



**HAL**  
open science

# A change of variable formula for the 2D fractional Brownian motion of Hurst index bigger or equal to $1/4$

Ivan Nourdin

► **To cite this version:**

Ivan Nourdin. A change of variable formula for the 2D fractional Brownian motion of Hurst index bigger or equal to  $1/4$ . 2008. hal-00287990v1

**HAL Id: hal-00287990**

**<https://hal.science/hal-00287990v1>**

Preprint submitted on 13 Jun 2008 (v1), last revised 3 Oct 2008 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A change of variable formula for the 2D fractional Brownian motion of Hurst index bigger or equal to $1/4$

by Ivan Nourdin\*  
University of Paris VI

Actual version: 13th June 2008

**Abstract:** We prove a change of variable formula for the 2D fractional Brownian motion of index  $H$  bigger or equal to  $1/4$ . For  $H$  strictly bigger than  $1/4$ , our formula coincides with that obtained by using the rough paths theory. For  $H = 1/4$  (the more interesting case), there is an additional term that is a classical Wiener integral against an independent standard Brownian motion.

**Key words:** Fractional Brownian motion; weak convergence; change of variable formula.

**2000 Mathematics Subject Classification:** 60F05, 60H05, 60G15, 60H07.

## 1 Introduction and main result

In [4], Coutin and Qian have shown that the rough paths theory of Lyons [11] can be applied to the 2D fractional Brownian motion  $B = (B^{(1)}, B^{(2)})$  under the condition that its Hurst parameter  $H$  (supposed to be the same for the two components) is *strictly* bigger than  $1/4$ . Since this seminal work, several authors have recovered this fact by using different routes (see e.g. Feyel and de La Pradelle [7], Friz and Victoir [8] or Unterberger [17] to cite but a few). On the other hand, it is still an open problem to bypass this restriction on  $H$ .

Rough paths theory is purely deterministic in essence. Actually, its random aspect comes only when it is applied to a singular path of a given *stochastic* process (like a Brownian motion, a fractional Brownian motion, etc.). In particular, *it does not allow to produce a new alea*. As such, the second point of Theorem 1.2 just below shows, in a sense, that it seems difficult to reach the case  $H = 1/4$  by using exclusively the tools of the rough paths theory.

Before stating our main result, we need the following definition:

**Definition 1.1** *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function belonging in  $\mathcal{C}^1$ , and fix  $t > 0$ . Provided it exists, we define  $\int_0^t \nabla f(B_s) \cdot dB_s$  as the limit in probability, as  $n \rightarrow \infty$ , of*

$$\begin{aligned} I_n(t) &:= \sum_{k=0}^{\lfloor nt \rfloor - 1} \frac{\frac{\partial f}{\partial x}(B_{k/n}^{(1)}, B_{k/n}^{(2)}) + \frac{\partial f}{\partial x}(B_{(k+1)/n}^{(1)}, B_{k/n}^{(2)})}{2} (B_{(k+1)/n}^{(1)} - B_{k/n}^{(1)}) \\ &+ \sum_{k=0}^{\lfloor nt \rfloor - 1} \frac{\frac{\partial f}{\partial y}(B_{k/n}^{(1)}, B_{k/n}^{(2)}) + \frac{\partial f}{\partial y}(B_{k/n}^{(1)}, B_{(k+1)/n}^{(2)})}{2} (B_{(k+1)/n}^{(2)} - B_{k/n}^{(2)}). \end{aligned} \quad (1.1)$$

---

\*Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Boîte courrier 188, 4 Place Jussieu, 75252 Paris Cedex 5, France, [ivan.nourdin@upmc.fr](mailto:ivan.nourdin@upmc.fr)

If we only have that the random couple  $((B_s)_{s \in [0,t]}, I_n(t))$  converges in law in  $\mathcal{C}([0,t]) \times \mathbb{R}$ , we denote the limit by

$$\left( (B_s)_{s \in [0,t]}, \int_0^t \nabla f(B_s) \cdot d^* B_s \right).$$

Our main result is as follows:

**Theorem 1.2** *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function belonging in  $\mathcal{C}^8$  and verifying  $(\mathbf{H}_8)$ , see (3.14) below. Let also  $B = (B^{(1)}, B^{(2)})$  denote a 2D fractional Brownian motion of Hurst index  $H \in (0, 1)$ , and let  $t > 0$  be a fixed time.*

1. *If  $H > 1/4$  then  $\int_0^t \nabla f(B_s) \cdot dB_s$  is well-defined, and we have*

$$f(B_t) = f(0) + \int_0^t \nabla f(B_s) \cdot dB_s. \quad (1.2)$$

2. *If  $H = 1/4$  then only  $\int_0^t \nabla f(B_s) \cdot d^* B_s$  is well-defined, and we have*

$$f(B_t) \stackrel{\text{Law}}{=} f(0) + \int_0^t \nabla f(B_s) \cdot d^* B_s + \frac{\sigma_{1/4}}{\sqrt{2}} \int_0^t \frac{\partial^2 f}{\partial x \partial y}(B_s) dW_s. \quad (1.3)$$

*Here,  $\sigma_{1/4}$  is the universal constant defined by (1.5) (see below), while  $\int_0^t \frac{\partial^2 f}{\partial x \partial y}(B_s) dW_s$  denotes a classical Wiener integral with respect to a (1D) standard Brownian motion  $W$  independent of  $B$ .*

3. *If  $H < 1/4$  then the integral  $\int_0^t B_s \cdot d^* B_s$  does not exist. Therefore, it is not possible to write a change of variable formula for  $B_t^{(1)} B_t^{(2)}$  using the integral defined in Definition 1.1.*

**Remark 1.3** 1. Due to the definition of  $\int_0^t \nabla f(B_s) \cdot d^* B_s$ , we see that we can freely move each component in (1.3) from the right hand side to the left (or from the left hand side to the right).

2. When  $\beta$  denotes a one-dimensional fractional Brownian motion with Hurst index in  $(0, 1/2)$ , it is easily checked, for any fixed  $t > 0$ , that  $\sum_{k=0}^{\lfloor nt \rfloor - 1} \beta_{k/n} (\beta_{(k+1)/n} - \beta_{k/n})$  does not converge in law. (Indeed, on the one hand, we have

$$\beta_{\lfloor nt \rfloor / t}^2 = \sum_{k=0}^{\lfloor nt \rfloor - 1} (\beta_{(k+1)/n}^2 - \beta_{k/n}^2) = 2 \sum_{k=0}^{\lfloor nt \rfloor - 1} \beta_{k/n} (\beta_{(k+1)/n} - \beta_{k/n}) + \sum_{k=0}^{\lfloor nt \rfloor - 1} (\beta_{(k+1)/n} - \beta_{k/n})^2$$

and, on the other hand, it is well-known that

$$n^{2H-1} \sum_{k=0}^{\lfloor nt \rfloor - 1} (\beta_{(k+1)/n} - \beta_{k/n})^2 \xrightarrow[n \rightarrow \infty]{L^2} t.$$

These two facts imply immediately that

$$\sum_{k=0}^{\lfloor nt \rfloor - 1} \beta_{k/n} (\beta_{(k+1)/n} - \beta_{k/n}) = \frac{1}{2} \left( \beta_{\lfloor nt \rfloor / t}^2 - \sum_{k=0}^{\lfloor nt \rfloor - 1} (\beta_{(k+1)/n} - \beta_{k/n})^2 \right)$$

does not converge in law). That is why we need to introduce a “symmetric” part in our definition of the Riemann sum (1.1). Indeed, if  $H > 1/6$ , then  $\sum_{k=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} (f(\beta_{k/n}) + f(\beta_{(k+1)/n})) (\beta_{(k+1)/n} - \beta_{k/n})$  converges in  $L^2$  for any regular enough function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , see [9] and [3].

3. We stress that it is still an open problem to know if each integral  $\int_0^t \frac{\partial f}{\partial x}(B_s) d^{(*)} B_s^{(1)}$  and  $\int_0^t \frac{\partial f}{\partial y}(B_s) d^{(*)} B_s^{(2)}$  could be defined separately. Indeed, in the first two points of Theorem 1.2, we “only” prove that their sum, that is  $\int_0^t \nabla f(B_s) \cdot d^{(*)} B_s$ , is well-defined.
4. Let us give a quicker proof of (1.3) in the particular simple case where  $f(x, y) = xy$ . Indeed, let  $\beta$  be a one-dimensional fractional Brownian motion of index  $1/4$ . The classical Breuer-Major’s theorem [1] yields:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor n \cdot \rfloor - 1} (\sqrt{n}(\beta_{(k+1)/n} - \beta_{k/n})^2 - 1) \stackrel{\text{Law}}{=} \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor n \cdot \rfloor - 1} ((\beta_{k+1} - \beta_k)^2 - 1) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sigma_{1/4} W. \quad (1.4)$$

Here,  $W$  denotes a standard Brownian motion independent of  $\beta$  (note that the independence<sup>†</sup> is an immediate consequence of the central limit theorem for multiple integrals proved in Peccati and Tudor [16]), the constant  $\sigma_{1/4}$  is defined by

$$\sigma_{1/4} := \sqrt{\frac{1}{2} \sum_{k \in \mathbb{Z}} \left( \sqrt{|k+1|} + \sqrt{|k-1|} - 2\sqrt{|k|} \right)^2} < \infty, \quad (1.5)$$

and the convergence in law holds in the Skorohod space  $\mathcal{D}$  of càdlàg functions on  $[0, \infty)$ . Now, let  $\tilde{\beta}$  be another fractional Brownian motion of index  $1/4$ , independent of  $\beta$ . From (1.4), we get

$$\left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (\sqrt{n}(\beta_{(k+1)/n} - \beta_{k/n})^2 - 1), \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (\sqrt{n}(\tilde{\beta}_{(k+1)/n} - \tilde{\beta}_{k/n})^2 - 1) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{D}^2} \sigma_{1/4} (W, \tilde{W})$$

for  $(W, \tilde{W})$  a 2D standard Brownian motion, independent of the 2D fractional Brownian motion  $(\beta, \tilde{\beta})$ . In particular, by difference, we have

$$\frac{1}{2} \sum_{k=0}^{\lfloor n \cdot \rfloor - 1} ((\beta_{(k+1)/n} - \beta_{k/n})^2 - (\tilde{\beta}_{(k+1)/n} - \tilde{\beta}_{k/n})^2) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \frac{\sigma_{1/4}}{2} (W - \tilde{W}) \stackrel{\text{Law}}{=} \frac{\sigma_{1/4}}{\sqrt{2}} W.$$

Now, set  $B^{(1)} = (\beta + \tilde{\beta})/\sqrt{2}$  and  $B^{(2)} = (\beta - \tilde{\beta})/\sqrt{2}$ . It is easily checked that  $B^{(1)}$  and  $B^{(2)}$  still are two independent fractional Brownian motions of index  $1/4$ . Moreover, we deduce from the previous convergence that

$$\sum_{k=0}^{\lfloor n \cdot \rfloor - 1} (B_{(k+1)/n}^{(1)} - B_{k/n}^{(1)})(B_{(k+1)/n}^{(2)} - B_{k/n}^{(2)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \frac{\sigma_{1/4}}{\sqrt{2}} W, \quad (1.6)$$

---

<sup>†</sup>More precisely, this means that the couple  $\left( \beta, \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor n \cdot \rfloor - 1} (\sqrt{n}(\beta_{(k+1)/n} - \beta_{k/n})^2 - 1) \right)$  converges in law to  $(\beta, \sigma_{1/4} W)$  with  $\beta$  and  $W$  independent.

with  $B^{(1)}$ ,  $B^{(2)}$  and  $W$  independent. On the other hand, we have, for any  $a, b, c, d \in \mathbb{R}$ :

$$bd - ac = a(d - c) + c(b - a) + (b - a)(d - c).$$

Choosing  $a = B_{k/n}^{(1)}$ ,  $b = B_{(k+1)/n}^{(1)}$ ,  $c = B_{k/n}^{(2)}$  and  $d = B_{(k+1)/n}^{(2)}$ , and summing for  $k$  over  $0, \dots, \lfloor nt \rfloor - 1$ , we obtain

$$\begin{aligned} B_{\lfloor nt \rfloor / n}^{(1)} B_{\lfloor nt \rfloor / n}^{(2)} &= \sum_{k=0}^{\lfloor nt \rfloor - 1} B_{k/n}^{(1)} (B_{(k+1)/n}^{(2)} - B_{k/n}^{(2)}) + B_{k/n}^{(2)} (B_{(k+1)/n}^{(1)} - B_{k/n}^{(1)}) \\ &+ \sum_{k=0}^{\lfloor nt \rfloor - 1} (B_{(k+1)/n}^{(1)} - B_{k/n}^{(1)}) (B_{(k+1)/n}^{(2)} - B_{k/n}^{(2)}). \end{aligned} \quad (1.7)$$

Thus, passing to the limit using (1.6), we get the desired conclusion in (1.3), in the particular case where  $f(x, y) = xy$ . Note that the second term in the right-hand side of (1.7) is the discrete analogue of the *2-covariation* introduced by Errami and Russo in [6].

5. We could prove (1.3) at a functional level (actually, it has been done for  $f(x, y) = xy$  in the proof just below). But, in order to keep the length of this paper within limits, we defer to future analysis this rather technical investigation.
6. In a very recent work, Nourdin and Réveillac [14] proved the following result (see also Burdzy and Swanson [2] for similar results in the case where  $\beta$  is replaced by the solution of the stochastic heat equation driven by a space/time white noise). If  $\beta$  denotes a one-dimensional fractional Brownian motion of index  $1/4$  and if  $g : \mathbb{R} \rightarrow \mathbb{R}$  is regular enough, then

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(\beta_{k/n}) (\sqrt{n}(\beta_{(k+1)/n} - \beta_{k/n})^2 - 1) \xrightarrow[n \rightarrow \infty]{\text{Law}} \frac{1}{4} \int_0^1 g''(\beta_s) ds + \sigma_{1/4} \int_0^1 g(\beta_s) dW_s \quad (1.8)$$

for  $W$  a standard Brownian motion independent of  $\beta$ . Compare with Proposition 3.3 below. In particular, for  $g$  chosen identically one in (1.8), note that it agrees with (1.4).

7. The fractional Brownian motion of index  $1/4$  has a remarkable physical interpretation in terms of particle systems. Indeed, if one considers an infinite number of particles, initially placed on the real line according to a Poisson distribution, performing independent Brownian motions and undergoing “elastic” collisions, then the trajectory of a fixed particle (after rescaling) converges to a fractional Brownian motion of index  $1/4$ . See Harris [10] for heuristic arguments, and Dürr, Goldstein and Lebowitz [5] for precise results.

Now, the rest of the note is entirely devoted to the proof of Theorem 1.2. The Section 2 contains some preliminaries and fixes the notation. Some technical results are postponed in Section 3. Finally, the proof of Theorem 1.2 is done in Section 4.

## 2 Preliminaries and notation

We shall now provide a short description of the tools of Malliavin calculus that will be needed in the following sections. The reader is referred to the monographs [12] and [15] for any unexplained notion or result.

Let  $B = (B_t^{(1)}, B_t^{(2)})_{t \in [0, T]}$  be a 2D fractional Brownian motion with Hurst parameter belonging in  $(0, 1/2)$ . We denote by  $\mathcal{H}$  the Hilbert space defined as the closure of the set of step  $\mathbb{R}^2$ -valued functions on  $[0, T]$ , with respect to the scalar product induced by

$$\langle (\mathbf{1}_{[0, t_1]}, \mathbf{1}_{[0, t_2]}), (\mathbf{1}_{[0, s_1]}, \mathbf{1}_{[0, s_2]}) \rangle_{\mathcal{H}} = R_H(t_1, s_1) + R_H(t_2, s_2), \quad s_i, t_i \in [0, T], \quad i = 1, 2,$$

where  $R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$ . Also,  $\mathfrak{H}$  denotes the Hilbert space defined as the closure of the set of step  $\mathbb{R}$ -valued functions on  $[0, T]$ , with respect to the scalar product induced by

$$\langle \mathbf{1}_{[0, t]}, \mathbf{1}_{[0, s]} \rangle_{\mathfrak{H}} = R_H(t, s), \quad s, t \in [0, T].$$

Consider the set of all smooth cylindrical random variables, i.e. of the form

$$F = f(B(\varphi_1), \dots, B(\varphi_k)), \quad \varphi_i \in \mathcal{H}, \quad i = 1, \dots, k, \quad (2.9)$$

where  $f \in \mathcal{C}^\infty(\mathbb{R}^k; \mathbb{R})$  is bounded with bounded derivatives. The derivative operator  $D$  of a smooth cylindrical random variable of the above form is defined as the  $\mathcal{H}$ -valued random variable

$$DF = \sum_{i=1}^k \frac{\partial f}{\partial x_i}(B(\varphi_1), \dots, B(\varphi_k)) \varphi_i.$$

By iteration, one can define the  $m$ th derivative  $D^m F$  (which is an element of  $L^2(\Omega, \mathcal{H}^{\otimes m})$ ) for  $m \geq 2$ . As usual, for any  $m \geq 1$ , the space  $\mathbb{D}^{m,2}$  denotes the closure of the set of smooth random variables with respect to the norm  $\|\cdot\|_{m,2}$  defined by the relation

$$\|F\|_{m,2}^2 = E|F|^2 + \sum_{i=1}^m E\|D^i F\|_{\mathcal{H}^{\otimes i}}^2.$$

In particular, if  $D_{B^{(i)}} F$  designates the Malliavin derivative with respect to  $B^{(i)}$ , we have

$$D_{B^{(i)}} B_t^{(j)} = \delta_{ij} \mathbf{1}_{[0, t]} \quad \text{for } i, j \in \{1, 2\}, \text{ and } \delta_{ij} \text{ the Kronecker symbol.}$$

The derivative  $D$  verifies the chain rule. Precisely, if  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is in  $\mathcal{C}_b^1$  and if  $F_i, i = 1, \dots, n$ , are in  $\mathbb{D}^{1,2}$ , then  $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$  and

$$D\varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) DF_i.$$

The  $m$ th derivative  $D^m$  verifies the Leibnitz rule. That is, for any  $F, G \in \mathbb{D}^{m,2}$  such that  $FG \in \mathbb{D}^{m,2}$ , we have

$$D_{t_1, \dots, t_m}^m (FG) = \sum D_I^{|I|} (F) D_{I^c}^{m-|I|} (G), \quad t_i \in [0, T], \quad i = 1, \dots, m, \quad (2.10)$$

where the sum runs over any subset  $I$  of  $\{t_1, \dots, t_m\}$ ,  $|I|$  denoting the cardinality of  $I$ .

The divergence operator  $\delta$  is the adjoint of the derivative operator. If a random variable  $u \in L^2(\Omega, \mathcal{H})$  belongs to  $\text{dom}\delta$ , the domain of the divergence operator, then  $\delta(u)$  is defined by the duality relationship

$$E(F\delta(u)) = E\langle DF, u \rangle_{\mathcal{H}}$$

for every  $F \in \mathbb{D}^{1,2}$ .

For every  $q \geq 1$ , let  $\mathcal{H}_q$  be the  $q$ th Wiener chaos of  $B$ , that is, the closed linear subspace of  $L^2(\Omega, \mathcal{A}, P)$  generated by the random variables  $\{H_q(B(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$ , where  $H_q$  is the  $q$ th Hermite polynomial given by  $H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q} (e^{-x^2/2})$ . The mapping

$$I_q(h^{\otimes q}) = H_q(B(h)) \tag{2.11}$$

provides a linear isometry between the symmetric tensor product  $\mathcal{H}^{\odot q}$  and  $\mathcal{H}_q$ . The following duality formula holds

$$E(FI_q(f)) = E(\langle D^q F, f \rangle_{\mathcal{H}^{\otimes q}}),$$

for any  $f \in \mathcal{H}^{\odot q}$  and  $F \in \mathbb{D}^{q,2}$ . In particular, we have

$$E(FI_q^{(i)}(f)) = E(\langle D_{B^{(i)}}^q F, f \rangle_{\mathfrak{H}^{\otimes q}}), \quad i = 1, 2, \tag{2.12}$$

for any  $f \in \mathfrak{H}^{\odot q}$  and  $F \in \mathbb{D}^{q,2}$ , where, for simplicity, we write  $I_q^{(1)}(f)$  – resp.  $I_q^{(2)}(f)$  – for  $I_q((f, 0))$  – resp.  $I_q((0, f))$ .

Finally, we mention the following particular case (actually, the only one we will need in the sequel) of the classical multiplication formula: if  $f, g \in \mathfrak{H}$ ,  $q \geq 1$  and  $i \in \{1, 2\}$ , then

$$I_q^{(i)}(f^{\otimes q}) I_q^{(i)}(g^{\otimes q}) = \sum_{r=0}^q r! \binom{q}{r}^2 I_{2q-2r}^{(i)}(f^{\otimes q-r} \otimes g^{\otimes q-r}) \langle f, g \rangle_{\mathfrak{H}}^r. \tag{2.13}$$

### 3 Some technical results

In this section, we collect some crucial results for the proof of (1.3), the only case which is difficult.

Here and in the rest of the paper, we set

$$\Delta B_{k/n}^{(i)} := B_{(k+1)/n}^{(i)} - B_{k/n}^{(i)}, \quad \delta_{k/n} := \mathbf{1}_{[k/n, (k+1)/n]} \quad \text{and} \quad \varepsilon_u := \mathbf{1}_{[0, u]},$$

for any  $i \in \{1, 2\}$ ,  $k \in \{0, \dots, n-1\}$  and  $u \in [0, 1]$ .

In the sequel, for  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  belonging to  $\mathcal{C}^q$ , we will need assumption of the type:

$$(\mathbf{H}_q) \quad \sup_{s \in [0, 1]} E \left( \left| \frac{\partial^{a+b} g}{\partial x^a \partial y^b} (B_s^{(1)}, B_s^{(2)}) \right|^p \right) < \infty \quad \text{for any } p \geq 1 \text{ and integers } a, b \geq 0 \text{ s.t. } a + b \leq q. \tag{3.14}$$

We begin by the following technical lemma:

**Lemma 3.1** *Let  $\beta$  be a 1D fractional Brownian motion of Hurst index  $1/4$ . We have*

(i)  $|E(\beta_r(\beta_t - \beta_s))| \leq \sqrt{|t - s|}$  for any  $0 \leq r, s, t \leq 1$ ,

$$(ii) \sum_{k,l=0}^{n-1} \left| \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \stackrel{n \rightarrow \infty}{=} O(n),$$

$$(iii) \sum_{k,l=0}^{n-1} \left| \langle \delta_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^r \right| \stackrel{n \rightarrow \infty}{=} O(n^{1-r/2}) \text{ for any } r \geq 1,$$

$$(iv) \sum_{k=0}^{n-1} \left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}} \right| \stackrel{n \rightarrow \infty}{=} O(1),$$

$$(v) \sum_{k=0}^{n-1} \left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 - \frac{1}{4n} \right| \stackrel{n \rightarrow \infty}{=} O(1/\sqrt{n}).$$

**Proof of Lemma 3.1.**

(i) We have

$$E(\beta_r(\beta_t - \beta_s)) = \frac{1}{2}(\sqrt{t} - \sqrt{s}) + \frac{1}{2} \left( \sqrt{|s - r|} - \sqrt{|t - r|} \right).$$

Using the classical inequality  $|\sqrt{|b|} - \sqrt{|a|}| \leq \sqrt{|b - a|}$ , the desired result follows.

(ii) Observe that

$$\langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} = \frac{1}{2\sqrt{n}} \left( \sqrt{k+1} - \sqrt{k} - \sqrt{|k+1-l|} + \sqrt{|k-l|} \right).$$

Consequently, for any fixed  $l \in \{0, \dots, n-1\}$ , we have

$$\begin{aligned} \sum_{k=0}^{n-1} \left| \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| &\leq \frac{1}{2} + \frac{1}{2\sqrt{n}} \left( \sum_{k=0}^{l-1} \sqrt{l-k} - \sqrt{l-k-1} \right. \\ &\quad \left. + 1 + \sum_{k=l+1}^{n-1} \sqrt{k-l+1} - \sqrt{k-l} \right) \\ &= \frac{1}{2} + \frac{1}{2\sqrt{n}} (\sqrt{l} + \sqrt{n-l}) \end{aligned}$$

from which we deduce that  $\sup_{0 \leq l \leq n-1} \sum_{k=0}^{n-1} \left| \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \stackrel{n \rightarrow \infty}{=} O(1)$ . It follows that

$$\sum_{k,l=0}^{n-1} \left| \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \leq n \sup_{0 \leq l \leq n-1} \sum_{k=0}^{n-1} \left| \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \stackrel{n \rightarrow \infty}{=} O(n).$$



(iii) We have, by noting  $\rho(x) = \frac{1}{2}(\sqrt{|x+1|} + \sqrt{|x-1|} - 2\sqrt{|x|})$ :

$$\sum_{k,l=0}^{n-1} \left| \langle \delta_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^r \right| = n^{-r/2} \sum_{k,l=0}^{n-1} |\rho^r(l-k)| \leq n^{1-r/2} \sum_{k \in \mathbb{Z}} |\rho^r(k)|.$$

Since  $\sum_{k \in \mathbb{Z}} |\rho^r(k)| < \infty$  if  $r \geq 1$ , the desired conclusion follows.

(iv) is a consequence of the following identity combined with a telescopic sum argument:

$$\left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}} \right| = \frac{1}{2\sqrt{n}} (\sqrt{k+1} - \sqrt{k}).$$

(v) We have

$$\left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 - \frac{1}{4n} \right| = \frac{1}{4n} (\sqrt{k+1} - \sqrt{k}) \left| \sqrt{k+1} - \sqrt{k} - 2 \right|.$$

Thus, the desired bound is immediately checked by combining a telescoping sum argument with the fact that

$$\left| \sqrt{k+1} - \sqrt{k} - 2 \right| = \left| \frac{1}{\sqrt{k+1} + \sqrt{k}} - 2 \right| \leq 2.$$

□

Also the following lemma will be useful in the sequel:

**Lemma 3.2** *Let  $\alpha \geq 0$  and  $q \geq 2$  be two positive integers, let  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  be any function belonging to  $\mathcal{C}^{2q}$  and verifying  $(\mathbf{H}_{2q})$  defined by (3.14), and let  $B = (B^{(1)}, B^{(2)})$  be a 2D fractional Brownian motion of Hurst index  $1/4$ . Set*

$$V_n = n^{-q/4} \sum_{k=0}^{n-1} g(B_{k/n}^{(1)}, B_{k/n}^{(2)}) (\Delta B_{k/n}^{(1)})^\alpha H_q(n^{1/4} \Delta B_{k/n}^{(2)}),$$

where  $H_q$  denotes the  $q$ th Hermite polynomial defined by  $H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q} (e^{-x^2/2})$ . Then, we have the following bound:

$$E(|V_n|^2) = O(n^{1-q/2-\alpha/2}) \quad \text{as } n \rightarrow \infty. \quad (3.15)$$

**Proof of Lemma 3.2.** We can write

$$\begin{aligned}
E(|V_n|^2) &= n^{-q/2} \sum_{k,l=0}^{n-1} E[g(B_{k/n}^{(1)}, B_{k/n}^{(2)})g(B_{l/n}^{(1)}, B_{l/n}^{(2)})(\Delta B_{k/n}^{(1)})^\alpha (\Delta B_{l/n}^{(1)})^\alpha \\
&\quad \times H_q(n^{1/4}\Delta B_{k/n}^{(2)})H_q(n^{1/4}\Delta B_{l/n}^{(2)})] \\
&\stackrel{(2.11)}{=} \sum_{k,l=0}^{n-1} E[g(B_{k/n}^{(1)}, B_{k/n}^{(2)})g(B_{l/n}^{(1)}, B_{l/n}^{(2)})(\Delta B_{k/n}^{(1)})^\alpha (\Delta B_{l/n}^{(1)})^\alpha I_q^{(2)}(\delta_{k/n}^{\otimes q})I_q^{(2)}(\delta_{l/n}^{\otimes q})] \\
&\stackrel{(2.13)}{=} \sum_{r=0}^q r! \binom{q}{r}^2 \sum_{k,l=0}^{n-1} E[g(B_{k/n}^{(1)}, B_{k/n}^{(2)})g(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \\
&\quad \times (\Delta B_{k/n}^{(1)})^\alpha (\Delta B_{l/n}^{(1)})^\alpha I_{2q-2r}^{(2)}(\delta_{k/n}^{\otimes q-r} \otimes \delta_{l/n}^{\otimes q-r})] \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^r \\
&\stackrel{(2.12)}{=} \sum_{r=0}^q r! \binom{q}{r}^2 \sum_{k,l=0}^{n-1} E \left\langle D_{B^{(2)}}^{2q-2r} \left( g(B_{k/n}^{(1)}, B_{k/n}^{(2)})g(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right. \right. \\
&\quad \left. \left. \times (\Delta B_{k/n}^{(1)})^\alpha (\Delta B_{l/n}^{(1)})^\alpha \right), \delta_{k/n}^{\otimes q-r} \otimes \delta_{l/n}^{\otimes q-r} \right\rangle_{\mathfrak{H}^{\otimes 2q-2r}} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^r \\
&\stackrel{(2.10)}{=} \sum_{r=0}^q r! \binom{q}{r}^2 \sum_{a+b=2q-2r} \sum_{k,l=0}^{n-1} E \left( \frac{d^a g}{dy^a}(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \frac{d^b g}{dy^b}(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right. \\
&\quad \left. \times (\Delta B_{k/n}^{(1)})^\alpha (\Delta B_{l/n}^{(1)})^\alpha \right) (2q-2r)! \left\langle \varepsilon_{k/n}^{\otimes a} \tilde{\otimes} \varepsilon_{l/n}^{\otimes b}, \delta_{k/n}^{\otimes q-r} \otimes \delta_{l/n}^{\otimes q-r} \right\rangle_{\mathfrak{H}^{\otimes 2q-2r}} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^r. \tag{3.16}
\end{aligned}$$

Now, observe that, uniformly in  $k, l \in \{0, \dots, n-1\}$ :

$$\left\langle \varepsilon_{k/n}^{\otimes a} \tilde{\otimes} \varepsilon_{l/n}^{\otimes b}, \delta_{k/n}^{\otimes q-r} \otimes \delta_{l/n}^{\otimes q-r} \right\rangle_{\mathfrak{H}^{\otimes 2q-2r}} \underset{n \rightarrow \infty}{=} O(n^{-(q-r)}), \text{ see Lemma 3.1 (i),}$$

$$\left| E \left( \frac{d^a g}{dy^a}(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \frac{d^b g}{dy^b}(B_{l/n}^{(1)}, B_{l/n}^{(2)}) (\Delta B_{k/n}^{(1)})^\alpha (\Delta B_{l/n}^{(1)})^\alpha \right) \right| \underset{n \rightarrow \infty}{=} O(n^{-\alpha/2}), \text{ use } (\mathbf{H}_{2q}),$$

and, also:

$$\sum_{k,l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^r = O(n^{1-r/2}) \text{ for any fixed } r \geq 1, \text{ see Lemma 3.1 (iii).}$$

Therefore, the desired conclusion is obtained by combining these three observations with (3.16).  $\square$

The independent Brownian motion appearing in (1.3) will come from the following proposition.

**Proposition 3.3** *Let  $(\beta, \tilde{\beta})$  be a 2D fractional Brownian motion of Hurst index  $1/4$ . Assume that  $g, \tilde{g} : \mathbb{R}^2 \rightarrow \mathbb{R}$  are two functions belonging in  $\mathcal{C}^4$  and verifying  $(\mathbf{H}_4)$  defined by (3.14). Then*

$$\begin{aligned}
(G_n, \tilde{G}_n) &:= \left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(\beta_{k/n}, \tilde{\beta}_{k/n}) (\sqrt{n}(\Delta\beta_{k/n})^2 - 1), \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \tilde{g}(\beta_{k/n}, \tilde{\beta}_{k/n}) (\sqrt{n}(\Delta\tilde{\beta}_{k/n})^2 - 1) \right) \\
&\xrightarrow[n \rightarrow \infty]{\text{stably}} \left( \sigma_{1/4} \int_0^1 g(\beta_s, \tilde{\beta}_s) dW_s + \frac{1}{4} \int_0^1 \frac{\partial^2 g}{\partial x^2}(\beta_s, \tilde{\beta}_s) ds, \sigma_{1/4} \int_0^1 \tilde{g}(\beta_s, \tilde{\beta}_s) d\tilde{W}_s + \frac{1}{4} \int_0^1 \frac{\partial^2 \tilde{g}}{\partial y^2}(\beta_s, \tilde{\beta}_s) ds \right),
\end{aligned}$$

where  $(W, \widetilde{W})$  is a 2D standard Brownian motion independent of  $(\beta, \widetilde{\beta})$ , and  $\sigma_{1/4}$  is defined by (1.5).

*Remark.* In the particular case where  $g(x, y) = g(x)$  and  $\widetilde{g}(x, y) = \widetilde{g}(y)$ , the conclusion of the lemma follows directly from (1.8).

**Proof.** Our proof uses the tools developed in Nourdin and Nualart [13], and is actually a literal extension of the proof of (1.8) contained in [14].

First, following step by step the computations made in [14], see more precisely (3.1) and (3.2) therein, and taking into account that  $\beta$  and  $\widetilde{\beta}$  are independent, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} E(G_n^2) &= \sigma_{1/4}^2 \int_0^1 E(g^2(\beta_s, \widetilde{\beta}_s)) ds + \frac{1}{16} E \left( \int_0^1 \frac{\partial^2 g}{\partial x^2}(\beta_s, \widetilde{\beta}_s) ds \right)^2 \\ \lim_{n \rightarrow \infty} E(\widetilde{G}_n^2) &= \sigma_{1/4}^2 \int_0^1 E(\widetilde{g}^2(\beta_s, \widetilde{\beta}_s)) ds + \frac{1}{16} E \left( \int_0^1 \frac{\partial^2 \widetilde{g}}{\partial y^2}(\beta_s, \widetilde{\beta}_s) ds \right)^2. \end{aligned} \quad (3.17)$$

Since the sequences  $(G_n)$  and  $(\widetilde{G}_n)$  are bounded in  $L^2(\Omega)$ , the sequence  $(G_n, \widetilde{G}_n, (\beta_t, \widetilde{\beta}_t)_{t \in [0,1]})$  is tight in  $\mathbb{R}^2 \times \mathcal{C}([0,1]^2)$ . Assume that  $(G_\infty, \widetilde{G}_\infty, (\beta_t, \widetilde{\beta}_t)_{t \in [0,1]})$  denotes the limit in law of a certain subsequence of  $(G_n, \widetilde{G}_n, (\beta_t, \widetilde{\beta}_t)_{t \in [0,1]})$ , denoted again by  $(G_n, \widetilde{G}_n, (\beta_t, \widetilde{\beta}_t)_{t \in [0,1]})$ .

We have to prove that

$$\begin{aligned} (G_\infty, \widetilde{G}_\infty) \stackrel{\text{Law}}{=} & \left( \sigma_{1/4} \int_0^1 g(\beta_s, \widetilde{\beta}_s) dW_s + \frac{1}{4} \int_0^1 \frac{\partial^2 g}{\partial x^2}(\beta_s, \widetilde{\beta}_s) ds, \right. \\ & \left. \sigma_{1/4} \int_0^1 \widetilde{g}(\beta_s, \widetilde{\beta}_s) d\widetilde{W}_s + \frac{1}{4} \int_0^1 \frac{\partial^2 \widetilde{g}}{\partial y^2}(\beta_s, \widetilde{\beta}_s) ds \right), \end{aligned}$$

or more precisely that

$$\begin{aligned} & E \left( e^{i\lambda G_\infty + i\widetilde{\lambda} \widetilde{G}_\infty} \mid (\beta_t, \widetilde{\beta}_t)_{t \in [0,1]} \right) \\ &= \exp \left\{ i\frac{\lambda}{4} \int_0^1 \frac{\partial^2 g}{\partial x^2}(\beta_s, \widetilde{\beta}_s) ds + i\frac{\widetilde{\lambda}}{4} \int_0^1 \frac{\partial^2 \widetilde{g}}{\partial y^2}(\beta_s, \widetilde{\beta}_s) ds - \frac{\lambda^2 \sigma_{1/4}^2}{2} \int_0^1 g^2(\beta_s, \widetilde{\beta}_s) ds \right. \\ & \quad \left. - \frac{\widetilde{\lambda}^2 \sigma_{1/4}^2}{2} \int_0^1 \widetilde{g}^2(\beta_s, \widetilde{\beta}_s) ds \right\} \end{aligned} \quad (3.18)$$

for any  $\lambda, \widetilde{\lambda} \in \mathbb{R}$ . This will be done by showing that, for any r.v.  $\xi$  of the form (2.9) and  $\lambda, \widetilde{\lambda} \in \mathbb{R}$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\partial}{\partial \lambda} \phi_n(\lambda, \widetilde{\lambda}) &= E \left\{ e^{i\lambda G_\infty + i\widetilde{\lambda} \widetilde{G}_\infty} \xi \left( \frac{i}{4} \int_0^1 \frac{\partial^2 g}{\partial x^2}(\beta_s, \widetilde{\beta}_s) ds - \lambda \sigma_{1/4}^2 \int_0^1 g^2(\beta_s, \widetilde{\beta}_s) ds \right) \right\} \\ \lim_{n \rightarrow \infty} \frac{\partial}{\partial \widetilde{\lambda}} \phi_n(\lambda, \widetilde{\lambda}) &= E \left\{ e^{i\lambda G_\infty + i\widetilde{\lambda} \widetilde{G}_\infty} \xi \left( \frac{i}{4} \int_0^1 \frac{\partial^2 \widetilde{g}}{\partial y^2}(\beta_s, \widetilde{\beta}_s) ds - \widetilde{\lambda} \sigma_{1/4}^2 \int_0^1 \widetilde{g}^2(\beta_s, \widetilde{\beta}_s) ds \right) \right\} \end{aligned} \quad (3.19)$$

where

$$\phi_n(\lambda, \tilde{\lambda}) := E \left( e^{i\lambda G_n + i\tilde{\lambda} \tilde{G}_n} \xi \right), \quad n \geq 1.$$

Let us make precise this argument. Since  $(G_n), (\tilde{G}_n)$  are bounded in  $L^2$  and  $(G_\infty, \tilde{G}_\infty, (\beta_t, \tilde{\beta}_t)_{t \in [0,1]})$  is the limit in law of  $(G_n, \tilde{G}_n, (\beta_t, \tilde{\beta}_t)_{t \in [0,1]})$ , we have that

$$\begin{aligned} E \left( G_\infty \xi e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty} \right) &= \lim_{n \rightarrow \infty} E \left( G_n \xi e^{i\lambda G_n + i\tilde{\lambda} \tilde{G}_n} \right) = \lim_{n \rightarrow \infty} \frac{\partial}{\partial \lambda} \phi_n(\lambda, \tilde{\lambda}), \\ E \left( \tilde{G}_\infty \xi e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty} \right) &= \lim_{n \rightarrow \infty} E \left( \tilde{G}_n \xi e^{i\lambda G_n + i\tilde{\lambda} \tilde{G}_n} \right) = \lim_{n \rightarrow \infty} \frac{\partial}{\partial \tilde{\lambda}} \phi_n(\lambda, \tilde{\lambda}), \end{aligned}$$

for all  $\lambda, \tilde{\lambda} \in \mathbb{R}$ , and for every  $\xi$  of the form (2.9). Furthermore, because we will prove that the convergence (3.19) holds for every  $\xi$  of the form (2.9), the conditional characteristic function  $(\lambda, \tilde{\lambda}) \mapsto E \left( e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty} | (\beta_t, \tilde{\beta}_t)_{t \in [0,1]} \right)$  satisfies the following linear system of ordinary differential equations:

$$\begin{aligned} &\frac{\partial}{\partial \lambda} E \left( e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty} | (\beta_t, \tilde{\beta}_t)_{t \in [0,1]} \right) \\ &= E \left( e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty} | (\beta_t, \tilde{\beta}_t)_{t \in [0,1]} \right) \times \left[ \frac{i}{4} \int_0^1 \frac{\partial^2 g}{\partial x^2}(\beta_s, \tilde{\beta}_s) ds - \lambda \sigma_{1/4}^2 \int_0^1 g^2(\beta_s, \tilde{\beta}_s) ds \right] \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial \tilde{\lambda}} E \left( e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty} | (\beta_t, \tilde{\beta}_t)_{t \in [0,1]} \right) \\ &= E \left( e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty} | (\beta_t, \tilde{\beta}_t)_{t \in [0,1]} \right) \times \left[ \frac{i}{4} \int_0^1 \frac{\partial^2 \tilde{g}}{\partial y^2}(\beta_s, \tilde{\beta}_s) ds - \lambda \sigma_{1/4}^2 \int_0^1 \tilde{g}^2(\beta_s, \tilde{\beta}_s) ds \right]. \end{aligned}$$

By solving it, we obtain (3.18), which yields the desired conclusion.

Thus, it remains to show (3.19). We will only prove the first limit, the second being obviously completely similar. Exactly as in [14] (and by taking into account of the independence between  $\beta$  and  $\tilde{\beta}$ ), see more precisely the proof of (3.12) therein, here we can write:

$$\begin{aligned} \frac{\partial}{\partial \lambda} \phi_n(\lambda, \tilde{\lambda}) &= -2\lambda \sum_{k,l=0}^{n-1} E(g(\beta_{k/n}, \tilde{\beta}_{k/n})g(\beta_{l/n}, \tilde{\beta}_{l/n})e^{i\lambda G_n + i\tilde{\lambda} \tilde{G}_n} \xi) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \\ &\quad + i \sum_{k=0}^{n-1} E \left( \frac{\partial^2 g}{\partial x^2}(\beta_{k/n}, \tilde{\beta}_{k/n}) e^{i\lambda G_n + i\tilde{\lambda} \tilde{G}_n} \xi \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 + i \sum_{k=0}^{n-1} r_{k,n}, \end{aligned} \quad (3.20)$$

with  $\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} r_{k,n} = 0$ .

Moreover, using once again the same argument allowing to prove (3.14)-(3.15) in [14], we have, on the one hand

$$\begin{aligned} &\lim_{n \rightarrow \infty} -2\lambda \sum_{k,l=0}^{n-1} E(g(\beta_{k/n}, \tilde{\beta}_{k/n})g(\beta_{l/n}, \tilde{\beta}_{l/n})e^{i\lambda G_n + i\tilde{\lambda} \tilde{G}_n} \xi) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \\ &= -\sigma_{1/4}^2 \lambda \int_0^1 E(g^2(\beta_s, \tilde{\beta}_s) e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty} \xi) ds, \end{aligned}$$

and, on the other hand:

$$\begin{aligned} \lim_{n \rightarrow \infty} i \sum_{k=0}^{n-1} E \left( \frac{\partial^2 g}{\partial x^2}(\beta_{k/n}, \tilde{\beta}_{k/n}) e^{i\lambda G_n + i\tilde{\lambda} \tilde{G}_n \xi} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right) \\ = \frac{i}{4} \int_0^1 E \left( \frac{\partial^2 g}{\partial x^2}(\beta_s, \tilde{\beta}_s) e^{i\lambda G_\infty + i\tilde{\lambda} \tilde{G}_\infty \xi} \right) ds. \end{aligned}$$

Combined with (3.20), this shows the first limit in (3.19), so that the proof of Proposition 3.3 is done.  $\square$

## 4 Proof of Theorem 1.2

We are now in position to prove Theorem 1.2.

**Proof of the third point** (case  $H < 1/4$ ). Using exactly the same proof as that contained in the fourth point of Remark 1.3, we have that, for any  $H \in (0, 3/4)$ :

$$n^{2H-1/2} \sum_{k=0}^{n-1} \Delta B_{k/n}^{(1)} \Delta B_{k/n}^{(2)} \xrightarrow[n \rightarrow \infty]{\text{Law}} \mathcal{N}(0, \sigma_H^2), \quad (4.21)$$

with  $\sigma_H \neq 0$ . But, see (1.7), the existence of  $\int_0^t B_s \cdot dB^* B_s$  would imply in particular that  $\sum_{k=0}^{n-1} \Delta B_{k/n}^{(1)} \Delta B_{k/n}^{(2)}$  converges in law as  $n \rightarrow \infty$ , which is in contradiction with (4.21) for  $H < 1/4$ . Therefore, the proof of the third point is done.

**Proof of the second point** (case  $H = 1/4$ ). For the simplicity of the exposition, we assume that  $t = 1$ , the general case being of course similar up to cumbersome notations. For any  $a, b, c, d \in \mathbb{R}$ , by the classical Taylor formula, we can expand  $f(b, d)$  as (compare with (1.7)):

$$\begin{aligned} & f(a, c) + \partial_1 f(a, c)(b - a) + \partial_2 f(a, c)(d - c) + \frac{1}{2} \partial_{11} f(a, c)(b - a)^2 + \frac{1}{2} \partial_{22} f(a, c)(d - c)^2 \\ & + \frac{1}{6} \partial_{111} f(a, c)(b - a)^3 + \frac{1}{6} \partial_{222} f(a, c)(d - c)^3 + \frac{1}{24} \partial_{1111} f(a, c)(b - a)^4 + \frac{1}{24} \partial_{2222} f(a, c)(d - c)^4 \\ & + \partial_{12} f(a, c)(b - a)(d - c) + \frac{1}{2} \partial_{112} f(a, c)(b - a)^2(d - c) + \frac{1}{2} \partial_{122} f(a, c)(b - a)(d - c)^2 \\ & + \frac{1}{6} \partial_{1112} f(a, c)(b - a)^3(d - c) + \frac{1}{4} \partial_{1122} f(a, c)(b - a)^2(d - c)^2 + \frac{1}{6} \partial_{1222} f(a, c)(b - a)(d - c)^3 \end{aligned} \quad (4.22)$$

plus a remainder term. Here, as usual, the notation  $\partial_{1\dots 12\dots 2} f$  (where the index 1 is repeated  $k$  times and the index 2 is repeated  $l$  times) means that  $f$  is differentiated  $k$  times w.r.t. the first component and  $l$  times w.r.t. the second one. Combining (4.22) with the following identity, available for any  $h : \mathbb{R} \rightarrow \mathbb{R}$  belonging in  $\mathcal{C}^4$ :

$$\begin{aligned} & h'(a)(b - a) + \frac{1}{2} h''(a)(b - a)^2 + \frac{1}{6} h'''(a)(b - a)^3 + \frac{1}{24} h''''(a)(b - a)^4 \\ = & \frac{h'(a) + h'(b)}{2} (b - a) - \frac{1}{12} h'''(a)(b - a)^3 - \frac{1}{24} h''''(a)(b - a)^4 + \text{some remainder} \end{aligned}$$

we get that  $f(b, d)$  can also be expanded as

$$\begin{aligned}
& f(a, c) + \frac{1}{2}(\partial_1 f(a, c) + \partial_1 f(b, c))(b - a) - \frac{1}{12}\partial_{111}f(a, c)(b - a)^3 - \frac{1}{24}\partial_{1111}f(a, c)(b - a)^4 \\
& + \frac{1}{2}(\partial_2 f(a, c) + \partial_2 f(a, d))(d - c) - \frac{1}{12}\partial_{222}f(a, c)(d - c)^3 - \frac{1}{24}\partial_{2222}f(a, c)(d - c)^4 \\
& + \partial_{12}f(a, c)(b - a)(d - c) + \frac{1}{2}\partial_{112}f(a, c)(b - a)^2(d - c) + \frac{1}{2}\partial_{122}f(a, c)(b - a)(d - c)^2 \\
& + \frac{1}{6}\partial_{1112}f(a, c)(b - a)^3(d - c) + \frac{1}{4}\partial_{1122}f(a, c)(b - a)^2(d - c)^2 + \frac{1}{6}\partial_{1222}f(a, c)(b - a)(d - c)^3
\end{aligned} \tag{4.23}$$

plus a remainder term.

By setting  $a = B_{k/n}^{(1)}$ ,  $b = B_{(k+1)/n}^{(1)}$ ,  $c = B_{k/n}^{(2)}$  and  $d = B_{(k+1)/n}^{(2)}$  in (4.23), and by summing the obtained expression for  $k$  over  $0, \dots, n-1$ , we deduce that the conclusion in Theorem 1.2 is a consequence of the following convergences:

$$S_n^{(1)} := \sum_{k=0}^{n-1} \partial_{111}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) (\Delta B_{k/n}^{(1)})^3 \xrightarrow[n \rightarrow \infty]{L^2} -\frac{3}{2} \int_0^1 \partial_{1111}f(B_s^{(1)}, B_s^{(2)}) ds \tag{4.24}$$

$$S_n^{(2)} := \sum_{k=0}^{n-1} \partial_{1111}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) (\Delta B_{k/n}^{(1)})^4 \xrightarrow[n \rightarrow \infty]{L^2} 3 \int_0^1 \partial_{1111}f(B_s^{(1)}, B_s^{(2)}) ds \tag{4.25}$$

$$S_n^{(3)} := \sum_{k=0}^{n-1} \partial_{222}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) (\Delta B_{k/n}^{(2)})^3 \xrightarrow[n \rightarrow \infty]{L^2} -\frac{3}{2} \int_0^1 \partial_{2222}f(B_s^{(1)}, B_s^{(2)}) ds \tag{4.26}$$

$$S_n^{(4)} := \sum_{k=0}^{n-1} \partial_{2222}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) (\Delta B_{k/n}^{(2)})^4 \xrightarrow[n \rightarrow \infty]{L^2} 3 \int_0^1 \partial_{2222}f(B_s^{(1)}, B_s^{(2)}) ds \tag{4.27}$$

$$\begin{aligned}
S_n^{(5)} := \sum_{k=0}^{n-1} \partial_{12}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(1)} \Delta B_{k/n}^{(2)} & \xrightarrow[n \rightarrow \infty]{\text{stably}} \frac{\sigma_{1/4}}{\sqrt{2}} \int_0^1 \partial_{12}f(B_s^{(1)}, B_s^{(2)}) dW_s \\
& + \frac{1}{4} \int_0^1 \partial_{1122}f(B_s^{(1)}, B_s^{(2)}) ds
\end{aligned} \tag{4.28}$$

$$S_n^{(6)} := \sum_{k=0}^{n-1} \partial_{112}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) (\Delta B_{k/n}^{(1)})^2 \Delta B_{k/n}^{(2)} \xrightarrow[n \rightarrow \infty]{L^2} -\frac{1}{2} \int_0^1 \partial_{1122}f(B_s^{(1)}, B_s^{(2)}) ds \tag{4.29}$$

$$S_n^{(7)} := \sum_{k=0}^{n-1} \partial_{122}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(1)} (\Delta B_{k/n}^{(2)})^2 \xrightarrow[n \rightarrow \infty]{L^2} -\frac{1}{2} \int_0^1 \partial_{1122}f(B_s^{(1)}, B_s^{(2)}) ds \tag{4.30}$$

$$S_n^{(8)} := \sum_{k=0}^{n-1} \partial_{1122}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) (\Delta B_{k/n}^{(1)})^2 (\Delta B_{k/n}^{(2)})^2 \xrightarrow[n \rightarrow \infty]{L^2} \int_0^1 \partial_{1122}f(B_s^{(1)}, B_s^{(2)}) ds \tag{4.31}$$

$$S_n^{(9)} := \sum_{k=0}^{n-1} \partial_{1112}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) (\Delta B_{k/n}^{(1)})^3 \Delta B_{k/n}^{(2)} \xrightarrow[n \rightarrow \infty]{\text{Prob}} 0 \tag{4.32}$$

$$S_n^{(10)} := \sum_{k=0}^{n-1} \partial_{1222}f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(1)} (\Delta B_{k/n}^{(2)})^3 \xrightarrow[n \rightarrow \infty]{\text{Prob}} 0. \tag{4.33}$$

Note that the term corresponding to the remainder term in (4.23) converges in probability to zero due to the fact that  $B$  has a finite quartic variation.

*Proof of (4.24), (4.26), (4.29) and (4.30).* By Lemma 3.2 with  $q = 3$  and  $\alpha = 0$ , and by using the basic fact that

$$(\Delta B_{k/n}^{(2)})^3 = n^{-3/4} H_3(n^{1/4} \Delta B_{k/n}^{(2)}) + \frac{3}{\sqrt{n}} \Delta B_{k/n}^{(2)}, \quad (4.34)$$

we immediately see that (4.26) is a consequence of the following convergence:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(2)} \xrightarrow[n \rightarrow \infty]{L^2} -\frac{1}{2} \int_0^1 \partial_{2222} f(B_s^{(1)}, B_s^{(2)}) ds. \quad (4.35)$$

So, let us prove (4.35). We have, on one hand:

$$\begin{aligned} & E \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(2)} \right|^2 \\ &= \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \Delta B_{k/n}^{(2)} \Delta B_{l/n}^{(2)} \right) \\ &= \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) I_2^{(2)}(\delta_{k/n} \otimes \delta_{l/n}) \right) \\ &+ \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \delta_{k/n}, \delta_{l/n} \rangle \mathfrak{H} \right) \\ &= \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle \mathfrak{H} \langle \varepsilon_{k/n}, \delta_{l/n} \rangle \mathfrak{H} \right) \\ &+ \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle \mathfrak{H} \langle \varepsilon_{l/n}, \delta_{l/n} \rangle \mathfrak{H} \right) \\ &+ \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \varepsilon_{l/n}, \delta_{k/n} \rangle \mathfrak{H} \langle \varepsilon_{k/n}, \delta_{l/n} \rangle \mathfrak{H} \right) \\ &+ \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \varepsilon_{l/n}, \delta_{k/n} \rangle \mathfrak{H} \langle \varepsilon_{l/n}, \delta_{l/n} \rangle \mathfrak{H} \right) \\ &+ \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \delta_{k/n}, \delta_{l/n} \rangle \mathfrak{H} \right) \\ &= a^{(n)} + b^{(n)} + c^{(n)} + d^{(n)} + e^{(n)}. \end{aligned}$$

Using Lemma 3.1 (i) and (ii), we have that  $a^{(n)}$ ,  $c^{(n)}$  and  $d^{(n)}$  tends to zero as  $n \rightarrow \infty$ . Using Lemma 3.1 (iii), we have that  $e^{(n)}$  tends to zero as  $n \rightarrow \infty$ . Finally, observe that

$$\begin{aligned} b^{(n)} &= \frac{1}{4n^2} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right) \\ &\quad - \frac{1}{2n\sqrt{n}} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right) \left( \langle \varepsilon_{l/n}, \delta_{l/n} \rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}} \right) \\ &\quad + \frac{1}{n} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right) \left( \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}} \right) \langle \varepsilon_{l/n}, \delta_{l/n} \rangle_{\mathfrak{H}}. \end{aligned}$$

Therefore, using Lemma 3.1 (i) and (iv), we have

$$E \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(2)} \right|^2 = E \left| \frac{1}{2n} \sum_{k=0}^{n-1} \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \right|^2 + o(1). \quad (4.36)$$

On the other hand, we have

$$\begin{aligned} &E \left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(2)} \times \frac{-1}{2n} \sum_{l=0}^{n-1} \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right) \\ &= -\frac{1}{2n\sqrt{n}} \sum_{k,l=0}^{n-1} E \left( \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \Delta B_{k/n}^{(2)} \right) \\ &= -\frac{1}{2n\sqrt{n}} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \\ &\quad - \frac{1}{2n\sqrt{n}} \sum_{k,l=0}^{n-1} E \left( \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \\ &= \frac{1}{4n^2} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right) \\ &\quad - \frac{1}{2n\sqrt{n}} \sum_{k,l=0}^{n-1} E \left( \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right) \left( \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}} \right) \\ &\quad - \frac{1}{2n\sqrt{n}} \sum_{k,l=0}^{n-1} E \left( \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \end{aligned}$$

We immediately have that the second (see Lemma 3.1 (iv)) and the third (see Lemma 3.1 (ii)) terms in the previous expression tends to zero as  $n \rightarrow \infty$ . That is

$$\begin{aligned} &E \left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(2)} \times \frac{-1}{2n} \sum_{l=0}^{n-1} \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \right) \\ &= E \left| \frac{1}{2n} \sum_{k=0}^{n-1} \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \right|^2 + o(1). \quad (4.37) \end{aligned}$$



We have proved, see (4.36) and (4.37), that

$$E \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(2)} + \frac{1}{2n} \sum_{k=0}^{n-1} \partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \right| \xrightarrow{n \rightarrow \infty} 0.$$

This implies (4.35).

The proof of (4.24) follows directly from (4.26) by interverting the roles played by  $B^{(1)}$  and  $B^{(2)}$ . On the other hand, by combining Lemma 3.2 with the following basic identity:

$$(\Delta B_{k/n}^{(2)})^2 = \frac{1}{\sqrt{n}} H_2(n^{1/4} \Delta B_{k/n}^{(2)}) + \frac{1}{\sqrt{n}},$$

we see that (4.30) is also a direct consequence of (4.35). Finally, (4.29) is obtained from (4.30) by interverting the roles played by  $B^{(1)}$  and  $B^{(2)}$ .

*Proof of (4.25), (4.27) and (4.31).* By combining Lemma 3.2 with the identity

$$(\Delta B_{k/n}^{(1)})^4 = \frac{1}{n} H_4(n^{1/4} \Delta B_{k/n}^{(1)}) + \frac{6}{n} H_2(n^{1/4} \Delta B_{k/n}^{(1)}) + \frac{3}{n},$$

we see that (4.27) is easily obtained through a Riemann sum argument. We can use the same arguments in order to prove (4.25). Finally, to obtain (4.31), it suffices to combine Lemma 3.2 with the identity

$$(\Delta B_{k/n}^{(1)})^2 (\Delta B_{k/n}^{(2)})^2 = \frac{1}{n} + \frac{1}{\sqrt{n}} (\Delta B_{k/n}^{(1)})^2 H_2(n^{1/4} \Delta B_{k/n}^{(2)}) + \frac{1}{n} H_2(n^{1/4} \Delta B_{k/n}^{(1)}).$$

*Proof of (4.32) and (4.33).* We only prove (4.33), the proof of (4.32) being obtained from (4.33) by interverting the roles played by  $B^{(1)}$  and  $B^{(2)}$ . By combining (4.34) with Lemma 3.2, it suffices to prove that

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{1222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(1)} \Delta B_{k/n}^{(2)} \xrightarrow[n \rightarrow \infty]{\text{Prob}} 0.$$

But this last convergence follows directly from Lemma 3.3. Therefore, the proof of (4.33) is done.

*Proof of (4.28).* We combine Proposition 3.3 with the idea developed in the third comment that we have addressed just after the statement of Theorem 1.2. Indeed, we have

$$\begin{aligned} & \sum_{k=0}^{n-1} \partial_{12} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(1)} \Delta B_{k/n}^{(2)} \\ = & \frac{1}{2\sqrt{n}} \sum_{k=0}^{n-1} \partial_{12} f\left(\frac{\beta_{k/n} + \tilde{\beta}_{k/n}}{\sqrt{2}}, \frac{\beta_{k/n} - \tilde{\beta}_{k/n}}{\sqrt{2}}\right) \left(\sqrt{n} (\Delta \beta_{k/n})^2 - 1\right) \\ & - \frac{1}{2\sqrt{n}} \sum_{k=0}^{n-1} \partial_{12} f\left(\frac{\beta_{k/n} + \tilde{\beta}_{k/n}}{\sqrt{2}}, \frac{\beta_{k/n} - \tilde{\beta}_{k/n}}{\sqrt{2}}\right) \left(\sqrt{n} (\Delta \tilde{\beta}_{k/n})^2 - 1\right) \end{aligned}$$

for  $\beta = (B^{(1)} + B^{(2)})/\sqrt{2}$  and  $\tilde{\beta} = (B^{(1)} - B^{(2)})/\sqrt{2}$ . Note that  $(\beta, \tilde{\beta})$  is also a 2D fractional Brownian motion of Hurst index  $1/4$ . Thus, using Proposition 3.3 with  $g(x, y) = \tilde{g}(x, y) = f\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)$ , we obtain that

$$\begin{aligned} & \sum_{k=0}^{n-1} \partial_{12} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(1)} \Delta B_{k/n}^{(2)} \\ \xrightarrow[n \rightarrow \infty]{\text{stably}} & \frac{\sigma_{1/4}}{2} \int_0^1 \partial_{12} f(B_s^{(1)}, B_s^{(2)}) d(W - \tilde{W})_s + \frac{1}{4} \int_0^1 \partial_{1122} f(B_s^{(1)}, B_s^{(2)}) ds \\ \stackrel{\text{Law}}{=} & \frac{\sigma_{1/4}}{\sqrt{2}} \int_0^1 \partial_{12} f(B_s^{(1)}, B_s^{(2)}) dW_s + \frac{1}{4} \int_0^1 \partial_{1122} f(B_s^{(1)}, B_s^{(2)}) ds, \end{aligned}$$

for  $(W, \tilde{W})$  a 2D standard Brownian motion independent of  $(\beta, \tilde{\beta})$ . The proof of (4.28) is done.

**Proof of the first point** (case  $H > 1/4$ ). This proof can be done by following exactly the same strategy than in the step above. The only difference is that, using a version of Lemma 3.2 together with computations similar to that allowing to obtain (4.35), the limits in (4.24)–(4.31) are, here, *all equal to zero* (for the sake of simplicity, the technical details are left to the reader). Therefore, we deduce (1.2) by using (4.23).  $\square$

## References

- [1] P. Breuer and P. Major (1983): *Central limit theorems for nonlinear functionals of Gaussian fields*. J. Multivariate Anal. **13** (3), 425-441.
- [2] K. Burdzy and J. Swanson (2008): *A change of variable formula with Itô correction term*. Preprint (available on ArXiv).
- [3] P. Cheridito and D. Nualart (2005): *Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter  $H$  in  $(0, 1/2)$* . Ann. Inst. H. Poincaré Probab. Statist. **41**, 1049-1081.
- [4] L. Coutin and Z. Qian (2002): *Stochastic rough path analysis and fractional Brownian motion*. Probab. Theory Relat. Fields **122**, 108-140.
- [5] D. Dürr, S. Goldstein and J.L. Lebowitz (1985): *Asymptotics of particle trajectories in infinite one-dimensional systems with collision*. Comm. Pure Appl. Math. **38**, 573-597.
- [6] M. Errami and F. Russo (2003):  *$n$ -covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes*. Stoch. Proc. Appl. **104**, 259-299.
- [7] D. Feyel and A. de La Pradelle (2006): *Curvilinear integrals along enriched paths*. Electron. J. Probab. **11**, 860-892.
- [8] P. Friz and N. Victoir (2009): *Differential equations driven by Gaussian signals I*. Ann. Inst. H. Poincaré Probab. Statist., to appear (available on ArXiv).

- [9] M. Gradinaru, I. Nourdin, F. Russo and P. Vallois (2005): *m-order integrals and Itô's formula for non-semimartingale processes; the case of a fractional Brownian motion with any Hurst index*. Ann. Inst. H. Poincaré Probab. Statist. **41**, 781-806.
- [10] T.E. Harris (1965): *Diffusions with collisions between particles*. J. Appl. Probab. **2**, 323-338.
- [11] T. Lyons (1998): *Differential equations driven by rough signals*. Rev. Mat. Iberoamericana **14** (2), 215-310.
- [12] P. Malliavin (1997): *Stochastic analysis*. Springer-Verlag, Berlin, Heidelberg, New-York.
- [13] I. Nourdin and D. Nualart (2008): *Central limit theorems for multiple Skorohod integrals*. Preprint (available on ArXiv).
- [14] I. Nourdin and A. Réveillac (2008): *Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case  $H = 1/4$* . Preprint (available on ArXiv).
- [15] D. Nualart (2006): *The Malliavin calculus and related topics of Probability and Its Applications*. Springer Verlag, Berlin, Second edition, 2006.
- [16] G. Peccati and C.A. Tudor (2005): *Gaussian limits for vector-valued multiple stochastic integrals*. Sm. Probab. **38**, 247-262, Lecture Notes in Math., 1857, Springer, Berlin.
- [17] J. Unterberger (2008): *Stochastic calculus for fractional Brownian motion with Hurst exponent  $H > 1/4$ : a rough path method by analytic extension*. Ann. Probab., to appear.