λ) .

Now if $\tilde{w} \geq 0$ is another supersolution, it follows easily from Lemma 0.1 that $u_0 \leq \tilde{w}$ and $u_n \leq \tilde{w}$ for all $n = 1, 2, \dots$. Passing to the limit, it follows that $u \leq \tilde{w}$.

□

With these definitions in mind, we investigate the existence, uniqueness and regularity of solutions of (P_λ) :

1.1.2 Main results

Theorem 1 (existence and optimal regularity). *Assume (0.1), (0.4), (0.6), (0.7) hold and let $\zeta_0 = G(1)$ solve (0.2), $\zeta_1 = G(b)$ solve (0.5) with $G = \left(-\Delta - \frac{c}{|x|^2}\right)^{-1}$ defined in Lemma 1.2.*

- **Either** there exist constants $\epsilon > 0$, $C > 0$ such that

$$\int_{\Omega} f(\epsilon\zeta_1)\zeta_0 < \infty \quad \text{and} \quad G(f(\epsilon\zeta_1)) \leq C\zeta_1 \quad \text{a.e.} \quad (0.8)$$

Then there exists $\lambda^* > 0$ depending on n , $a(x)$, f , $b(x)$ such that

- if $\lambda < \lambda^*$ then (P_λ) has a minimal weak solution u .

Furthermore, for some constant $C > 0$ independent of $x \in \Omega$, we have

$$\zeta_1 \leq u \leq C\zeta_1 \quad \text{a.e. in } \Omega$$

- if $\lambda = \lambda^*$ then (P_λ) has a minimal weak solution,
- if $\lambda > \lambda^*$ then (P_λ) has no solution, even in the weak sense and there is complete blow-up.
- **Or** (0.8) holds for no $\epsilon > 0$, $C > 0$. Then, given any $\lambda > 0$, (P_λ) has no solution, even in the weak sense, and there is complete blow-up.

This result requires the following definitions :

Definition 0.1. Let $\{a_n(x)\}$, $\{b_n(x)\}$ and $\{f_n\}$ be increasing sequences of bounded smooth functions converging pointwise respectively to $a(x)$, $b(x)$ and f and let \underline{u}_n be the minimal nonnegative solution of

$$\begin{cases} -\Delta \underline{u}_n - a_n \underline{u}_n = f_n(\underline{u}_n) + \lambda b_n & \text{in } \Omega \\ \underline{u}_n = 0 & \text{on } \partial\Omega \end{cases} \quad (P_n)$$

We say that there is **complete blow-up** in (P_λ) if, given any such $\{a_n(x)\}$, $\{b_n(x)\}$, $\{f_n\}$ and $\{\underline{u}_n\}$,

$$\frac{\underline{u}_n(x)}{\delta(x)} \rightarrow +\infty \text{ uniformly on } \Omega,$$

where $\delta(x) := \text{dist}(x, \partial\Omega)$.

Definition 0.2. The parameter λ^* is called the **extremal parameter** of the family of equations $\{(P_\lambda)\}_\lambda$ and the corresponding solution u_{λ^*} is called the **extremal solution**.

Theorem 2 (uniqueness of stable solutions). *Make the same assumptions as in Theorem 1. If it exists, let u_λ denote the minimal nonnegative (weak) solution of (P_λ) . If $0 < \lambda < \lambda^*$,*

- u_λ is stable
- Assume that $f(\zeta_1) + \lambda b \in H^*$. Then $u_\lambda \in H$ and u_λ is the only stable (weak) solution of (P_λ) belonging to H .

If $\lambda = \lambda^*$,

- u_{λ^*} is stable
- Assume $b \in L^p$ for some $p > n$. Then u_{λ^*} is the only weak solution of (P_{λ^*}) .

Stability is defined as follows :

Definition 0.4. *We say that u is **stable** if the generalized first eigenvalue $\lambda_1(u)$ of the linearized operator of equation (P_λ) is nonnegative, i.e., if*

$$\lambda_1(u) := \inf\{J(\phi) : \phi \in C_c^\infty(\Omega) \setminus \{0\}\} \geq 0$$

where

$$J(\phi) = \frac{\int_\Omega |\nabla \phi|^2 - \int_\Omega a(x)\phi^2 - \int_\Omega f'(u)\phi^2}{\int_\Omega \phi^2}$$

The proof of Theorem 1 is presented in section 2 , whereas Theorem 2 is proved in section 3. Applications can be found in the remaining sections 4, 5 and 6.

1.2 Preliminary : linear theory

We construct here a few basic tools to be used later on and start out with the L^2 theory.

Lemma 1.1. *Suppose (0.1) holds and let $b \in H^* \cap L^1_\delta(\Omega)$.*

There exists a unique $u \in H$, weak solution of

$$\begin{cases} -\Delta u - a(x)u = b \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \quad (1.1)$$

Furthermore,

$$\|u\|_H = \|b\|_{H^*} \quad (1.2)$$

$$b \geq 0 \text{ in the sense of distributions} \Rightarrow u \geq 0 \text{ a.e.} \quad (1.3)$$

Proof. By working with b^+ , b^- we can assume that $b \geq 0$. It follows from Lax-Milgram lemma that there exists a unique $u \in H$ such that

$$(u|\phi)_H = \langle b, \phi \rangle_{H^*, H} \quad \text{for all } \phi \in H.$$

Furthermore, (1.2) holds. We now show that u solves (1.1) in the weak sense : Let

$$a_k(x) = \min(a(x), k), \quad k > 0$$

and u_k be the solution to (1.1) with the potential $a(x)$ replaced by the potential $a_k(x)$. Then it is easy to check that u_k is nondecreasing in k , and converges to u in $L^2(\Omega)$. Now take $\phi \in C^2(\overline{\Omega})$, $\phi = 0$ on $\partial\Omega$. Then

$$\int_{\Omega} u_k(-\Delta\phi) = \int_{\Omega} a_k(x)u_k\phi + b\phi,$$

and note that here all the integrals are finite. By taking in particular $\phi = \phi_0$ to be the solution of

$$\begin{cases} -\Delta\phi_0 = 1 \text{ in } \Omega \\ \phi_0 = 0 \text{ on } \partial\Omega \end{cases}$$

, and using Fatou's lemma, we see that $\int_{\Omega} a(x)u\phi_0$ exists and is finite. Given any $\phi \in C^2(\overline{\Omega})$, $\phi = 0$ on $\partial\Omega$, we can find $C > 0$ so that $|\phi| \leq C\phi_0$. It follows that we can pass to the limit in the equation satisfied by u_k and conclude that u solves (1.1) in the weak sense.

Next we show that if $\tilde{u} \in H$ is another weak solution of (1.1) then $\tilde{u} = u$. By definition of H , there exists $u_n \in C_c^\infty(\Omega)$ such that $u_n \rightarrow \tilde{u}$ in H (and a fortiori in $L^1(\Omega)$). Hence for $\phi \in C_c^\infty(\Omega)$,

$$(\tilde{u}|\phi)_H = \lim_{n \rightarrow \infty} (u_n|\phi)_H$$

Using integration by parts and the fact that $u_n \rightarrow \tilde{u}$ in $L^1(\Omega)$,

$$(u_n|\phi)_H = \int_{\Omega} u_n \left(-\Delta\phi - \frac{c}{|x|^2}\phi \right) \rightarrow \int_{\Omega} \tilde{u} \left(-\Delta\phi - \frac{c}{|x|^2}\phi \right) = \langle b, \phi \rangle_{H^*,H}$$

So that

$$(\tilde{u}|\phi)_H = \langle b, \phi \rangle_{H^*,H} \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

By density, the equality holds for all $\phi \in H$ and $\tilde{u} = u$ by Lax-Milgram lemma.

Finally we show (1.3). Let $b \in H^*$, $b \geq 0$ (in the sense of distributions) and $u \in H$ be the corresponding solution of (1.1).

By definition of H , there exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ converging to u in H . Letting $b_n = -\Delta u_n - \frac{c}{|x|^2}u_n$, it follows that $b_n \in H^*$ and $b_n \rightarrow b$ in H^* .

Now, $u_n \in H_0^1(\Omega) \Rightarrow u_n^- \in H_0^1(\Omega)$ and integrating the equation satisfied by u_n against u_n^- yields

$$-\|u_n^-\|_H^2 = \langle b_n, u_n^- \rangle_{H^*,H}$$

To pass to the limit in this last equation, we just need to prove that $\{u_n^-\}$ remains bounded in H .

$$\begin{aligned} \|u_n^-\|_H^2 &= \int_{\Omega} |\nabla u_n^-|^2 - \int_{\Omega} a(x)(u_n^-)^2 \\ &= \int_{\Omega} |\nabla u_n^-|^2 - \int_{\Omega} a(x)u_n^2 + \int_{\Omega} a(x)(u_n^+)^2 \\ &\leq \int_{\Omega} |\nabla u_n^-|^2 - \int_{\Omega} a(x)u_n^2 + \int_{\Omega} |\nabla u_n^+|^2 = \int_{\Omega} |\nabla u_n|^2 - \int_{\Omega} a(x)u_n^2 \\ &= \|u_n\|_H^2 \end{aligned} \tag{1.4}$$

where we've used (0.1) in the inequality.

□

Remark. Observe in passing that ζ_0 solving (0.3) is the minimal nonnegative weak solution of (0.2) : by the previous lemma (and its proof), ζ_0 is indeed a weak solution of (0.2). If u denotes the minimal nonnegative weak solution of (0.2), and u_k the solution

of (0.2) when a is replaced by $a_k = \min(a, k)$, it can be shown as above that $u_k \rightarrow u$ in $L^1(\Omega)$ and that $\{u_k\}_k$ remains bounded in H so that $u \in H$ and $u = \zeta_0$.

Lemma 1.2. *Let $b \in L^1(\Omega, \delta(x)dx)$ with $b \geq 0$ a.e. and $b \not\equiv 0$. The equation*

$$\begin{cases} -\Delta u - \frac{c}{|x|^2}u = b \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \quad (1.5)$$

has a nonnegative weak solution $u \in L^1(\Omega)$ (which may not be unique) if and only if

$$\int_{\Omega} b(x)\zeta_0 dx < \infty \quad (1.6)$$

where ζ_0 denotes the solution of (0.3). We then denote the minimal nonnegative weak solution u of (1.5) by

$$u = G(b)$$

Proof.

Suppose first that $\int_{\Omega} b(x)\zeta_0 dx < \infty$ and let $b_n = \min(n, b)$ for $n \in \mathbb{N}$.

By Lemma 1.1, there exists a unique $v_n \in H$, $v_n \geq 0$, solving (1.5) with b_n in place of b and, testing with ζ_0 in (1.5) and with v_n in (0.3), we obtain

$$\int_{\Omega} b_n \zeta_0 = (v_n | \zeta_0)_H = \int_{\Omega} v_n$$

Hence,

$$\|v_n\|_{L^1} \leq \int_{\Omega} b \zeta_0$$

Testing with z , solving

$$\begin{cases} -\Delta z = 1 \text{ in } \Omega \\ z = 0 \text{ on } \partial\Omega \end{cases} \quad (1.7)$$

we get

$$(v_n | z)_H = \int_{\Omega} b_n z = \int_{\Omega} v_n - \int_{\Omega} a(x)v_n z$$

So that

$$\int_{\Omega} a(x)v_n \delta \leq C \int_{\Omega} b \zeta_0$$

Observe that Lemma 1.1 implies that v_n is nondecreasing and using a standard monotone convergence argument, it follows that $v = \lim v_n$ (weakly) solves (1.5). By Lemma

0.2, we can then construct the minimal nonnegative weak solution $u = G(b)$. Conversely, suppose $v \geq 0$ is a weak solution of (1.5) and assume for now that $b \in L^\infty$ and that v is minimal. $\|b\|_{L^\infty} \zeta_0$ is then a supersolution of (1.5), hence $v \leq C\zeta_0$. Also, as in the proof of Lemma 1.1, we can show that v is an H solution. Next, take a sequence of bounded functions Φ_n increasing pointwise to $a(x)\zeta_0$ and let v_n be the solution of

$$\begin{cases} -\Delta v_n = \Phi_n & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega \end{cases} \quad (1.8)$$

Testing with v_n , we obtain

$$\int_{\Omega} b v_n = (v|v_n)_H = \int_{\Omega} v(\Phi_n - a(x)v_n) \quad (1.9)$$

Since $a(x)\zeta_0 \in L^1_\delta$, $\phi_n \nearrow a(x)\zeta_0$ in L^1_δ and, by Lemma 0.1, $v_n \nearrow v := \zeta_0 - z$ in L^1 , with z solving (1.7).

Now, $\Phi_n \leq a(x)\zeta_0$ and $v_n \leq v \leq \zeta_0$, hence

$$|v(\Phi_n - a(x)v_n)| \leq 2a(x)\zeta_0^2$$

Suppose in addition that $\int_{\Omega} a(x)\zeta_0^2 < \infty$ so that we can apply Lebesgue's theorem and pass to the limit in (1.9) :

$$\int_{\Omega} b(\zeta_0 - z) = \int_{\Omega} v[a(x)\zeta_0 - a(x)(\zeta_0 - z)] = \int_{\Omega} a(x)vz$$

Hence,

$$\int_{\Omega} b\zeta_0 \leq \|z\|_{C^1} \left(\int_{\Omega} b\delta + \int_{\Omega} a(x)v\delta \right) \quad (1.10)$$

We made two auxiliary assumptions to arrive to this result. First, we assumed that $b \in L^\infty$. If this is not true, we can replace b by $b_n = \min(b, n)$, apply (1.10) to b_n and let $n \rightarrow \infty$. We also assumed that $\int_{\Omega} a(x)\zeta_0^2 < \infty$. If not, replace $a(x)$ by $a_\epsilon(x) := (1 - \epsilon)a(x)$ and ζ_0 by ζ_ϵ the solution of (0.3) with a_ϵ in place of a . Multiply (0.1) by $(1 - \epsilon)$ to obtain

$$\int_{\Omega} |\nabla u|^2 - \int_{\Omega} a_\epsilon(x)u^2 \geq \epsilon \int_{\Omega} |\nabla u|^2 \quad \text{for all } u \in C_c^\infty(\Omega)$$

The space H corresponding to the potential a_ϵ is therefore good old $H_0^1(\Omega)$ and, by construction, $\zeta_\epsilon \in H_0^1(\Omega)$. Using the above inequality, it follows that

$$\int_{\Omega} a_\epsilon \zeta_\epsilon^2 < \infty$$

We can therefore apply (1.10) with ζ_ϵ in place of ζ_0 and let $\epsilon \rightarrow 0$. \square

1.3 Existence vs. complete blow-up

In this section, we will prove existence or nonexistence of weak solutions of (P_λ) , using the tools we have just constructed and monotonicity arguments.

The following result, due to [BC], for which a proof in our context can be taken from [D], proves the blow-up results of Theorem 1 provided nonexistence of weak solutions is established :

Lemma 2.0. *Fix $\lambda > 0$ Suppose (P_λ) has no weak solution. Then there is complete blow-up in (P_λ)*

Next, we extend a technical result of [BC] :

Lemma 2.1. *Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 , concave function such that for some $C > 0$,*

$$0 \leq \Phi' \leq C$$

Let $h, k \in L_{loc}^1(\Omega)$, $h, k \geq 0$, $k \not\equiv 0$, satisfy (1.6) so that $u = G(h)$ and $v = G(k)$ are well-defined. Then, letting $w = v\Phi(u/v)$,

$$w \in L^1(\Omega) , a(x)w \in L_{\delta}^1 \text{ and}$$

$$-\Delta w - \frac{c}{|x|^2} w \geq \Phi'(u/v)(h - k) + \Phi(1)k \quad \text{in the weak sense} \quad (2.1)$$

Proof. Suppose first $u, v \in C^2(\bar{\Omega})$, $v > 0$ in Ω and $\Phi \in C^2(\mathbb{R})$. Applying Lemma 5.3 in [BC], it follows that a.e. in Ω ,

$$\begin{aligned} -\Delta w - \frac{c}{|x|^2} w &\geq \Phi'(u/v)(-\Delta u) + [\Phi(u/v) - \Phi'(u/v)u/v] (-\Delta v) - a(x)\Phi(u/v)v \\ &\geq \Phi'(u/v)h + [\Phi(u/v) - \Phi'(u/v)u/v] k \\ &\geq \Phi'(u/v)(h - k) + [\Phi(u/v) - \Phi'(u/v)u/v + \Phi'(u/v)] k \end{aligned}$$

Since Φ is concave,

$$\Phi(s) + (1-s)\Phi'(s) \geq \Phi(1) \quad \text{for all } s \in \mathbb{R}$$

Hence,

$$-\Delta w - \frac{c}{|x|^2}w \geq \Phi'(u/v)(h-k) + \Phi(1)k \quad \text{a.e. in } \Omega \quad (2.2)$$

Since Φ' is bounded, we see, as in [BC], that

$$|v\Phi(u/v)| = |v(\Phi(u/v) - \Phi(0)) + \Phi(0)v| \leq C(u+v) \quad (2.3)$$

Hence, w vanishes on $\partial\Omega$ and integrating by parts, (2.2) holds in the weak sense. By approximation of Φ , we can also say that (2.2) holds even when Φ is only C^1 . Finally observe that all of the above computations still hold if u, v are merely $C^{1,\alpha}(\bar{\Omega})$. In the general case, let a_n, h_n, k_n be bounded functions increasing pointwise to a, h, k and u_n, v_n be the solutions of the associated equations. Also write $w_n = v_n\Phi(u_n/v_n)$. For n large enough, $k_n \neq 0$ since $k \neq 0$ and by the strong maximum principle, $v_n > 0$ in Ω . We can then apply (2.2) to obtain

$$-\Delta w_n - a_n(x)w_n \geq \Phi'(u_n/v_n)(h_n - k_n) + \Phi(1)k_n \quad \text{weakly} \quad (2.4)$$

Since $-\Delta v \geq 0$ in the sense of distributions and $v \geq 0$, it follows from the mean value formula that $v > 0$ a.e. in Ω , so that $v\Phi(u/v)$ is well defined a.e. Moreover, it is clear that $u_n \nearrow u$ in L^1 and that $a_n(x)u_n(x) \nearrow a(x)u(x)$ in L^1_δ and similarly for v . So that, using (2.3) and Lebesgue's theorem

$$w_n \rightarrow w \quad \text{in } L^1 \quad \text{and} \quad a_n(x)w_n \rightarrow a(x)w \quad \text{in } L^1_\delta$$

Since Φ' is bounded, we can also easily pass to the limit in the right-hand side of (2.4) and obtain (2.1).

□

The next lemma contains the heart of the proof.

Lemma 2.2. *Assume that $b \in L^1_{loc}(\Omega)$, $b \geq 0$ satisfies (0.4) and let $\zeta_1 = G(b)$. Suppose u is a weak solution of (P_1) . Then*

$$\int_{\Omega} f(\zeta_1)\zeta_0 < \infty \quad \text{and} \quad G(f(\zeta_1)) \leq C\zeta_1 \quad \text{where} \quad C = \int_1^{\infty} g(s)ds$$

Conversely, if

$$\int_{\Omega} f(2\zeta_1)\zeta_0 < \infty \quad \text{and} \quad G(f(2\zeta_1)) \leq \zeta_1$$

then (P_1) admits a weak solution u .

Proof. Suppose first that u is a weak solution of (P_1) and, recalling (0.6), define for $t \geq 1$

$$\Phi(t) = \int_1^t g(s)ds$$

and let $w = \Phi(u/\zeta_1)\zeta_1$. Observe that u is a supersolution of the equation satisfied by ζ_1 , so by minimality of ζ_1 , $u \geq \zeta_1$ and one can easily check that Lemma 2.1 applies with our choice of Φ , so that

$$-\Delta w - \frac{c}{|x|^2}w \geq g(u/\zeta_1)f(u) \geq f(\zeta_1)$$

Since $w \leq C\zeta_1$, $G(f(\zeta_1))$ can be constructed e.g. by monotone iteration, hence $\int_{\Omega} f(\zeta_1)\zeta_0 < \infty$ and we have

$$C\zeta_1 \geq w \geq G(f(\zeta_1))$$

Conversely, suppose that $G(f(2\zeta_1)) \leq \zeta_1$ and let $w = G(f(2\zeta_1)) + \zeta_1$. Then $w \leq 2\zeta_1$ and

$$-\Delta w - \frac{c}{|x|^2}w = f(2\zeta_1) + b \geq f(w) + b$$

So w is a supersolution of (P_1) and one can construct a weak solution, using a standard argument of monotone iteration.

□

The following two lemmas are technical.

Lemma 2.3. *Fix $\epsilon \in (0, 1)$ and for $t \in \mathbb{R}$ set $\tilde{g}(s) = \sup_{t \geq 1} g_t(s)$*

where

$$g_t(s) = \begin{cases} g(s) & \text{if } s \leq t \\ g(t) - \frac{1}{\epsilon}(s-t) & \text{if } s > t \end{cases}$$

Then $\tilde{g}(s)$ is Lipschitz continuous, satisfies (0.6), $s \rightarrow s\tilde{g}(s)$ is nonincreasing and

$$\int_1^\infty \tilde{g}(s)ds - \int_1^\infty g(s)ds \leq \epsilon/2$$

Proof. We first check the continuity of $\tilde{g}(t)$. Fix $1 \leq s < t$. We have

$$\tilde{g}(s) - \tilde{g}(t) = \sup_{r>1} g_r(s) - \tilde{g}(t).$$

Fix any $r > 1$. Then

$$g_r(s) - \tilde{g}(t) < g_r(s) - g_{\min\{r,s\}}(t).$$

For $r < s$ the right hand side is equal to $g_r(s) - g_r(t) = (t-s)/\epsilon$ and for $r \geq s$ it is equal to $g(s) - g_s(t) = (t-s)/\epsilon$ by the definition of $g_s(t)$. So, for any $r > 1$

$$g_r(s) - \tilde{g}(t) \leq (t-s)/\epsilon,$$

hence $0 \leq \tilde{g}(s) - \tilde{g}(t) \leq (t-s)/\epsilon$, which proves the continuity of $\tilde{g}(t)$.

Now we show that

$$\int_1^\infty \tilde{g}(s)ds - \int_1^\infty g(s)ds \leq \epsilon/2.$$

Indeed, if $g(t)$ is a step function, the answer is geometrically clear : \tilde{g} is then a piecewise linear map and the difference between the integrals is given by

$$\sum \frac{\epsilon}{2}[g]^2(t) \leq \frac{\epsilon}{2}g(1) \sum [g](t) = \frac{\epsilon}{2}.$$

where the sums are taken over all points t of discontinuity of g and where $[g](t)$ denotes the jump of g at t .

If g isn't a step function, since g is monotonous, we may approximate it with an increasing sequence of step functions $\{g_n(t)\}$ and denote $g_{n,s}(t)$ and $\tilde{g}_n(t)$ the corresponding functions, defined as for $g(t)$. On the one hand,

$$\int_1^\infty \tilde{g}_n(s)ds - \int_1^\infty g_n(s)ds \leq \epsilon/2$$

for any $n \in \mathbb{N}$ and on the other hand $g_{n,s}(t) \nearrow g_s(t)$, $\tilde{g}_n(t) \nearrow \tilde{g}(t)$, so the desired estimate is also true for $g(t)$.

It remains to prove the monotonicity of $sg_t(s)$. It suffices to show that for every $t > 1$ the function $sg_t(s)$ is nonincreasing. Since it is the case for $sg(s)$ we only have to check that $tg_t(t) > sg_t(s)$ for $s > t > 1$. In fact

$$sg_t(s) = tg(t) - (s-t)(g(t) - s/\epsilon) < tg(t) = tg_t(t)$$

since $\epsilon < 1$ and $s > 1$. □

Lemma 2.4. *Let $\mu \in (0, 1)$. There exists a C^1 , concave, bounded solution Ψ of*

$$\begin{cases} \Psi'(t) = g(t/\Psi(t)) \text{ for } t \geq 1 \\ \Psi(1) = \mu \end{cases}$$

Proof.

By the preceding lemma, up to replacing g by \tilde{g} , we may assume g Lipschitz continuous. By Cauchy's theorem, there exists a unique C^1 solution Ψ , which will be globally defined if we show that it is bounded.

Setting $\varphi(t) = t/\Psi(t)$, $\gamma = \varphi(1) = \mu^{-1}$ we obtain

$$\frac{\varphi'}{\varphi - \varphi^2 g(\varphi)} = \frac{1}{t}, \quad \text{i.e.} \quad \int_{\gamma}^{\varphi} \frac{ds}{s - s^2 g(s)} = \log t.$$

To show that $\varphi \geq t/c$ for some $c > 0$ it suffices to see that

$$\int_{\gamma}^{\varphi} \frac{ds}{s - s^2 g(s)} \leq \log \varphi + C \text{ for } \varphi > \gamma.$$

The above is equivalent to

$$\begin{aligned} \int_{\gamma}^{\varphi} ds \left(\frac{1}{s - s^2 g(s)} - \frac{1}{s} \right) &\leq C \text{ or} \\ \int_{\gamma}^{\varphi} ds \frac{g(s)}{1 - sg(s)} &\leq C \text{ for some } C > 0. \end{aligned}$$

Since $tg(t)$ is nonincreasing, by (0.6) we get for $\beta = 1 - \gamma g(\gamma) > 0$

$$\int_{\gamma}^{\varphi} ds \frac{g(s)}{1 - sg(s)} \leq \int_{\gamma}^{\varphi} ds \frac{g(s)}{\beta} < \infty.$$

Hence Ψ is bounded.

Finally, since g is nonincreasing, it follows from the equation that Ψ is concave if $t/\Psi(t)$ is nondecreasing, which holds true, as

$$\left(\frac{t}{\Psi}\right)' = \frac{\Psi - t\Psi'}{\Psi^2} \geq 0$$

since by (0.6), $\Psi(t)/t \geq g(t/\Psi) = \Psi'(t)$. □

This last lemma shows the estimate $u \leq C(\lambda)\zeta_1$ when $0 < \lambda < \lambda^*$.

Lemma 2.5. *Suppose there exists $\lambda^* > 0$ such that (P_{λ^*}) has a weak solution.*

Then for all $0 < \lambda < \lambda^$, (P_λ) has a solution u satisfying for some $C > 0$ (depending on λ) and for a.e. $x \in \Omega$ the following estimate*

$$\zeta_1 \leq u \leq C\zeta_1.$$

Proof. We let $\mu = \lambda/\lambda^*$, define Ψ as in Lemma 2.4. and let u^* denote a weak solution of (P_{λ^*}) , $v^* = \lambda^*\zeta_1$ and $w = v^*\Psi(u^*/v^*)$. Observe that $u^* \geq v^*$ by minimality of ζ_1 and apply Lemma 2.1 :

$$\begin{aligned} -\Delta w - \frac{c}{|x|^2}w &\geq \Psi'(u^*/v^*)f(u^*) + \mu\lambda^*b \\ &\geq g\left(\frac{u^*/v^*}{\Psi(u^*/v^*)}\right)f(u^*) + \lambda b \\ &\geq f(w) + \lambda b \end{aligned}$$

So the minimal solution u of (P_λ) is bounded by w and since Ψ is bounded, $u \leq Cv^* \leq C'\zeta_1$. □

We are now in a position to prove Theorem 1.

- Suppose first that (0.8) holds. We show that (P_λ) has a weak solution for $\lambda > 0$ small. It is then standard (see e.g. [D]) to show the existence of a finite $\lambda^* > 0$ such that (P_λ) has a weak solution if and only if $\lambda \leq \lambda^*$ and by Lemmas 2.0 and 2.5, we will have proven the first part of Theorem 1.

By Lemma 2.2, (P_λ) has a solution as soon as $G(f(2\lambda\zeta_1))$ exists and

$$G(f(2\lambda\zeta_1)) \leq \lambda\zeta_1 \quad (2.5)$$

By definition of g ,

$$f(2\lambda\zeta_1) \leq g\left(\frac{\epsilon}{2\lambda}\right) f(\epsilon\zeta_1)$$

By (0.8), $G(f(2\lambda\zeta_1))$ exists and, by minimality of $G(f(2\lambda\zeta_1))$,

$$G(f(2\lambda\zeta_1)) \leq g\left(\frac{\epsilon}{2\lambda}\right) G(f(\epsilon\zeta_1)) \leq Cg\left(\frac{\epsilon}{2\lambda}\right) \zeta_1$$

To prove that (2.5) holds for λ small, it is therefore enough to show that

$$\frac{1}{\lambda}g\left(\frac{\epsilon}{2\lambda}\right) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow 0$$

or that

$$\lim_{M \rightarrow \infty} Mg(M) = 0$$

Since $M \rightarrow Mg(M)$ is nonincreasing, the above limit is well-defined. But if we had $\lim_{M \rightarrow \infty} Mg(M) = C > 0$ then $g(M) \sim C/M$ near ∞ , contradicting $\int_1^\infty g(s)ds < \infty$.

This completes the proof of the first part of Theorem 1.

• We now prove the second part of Theorem 1 : we assume that for some $\lambda > 0$, (P_λ) has a weak solution u_λ and show that (0.8) must hold for some $C, \epsilon > 0$. By Lemma 2.2,

$$\int_{\Omega} f(\lambda\zeta_1)\zeta_0 < \infty \quad \text{and} \quad G(f(\lambda\zeta_1)) \leq C\lambda\zeta_1$$

So choosing $\epsilon = \lambda$ and $C' = C\lambda$, Theorem 1 is proved.

1.4 Proof of Theorem 2

Step 1. We first prove that u_λ is stable for $\lambda \in [0, \lambda^*]$. Consider for $n \in \mathbb{N}$ the minimal solution u_n of

$$\begin{cases} -\Delta u_n - a_n(x)u_n = f(u_n) + \lambda b_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \quad (3.1)$$

where $a_n(x) = \min(a(x), n)$ and $b_n = \min(b, n)$.

On the one hand, it is known that $\lambda_1(-\Delta - a_n(x) - f'(u_n)) \geq 0$. We briefly recall the proof of this fact : fix $p > n$ and consider the functional $F : \mathbb{R} \times W^{2,p}(\Omega) \rightarrow L^p(\Omega)$ defined by

$$F(\lambda, u) = -\Delta u - a_n(x)u - f(u) - \lambda b_n$$

It follows easily from (0.1) and the implicit function theorem that there exists a unique maximal curve $\lambda \in [0, \lambda^\#) \rightarrow u(\lambda)$ such that

$$F(\lambda, u(\lambda)) = 0 \quad \text{and} \quad F_u(\lambda, u(\lambda)) \in Iso(W^{2,p}, L^p).$$

If $0 < \lambda < \lambda^\#$, since u_n is the minimal solution of (3.1), $u_n \leq u(\lambda)$ and it follows by elliptic regularity that u_n is in the domain of F , so that $u_n = u(\lambda)$.

If $0 < \lambda < \lambda^*$, u_n is bounded (and hence in the domain of F) so that we must have $\lambda^\# = \lambda^*$ (otherwise we could extend the curve $u(\lambda)$ beyond $\lambda^\#$, contradicting its maximality).

So $\lambda_1(F_u(\lambda, u_n))$ never vanishes for $\lambda < \lambda^*$ and since by (0.1), $\lambda_1(F_u(0, 0)) > 0$, we conclude that $\lambda_1(-\Delta - a_n(x) - f'(u_n)) \geq 0$, for $0 \leq \lambda \leq \lambda^*$. On the other hand u_n increases with n to a solution of (P_λ) , and since $u_n \leq u_\lambda$, the limit is the minimal solution u_λ . Now by monotone convergence we conclude

$$\lambda_1(-\Delta - a(x) - f'(u_\lambda)) \geq 0.$$

Step 2. We now show that if $f(\zeta_1) + \lambda b \in H^*$ and $\lambda < \lambda^*$ then the minimal solution u_λ of (P_λ) belongs to H . We know by Theorem 1, that $u_\lambda \leq C\zeta_1$ so that

$$0 \leq f(u_\lambda) \leq f(C\zeta_1) \leq g(1/C)f(\zeta_1)$$

Hence, for $\phi \in C_c^\infty(\Omega)$,

$$\left| \int_{\Omega} (f(u_\lambda) + \lambda b)\phi \right| \leq \int_{\Omega} (Cf(\zeta_1) + \lambda b)|\phi| \leq C\|\phi\|_H$$

So that, $G := f(u_\lambda) + \lambda b \in H^*$. Letting $a_n(x) = \min(a(x), n)$ and u_n denote the solution of

$$\begin{cases} -\Delta u_n - a_n(x)u_n = G & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

We have

$$\|u_n\|_H^2 = \int_{\Omega} Gu_n + (a_n - a)u_n \leq \int_{\Omega} Gu_n \leq C\|u_n\|_H$$

Hence u_n is bounded in H and $u_{\lambda} \in H$.

Step 3. Next, following Brezis and Vazquez [BV], we prove that for $0 < \lambda < \lambda^*$ there is at most one stable solution belonging to H . Arguing by contradiction, let $u_1, u_2 \in H$ be two distinct stable solutions of (P_{λ}) . We may suppose that $u_1 = u_{\lambda}$ – the minimal solution. Then by the maximum principle $u_2 - u_1 > C\delta(x)$ for some $C > 0$. The stability for u_2 writes $\lambda_1(L - f'(u_2)) \geq 0$. We test this inequality against $u_2 - u_1$ (Note that we can do it, as $u_{1,2} \in H$). We obtain

$$\int_{\Omega} |\nabla(u_2 - u_1)|^2 - a(x)(u_2 - u_1)^2 \geq \int_{\Omega} f'(u_2)(u_2 - u_1)^2.$$

Since u_1 and u_2 are solutions, we also have

$$\int_{\Omega} |\nabla(u_2 - u_1)|^2 - a(x)(u_2 - u_1)^2 = \int_{\Omega} (f(u_2) - f(u_1))(u_2 - u_1).$$

Hence

$$\int_{\Omega} (f(u_2) - f(u_1))(u_2 - u_1) \geq \int_{\Omega} f'(u_2)(u_2 - u_1)^2$$

As f is convex $f(u_2) - f(u_1) \leq f'(u_2)(u_2 - u_1)$ and since $u_2 - u_1 > 0$ on Ω we arrive at

$$f(u_2) - f(u_1) = f'(u_2)(u_2 - u_1) \text{ a.e. in } \Omega \quad (3.2)$$

We claim that

$$f(t) = 0 \quad \text{for } t \in [0, \text{ess sup } u_2]. \quad (3.3)$$

and give two proofs for it. The first one is elementary but assumes additional regularity on f whereas the second one achieves full generality at the expense of simplicity. Note that once (3.3) is proven, we obtain a contradiction with Lemma 1.1, since u_1 and u_2 would both solve

$$\begin{cases} -\Delta u - a(x)u = b \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

• **Proof of (3.3) when f' is Lipschitz.**

By convexity of f we conclude from (3.2) that f is affine between $u_1(x)$ and $u_2(x)$ for almost every $x \in \Omega$. Setting for $\epsilon \in (0, 1)$, $v = \epsilon u_1 + (1 - \epsilon)u_2$, the above implies that $f''(v(x))$ exists for a.e. $x \in \Omega$ and $f''(v(x)) = 0$ a.e. in Ω .

Using Gagliardo-Nirenberg inequalities, it is clear that $v \in W_{loc}^{1,1}(\Omega)$ since $\Delta v \in L_{\delta}^1(\Omega)$. So that after multiplying $f''(v(x))$ by ∇v , we may apply the chain rule and obtain $\nabla(f'(v)) = 0$ a.e. in Ω , yielding

$$f'(v) = C \quad \text{a.e. in } \Omega$$

Repeating this procedure, we obtain

$$f(v) = Cv + D \quad \text{a.e. in } \Omega$$

By convexity of f , this implies in turn that

$$f(t) = Ct + D \quad \text{for } t \in [\text{ess inf } v, \text{ess sup } v].$$

By Lemma 3.2, $\text{ess inf } v = 0$. Since $f(0) = f'(0) = 0$, it follows that $f \equiv 0$ between 0 and $\text{ess sup } v$. Since $\epsilon \in (0, 1)$ is arbitrary, $f(t) = 0$ for $t \in [0, \text{ess sup } u_2]$. **Remark.**

This proof works only when f' is Lipschitz. Indeed, one can construct a nonconstant, monotone and continuous function g such that $g' = 0$ a.e. (see e.g. [R] p. 144-145) and choose $f' = g$.

• **Proof of (3.3) without assuming that f' is Lipschitz.**

Recall that we have also $u_2(x) - u_1(x) \geq C\delta(x)$ for a.e. $x \in \Omega$. Hence, by (3.2), we can fix representatives of u_1 and u_2 such that for every $x \in \Omega$ either f is affine in $(u_1(x), u_2(x))$ and $u_2(x) - u_1(x) \geq C\delta(x)$, or $u_1(x) = u_2(x) = 0$. Setting

$$A = \bigcup_{x \in \Omega} (u_1(x), u_2(x))$$

we claim that $A \supseteq (\text{ess inf } u_1, \text{ess sup } u_2)$. For this we need the following lemma:

Lemma 3.1. *Let $v \in W_{loc}^{1,1}(\Omega)$, where Ω is a connected domain. Then, for any representative of v , $v(\Omega)$ is dense in $[\text{ess inf } v, \text{ess sup } v]$*

Proof. Recall Stampacchia's theorem (see e.g. [GT]) asserting that if $w \in W_{loc}^{1,1}(\Omega)$ then $\nabla w = 0$ a.e. on any set where w is a constant. In particular, if $w \in W_{loc}^{1,1}(\Omega; \{0, 1\})$, w is constant. Suppose by contradiction that there exists a non-void open interval $I \subset [\text{ess inf } v, \text{ess sup } v]$ such that $v(\Omega) \cap I = \emptyset$.

Consider a function $s : \mathbb{R} \setminus I \rightarrow \{0, 1\}$ defined by $s(x) = 0$ for $x \leq \inf I$, $s(x) = 1$ elsewhere. Then $s(x)$ is regular and $s \circ v \in W_{loc}^{1,1}(\Omega, \{0, 1\})$. We obtain a contradiction since $s \circ v$ is not a.e. constant due to

$$\text{ess inf } v \leq \inf I < \sup I \leq \text{ess sup } v.$$

□

Now we prove that $A \supseteq (\text{ess inf } u_1, \text{ess sup } u_2)$. Using Gagliardo-Nirenberg inequalities, we know that $\nabla u_1 \in L_{loc}^1(\Omega)$ (since $\Delta u_1 \in L_{\delta}^1$). Choose a sequence $\{\Omega_n\}$ of connected subdomains of Ω such that $\Omega_n \subseteq \Omega_{n+1}$, $\bar{\Omega}_n \subset \Omega$ and $\cup \Omega_n = \Omega$. Set

$$A_n = \bigcup_{x \in \Omega_n} (u_1(x), u_2(x)),$$

it suffices to show that

$$A_n \supseteq I_n = (\text{ess inf }_{\Omega_n} u_1, \text{ess sup }_{\Omega_n} u_2).$$

Let $c_n > 0$ be such that $u_2 - u_1 > c_n$ on Ω_n (note that such a constant exists, since $\text{dist}(\Omega_n, \partial\Omega) > 0$.) It is clear that $A_n \cap I_n \neq \emptyset$.

Choose a connected component (i.e. an interval) A' of A_n such that $A' \cap I_n \neq \emptyset$. We show that $A' \supseteq I_n$. Indeed, if $\inf A' > \inf I_n$ then by Lemma 3.1 there exists $x \in \Omega$ such that $u_1(x) \in (\inf I_n, \inf A')$ and $\inf A' - u_1(x) < c_n$. Then $(u_1(x), u_2(x))$ intersects, but is not contained in A' , which contradicts the maximality of the connected component A' . Hence, going back we find that f is affine in $(\text{ess inf } u_1, \text{ess sup } u_2)$.

Assume temporarily that $\text{ess inf } u_1 = 0$. Since $f(0) = f'(0) = 0$, $f \equiv 0$ between 0 and $\text{ess sup } u_2$, which completes the proof of (3.3). So it only remains to prove that $\text{ess inf } u_1 = 0$. We prove this in the following lemma.

Lemma 3.2. *If $h \in L_{\delta}^1$, $u \in L^1(\Omega)$, $u \geq 0$ and for all $\phi \in C^2(\bar{\Omega})$, $\phi|_{\partial\Omega} = 0$,*

$$\int_{\Omega} u(-\Delta\phi) = \int_{\Omega} h\phi \tag{3.4}$$

then $\text{ess inf } u = 0$.

Proof. Assume by contradiction that $u \geq \epsilon > 0$ a.e. in Ω . Let ρ_n be a standard mollifier and, extending u and h by 0 in $\mathbb{R}^n \setminus \Omega$, let $u_n = u * \rho_n$ and $h_n = h * \rho_n$. On the one hand, there exists $\alpha > 0$ such that for n large enough

$$u_n \geq \alpha\epsilon \quad \text{everywhere in } \Omega$$

Indeed, since Ω is smooth, there exists $\alpha > 0$ such that for $x \in \Omega$,

$$u_n(x) \geq \epsilon \int_{\Omega \cap B_{1/n}(x)} \rho_n(x-y) dy \geq \alpha\epsilon \int_{B_{1/n}(x)} \rho_n(x-y) dy = \alpha\epsilon.$$

On the other hand, since $-\Delta u = h$ in $D'(\Omega)$, given $\omega \subset\subset \Omega$, we have for n large enough

$$-\Delta u_n = h_n \quad \text{everywhere in } \omega$$

Let ϕ solve

$$\begin{cases} -\Delta\phi = 1 & \text{in } \omega \\ \phi = 0 & \text{on } \partial\omega \end{cases} \quad (*)$$

and integrate by parts to obtain

$$\int_{\omega} h_n \phi - \int_{\omega} u_n = \int_{\partial\omega} u_n \partial_{\nu} \phi \leq \alpha\epsilon \int_{\partial\omega} \partial_{\nu} \phi = -\alpha\epsilon |\omega|$$

Now, $u_n \rightarrow u$ in L^1 and $h_n \rightarrow h$ in L^1_{δ} so that

$$\int_{\omega} h \phi - \int_{\omega} u \leq -\alpha\epsilon |\omega|$$

Choosing $\omega = \omega_k := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/k\}$ with $k \rightarrow \infty$, and writing ϕ_k the corresponding solution of (*), it is clear that $\phi_k \nearrow \phi$, where ϕ solves

$$\begin{cases} -\Delta\phi = 1 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

Passing to the limit and using (3.4) we obtain

$$0 \leq -\alpha\epsilon |\Omega|$$

and we have obtained the desired contradiction. \square

Step 4. Finally we prove uniqueness of the solution of (P_{λ^*}) . For the sake of contradiction, let u_1 and u_2 be two distinct solutions of (P_{λ^*}) , u_1 being the minimal solution. By the maximum principle $u_2 - u_1 > C\delta(x)$ for some $C > 0$. Set $v = \frac{u_1 + u_2}{2}$. Then

$$-\Delta v - \frac{c}{|x|^2}v = \frac{f(u_1) + f(u_2)}{2} + \lambda^*b \geq f\left(\frac{u_1 + u_2}{2}\right) + \lambda^*b = f(v) + \lambda^*b$$

by convexity of f . Hence v is a supersolution of (P_{λ^*}) and by Lemma 3.3, it is a solution.

Consequently, we have equality in the above expression and by convexity of f , we conclude that for almost every $x \in \Omega$, f is linear on $[u_1(x), u_2(x)]$. Arguing as before, we obtain the desired contradiction. Following the proof of Martel [M], in the next lemma we prove nonexistence of strict supersolutions for (P_{λ^*}) .

Lemma 3.3. *Suppose that $b \in L^p$ for some $p > N$ and let v be a supersolution of (P_{λ^*}) . Then v is a solution of (P_{λ^*}) .*

Proof. Let $\mu \in \mathcal{D}'(\Omega)$ defined by

$$\langle \mu, \phi \rangle = \int_{\Omega} v \left(-\Delta \phi - \frac{c}{|x|^2} \phi \right) - (f(v) + \lambda b) \phi \quad \text{for } \phi \in C_c^\infty(\Omega).$$

Since v is a supersolution, μ is a nonnegative Radon measure. Arguing by contradiction, suppose now that v is not a solution. This means that $\mu \neq 0$. Since $a(x)v + f(v) + \lambda b \in L^1_{\delta}(\Omega)$, we can construct (using Lemma 0.1) the solution $\xi \in L^1(\Omega)$ of

$$\begin{cases} -\Delta \xi = \mu & \text{in } \Omega \\ \xi = 0 & \text{on } \partial\Omega \end{cases}$$

By the weak maximum principle $\xi > \epsilon\delta(x)$ for some $\epsilon > 0$. On the other hand $b \in L^p$ for some $p > N$ implies $\eta = (-\Delta)^{-1}(b) \in C^{1,\alpha}$, hence $\eta < C\delta(x)$ in Ω for some $C < \infty$. Set

$$w = v + \epsilon C^{-1} \eta - \xi$$

Then $w < v$ and

$$-\Delta w = av + f(v) + (\lambda^* + \epsilon C^{-1})b > aw + f(w) + (\lambda^* + \epsilon C^{-1})b,$$

hence w is a supersolution to $(P_{\lambda^* + \epsilon C^{-1}})$ which contradicts the extremality of λ^* .

□

1.5 Application 1 : $a(x) = c/|x|^2$, $g(u) = u^p$

This equation was extensively studied in [D] and we showed there that, in a domain Ω containing the origin, (P_λ) has a weak solution (for small $\lambda > 0$) if and only if $c \leq c_0 := (n-2)^2/4$ and

$$1 < p < p_0 \quad \text{where} \quad p_0 = 1 + 2/a \quad \text{and} \quad a = \frac{n-2 - \sqrt{(n-2)^2 - 4c}}{2} > 0$$

In [D] we showed that for $b \in L^\infty$, $|x|^a \zeta_1 \in L^\infty$. So when $p < p_0$, $\zeta_1^p \sim |x|^{-a-2+\epsilon}$ for some $\epsilon > 0$ and $G(\zeta_1^p) \sim |x|^{-a}$ so that (0.8) is satisfied.

When $p \geq p_0$ however, ζ_1^p is at least of the order of $|x|^{-a-2}$ near 0 so that $G(\zeta_1^p)$ is at least of the order of $|x|^{-a} \ln(1/|x|)$ and (0.8) never holds. We give the details of a similar argument to prove the following new result :

Proposition 4.1. *Fix $0 < c \leq (n-2)^2/4$, $1 < p < p_0$, and $b \in L^\infty$. Let λ_p^* be the corresponding extremal parameter and $\epsilon = a(p_0 - p)$. Also define $\theta = \sqrt{(n-2)^2 - 4c}$. There exists a constant C , depending only on Ω , n , b such that*

$$C^{-1} \{\epsilon(\epsilon + \theta)\}^{\frac{1}{p-1}} \leq \lambda_p^* \leq C \{\epsilon(\epsilon + \theta)\}^{\frac{1}{p-1}}$$

Remark.

- The proposition strengthens the result of [D] by saying that solutions exist only for λ belonging to a shrinking interval $(0, \lambda_p^*]$ that eventually becomes empty when $p = p_0$.

- Though the transition is continuous, the rate of convergence of $(\lambda_p^*)^{p-1}$ to zero, jumps from ϵ when $c < c_0$ to ϵ^2 when $c = c_0$.

Proof.

To simplify notations, we write λ instead of λ_p^* and C for any constant depending only on Ω , n , b and call these constants universal. By Lemma 2.2 we have, for $C = \int_1^\infty s^{-p} ds$ (which is bounded by a universal constant since $p_0 \geq \frac{n+2}{n-2}$),

$$G((\lambda \zeta_1)^p) \leq C \lambda \zeta_1$$

Hence,

$$\lambda^{p-1} \leq C \lim_{x \rightarrow 0} (\zeta_1 / G(\zeta_1^p))(x) \quad (4.1)$$

Restrict to the case $\Omega = B_1$, the unit ball centered at the origin and $b \equiv 1$. It is then easy to check that

$$\zeta_1 = \zeta_0 := \frac{1}{2n+c} (|x|^{-a} - |x|^2) \quad (4.2)$$

and

$$\begin{aligned} \zeta_0^p &= \frac{1}{(2n+c)^p} |x|^{-ap} (1 - |x|^{a+2})^p \geq \frac{1}{(2n+c)^p} (|x|^{-ap} - p|x|^{2+a-ap}) \\ &\geq \frac{1}{(2n+c)^p} (|x|^{-ap} - |x|^\epsilon) =: k \end{aligned} \quad (4.3)$$

A computation then yields

$$\begin{aligned} (2n+c)^p G(k) &= \left(\frac{1}{\epsilon(\epsilon+\theta)} - \frac{p}{(a+\epsilon)(a+\epsilon+\theta)} \right) |x|^{-a} \\ &\quad - \frac{1}{\epsilon(\epsilon+\theta)} |x|^{-a+\epsilon} + \frac{p}{(a+\epsilon)(a+\epsilon+\theta)} |x|^{2+\epsilon} \end{aligned} \quad (4.4)$$

Combining (4.1), (4.2), (4.3) and (4.4), it follows that

$$\lambda^{p-1} \leq C(2n+c)^{p-1} \left[\frac{1}{\epsilon(\epsilon+\theta)} - \frac{p}{(a+\epsilon)(a+\epsilon+\theta)} \right]^{-1} \leq C\epsilon(\epsilon+\theta)$$

Conversely, applying Lemma 2.2, (P_λ) has a solution as soon as

$$G((2\lambda\zeta_0)^p) \leq \zeta_0$$

Hence,

$$(\lambda_p^*)^{p-1} \geq 2^{-p} \inf_{B_1} \zeta_0 / G(\zeta_0^p)$$

Now,

$$\zeta_0^p = \zeta_0^{p-1} \zeta_0 \leq C|x|^{-a(p-1)} \zeta_0 \leq Ck$$

Hence, we just need to estimate $\inf \zeta_0 / G(k)$. Starting from (4.2) and (4.4), and letting $A = (\epsilon(\epsilon+\theta))^{-1}$ and $r = |x|$, it follows that

$$G(k) / \zeta_0 \leq C \frac{A(1-r^\epsilon)}{1-r^{2+a}} \leq C \cdot A$$

This inequality provides the desired lower bound on $\zeta_0 / G(k)$ and hence on λ_p^* . When

$b \in L^\infty$ is arbitrary, we have, using Lemma 1.5 in [D],

$$C(\Omega) \left(\int_{\Omega} b \zeta_0 \right) \zeta_0 \leq \zeta_1 \leq \|b\|_{L^\infty} \zeta_0$$

so that all of the above estimates still hold (with new constants.) For a general domain Ω , let $r, R > 0$ be such that $B_r \subset \Omega \subset B_R$ and (extending b by 0 in $B_R \setminus \Omega$) observe that

$$\lambda^*(B_R) \leq \lambda^*(\Omega) \leq \lambda^*(B_r)$$

This follows from the fact that if u solves (P_λ) in Ω for some $\lambda > 0$, then u is a supersolution of (P_λ) in B_r , so that a solution of (P_λ) in B_r may be constructed.

□

1.6 Application 2 : $a(x) = c/\delta(x)^2$, $\Omega = B_1$

Hardy's inequality (0.1) holds for $0 < c \leq 1/4$. We show that $\zeta_0 \in L^\infty$, so that, for any perturbation $b \in L^\infty$ and any nonlinearity f satisfying our assumptions (0.5)..(0.7), (P_λ) has solutions for $\lambda > 0$ small.

Proposition 5.1. *Let $\Omega = B_1$, $b \in L^\infty(B_1)$, $0 < c \leq 1/4$ and $a(x) = c/\delta(x)^2 = c/(1 - |x|)^2$. Then*

$$\zeta_1 = G(b) \in L^\infty(B_1)$$

Proof. Without loss of generality, we restrict to the case $b \equiv 1$ and $c = 1/4$. By elliptic regularity, $\zeta_1 \in C^\infty(B_1)$ and $y(r) := \zeta_1(x)$ (where $r = |x|$) solves

$$y'' + \frac{n-1}{r}y' + \frac{1}{4(1-r)^2}y = -1$$

$r = 1$ is a regular singular point and the indicial equation reads :

$$s(s-1) + 1/4 = 0$$

The only root of this equation is $s = 1/2$ so by a theorem of Frobenius (see e.g. [T]), there exists a fundamental system of solutions to the homogeneous equation of the form

$$y_1 = \sqrt{1-r}A(r) \quad y_2 = \sqrt{1-r} \ln(1-r)B(r)$$

where A and B are analytic in a neighbourhood of $r = 1$. It follows from the Wronskian method that

$$y = C_1 y_1 + C_2 y_2 + y_1 \int_r^1 \frac{y_2}{W} + y_2 \int_r^1 \frac{y_1}{W}$$

where C_1, C_2 are constants and $W = y_2' y_1 - y_1' y_2$ is the associated Wronskian. From the expression of y_1, y_2 , it follows that y is bounded.

□

1.7 Application 3 : $a(x) = c/\text{dist}(x, \Sigma)^2$, $g(u) = u^p$

In this section, we let Σ be a smooth manifold of codimension $k \in \{3, \dots, n\}$ (with the convention that Σ is a point if $k = n$) contained in a compact subset of Ω . The letter d denotes the function

$$d(x) = \text{dist}(x, \Sigma).$$

For simplicity, we also let $b \equiv 1$. Finally we define

$$a = (k-2)/2 - \frac{1}{2} \sqrt{(k-2)^2 - 4c} \tag{6.1}$$

and

$$p_0 = 1 + 2/a \tag{6.2}$$

We will show that Hardy's inequality holds for the potential $a(x) = c/d(x)^2$ provided $c > 0$ is chosen small enough and $k \geq 3$. As mentioned in the introduction, we obtain the following critical exponent result :

Proposition 6.1. *If $1 < p < p_0$, condition (0.8) holds If $p > p_0$, condition (0.8) fails*

Remark. The case $p = p_0$ remains open.

Proof. The proof is organized as follows : we start out by constructing a system of coordinates that transforms Σ into a hyperplane and preserves $d(x) = \text{dist}(x, \Sigma)$. In that respect, since the case where Σ reduces to a single point was already treated in [D], we may assume that $k < n$.

Next, we divide the proof into several lemmas : we first prove Hardy's inequality and then compute successively Δd , ζ_0 and $G(\zeta_0^p)$. With these estimates, we can then easily prove Proposition 6.1. Since Σ is smooth, for $\beta > 0$ sufficiently small, say $\beta \leq \beta_0$, each $x \in \Omega_\beta$ has a unique projection $\pi(x) \in \Sigma$ such that $d(x) = |x - \pi(x)|$. Let N_1, \dots, N_k be an orthonormal family of vector fields which are orthogonal to the surface Σ (they are, at least locally, well defined). Then for each $x \in \Omega_\beta$ there exists a unique $\alpha = (\alpha_1(x), \dots, \alpha_k(x)) \in \mathbb{R}^k$ such that

$$x = \pi(x) + \sum_{i=1}^k \alpha_i(x) N_i(\pi(x))$$

and letting $|\cdot|$ denote the Euclidean norm in \mathbb{R}^k ,

$$d(x) = |\alpha| \tag{6.3}$$

Now fix a point $\sigma_0 \in \Sigma$ and suppose for simplicity that $\sigma_0 = 0$. Let

$$\sigma : \begin{cases} W \rightarrow \Sigma \subset \mathbb{R}^n \\ y \mapsto \sigma(y) \end{cases}$$

be a parametrization of Σ near $\sigma_0 = 0$, where W is a neighbourhood of 0 in \mathbb{R}^{n-k} . We may choose σ so that $\left\{ \frac{\partial \sigma}{\partial y_1} \Big|_{\sigma=0}, \dots, \frac{\partial \sigma}{\partial y_{n-k}} \Big|_{\sigma=0}, N_1 \Big|_{\sigma=0}, \dots, N_k \Big|_{\sigma=0} \right\}$ be a family of orthonormal vectors, which up to a rotation of \mathbb{R}^n we may assume to be the canonical basis.

It follows from the above discussion that there exist $\beta_0 > 0, V$ a neighbourhood of $\sigma_0 = 0$ in Ω (which may be assumed to be balanced, i.e. $\lambda V \subset V$ for all $|\lambda| < 1$), W a neighbourhood of 0 in \mathbb{R}^{n-k} and a diffeomorphism

$$J : \begin{cases} V \rightarrow W \times B_{\beta_0}^k \\ x \mapsto (y, \alpha) \end{cases} \tag{6.4}$$

where $B_{\beta_0}^k$ is the ball of radius β_0 in \mathbb{R}^k centered at the origin and (6.1) holds.

Observe that $J(0) = 0$. We claim that $J'(0) = Id$. Indeed if $H = J^{-1}$,

$$H(y, \alpha) = \sigma(y) + \sum_{i=1}^k \alpha_i N_i(\sigma(y))$$

and

$$H'(0) = \left(\frac{\partial \sigma}{\partial y_1} \Big|_{\sigma=0}, \dots, \frac{\partial \sigma}{\partial y_{n-k}} \Big|_{\sigma=0}, N_1 \Big|_{\sigma=0}, \dots, N_k \Big|_{\sigma=0} \right) = Id$$

Finally define

$$l : \begin{cases} \mathbb{R}^n \rightarrow \mathbb{R}^+ \\ (x_1, \dots, x_n) \mapsto \left(\sum_{i=n-k+1}^n x_i^2 \right)^{1/2} \end{cases} \quad (6.5)$$

With these notations (6.3) reads

$$d(x) = |\alpha| = l(J(x)) \quad (6.6)$$

Lemma 6.1. Hardy's inequality. *(0.1) holds for $a(x) = c/d(x)^2$ provided $c > 0$ is chosen small enough.*

Proof. Consider first a function $\phi \in C_c^\infty(V)$ with V as in (6.4) and let $\psi = \phi \circ J$. By the standard Hardy inequality, we have

$$\int_{\mathbb{R}^k} |\nabla_\alpha \psi(y, \cdot)|^2 d\alpha \geq \frac{(k-2)^2}{4} \int_{\mathbb{R}^k} \frac{\psi^2(y, \cdot)}{|\alpha|^2} d\alpha$$

Integrating with respect to y , we obtain

$$\int_{\mathbb{R}^n} |\nabla \psi|^2 \geq \frac{(k-2)^2}{4} \int_{\mathbb{R}^n} \frac{\psi^2}{l^2}$$

Changing coordinates, using (6.2) and the fact that $DJ \sim Id$ in V , we obtain

$$\int_{\mathbb{R}^n} |\nabla \phi|^2 \geq c \int_{\mathbb{R}^n} \frac{\phi^2}{d^2}$$

where c can be chosen arbitrarily close to $(k-2)^2/4$ by shrinking V .

In the general case where $\phi \in C_c^\infty(\Omega)$, one just needs to use a partition of unity adapted to a coverage of Σ by neighbourhoods V where the above computation holds. Outside of this covering, d is bounded below and we therefore have for $c > 0$ sufficiently small

$$\int_{\Omega} |\nabla \phi|^2 \geq c \int_{\Omega} \frac{\phi^2}{d^2}$$

Taking $c > 0$ even smaller, we then obtain (0.1). Also observe that the above estimates yield the following inequality, in the spirit of [BM] : for all $\epsilon > 0$ there exists $\lambda \in \mathbb{R}$ such that

$$\int_{\Omega} |\nabla \phi|^2 + \lambda \int_{\Omega} \phi^2 \geq \left(\frac{(k-2)^2}{4} - \epsilon \right) \int_{\Omega} \frac{\phi^2}{d^2}$$

The interested reader will find refined versions of the Hardy inequality in [FT] and its references. Some geometric assumptions on Ω and Σ are however required. \square

Lemma 6.2. *Let $\Omega_{\beta} = \{x \in \Omega : d(x) < \beta\}$. Then*

$$\Delta d = \frac{k-1}{d}(1 + \eta) \quad \text{in } \Omega_{\beta}$$

where $\eta = \eta(x; \beta) \rightarrow 0$ uniformly in $x \in \Omega_{\beta}$ as $\beta \rightarrow 0$.

Proof. Use the notations of (6.4) and scale the coordinates, i.e. for $\epsilon = \beta/\beta_0 > 0$, $x \in V$ let

$$\tilde{x} = \epsilon x.$$

Since $J(0) = 0$ and $J'(0) = Id$,

$$J(\tilde{x}) = \tilde{x} + h(\tilde{x}) \tag{6.7}$$

where

$$h(\tilde{x}) = \int_0^1 (1-t)(J''(t\tilde{x}) \cdot \tilde{x} | \tilde{x}) dt = \epsilon^2 o(1) \quad \text{uniformly in } \tilde{x} \in \epsilon V. \tag{6.8}$$

Using (6.6) and (6.7), we obtain

$$\frac{\partial d}{\partial \tilde{x}_i} = \frac{\partial l}{\partial z_j} \left(\delta_{ij} + \frac{\partial h_j}{\partial \tilde{x}_i} \right)$$

and

$$\frac{\partial^2 d}{\partial \tilde{x}_i^2} = \frac{\partial^2 l}{\partial z_j^2} \left(\delta_{ij} + \frac{\partial h_j}{\partial \tilde{x}_i} \right)^2 + \frac{\partial^2 h_j}{\partial \tilde{x}_i^2} \frac{\partial l}{\partial z_j}$$

With (6.8), one can show that $\nabla h = \epsilon o(1)$ and that $\nabla^2 h = o(1)$. It's also easy to check from (6.5) that $\nabla l \in L^{\infty}$ and $l \nabla^2 l \in L^{\infty}$ so that we finally obtain

$$\Delta d = \Delta l \cdot (1 + \epsilon o(1)) \quad \text{uniformly in } \epsilon V \tag{6.9}$$

Now by a straightforward computation, we have that

$$\Delta l = \frac{k-1}{l} \quad (6.10)$$

Finally,

$$d(\tilde{x}) = l(J(\tilde{x})) = l(\tilde{x})(1 + g(\tilde{x})) \quad (6.11)$$

where

$$g(\tilde{x}) = \frac{1}{l(\tilde{x})} \int_0^1 \nabla l(\tilde{x} + th(\tilde{x})) \cdot h(\tilde{x}) dt = \epsilon o(1) \quad \text{uniformly in } \tilde{x} \in \epsilon V,$$

as follows from (6.8) and the fact that $\nabla l \in L^\infty$. Collecting (6.9), (6.10) and (6.11), we obtain the desired result in ϵV , which remains true in a neighbourhood Ω_β of Σ by using a finite covering of Σ for which the above computations hold. \square

Lemma 6.3. *For all $\epsilon > 0$, there exists $C > 0$ such that*

$$C^{-1}d^{-a+\epsilon}\delta \leq \zeta_0 \leq Cd^{-a-\epsilon}$$

Proof. First observe that we just need to prove the estimates in a neighbourhood of Σ and apply elliptic regularity elsewhere. Define now

$$P(X) = X(X-1) + (k-1)X + c$$

and observe that $-a$ (defined in (6.1)) is the larger root of P . Next, fix $\epsilon > 0$ and define

$$w = Cd^{-a-\epsilon}$$

for some constant C to be chosen later on. A simple computation and Lemma 6.2 yield

$$-\Delta w - \frac{c}{d^2}w = -CP(-a-\epsilon)d^{-a-\epsilon-2}(1+\eta)$$

By choosing $\beta > 0$ small and C large enough, it follows that

$$\begin{cases} -\Delta w - \frac{c}{d^2}w \geq 1 & \text{in } \Omega_\beta \\ w \geq \zeta_0 & \text{on } \partial\Omega_\beta \end{cases}$$

and by the maximum principle (apply e.g. Lemma 1.1 to $(w - \zeta_0)^-$) we obtain the desired upper bound. The lower bound is obtained in the exact same manner. \square

Lemma 6.4. *For all $\epsilon > 0$ there exists $C > 0$ such that*

$$G(d^{-a-2+\epsilon}) \leq C\zeta_0 \quad \text{and} \quad G(d^{-a-2-2\epsilon}) \geq Cd^{-a-\epsilon}\delta$$

. The proof is analogous to that of the previous lemma and we skip it. \square **Proof of**

Proposition 6.1 continued. Recall (6.2) and given $p < p_0$, fix $\epsilon > 0$ so small that $p(a + \epsilon) < a + 2 - \epsilon$. By Lemma 6.3,

$$\zeta_0^p \leq Cd^{-a-2+\epsilon}$$

And by Lemma 6.4,

$$G(\zeta_0^p) \leq C\zeta_0$$

Conversely if $p > p_0$, let $\epsilon_0 > 0$ be such that $p(a - \epsilon_0) > a + 2 + 2\epsilon_0$. By Lemma 6.3,

$$\zeta_0^p \geq Cd^{-a-2-2\epsilon_0}\delta^p$$

And by Lemma 6.4,

$$G(\zeta_0^p) \geq Cd^{-a-\epsilon_0}\delta$$

Applying Lemma 6.3 with $\epsilon < \epsilon_0$, we obtain that (0.8) can never hold. \square

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