On the distance between separatrices for the discretized pendulum equation

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abstract

We consider the discretization

$$q(t+\varepsilon)+q(t-\varepsilon)-2q(t)=\varepsilon^2\sin\big(q(t)\big),$$

$\varepsilon>0$ a small parameter, of the pendulum equation $q''=\sin(q)$; in system form, we have the discretization

$$q(t+\varepsilon)-q(t)=\varepsilon p(t+\varepsilon),\ p(t+\varepsilon)-p(t)=\varepsilon\sin\big(q(t)\big).$$

of the system

$$q'=p,\ p'=\sin(q).$$

The latter system of ordinary differential equations has two saddle points at $A=(0,0)$, $B=(2\pi,0)$ and near both, there exist stable and unstable manifolds. It also admits a heteroclinic orbit connecting the stationary points B and A parametrised by $q_0(t)=4\arctan\big(e^{-t}\big)$ and which contains the stable manifold of this system at A as well as its unstable manifold at B. We prove that the stable manifold of the point A and the unstable manifold of the point B do not coincide for the discretization. More precisely, we show that the vertical distance between these two manifolds is exponentially small but not zero and in particular we give an asymptotic estimate of this distance. For this purpose we use a method adapted from the article of Schäfke-Volkmer [10] using formal series and accurate estimates of the coefficients. Our result is similar to that of Lazutkin et. al. [9]; our method of proof, however, is quite different.

Keywords: Difference equation; Manifolds; Linear operator; Formal solution; Gevrey asymptotic; Quasi-solution

1 Introduction

We consider the following difference equation

$$q(t+\varepsilon)+q(t-\varepsilon)-2q(t)=\varepsilon^2\sin\big(q(t)\big). \tag{1.1}$$

This second order equation is a discretization of the pendulum equation $q'' = \sin(q)$. It is equivalent to the following system of first order difference equations

$$\begin{cases} q(t+\varepsilon) = q(t) + \varepsilon p(t+\varepsilon), \\ p(t+\varepsilon) = p(t) + \varepsilon \sin\big(q(t)\big). \end{cases} \tag{1.2}$$

which can be considered as a discretization of the system

$$\begin{cases} q' = p, \\ p' = \sin(q). \end{cases} \tag{1.3}$$

The latter system has two saddle points at $A = (0,0)$, $B = (2\pi, 0)$ and there exist stable and unstable manifolds. For the discretized equation (1.2) and sufficiently small $\varepsilon > 0$, these manifolds still exist.

The system (1.3) has $\big(q_0(t), q_0'(t)\big)$, where $q_0(t) = 4\arctan\big(e^{-t}\big)$, as a heteroclinic orbit connecting the stationary points B and A; it is a parametrisation of the curve $p = -2\sin(q/2)$ and contains the stable manifold of (1.3) at the point A as well as its unstable manifold at B. This curve, together with $p = 2\sin(q/2)$, separates regions with periodic orbits from regions with non-periodic orbits and is therefore often called a *separatrix*. Our purpose is study the behavior of this separatrix under discretization of the equation – it turns out that there is no longer a heteroclinic orbit for system (1.2) and its the stable manifold at A and the unstable manifold at B no longer coincide. More precisely, we want to estimate the distance between the stable manifold $W^-_{s,\varepsilon}$ of (1.2) at A and the unstable manifold $W^+_{u,\varepsilon}$ of (1.2) at B as a function of the parameter ε.

Lazutkin et. al. [9], Gelfreich [4], (see also Lazutkin [7][8]) had given an asymptotic estimate of the splitting angle between the manifolds. Starting from a heteroclinic solution of the differential equation, they study the behavior of analytic solutions of the difference equation in the neighbourhood of its singularities $t = \pm\frac{\pi}{2}i$.

We show that the distance between these two manifolds is exponentially small but not zero and we give an asymptotic estimate of this distance. This result is similar to that of Lazutkin et. al. [9]; our method of proof, however, is quite different.

We use a method adapted from the article of Schäfke-Volkmer [10] using a formal power series solution and accurate estimates of the coefficients. This method was adapted for the logistic equation in Sellama[11]. It turns out that the adaptation of this method for the pendulum equation is more difficult than in the case of the logistic equation.

We will show

Theorem 1.1. *Given any positive t_0, it is known that for sufficiently smal $\varepsilon_0 > 0$ and all $t \in]-\infty, t_0]$ there is exactly one one point $w^+_{u,\varepsilon}(t) = (q_0(t), \tilde{p}^+_{u,\varepsilon}(t))$ on the stable unstable manifold having first coordinate $q_0(t)$. There exist constants $\alpha \neq 0$, such that for any positive t_0*

$$\operatorname{dist}_v\big(w^+_{s,\varepsilon}(t), W^-_{s,\varepsilon}\big) = \frac{4\pi\alpha}{\varepsilon^2}\cosh(t)\sin\Big(\frac{2\pi t}{\varepsilon}\Big)e^{-\frac{\pi^2}{\varepsilon}} + O\Big(\frac{1}{\varepsilon}e^{-\frac{\pi^2}{\varepsilon}}\Big), \quad \text{as } \varepsilon \searrow 0,$$

uniformly for $-t_0 < t < t_0$, *where* $\mathrm{dist}_v\big(P, W^-_{u,\varepsilon}\big)$ *denotes the vertical distance of a point* P *from the unstable manifolds* $W^-_{u,\varepsilon}$.

This result corresponds to the result of Lazutkin et. al. [9] as the angle between the manifolds at an intersection point is asymptotically equivalent to $\frac{1}{q_0'(t)}\frac{d}{dt}\,\mathrm{dist}_v\big(w^+_{s,\varepsilon}(t), W^-_{u,\varepsilon}\big)$, but we do not want to give any detail here.

Our proof uses the following steps. First, we construct a formal solution for the difference equation (1.1) in the form of a power series in $d = 2\mathrm{arsinh}(\varepsilon/2)$, whose coefficients are polynomials in $u = \tanh(dt/\varepsilon)$. This is done in section 2; the introduction of d is necessary because polynomials are desired as coefficients. Then, we give asymptotic approximations of these coefficients using appropriate norms on spaces of polynomials. To that purpose we introduce operators on polynomials series. In section 6 we use the truncated Laplace transform to construct a function which satisfies (1.1) except for an exponentially small error. The next and last step is to give an asymptotic estimate for the distance of some point of the stable manifold from the unstable manifold. A calculation shows that $\alpha = 89.0334$ and therefore $4\pi\alpha = 1118.8267$ (See Remark 5.4); the corresponding constants of Lazutkin have already been calculated with high precision (See Lazutkin et. al. [9]). A *proof* that $\alpha \neq 0$ as in [10] or [11] would be possible. Y.B. Suris [12] had shown that $\alpha \neq 0$.

2 Formal solutions

The purpose of this section is to find a convenient formal solution for equation (1.1). First, we need some preparations. We put

$$\begin{aligned} u: &= \tanh\left(\frac{d}{\varepsilon}t\right), \\ q_{0d}(t): &= 4\arctan\left(\exp\left(-\frac{d}{\varepsilon}t\right)\right), \\ q_d(t) &= \sqrt{1-u^2}A_d(u) + q_{0d}(t), \quad A_d(u) = \sum_{n=1}^{\infty} A_n(u)d^n \end{aligned}$$

for a formal solution of (1.1), where $d = \varepsilon + \sum_{n=3}^{\infty} d_n\varepsilon^n$ is a formal powers series in ε to be determined.

Remark. The linearization of equation (1.1) at the point A gives the following equation

$$Z(t+\varepsilon) + Z(t-\varepsilon) - 2Z(t) = \varepsilon^2 Z(t).$$

The parameter d is such that $Z(t) = e^{-dt}$ is a solution of this equation, therefore ε and d are coupled by the relation $d = 2\mathrm{arcsinh}(\varepsilon/2)$.

By Taylor expansion, we obtain

$$q_{0d}(t+\varepsilon)+q_{0d}(t-\varepsilon)-2q_{0d}(t)=2\sum_{n=1}^{+\infty}\frac{1}{(2n)\,!}q_{0d}^{(2n)}(t)\,\varepsilon^{2n}, \tag{2.1}$$

where $\dfrac{2}{(2n)!}q_{0d}^{(2n)}(t)\varepsilon^{2n}/d^{2n}$ is an odd polynomial $I_{2n-1}(u)$ multiplied by $\sqrt{1-u^2}$; we find $I_{2n-1}(1)=4/(2n)!$.

Using $\cos(q_{0d})=2u^2-1,\ \sin(q_{0d})=2u\sqrt{1-u^2}$, we can express our equation (1.1) in the form

$$A_d(T^+)\sqrt{\frac{1-(T^+)^2}{1-u^2}}+A_d(T^-)\sqrt{\frac{1-(T^-)^2}{1-u^2}}-2A_d(u)=f\big(\varepsilon,u,A_d(u)\big) \tag{2.2}$$

or equivalently

$$\frac{A_d(T^+)}{\cosh(d)+u\sinh(d)}+\frac{A_d(T^-)}{\cosh(d)-u\sinh(d)}-2A_d(u)=f\big(\varepsilon,u,A_d(u)\big) \tag{2.3}$$

where

$$\begin{aligned}
f\big(\varepsilon,u,A_d(u)\big) &= \varepsilon^2\left(2u\cos\left(A_d(u)\sqrt{1-u^2}\right)+\frac{2u^2-1}{\sqrt{1-u^2}}\sin\left(A_d(u)\sqrt{1-u^2}\right)\right)\\
&- \sum_{n=1}^{+\infty}I_{2n-1}(u)\,d^{2n},\\
T^+ &= T^+(d,u)=\frac{u+\tanh(d)}{1+u\tanh(d)}=\tanh\Big(\frac{d}{\varepsilon}(t+\varepsilon)\Big),\\
T^- &= T^-(d,u)=\frac{u-\tanh(d)}{1-u\tanh(d)}=\tanh\Big(\frac{d}{\varepsilon}(t-\varepsilon)\Big).
\end{aligned}$$

As $u\to 1$, the expressions T^+ and T^- reduce to 1, the denominators in (2.3) simplify to $e^{\pm d}$ and hence equation (2.3) reduces to

$$(e^{-d}+e^{d}-2)A_d(1)=\varepsilon^2(2+A_d(1))-4(\cosh(d)-1).$$

This is equivalent to $(2\cosh(d)-2-\varepsilon^2)(2+A_d(1))=0$ and hence we have necessarily $\varepsilon=2\sinh(d/2)$ if we want a formal solution such that the coefficients have limits as $u\to 1$.

Theorem 2.1. ***(On the formal solution)*** *If $\varepsilon=2\sinh(d/2)$, then equation (2.2) has a unique formal solution of the form*

$$A_d(u)=\sum_{n=1}^{+\infty}A_{2n-1}(u)d^{2n}, \tag{2.4}$$

where $A_{2n-1}(u)$ are odd polynomials of degree $\le 2n-1$.

Remark: A similar formal solution was found using another method in [12].
Proof. We will use the Induction Principle to show that there exist unique odd polynomials $A_1, A_3, A_5 ... A_{2n-1}$ such that

$$Z_n(d,u) = \sum_{k=1}^{n} A_{2k-1}(u) d^{2k} \tag{2.5}$$

satisfy

$$R_n(d,u) = O(d^{2n+4}) \tag{2.6}$$

where

$$\begin{aligned} R_n(d,u) &= Z_{n,d}\big(T^+\big)\sqrt{\frac{1-(T^+)^2}{1-u^2}} + Z_{n,d}\big(T^-\big)\sqrt{\frac{1-(T^-)^2}{1-u^2}} \\ &- 2Z_{n,d}(u) - f\big(\varepsilon, u, Z_{n,d}(u)\big) \end{aligned} \tag{2.7}$$

For $n = 1$, a short calculation shows that we must have $A_1(u) = -\frac{1}{4}u$ and hence $Z_{1,d}(u) = -\frac{1}{4}ud^2$. We obtain

$$R_1(d,u) = (\frac{-91}{48}u^5 + \frac{137}{48}u^3 - \frac{23}{24})d^6 + O(d^8).$$

Suppose now that there exists $A_1, A_3, A_5 ... A_{2n-1}$ such that

$$Z_n(d,u) = \sum_{k=1}^{n} A_{2k-1}(u) d^{2k} \tag{2.8}$$

satisfies (2.6), (2.7). We show that there is a unique polynomial $A_{2n+1}(u)$ such that

$$Z_{n+1}(d,u) = Z_n(d,u) + A_{2n+1}(u) d^{2n+2} \tag{2.9}$$

satisfies (2.6). We put

$$R_n(d,u) = R_{2n+3}(u) d^{2n+4} + O\big(d^{2n+6}\big) \tag{2.10}$$

where $R_{2n+3}(u)$ is odd and $\deg(R_{2n+3}(u)) \leq 2n+3$.

We substitute $Z_{n+1}(d,u)$ in equation (2.7). Using Taylor expansion, (2.9), (2.10) and $\varepsilon = 2\sinh(d/2)$, we obtain

$$\begin{aligned} &Z_{n+1,d}\big(T^+\big)\sqrt{\frac{1-(T^+)^2}{1-u^2}} - Z_{n+1,d}\big(T^-\big)\sqrt{\frac{1-(T^-)^2}{1-u^2}} - 2Z_{n+1,d}(u) - \\ &f\big(\varepsilon, u, Z_{n+1,d}\big) = \Big[(u^4 - 2u^2 + 1)A''_{2n+1}(u) + (4u^3 - 4u)A'_{2n+1}(u) + \\ &R_{2n+3}(u)\Big] d^{2n+4} + O\big(d^{2n+6}\big) \end{aligned}$$

We notice that (2.10) is satisfied if only if

$$\big[(1-u^2)^2 A'_{2n+1}(u)\big]' + R_{2n+3}(u) = 0 \tag{2.11}$$

This differential equation has a unique solution vanishing at $u = 0$ without singularity at $u = 1$, namely

$$A_{2n+1}(u) = -\int_0^u \frac{\int_1^t R_{2n+3}(s)ds}{(1-t^2)^2} dt. \tag{2.12}$$

We now show that this solution is an odd polynomial of u. It is clear that $\int_1^t R_{2n+3}(s)ds$ vanishes for $t = 1$ and as $R_{2n+3}(s)$ is odd, it also vanishes for $t = -1$. It suffices to show that $R_{2n+3}(s)$ also vanishes at $t = \pm 1$. Indeed, taking the limit of (2.7) as $u \to 1$ as we did for (2.3) and using

$$\lim_{u\to 1} f\big(\varepsilon, u, Z(d,u)\big) \quad = \quad \varepsilon^2 Z(d,1)$$

we obtain

$$R_{2n+3}(1)d^{2n+4} = \left(e^d + e^{-d} - 2 - \varepsilon^2\right) Z(d,1) + O(d^{2n+6}).$$

By our choice of $\varepsilon = 2\sinh(d/2)$, we obtain $R_{2n+3}(1)d^{2n+4} = O(d^{2n+6})$. Consequently $R_{2n+3}(1) = 0$. As $R_{2n+3}(u)$ is odd, we also have $R_{2n+3}(-1) = -R_{2n+3}(1) = 0$. This proves that $A_{2n+1}(u)$ is an odd polynomial of degree$\big(A_{2n+1}(u)\big) \le 2n+1$ and $A_{2n+1}(0) = 0$.

The first polynomials $A_{2n-1}(u)$ with $n > 0$ can be calculated using Maple.

n	1	2	3
$A_{2n-1}(u)$	$-\frac{1}{4}u$	$\left(\frac{91}{864}u^3 - \frac{47}{576}u\right)$	$\left(-\frac{319}{2880}u^5 + \frac{185}{1152}u^3 - \frac{3703}{69120}u\right)$

Now, we ntroduce the operators $\mathcal{C}_2, \mathcal{C}, \mathcal{S}_2, \mathcal{S}$ defined by

$$\begin{aligned}
\mathcal{C}(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+\frac{1}{2}}) + Z(d,T^{-\frac{1}{2}})\big) \\
\mathcal{S}(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+\frac{1}{2}}) - Z(d,T^{-\frac{1}{2}})\big) \\
\mathcal{C}_1(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+}) + Z(d,T^{-})\big) \\
\mathcal{S}_1(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+}) - Z(d,T^{-})\big)
\end{aligned} \tag{2.13}$$

where $T^{+\frac{1}{2}} = T^+(\frac{d}{2}, u)$, $T^{-\frac{1}{2}} = T^-(\frac{d}{2}, u)$ and $Z(d,u)$ is a formal power series in d whose coefficients are polynomials. We can show that

$$\begin{aligned}
\mathcal{C}_1 &= 2\mathcal{S}^2 + Id \\
\mathcal{S}_1 &= 2\mathcal{S}\mathcal{C}
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
\mathcal{C}_1(Q \cdot G) &= \mathcal{C}_1(Q)\mathcal{C}_1(G) + \mathcal{S}_1(Q)\mathcal{S}_1(G) \\
\mathcal{S}_1(Q \cdot G) &= \mathcal{C}_1(Q)\mathcal{S}_1(G) + \mathcal{S}_1(Q)\mathcal{C}_1(G)) \\
\mathcal{C}(Q \cdot G) &= \mathcal{C}(Q)\mathcal{C}(G) + \mathcal{S}(Q)\mathcal{S}(G) \\
\mathcal{S}(Q \cdot G) &= \mathcal{C}(Q)\mathcal{S}(G) + \mathcal{S}(Q)\mathcal{C}(G)
\end{aligned} \tag{2.15}$$

if Q, G are formal power series in d whose coefficients are polynomials of u.

3 Norms for polynomials and basis

In this section we recall some definitions and results of [10]. Using a certain suquence of polynomials. we define convenient norms on spaces of polynomials which satisfies some useful proprieties. We denote by

- $\mathcal{P}$ the set of all polynomial whose coefficents are complex
- $\mathcal{P}_n$ the spaces of all polynomials of degree less than or equal to n

Proposition 3.1. *[10]. We define the sequence of polynomials $\tau_n(u)$ by*

$$\tau_0(u) = 1,\ \tau_1(u) = u,\ \tau_{n+1}(u) = \frac{1}{n} D\tau_n(u) \text{ for } n \geq 1,$$

where the operator D is defined by

$$D := (1 - u^2)\frac{\partial}{\partial u}.$$

Then we have

1. $T^+(d, u) = \sum_{n=0}^{\infty} \tau_{n+1}(u) d^n$,
2. $\tau_n(u)$ *has exactly degree* n *and hence* $\tau_0(u), ..., \tau_n(u)$ *form a basis of* $\mathcal{P}_n$,
3. $\tau_n(\tanh(z)) = \frac{1}{(n-1)!}\big(\frac{d}{dz}\big)^{n-1}\big(\tanh(z)\big)$

Definition 3.2. *Let $p \in \mathcal{P}_n$. As $\tau_0(u), ..., \tau_n(u)$ form a basis of $\mathcal{P}_n$, we can write $p \in \mathcal{P}_n$ as*

$$p = \sum_{k=0}^{n} a_k \tau_k(u).$$

Then we define the norms

$$\|p\|_n = \sum_{i=0}^{n} |a_i| \left(\frac{\pi}{2}\right)^{n-i}. \tag{3.1}$$

Theorem 3.3. *[10]. Let n,m be positive integers and $p \in \mathcal{P}_n$, $q \in \mathcal{P}_m$. The norms (3.1) have the following properties:*

1. $\|Dp\|_{n+1} \leq n\|p\|_n$.
2. *If the constant term of p in the basis $\{\tau_0, \tau_1.., \tau_n\}$ is zero, we have*
$$\|p\|_n \leq \|Dp\|_{n+1}.$$
3. *There exists a constant M_2 such that $\|pq\|_{n+m} \leq M_2\|p\|_n\|q\|_m$.*
4. *There is a constant M_3 such that that for all $n > 1$, $|p(u)| \leq M_3 \left(\frac{2}{\pi}\right)^n \|p\|_n$ $(-1 \leq u \leq 1)$.*
5. *There is a constant M_4 such that for all $n > 1$ with $p(1) = p(-1) = 0$*
$$\left\| \frac{p}{\tau_2} \right\|_{n-2} \leq M_4\|p\|_n$$

4 Operators

In this section we will use some definitions of Schäfke-Volkmer[10] and adapt their results on operators on polynomial series to our context. Let

$$\mathcal{Q} := \left\{ Q(d,u) = \sum_{n=0}^{\infty} Q_n(u)d^n, \text{ where } Q_n(u) \in \mathcal{P}_n, \text{ for all } n \in \mathbb{N} \right\}.$$

By abuse of notation, let $\|Q\|_n = \|Q_n\|_n$ for a polynomial series

$$Q(d,u) = \sum_{n=0}^{\infty} Q_n(u)d^n.$$

Definition 4.1. *Let f be formal power series of z whose coefficients are complex. We define a linear operator $f(dD)$ on $\mathcal{Q}$ by*

$$f(dD)Q(d,u) = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} f_i D^i Q_{n-i}(u) \Big) d^n \tag{4.1}$$

where $f(z) = \sum_{i=0}^{\infty} f_i z^i$ and $Q \in \mathcal{Q}$.

By the above Definition and (1) of Proposition 3.1 we can show that

$$Q(d, T^+\big(\theta d, u)\big) = \big(\exp(\theta dD)Q\big)(d,u) \text{ for } \ Q \in \mathcal{Q} \text{ and all } \theta \in \mathbb{C}$$

Thus with (2.14) and (1) of Proposition 3.1 we obtain

$$\begin{array}{ll} \mathcal{C}(Q) = \cosh(\frac{d}{2}D)Q, & \mathcal{S}(Q) = \sinh(\frac{d}{2}D)Q \\ \mathcal{C}_1(Q) = \cosh(dD)Q, & \mathcal{S}_1(Q) = \sinh(dD)Q \end{array} \tag{4.2}$$

for polynomial series Q in $\mathcal{Q}$.

Remark. According to the definition of norms in (3.1), we have

$$\text{If} \quad Q \in \mathcal{Q}, \quad \text{then} \quad dQ \in \mathcal{Q} \quad \text{and} \quad \|dQ\|_n = \frac{\pi}{2}\|Q\|_{n-1} \quad \text{for all} \quad n \geq 1. \tag{4.3}$$

Theorem 4.2. *[10] Let $f(z)$ be formal power series having a radius of convergence greater than 2π and let k be a positve integer. There is a constant K such that: If Q is a polynomial series having the following property*

$$\|Q\|_n \leq \begin{cases} 0 & \text{for } n < k \\ M(n-k)\,!(2\pi)^{-n} & \text{for } n \geq k \end{cases}$$

where M is independent of n and $Q \in \mathcal{Q}$ then the polynomial series $f(dD)Q$ satisfies

$$\|f(dD)Q\|_n \leq \begin{cases} 0 & \text{for } n < k \\ MK(n-k)\,!(2\pi)^{-n} & \text{for } n \geq k \end{cases}$$

Now we define on $\mathcal{Q}$ the following operator

$$\mathcal{J} = \frac{\mathcal{S}}{dD}. \tag{4.4}$$

where the notation $\frac{\mathcal{S}}{dD}$ means simply $F(dD)$ with $F(z) = \dfrac{1}{z}\sinh(\frac{z}{2})$.

Lemma 4.3. *For each integer k there exist a positive constant K such that: If Q is a polynomial series with odd Q_n of degree at most n, $\|Q\|_n = 0$ for $n < k$ in case of positive k and*

$$\|dDQ\|_n \leq M(n-k)\,!(2\pi)^{-n} \quad \text{for } n \geq \max(0,k),$$

where M is independent of n, then the polynomial series $\mathcal{J}^{-1}(Q)$ satisfies

$$\|\mathcal{J}^{-1}(Q)\|_n \leq MK(2\pi)^{-n} \begin{cases} (n-k+1)\,! & \text{for } k \leq 1 \\ (n-1)\,!\log(n) & \text{for } k = 2 \\ (n-1)\,! & \text{for } k \geq 3 \end{cases}$$

Proof. We can see easily that $\mathcal{J}^{-1} = \pi\,\tilde{\mathcal{C}}^{-1} + g(dD)$, where $\tilde{\mathcal{C}} = \cosh(\frac{1}{4}dD)$ and $g(z)$ is analytic for $|z| < 4\pi$, and use the proof of [10].

We have $\mathcal{S} = dD\,\mathcal{J} = \mathcal{J}\,dD$, but using this relation for the inversion of $\mathcal{S}$ would give an insufficient result. Using of the formula

$$1 = \frac{2}{z}\sinh(\frac{z}{2}) + F(z)z, \text{ where } F(z) = z^{-2}(z - 2\sinh(\frac{z}{2}))$$

is an entire function, we obtain the relation

$$Q = 2\mathcal{J}Q + F(dD)\,dDQ \tag{4.5}$$

for polynomial series $Q \in \mathcal{Q}$. This will be essential in the proof of

Theorem 4.4. *For each integer k there exist a positive constant K such that: If Q is a polynomial series with odd Q_n of degree at most n, $\|Q\|_n = 0$ for $n < k$ in case of positive k, and*

$$\|\mathcal{S}(Q)\|_n \le M(n-k)\,!(2\pi)^{-n} \quad \textit{for } n \ge \max(0,k),$$

where M is independent of n, then the polynomial series Q satisfies

$$\|Q\|_n \le MK(2\pi)^{-n} \begin{cases} (n-k+1)\,! & \textit{for } k \le 1 \\ (n-1)\,!\log(n) & \textit{for } k = 2 \\ (n-1)\,! & \textit{for } k \ge 3 \end{cases}$$

Proof. By the preceding theorem, we have the wanted inequalities for $dDQ = \mathcal{J}^{-1}\mathcal{S}Q$ in the place of Q. Here we used again $\|dDZ\|_n \le (n-1)\|Z\|_{n-1}$ for any polynomial series $Z \in \mathcal{Q}$. Using theorem 4.2 implies the same for $F(dD)dDQ$ with the entire function F of (4.5) As $\|Z\|_n \le \|dDZ\|_{n+1}$ by theorem 3.3, we find the wanted inequalities (and even something better in the cases $k \ge 2$) also for $\mathcal{J}Q$ because $dD\mathcal{J} = \mathcal{S}$. Thus formula (4.5) yields the result □

In order to obtain an asymptotic approximation for the coefficients of the formal solution, we will need to reverse some operators. This is not possible for the operators $\mathcal{S}$ and dD on the set $\mathcal{Q}$, but we can define a subset $\mathcal{Q}^*$ of $\mathcal{Q}$ on which these operators have a right inverses.

If we define

$$\mathcal{Q}^* := \left\{ Q(d,u) = \sum_{n=1}^{\infty} P_n(u)d^n, \text{ where } P_n(u) \in \mathcal{P}_n^*, \text{ for all } n \ge 1 \right\}.$$

where $\mathcal{P}_n^*$ is the subspace of $\mathcal{P}_n$ defined by

$$\mathcal{P}_n^* := \{\, \sum_{i=0}^{n} \alpha_i\tau_i \in \mathcal{P}_n, |\ \alpha_0 = 0 \,\}$$

Then, the restrictions of the operators $dD, \mathcal{S}$ to $\mathcal{Q}^*$, denoted here by the same symbols

$$\begin{aligned} dD &: \mathcal{Q}^* \to (1-u^2)d^2\mathcal{Q} \\ \mathcal{S} &: \mathcal{Q}^* \to (1-u^2)d^2\mathcal{Q} \end{aligned}$$

are bijective. We denote by $\mathcal{T}$ the inverse of the restriction of $\mathcal{S}$ to $\mathcal{Q}^*$, , and we have

$$\mathcal{T}\mathcal{S} = \mathrm{Id} \quad \text{on} \quad \mathcal{Q}^*$$

Theorem 4.5. *[10] We consider a polynomial series*

$$Q_\alpha(d,u) = \sum_{n=2}^{\infty} \alpha_n (n-1)\,! \Big(\frac{i}{2\pi}\Big)^{n-1} \tau_n(u) d^n$$

where $\alpha_n = \mathcal{O}(n^{-k})$ as $n \to \infty$ with some integer $k \geq 2$. Let

$$\alpha := \frac{4}{\pi} \sum_{n=1}^{\infty} \alpha_n,$$

then the coefficients $\{\mathcal{T}(Q_\alpha)\}_n$ of $\mathcal{T}(Q_\alpha)$ satisfy

$$\Big\| \{\mathcal{T}(Q_\alpha)\}_n - \alpha(n-1)\,! \Big(\frac{i}{2\pi}\Big)^{n-1} \tau_n \Big\|_n = \mathcal{O}\Big((n-k)\,!(2\pi)^{-n}\Big)$$

as $n \to \infty$ for n.

Proof. The proof of this theorem is completely analogous to that of [10].

Theorem 4.6. *[10] Let k, l, p, q be integer with $p \geq k$ and $q \geq l$. Define m as the minimum of $k+q$ and $l+p$. then there is a constant K with the following property:*

If P and Q are polynomial series such that $\|P\|_n = 0$ for $n < p$, $\|Q\|_n = 0$ for $n < q$ and

$$\begin{aligned} \|P\|_n &\leq M_1 (n-k)\,!\,(2\pi)^{-n} \quad \text{for } n \geq p \\ \|Q\|_n &\leq M_2 (n-l)\,!\,(2\pi)^{-n} \quad \text{for } n \geq q \end{aligned}$$

then

$$\|PQ\|_n \leq K M_1 M_2 (n-m)\,!\,(2\pi)^{-n} \quad \text{for } n \geq p+q.$$

Remark 4.7. Observe that the results of this section can also be applied, if the constants M are replaced by any increasing sequence $(M_n)_{n\in N}$. In theorems 4.2 and 4.6 the first n terms of the resulting polynomial series only depend of the first n terms of the given series, so the "M" in the result simply has to be replaced by "M_n". In lemma 4.3 and theorem 4.4, the first n terms of the result depend of the first $n+1$ given terms, so "M" in the result has to be replaced by "M_{n+1}".

5 Asymptotic approximation of the coefficients of the formal solution

In this section we will estimate the coefficients of the formal solution obtained previously (section 2). The idea is to write equation (2.2) essentially in the form

$$V(d,u)\mathcal{S}(Q_1\mathcal{S}(Q_2\,A))(d,u) = g\big(d,u,A(d,u)\big), \tag{5.1}$$

where V, Q_1 and Q_2 are known polynomials of d and u and g is a certain function of d, u and A involving the operators $\mathcal{S}, \mathcal{C}$ and $\mathcal{J}$ multiplied by sufficiently high powers of d.

Thanks to this equation, we will estimate the coefficients of the formal solution using the results of the previous section. We show that the coefficients of this formal solution is Gevrey-1, more precisely $\|A\|_n = O\big(n\,!\,(2\pi)^{-n}\big)$.

5.1 Rewriting of equation (2.2)

Consider the decomposition

$$A(d,u) = U(d,u) + F(d,u) \tag{5.2}$$

where U is the initial part of A calculated before

$$U(d,u) = -\frac{1}{4}ud^2 + \left(\frac{91}{864}u^3 - \frac{47}{576}u\right)d^4 + \left(-\frac{319}{2880}u^5 + \frac{185}{1152}u^3 - \frac{3703}{69120}u\right)d^6.$$

We insert this into (2.3), with (2.14) and (2.15), and obtain

$$2\cosh(d)\cdot\mathcal{C}_1(F) - 2u\,\sinh(d)\cdot\mathcal{S}_1(F) = W_0\cdot F + f_1\Big(d,u,F(d,u)\Big) \tag{5.3}$$

where

$$\begin{aligned}
W_0 &= \Big(\cosh^2(d) - u^2\sinh^2(d)\Big)\Big[2 + (2u^2-1)\varepsilon^2\cos\Big(U\cdot\sqrt{1-u^2}\Big)\\
&- 2u\,\varepsilon^2\sin\Big(U\cdot\sqrt{1-u^2}\Big)\cdot\sqrt{1-u^2}\Big]\\
f_1\Big(d,u,F(d,u)\Big) &= \varepsilon^2\Big(\cosh^2(d) - u^2\sinh^2(d)\Big)\Big[\Big(\frac{2u^2-1}{\sqrt{1-u^2}}\sin\Big(U\cdot\sqrt{1-u^2}\Big)\\
&+ 2u\cos\Big(U\cdot\sqrt{1-u^2}\Big)\Big)\cos\Big(F(d,u)\sqrt{1-u^2}\Big) +\\
&\Big(\frac{2u^2-1}{\sqrt{1-u^2}}\cos\Big(U\cdot\sqrt{1-u^2}\Big) - 2u\sin\Big(U\cdot\sqrt{1-u^2}\Big)\Big)\times\\
&\Big(\sin\Big(F(d,u)\sqrt{1-u^2}\Big) - F(d,u)\sqrt{1-u^2}\Big)\Big]\\
&- \big(\cosh^2(d) - u^2\sinh^2(d)\big)\sum_{n=1}^{+\infty} I_{2n-1}(u)d^{2n} - 2\cosh(d)\mathcal{C}_1(U)\\
&- 2u\,\sinh(d)\mathcal{S}_1(U) - 2\Big(\cosh^2(d) - u^2\sinh^2(d)\Big)\cdot U.
\end{aligned}$$

Observ that f_1 has the form

$$\begin{aligned}
f_1\big(d,u,F(d,u)\big) &= y_0(d,u) + y_1(d,u)\sum_{n=1}^{\infty}\frac{1}{(2n)!}(1-u^2)^n F(d,u)^{2n} +\\
&\qquad y_2(d,u)\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}(1-u^2)^n F(d,u)^{2n+1},
\end{aligned}\tag{5.4}$$

where $y_n(d,u), n = 1,2,3$ are convergent polynomial series.

Now, we let

$$\begin{aligned}
F(d,u) &= Q(d,u)\cdot G(d,u),\\
J(d,u) &= Q_1(d,u)\cdot\mathcal{S}(G),
\end{aligned}\tag{5.5}$$

where G is a formal power series whose the first term contains d^8 and

$$\begin{aligned}
Q(d,u) &= 1 + \tfrac{1}{4}(1-u^2)d^2 + \Big(\tfrac{91}{432}u^4 - \tfrac{13}{48}u^2 + \tfrac{13}{216}\Big)d^4\\
&+ \Big(-\tfrac{319}{960}u^6 + \tfrac{1079}{1728}u^4 - \tfrac{937}{2880}u^2 + \tfrac{287}{8640}\Big)d^6,\\
Q_1(d,u) &= (u^2-1)d^2 + \tfrac{1}{4}(1-u^4)d^4 - \frac{5}{48}(1-u^2)\Big(\tfrac{4}{9}u^4 + u^2 + 1\Big)d^6\\
&+ (1-u^2)\Big(-\tfrac{367}{2160}u^6 + \tfrac{185}{432}u^4 - \frac{997}{4320}u^2\Big)d^8.
\end{aligned}\tag{5.6}$$

The choice of $Q_1(d,u)$ and $Q(d,u)$ depends in a precise way of the form of the equation (5.1) and has been determined using Maple.

Using (5.5), (5.6) and (2.15), we can rewrite equation (5.3) in the form

$$\begin{aligned} W_0Q + f_1\Big(d,u,F_d(u)\Big) &= \Big[2\cosh(d)\mathcal{C}_1(Q) - 2u\ \sinh(d)\mathcal{S}_1(Q)\Big]\mathcal{C}_1(G) \\ &+ \Big[2\cosh(d)\mathcal{S}_1(Q) - 2u\ \sinh(d)\mathcal{C}_1(Q)\Big]\mathcal{S}_1(G). \end{aligned}$$

Using (2.14), we obtain

$$V\cdot\mathcal{S}^2(G) + W\cdot\mathcal{SC}(G) = W_1G + f_2\Big(d,u,F_d(u)\Big) \tag{5.7}$$

where $f_2\Big(d,u,F_d(u)\Big) = \frac{1}{4}f_1\Big(d,u,F_d(u)\Big)$

$$\begin{aligned} V(d,u) &= \cosh(d)\mathcal{C}_1(Q) - u\ \sinh(d)\mathcal{S}_1(Q) \\ W(d,u) &= \cosh(d)\mathcal{S}_1(Q) - u\ \sinh(d)\mathcal{C}_1(Q) \\ W_1(d,u) &= \tfrac{1}{4}\Big(-2\cosh(d)\mathcal{C}_1(Q) + 2u\ \sinh(d)\mathcal{S}_1(Q) + W_0Q\Big) \end{aligned} \tag{5.8}$$

The calculation of the first terms of the series W_1 by Maple shows that the convergent polynomial series $W_1(d,u)$ begins with a term containing d^{10}.

Using (5.5) and (2.15), we find

$$\mathcal{S}(J) = \mathcal{S}\Big(Q_1\mathcal{S}(G)\Big) = \mathcal{C}(Q_1)\mathcal{S}^2(G) + \mathcal{S}(Q_1)\mathcal{SC}(G). \tag{5.9}$$

Using

$$\begin{aligned} V_1(d,u) &= 1 + (1-u^2)d^2 + \Big(-\frac{71}{432}u^4 - \frac{1}{12}u^2 + \frac{107}{432}\Big)d^4 \\ &+ \Big(\frac{1351}{2160}u^6 - \frac{193}{144}u^4 + \frac{49}{60}u^2 - \frac{11}{108}\Big)d^6, \end{aligned}$$

we obtain

$$\begin{aligned} V_1\cdot\mathcal{C}(Q_1) &= Q_1V + W_2 \\ V_1\cdot\mathcal{S}(Q_1) &= Q_1W + W_3 \end{aligned}$$

where $W_2(d,u)$ and $W_3(d,u)$ are convergent polynomials series beginning with d^{10}. With (5.7) and (5.9), this implies

$$V_1\cdot\mathcal{S}\Big(Q_1\mathcal{S}(G)\Big) = W_2\cdot\mathcal{S}^2(G) + W_3\mathcal{SC}(G) + Q_1W_1G + Q_1f_2\Big(d,u,Q_1G(d,u)\Big). \tag{5.10}$$

This allows us to prove the following theorem

Theorem 5.1.

$$\begin{aligned} G(d,u) &= \Big(\frac{\alpha}{d^2} + (\beta + \frac{\alpha}{3})\Big)\frac{u\,H_0(d,u)}{\tau_2(u)} - \big(\beta d + \frac{\alpha}{d}\big)H_2(d,u) \\ &+ \delta d H_1(d,u) + S(d,u) \end{aligned} \tag{5.11}$$

where α, β, δ are constants and the polynomial series H_0, H_1, H_2, S are defined by

$$\begin{aligned} H_0(d,u) : &= \sum_{\substack{n=10 \\ n\ even}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u)\,d^n \\ H_1(d,u) : &= \sum_{\substack{n=9 \\ n\ odd}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^{n+1} \tau_n(u)\,d^n \\ H_2(d,u) : &= \sum_{\substack{n=9 \\ n\ odd}}^{\infty} n\,!\Big(\frac{i}{2\pi}\Big)^{n+1} \tau_n(u)\,d^n \end{aligned} \tag{5.12}$$

and $S(d,u)$ is a polynomial series satisfying

$$\|S\|_n = \mathcal{O}\Big((n-3)!(2\pi)^n\Big).$$

To prove this theorem we need to make some overvaluations on the coefficients of the polynomial series $\mathcal{S}\big(Q_1\mathcal{S}(G)\big)$. This will be the subject of the following paragraph.

Remark: Observe that the series F, G are odd in u, even in d and beginning with d^8. The series J is even in u, odd in d and beginning with d^{11} . In the series F, G, A, J, the degree of the polynomial that is the coefficient of d^n is at most $n-1$; thus the results of section 4 can still be applied and $d^{-1}F, d^{-1}G, d^{-1}A, d^{-1}J \in \mathcal{Q}$

5.2 Upper bounds for the coefficients of $\mathcal{S}\big(Q_1\mathcal{S}(G)\big)$

In this paragraph, we will use equation (5.10), together with the definitions of V_1 and Q_1, J and G, $W_i, i = 1, 2, 3$, to prove

Lemma 5.2.

$$\big\|\frac{1}{d}\mathcal{S}\big(Q_1\mathcal{S}(G)\big)\big\|_n = \mathcal{O}\Big((n-8)\,!(2\pi)^{-n}\Big) \quad as\ n \to \infty.$$

Proof. We set

$$e_n := \frac{(2\pi)^n \big\|\mathcal{S}(J)\big\|_n}{(n-8)\,!} \quad \text{for} \quad n \geq 12 \tag{5.13}$$

We must show that $e_n = \mathcal{O}(n^{-1})$. In the sequel, we will use the following convention: if $a_n, n = 0, 1, ...$ is any sequence of positve real numbers, then

$$a_n^+ := \max(a_0, a_1, ...a_n) \text{ for all } n \geq 0$$

We have

$$\left\|\mathcal{S}(J)\right\|_n \leq e_n (n-8)\,!(2\pi)^{-n} \quad \text{for} \quad n \geq 12. \tag{5.14}$$

Using Theorem 4.4 and Remark 4.7, we obtain

$$\left\|J\right\|_n \leq K_1 e_{n+1}^+ (n-1)\,!(2\pi)^{-n}, \quad \text{for} \quad n \geq 11. \tag{5.15}$$

where K_1 denotes the constant associated with the operator $\mathcal{S}$ in Theorem 4.4, it is independent of the present context. In this proof $K_i, i = 1, ..9$ will always denote constants independent of n and the sequence e_n.

Using (5.5), we obtain

$$\left\|Q_1\mathcal{S}(G)\right\|_n \leq K_1 e_{n+1}^+ (n-1)\,!(2\pi)^{-n} \quad \text{for} \quad n \geq 11 \tag{5.16}$$

We use (5) of Theorem 3.3 and 4.6. Since $\dfrac{Q_1(d,u)}{\tau_2(u)d^2}$ is a convergent power series beginning with 1, there is a constant K_2 such that

$$\left\|\mathcal{S}(G)\right\|_n \leq K_2\, e_{n+3}^+ (n+1)\,!(2\pi)^{-n}, \quad \text{for} \quad n \geq 9. \tag{5.17}$$

Using again Theorem 4.4 (and remark 4.7) and the fact that $F = G/Q$ where Q is given in (5.6), we obtain

$$\begin{aligned} \left\|G\right\|_n &\leq K_3 e_{n+4}^+ (n+2)\,!(2\pi)^{-n} \quad \text{for} \quad n \geq 8, \\ \left\|F\right\|_n &\leq K_3 e_{n+4}^+ (n+2)\,!(2\pi)^{-n} \quad \text{for} \quad n \geq 8, \end{aligned} \tag{5.18}$$

where K_3 is a constant independent of n.

This together with theorem 4.6 implies that there are constants K_4, L such that for all $k \geq 2$

$$\left\|F^k\right\|_n \leq K_4 L_1^k f_n^{(k)} (n-5)\,!(2\pi)^{-n} \quad \text{for } n \geq 8k \tag{5.19}$$

where

$$\begin{aligned} f_n^{(2)} &= \sum_{i=8}^{n-8} e_{i+4}^+ e_{n-i+4}^+ \frac{(i+2)\,!\,(n-i+2)\,!}{(n-5)\,!}, \quad \text{for } n \geq 16, \\ f_n^{(k+1)} &= \sum_{i=8}^{n-8k} e_{i+4}^+ f_{n-i}^{(k)} \frac{(i+2)\,!\,(n-i-5)\,!}{(n-5)\,!}, \quad \text{for } n \geq 8(k+1), \end{aligned}$$

with $f_n^{(k)} := 0$ for $n < 8k$.

Using Theorems 4.2 and 4.6 and $W_i = O(d^{10}), i = 1, 2, 3$, we obtain

$$\Big\|W_2 \cdot \mathcal{S}^2(G)\Big\|_n \le K_5 e^+_{n-6}(n-9)\,!(2\pi)^{-n} \quad \text{for } n \ge 20 \tag{5.20}$$

$$\Big\|W_3\mathcal{SC}(G)\Big\|_n \le K_6 e^+_{n-6}(n-9)\,!(2\pi)^{-n} \quad \text{for } n \ge 19, \tag{5.21}$$

$$\Big\|Q_1 W_1 G\Big\|_n \le K_7 e^+_{n-8}(n-10)\,!(2\pi)^{-n} \quad \text{for } n \ge 20, \tag{5.22}$$

$$\Big\|Q_1 f_2\Big(d, u, F_d(u)\Big)\Big\|_n \le K_8(1 + \sum_{k\ge 2}\frac{L_2^k}{k!} f^{(k)+}_{n-4})(n-9)\,!(2\pi)^{-n} \quad \text{for } n \ge 12, \tag{5.23}$$

Now, let us take the equation (5.10)

$$\Big\|V_1 \cdot \mathcal{S}(J)\Big\|_n \le \Big\|W_2 \cdot \mathcal{S}^2(G)\Big\|_n + \Big\|W_3\mathcal{SC}(G)\Big\|_n + \Big\|Q_1 W_1 G\Big\|_n + \Big\|Q_1 f_2\Big(d, u, F_d(u)\Big)\Big\|_n$$

Using (5.20), (5.21), (5.22) and (5.23), we obtain

$$\Big\|V_1 \cdot \mathcal{S}(J)\Big\|_n \le K_9\Big(1 + e^+_{n-6} + \sum_{k\ge 2}\frac{L_2^k}{k!} f^{(k)+}_{n-4}\Big)(n-9)!\,(2\pi)^{-n} \tag{5.24}$$

Since, V_1 is a convergent polynomial series begins with 1 , we also have

$$\Big\|\mathcal{S}(J)\Big\|_n \le K_{10}\Big(1 + e^+_{n-6} + \sum_{k\ge 2}\frac{L_2^k}{k!} f^{(k)+}_{n-4}\Big)(n-9)!\,(2\pi)^{-n} \tag{5.25}$$

Using (5.13), we obtain

$$e^+_n \le \frac{K}{n}\Big(1 + e^+_{n-6} + \sum_{k\ge 2}\frac{L_2^k}{k!} f^{(k)+}_{n-4}\Big) \quad \text{for } n \ge 12 \tag{5.26}$$

lem5.2

Lemma 5.3. *Under the condition (5.26), we have* $e_n = \mathcal{O}(n^{-1})$ *as* $n \to \infty$.

Proof. Let $K_1 \ge 10\,! e^+_{12}$ an arbitrary number. We assume that

$$e_n \le \frac{K_1(n+p-1)\,!}{(n-2)\,!(p+11)\,!} \qquad \text{for } 12 \le n \le N-4 \tag{5.27}$$

with some $p \ge -1$, $N \ge 16$. This gives for $16 \le n \le N$

$$(n-5)\,! f^{(2)}_n \le K_1^2 \sum_{i=8}^{n-8} \frac{(i+p+3)\,!(n+p-i+3)\,!}{\big((p+11)\,!\big)^2}.$$

The first and last term of the above sum are the largest, so we can easily estimate

$$\sum_{i=8}^{n-8}(i+p+3)\,!(n+p-i+3)\,! \le (p+11)\,!(n+p-4)\,! \quad .$$

We obtain

$$f_n^{(2)} \le K_1^2 \frac{(n+p-4)\,!}{(p+11)\,!(n-5)\,!} \qquad \text{for } 16 \le n \le N.$$

In a similar way, we can prove by induction that

$$f_n^{(k)} \le K_1^k \frac{(n+p-4)\,!}{(p+11)\,!(n-5)\,!} \le K_1^k \frac{(n+p-1)\,!}{(p+11)\,!(n-2)\,!} \qquad \text{for } 8k \le n \le N.$$

Using the assumption of the lemma, we obtain

$$e_n \le \frac{K}{n} e^{(1+L_2)K_1} \frac{(n+p-1)\,!}{(p+11)\,!(n-2)\,!} \qquad \text{for } 16 \le n \le N.$$

Now we choose $N_0 \ge 16$ so large that $\frac{K \exp((1+L_2)K_1)}{N_0} \le K_1$ and then p so large that (5.27) holds for $N = N_0$. In a first step, our considerations imply by induction over N that (5.27) holds for *all* N and hence

$$e_n = \mathcal{O}\left(\frac{(n+p-1)\,!}{(n-2)\,!} \right) \text{ as } n \to \infty$$

for this possibly large value of p.

As K_1 is arbitrary in (5.27), we have also shown for any $p \ge -1$ that

$$e_n = \mathcal{O}\left(\frac{(n+p-1)\,!}{(n-2)\,!} \right) \text{ as } n \to \infty$$

implies that

$$e_n = \mathcal{O}\left(\frac{(n+p-2)\,!}{(n-2)\,!} \right) \text{ as } n \to \infty \quad .$$

Consequently the last assertion is proved for $p = -1$ and we have shown

$$e_n = \mathcal{O}\big(n^{-1}\big) \text{ as } n \to \infty \quad .$$

Finally we have proved that

$$\big\|\mathcal{S}(J)\big\|_n = \mathcal{O}\big((n-9)\,!(2\pi)^{-n}\big) \text{ as } n \to \infty$$

and hence that

$$\big\|\frac{1}{d}\mathcal{S}(J)\big\|_n = \mathcal{O}\big((n-8)\,!(2\pi)^{-n}\big) \text{ as } n \to \infty$$

which completes the proof of the lemma. $\square$

5.3 Proof of theorem 5.1

Let $E := \frac{1}{d}\mathcal{S}(J) = \mathcal{S}(d^{-1}J)$. The polynomial series E is odd in d and its coefficients are odd in u. We partition it

$$\begin{aligned} E_n(u) &= \alpha_n(n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1}\tau_n(u) + \beta_{n-2}(n-3)!\Big(\frac{i}{2\pi}\Big)^{n-3}\tau_{n-2}(u) + \\ &\quad \gamma_{n-4}(n-5)!\Big(\frac{i}{2\pi}\Big)^{n-5}\tau_{n-4}(u) + \overline{E}_{n-6}(u) \end{aligned} \tag{5.28}$$

for odd $n \geq 11$, where α_n , β_n and γ_n are real numbers and also $\overline{E}_n$ have at most degree n for all n. For the whole series E this is equivalent to

$$\mathcal{S}(d^{-1}J) = E = E_1 + d^2E_2 + d^4E_3 + d^6\overline{E} \tag{5.29}$$

where

$$\begin{aligned} E_1 &= \sum_{n=11}^{+\infty} \alpha_n(n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1}\tau_n(u)d^n \\ E_2 &= \sum_{n=9}^{+\infty} \beta_n(n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1}\tau_n(u)d^n \\ E_3 &= \sum_{n=7}^{+\infty} \gamma_n(n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1}\tau_n(u)d^n \\ \overline{E} &= \sum_{n=5}^{+\infty} \overline{E}_n(u)d^n \end{aligned}$$

Lemma 5.2 implies that

$\alpha_n = \mathcal{O}(n^{-7})$, $\beta_n = \mathcal{O}(n^{-5})$, $\gamma_n = \mathcal{O}(n^{-3})$ and $\|\overline{E}_n\|_n = \mathcal{O}\big((n-2)!(2\pi)^{-n}\big)$.

Applying $\mathcal{T}$ to (5.29) we obtain

$$\frac{1}{d}J = \mathcal{T}(E_1) + d^2\mathcal{T}(E_2) + d^4\mathcal{T}(E_3) + d^6\mathcal{T}(\overline{E}) \tag{5.30}$$

To the first three summands we apply Theorem 4.5. Thus we obtain

$$\begin{aligned} \Big\|\{\mathcal{T}(E_1)\}_n - \alpha(n-1)!\Big(\frac{i}{2\pi}\Big)^n\tau_n(u)\Big\|_n &= \mathcal{O}\big((n-7)!(2\pi)^{-n}\big) \\ \Big\|\{\mathcal{T}(E_2)\}_n - \beta(n-1)!\Big(\frac{i}{2\pi}\Big)^n\tau_n(u)\Big\|_n &= \mathcal{O}\big((n-5)!(2\pi)^{-n}\big) \\ \Big\|\{\mathcal{T}(E_3)\}_n - \gamma(n-1)!\Big(\frac{i}{2\pi}\Big)^n\tau_n(u)\Big\|_n &= \mathcal{O}\big((n-3)!(2\pi)^{-n}\big) \end{aligned}$$

where

$$\alpha = \frac{4}{\pi}\sum_{\substack{n=11 \\ n\ odd}}^{\infty}\alpha_n,\ \beta = \frac{4}{\pi}\sum_{\substack{n=9 \\ n\ odd}}^{\infty}\beta_n, \gamma = \frac{4}{\pi}\sum_{\substack{n=7 \\ n\ odd}}^{\infty}\gamma_n.$$

To the last part of (5.30) we apply Theorem 4.4 and obtain

$$\Big\|\{\mathcal{T}(\overline{E})\}_n\Big\|_n \;=\; \mathcal{O}\big((n-1)\,!(2\pi)^{-n}\log(n)\big),$$

thus altogether

$$\begin{aligned}\Big\|\{(\frac{1}{d}J)\}_n \;-\; & \alpha(n-1)\,!\Big(\frac{i}{2\pi}\Big)^n\tau_n-\beta(n-3)\,!\Big(\frac{i}{2\pi}\Big)^{n-2}\tau_{n-2}- \\ & \gamma(n-5)\,!\Big(\frac{i}{2\pi}\Big)^{n-4}\tau_{n-4}\Big\|_n=\mathcal{O}\Big((n-7)\,!(2\pi)^{-n}\log(n)\Big).\end{aligned}$$

Using (4.3) we obtain

$$\begin{aligned}\Big\|\{J\}_n \;-\; & \alpha(n-2)\,!\Big(\frac{i}{2\pi}\Big)^{n-1}\tau_{n-1}-\beta(n-4)\,!\Big(\frac{i}{2\pi}\Big)^{n-3}\tau_{n-3} \\ -\; & \gamma(n-6)\,!\Big(\frac{i}{2\pi}\Big)^{n-5}\tau_{n-5}\Big\|_n=\mathcal{O}\Big((n-8)\,!(2\pi)^{-n}\log(n)\Big)\quad .\end{aligned}$$

Remark 5.4. *The asymptotic of J_n gives a good approximation of α; its suffice to calculate , using a formal calculation software (for example: Pari), the first* 40 *terms of $A(d,u)$ by the recurrence of Section 2 and to evaluate the highest coefficients of J_n to get the approximation $\alpha = 89.0334$.*

Next we observe that $J = Q_1\mathcal{S}(G)$, where Q_1 is given in (5.6) Using part 5. of Theorem 3.3, we obtain

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$$\begin{aligned}\Big\|\big\{\mathcal{S}(G)\big\}_n \;+\; & \alpha n\,!\Big(\frac{i}{2\pi}\Big)^{n+1}\frac{\tau_{n+1}}{\tau_2}+\big(\beta-\alpha p_2(u)\big)(n-2)\,!\Big(\frac{i}{2\pi}\Big)^{n-1}\frac{\tau_{n-1}}{\tau_2} \\ & +\big(\gamma-\beta p_2(u)-\alpha p_4(u)\big)(n-4)\,!\Big(\frac{i}{2\pi}\Big)^{n-3}\frac{\tau_{n-3}}{\tau_2}\Big\|_n \\ =\; & \mathcal{O}\Big((n-6)\,!(2\pi)^{-n}\log(n)\Big) \qquad (5.31)\end{aligned}$$

where

$$\begin{aligned}p_2(u) &= -\frac{1}{4}-\frac{u^2}{4}=-\frac{1}{2}+\frac{\tau_2}{4} \\ p_4(u) &= \frac{1}{24}-\frac{u^2}{48}-\frac{7u^4}{432}=\frac{1}{216}+\frac{55}{1296}\tau_2+\frac{7}{432}\tau_4.\end{aligned}$$

Remark: Observe that the approximation (5.31) of the coefficients $\{\mathcal{S}(G)\}_n$ is polynomial. Indeed; the polynomials $\tau_n(u), n \geq 2$ are divisible by $\tau_2(u)$.

In order to find an asymptotic estimation for the coefficients of the formal solution, we need to apply the inverse of operator $\mathcal{S}$. To this purpose, we show the following lemma

Lemma 5.5. *If H_0, H_1, H_2 are the polynomial series defined in (5.12). Then*

- *1 If we define the operator $\mathcal{C}_{\frac{1}{4}} = \cosh\Big(\frac{dD}{4}\Big)$, then the polynomial series $\mathcal{C}_{\frac{1}{4}}(H_0), \mathcal{C}_{\frac{1}{4}}(H_1)$ are converging.*
- *2. the polynomial series $\mathcal{S}(H_0), \mathcal{S}(H_1)$ are converging.*
- *3. $\mathcal{C}(H_0) = -H_0 + \mu_1(d,u)$, where $\mu_1(d,u)$ is a convergent series.*
- *4. $\mathcal{S}(H_2) = \frac{1}{2}H_0 + \mu_2(d,u)$, where $H_2 = d\frac{\partial}{\partial d}H_1$ and $\mu_2(d,u)$ is a convergent series .*
- *5. $\mathcal{S}\Big(\frac{H_0}{\tau_2(u)}\Big) = -u\sinh(d)\frac{H_0}{\tau_2(u)} + \mu_3(d,u)$, where $\mu_3(d,u)$ is a convergent series .*
- *6. $\mathcal{S}\Big(\frac{u\,H_0}{\tau_2(u)}\Big) = -\frac{1}{2}(u^2+1)\sinh(d)\frac{H_0}{\tau_2(u)} + \mu_4(d,u)$, where $\mu_4(d,u)$ is a convergent series .*
- *7. $\mathcal{S}\Big(\frac{u\,H_0}{\sinh(d)\tau_2(u)} - H_2\Big) = -\frac{H_0}{\tau_2(u)} + \mu_5(d,u)$, where $\mu_5(d,u)$ is a convergent series .*

Proof. (1)- We have

$$\mathcal{C}_{\frac{1}{4}}(H_0) = \sum_{\substack{m=0 \\ m\ even}}^{\infty} \frac{1}{4^m m!} d^m D^m \Big(\sum_{\substack{n=10 \\ n\ even}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u)\, d^n \Big)$$

using the definition of the operator D in Proposition 3.1, we obtain

$$\begin{aligned} \mathcal{C}_{\frac{1}{4}}(H_0) &= \sum_{m,n} \frac{1}{4^m m!}(n+m-1)\,!\Big(\frac{i}{2\pi}\Big)^n \tau_{n+m}(u)\, d^{n+m} \\ &= \sum_{\substack{k=10 \\ k\ even}}^{\infty} \gamma_k (k-1)!\Big(\frac{i}{2\pi}\Big)^k \tau_k d^k \end{aligned} \tag{5.32}$$

where

$$\gamma_k := \sum_{\substack{m=0 \\ m\ even}}^{k-10} \frac{1}{4^m m!}\Big(\frac{2\pi}{i}\Big)^m.$$

Hence

$$\gamma_k = \sum_{\substack{m=0 \\ m\ even}}^{\infty} \frac{1}{4^m m!}\Big(\frac{2\pi}{i}\Big)^m - \sum_{\substack{m=k-8 \\ m\ even}}^{\infty} \frac{1}{4^m m!}\Big(\frac{2\pi}{i}\Big)^m.$$

Using

$$\sum_{\substack{m=0\\ m\ even}}^{\infty} \frac{1}{4^m m!}(\frac{2\pi}{i})^m = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!}(\frac{\pi}{2})^{2l} = \cos(\frac{\pi}{2}) = 0,$$

we find

$$\gamma_k = -\sum_{\substack{m=k-8\\ m\ even}}^{\infty} \frac{1}{m!}(\frac{\pi}{2i})^m$$

which implies

$$\begin{aligned} |\gamma_k| &\leq \frac{1}{(k-8)!}\Big(\frac{\pi}{2}\Big)^{k-8} \sum_{\substack{m=0\\ m\ even}}^{\infty} \frac{1}{m!}(\frac{\pi}{2})^m \\ &\leq \frac{\cosh(\pi/2)}{(k-8)!}\Big(\frac{\pi}{2}\Big)^{k-8} \end{aligned}$$

This with (5.32) imply that $\mathcal{C}_{\frac{1}{4}}(H_0) = \mu(d,u)$ is convergent. For $\mathcal{C}_{\frac{1}{4}}(H_1)$, we can use the same method.

(2)- As $\mathcal{S} = 2\mathcal{S}_{\frac{1}{4}}\mathcal{C}_{\frac{1}{4}}$, where $\mathcal{S}_{\frac{1}{4}} = \sinh\Big(\frac{dD}{4}\Big)$, we obtain using (1),

$$\mathcal{S}(H_0) = 2\mathcal{S}_{\frac{1}{4}}\Big(\mathcal{C}_{\frac{1}{4}}(H_0)\Big) = 2\mathcal{S}_{\frac{1}{4}}(\mu)$$

This implies that $\mathcal{S}(H_0)$ is convergent. For $\mathcal{S}(H_1)$, we can use the same method.

(3)- We have $\mathcal{C}(H_0) = (2\mathcal{C}^2_{\frac{1}{4}} - Id)H_0 = -H_0 + 2\mathcal{C}^2_{\frac{1}{4}}(H_0)$. This with (1) imply

$$\mathcal{C}(H_0) = -H_0 + \mu_1(d,u)$$

where $\mu_1(d,u)$ is a convergent series.

(4)- We differentiate the equation $\mathcal{S}(H_1) = C_1$ with respect to d. As

$$z\frac{d}{dz}\Big(\sinh(z/2)\Big) = \frac{z}{2}\cosh(z/2),$$

we obtain

$$\mathcal{S}(H_2) + \frac{1}{2}\mathcal{C}(dDH_1) = d\frac{\partial C_1}{\partial d}$$

Using (3) of this lemma, we obtain

$$\mathcal{S}(H_2) = \frac{1}{2}dDH_1 + \mu_2(d,u) = \frac{1}{2}H_0 + \mu_2(d,u),$$

where $\mu_2(d,u)$ is a convergent series.

(5)- Using (2.15) and (2), (3) of this lemma we obtain

$$\begin{aligned}\mathcal{S}(H_0)=\mathcal{S}\Big(\tau_2(u)\frac{H_0}{\tau_2(u)}\Big) &= \mathcal{S}\big(\tau_2(u)\big)\mathcal{C}\Big(\frac{H_0}{\tau_2}\Big)+\mathcal{C}(\tau_2(u))\mathcal{S}\Big(\frac{H_0}{\tau_2}\Big)\\ &= \mu_3(d,u),\\ \mathcal{C}(H_0)=\mathcal{C}\Big(\tau_2(u)\frac{H_0}{\tau_2(u)}\Big) &= \mathcal{C}\big(\tau_2(u)\big)\mathcal{C}\Big(\frac{H_0}{\tau_2}\Big)+\mathcal{S}(\tau_2(u))\mathcal{S}\Big(\frac{H_0}{\tau_2}\Big)\\ &= -H_0.\end{aligned}$$

This implis

$$\begin{aligned}\mathcal{S}\Big(\frac{H_0}{\tau_2}\Big) &= \frac{\mathcal{S}\big(\tau_2(u)\big)}{\mathcal{C}\big(\tau_2(u)\big)^2-\mathcal{S}\big(\tau_2(u)\big)^2}H_0+\mu_4(d,u)\\ &= -u\sinh(d)\frac{H_0}{\tau_2(u)}+\mu_4(d,u),\end{aligned}$$

where $\mu_3(d,u)$, $\mu_4(d,u)$ are converging series.

(6)- The proof of (6) is similar to that of (5).

(7)- Using (4) and (6), we obtain

$$\begin{aligned}\mathcal{S}\Big(\frac{u\,H_0}{\sinh(d)\tau_2(u)}-H_2\Big) &= \Big(-\frac{1}{2}(u^2+1)-\frac{1}{2}\tau_2(u)\Big)\frac{H_0}{\tau_2(u)}+\mu_5(d,u)\\ &= -\frac{H_0}{\tau_2(u)}+\mu_5(d,u),\end{aligned}$$

where $\mu_5(d,u)$ is a convergent series. This completes the proof of Lemma.

Using the definition of H_0 in (5.12) and (5.31), we can rewrite $\frac{1}{d}\mathcal{S}(G)$ in the form

$$\frac{1}{d}\mathcal{S}(G) = -\alpha\frac{H_0}{d^2\tau_2(u)}-(\beta+\frac{1}{2}\alpha)\frac{H_0}{\tau_2(u)}+\frac{1}{4}\alpha H_0+\overline{X}, \qquad (5.33)$$

where

$$\big\|\overline{X}\big\|_n=\mathcal{O}\Big((n-3)\,!(2\pi)^{-n}\Big).$$

Using (2) and (5) of the previous Lemma and applying also the inverse of the operator $\mathcal{S}$ in (5.33), we obtain

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$$\frac{1}{d}G=\frac{1}{d^2}\Big(\alpha+(\beta+\frac{1}{2}\alpha)d^2\Big)\Big[\frac{1}{\sinh(d)}\frac{u\,H_0}{\tau_2(u)}-H_2\Big]+\frac{1}{2}\alpha H_2+\delta H_1+\overline{X}_1 \quad (5.34)$$

where, δ is a constant and

$$H_2(d,u) = d\frac{\partial H_1}{\partial d}(d,u) = \sum_{\substack{n=9\\ n\ odd}}^{\infty} n\,!\Big(\frac{i}{2\pi}\Big)^{n+1}\tau_n(u)\,d^n \tag{5.35}$$

$$\big\|\overline{X}_1\big\|_n = \mathcal{O}\Big((n-2)\,!(2\pi)^{-n}\Big) \tag{5.36}$$

$$\delta = \frac{4}{\pi}\sum_{n=1}^{\infty}\delta_n, \tag{5.37}$$

with $\delta_n = O(n^{-2})$. In the expression of $\frac{1}{d}G$, the term δH_1 comes from the fact that the series X can be written

$$X(d,u) = D_1(d,u) + d^2 D_2(d,u)$$

where

$$D_1(d,u) = \sum_{n=8}^{\infty}\delta_n(n-1)!\Big(\frac{i}{2\pi}\Big)^n\tau_n(u)d^n$$

$$\big\|D_2\big\|_n = \mathcal{O}\Big((n-1)\,!(2\pi)^{-n}\Big)$$

if we apply theorem 4.5 on the series $D_1(d,u)$ and Theorem 4.4 on $D_1(d,u)$, the term δH_1 appears in the expression of $\frac{1}{d}G$.

Since $\dfrac{1}{\sinh(d)} = d^{-1} - \dfrac{d}{6} + \mathcal{O}(d^3)$, we obtain

$$\begin{aligned} G(d,u) &= \Big(\frac{\alpha}{d^2} + (\beta + \frac{\alpha}{3})\Big)\frac{u\,H_0(d,u)}{\tau_2(u)} - \big(\beta d + \frac{\alpha}{d}\big)H_2(d,u) \\ &+ \delta d H_1(d,u) + S(d,u) \end{aligned} \tag{5.38}$$

where $\|S\|_n = \mathcal{O}\Big((n-3)!(2\pi)^n\Big)$ $\square$

Observe that in $d^{-2}\dfrac{u\,H_0}{\tau_2(u)}$, $d^{-1}H_2$ the degree of the coefficients of d^n exceeds n. This is due to the fact that the expressions $u\frac{\tau_{n+2}}{\tau_2} - \tau_{n+1}$ etc., which are of degree $n-1$, were split.

It is not necessary (but would not be difficult) to write down asymptotic approximations for the coefficients of F, because equations (5.5) and (5.3) can be used. This completes the proof of the theorem 5.1 $\square$

6 Functions and quasi-solutions

So far, we have shown that equation (2.2) has a formal solution and we have found an asymptotic approximation of the coefficients of the formal solution. We will use

this to construct a quasi-solution, i.e. a function that satisfies equation (2.2) except for some exponentially small error. To that purpose, we define the functions

$$H_n(u) : \;=\; (n-1)\,!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u) \tag{6.1}$$

and

$$h_0(t,u) : \;=\; \sum_{\substack{n=10\\ n\ even}}^{\infty} H_n(u)\frac{t^{n-1}}{(n-1)\,!} \tag{6.2}$$

$$h_1(t,u) : \;=\; \frac{i}{2\pi}\sum_{\substack{n=9\\ n\ odd}}^{\infty} H_n(u)\frac{t^{n-1}}{(n-1)\,!} \tag{6.3}$$

$$h_2(t,u) : \;=\; \frac{i}{2\pi}\sum_{\substack{n=9\\ n\ odd}}^{\infty} n\,H_n(u)\frac{t^{n-1}}{(n-1)\,!}\quad . \tag{6.4}$$

This means that

$$\begin{aligned} h_0(t,u) &= \frac{i}{2\pi}\sum_{\substack{n=1\\ n\ odd}}^{\infty}\Big(\frac{i}{2\pi}\Big)^n \tau_{n+1}(u)\,t^n - \Big(H_2\,t + H_4\frac{t^3}{3!} + H_6\,\frac{t^5}{5!} + H_8\,\frac{t^7}{7!}\Big) \\ h_1(t,u) &= \big(\frac{i}{2\pi}\big)^2\sum_{\substack{n=0\\ n\ even}}^{\infty}\Big(\frac{i}{2\pi}\Big)^n \tau_{n+1}(u)t^n - \frac{i}{2\pi}\Big(H_1 + H_3\frac{t^2}{2!} + H_5\,\frac{t^4}{4!}7H_7\,\frac{t^6}{6!}\Big) \end{aligned} \tag{6.5}$$

Using part 4. of the proposition (3.1), we obtain

$$\frac{i}{2\pi}\sum_{\substack{n=1\\ n\ odd}}^{\infty}\Big(\frac{i}{2\pi}\Big)^n \tau_{n+1}(u)\,t^n = \frac{i}{2\pi}\sum_{\substack{n=1\\ n\ odd}}^{\infty}\frac{1}{n\,!}\Big(\frac{it}{2\pi}\Big)^n\frac{d^n}{d^n\xi}\Big(\tanh(\xi)\Big)$$

This is obviously the difference of two Taylor expansion and thus we can write

$$\begin{aligned} h_0(t,u) &= \frac{i}{4\pi}\Big[\tanh\big(\xi+\frac{it}{2\pi}\big) - \tanh\big(\xi-\frac{it}{2\pi}\big)\Big] \\ &- \Big(H_2\,t + H_4\frac{t^3}{3!} + H_6\,\frac{t^5}{5!} + H_8\,\frac{t^7}{7!}\Big). \end{aligned} \tag{6.6}$$

Similarly,

$$\begin{aligned} h_1(t,u) &= -\frac{1}{4\pi^2}\sum_{\substack{n=0\\ n\ even}}^{\infty}\frac{1}{n\,!}\Big(\frac{it}{2\pi}\Big)^n\frac{d^n}{d^n\xi}\Big(\tanh(\xi)\Big) \\ &- \frac{i}{2\pi}\Big(H_1 + H_3\frac{t^2}{2!} + H_5\,\frac{t^4}{4!}7H_7\,\frac{t^6}{6!}\Big) \end{aligned}$$

or equivalently

$$\begin{aligned} h_1(t,u) &= \frac{-1}{8\pi^2}\Big[\tanh\big(\xi+\frac{it}{2\pi}\big)+\tanh\big(\xi-\frac{it}{2\pi}\big)\Big] \\ &- \frac{i}{2\pi}\Big(H_1+H_3\frac{t^2}{2!}+H_5\,\frac{t^4}{4!}7H_7\,\frac{t^6}{6!}\Big). \end{aligned} \tag{6.7}$$

The functional equations for the trigonometric and hyperbolic functions imply that

$$\begin{aligned} h_0(t,u) &= -\frac{1}{2\pi}\frac{(1-u^2)\,\sin\big(\frac{t}{2\pi}\big)\cos\big(\frac{t}{2\pi}\big)}{\cos\big(\frac{t}{2\pi}\big)^2+u^2\sin\big(\frac{t}{2\pi}\big)^2} \\ &+ \frac{\tau_2}{4\pi^2}t-\frac{\tau_4}{16\pi^4}t^3+\frac{\tau_6}{64\pi^6}t^5-\frac{\tau_8}{256\pi^8}t^7 \\ h_1(t,u) &= -\frac{1}{4\pi^2}\frac{(u-u^3)\sin\big(\frac{t}{2\pi}\big)^2}{\cos\big(\frac{t}{2\pi}\big)^2+u^2\sin\big(\frac{t}{2\pi}\big)^2} \\ &+ \frac{\tau_1}{4\pi^2}-\frac{\tau_3}{16\pi^4}t^2+\frac{\tau_5}{64\pi^6}t^4-\frac{\tau_7}{256\pi^8}t^6 \end{aligned} \tag{6.8}$$

For fixed real u the functions $h_k(.,u), k=0,1,2$ are analytic in $|t|<\rho$, where $\rho=\pi^2$. In the subsequent definition, we consider real values of u, $0<u\leq 1$, here $h_k, k=0,1,2$ are also analytic with respect to t on the positive real axis.

We define the functions $\mathcal{H}_k(d,u), k=0..3$, by

$$\begin{aligned} \mathcal{H}_0(d,u): &= \int_0^{+\infty} e^{-\frac{t}{d}}h_0(t,u)d\,t \text{ for } 0<u\leq 1 \\ \mathcal{H}_1(d,u): &= \int_0^{+\infty} e^{-\frac{t}{d}}h_1(t,u)d\,t \text{ for } 0<u\leq 1 \\ \mathcal{H}_2(d,u): &= \int_0^{+\infty} e^{-\frac{t}{d}}h_2(t,u)d\,t \text{ for } 0<u\leq 1. \end{aligned} \tag{6.9}$$

We have $\mathcal{H}_2(d,u)=d\dfrac{\partial\mathcal{H}_1}{\partial d}(d,u)$. Indeed

$$\begin{aligned} d\frac{\partial\mathcal{H}_1}{\partial d}(d,u) &= \int_0^{+\infty}\Big(\frac{1}{d}e^{-\frac{t}{d}}\Big)t\cdot h_1(t,u)d\,t \\ &= -\int_0^{+\infty}\frac{\partial}{\partial t}\Big(e^{-\frac{t}{d}}\Big)\cdot t\cdot h_1(t,u)d\,t \\ &= \int_0^{+\infty}e^{-\frac{t}{d}}\big(h_1(t,u)+t\cdot\frac{\partial}{\partial t}h_1(t,u)\big)d\,t \\ &= \int_0^{+\infty}e^{-\frac{t}{d}}h_2(t,u)d\,t \end{aligned}$$

The functions $\mathcal{H}_k(d,.)$ are real analytic; they can be continued analytically to the interval $-1<u\leq 1$ in the following way. Choose some positive number

M and let Γ_1 the path consisting of the segment from 0 to Mi and of the ray $t \mapsto t + Mi, t \geq 0$. Let Γ_2 the symmetric path that could also be obtained using $-M$ instead of M. Recalling (6.6), we can also define

$$\begin{aligned} \mathcal{H}_0(d,u) : &= \frac{i}{4\pi}\left[\int_{\Gamma_2} e^{-\frac{t}{d}} \tanh\left(\xi + \frac{it}{2\pi}\right) dt - \int_{\Gamma_1} e^{-\frac{t}{d}} \tanh\left(\xi - \frac{it}{2\pi}\right) dt\right] \\ &+ \mu_0(d,u), \end{aligned} \tag{6.10}$$

where

$$\mu_0(d,u) := \frac{1}{4\pi^2}\tau_2(u)d^2 - \frac{3}{8\pi^4}\tau_4(u)d^4 + \frac{15}{8\pi^6}\tau_6(u)d^6 - \frac{315}{16\pi^8}\tau_8(u)d^8,$$

for $-\tanh(\frac{2}{\pi}M) < u \leq 1$, where $\xi = \operatorname{artanh}(u)$, because the singularities of tanh are $i(\frac{\pi}{2} + n\pi)$, n integer. As M is arbitrary, this defines the analytic continuation of $\mathcal{H}_0(d,.)$ for $-1 < u \leq 1$. Similarly, the real analytic continuations of $\mathcal{H}_k, k = 1,2$ are defined.

In the sequel, we use the operator $\mathcal{C}, \mathcal{S}$ also for functions.

Lemma 6.1. *Consider the functions $\mathcal{H}_k(d,u), k = 0..2$, defined in (6.9). Then, for $-1 < u \leq 1$*

- *1 For $k = 0, 1$,*

$$\mathcal{H}_k(d, T^{\pm\frac{1}{2}}) = -\mathcal{H}_k(d,u) + \mu_k^{\pm}(d,u), \tag{6.11}$$

  where $T^{+\frac{1}{2}}, T^{-\frac{1}{2}}$ are defined in (2.13) and the functions $\mu_k^{\pm}(d,u), k = 0,1$, are analytic, beginning with d^{10}, resp d^9

- *2. For $k = 0,1$, $\mathcal{S}(\mathcal{H}_k) = \mu_k(d,u)$, where the functions $\mu_k(d,u), k = 0,1$, are analytic, beginning with d^{11}, resp d^{10} .*

- *3. For $k = 0,1$, $\mathcal{C}(\mathcal{H}_k) = -\mathcal{H}_k(d,u) + \lambda_k(d,u)$, where the functions $\lambda_k(d,u), k = 0,1$, are analytic, beginning with d^{10}, resp d^9.*

- *4. $\mathcal{S}(\mathcal{H}_2) = \frac{1}{2}\mathcal{H}_0(d,u) + \mu_2(d,u)$, where $\mu_2(d,u)$ is a analytic function, beginnings with d^{10} .*

- *5. $\mathcal{S}\left(\frac{u\,\mathcal{H}_0}{\tau_2(u)}\right) = -\frac{1}{2}(u^2+1)\sinh(d)\frac{\mathcal{H}_0}{\tau_2(u)} + \mu_4(d,u)$, where the function $\mu_4(d,u)$ is analytic, beginnings with d^{11} .*

Proof. (1)- For $k = 0$ we replace u by $T^{+\frac{1}{2}}$ in (6.6). Using (6.10) and $\xi(T^{+\frac{1}{2}}) = \xi(u) + \frac{1}{2}d$ we obtain for $0 < u \leq 1$

$$\mathcal{H}_0(d, T^{+\frac{1}{2}}) = \int_0^{+\infty} e^{-\frac{t}{d}} h_0(t, T^{+\frac{1}{2}}) dt = \frac{i}{4\pi}\mathcal{I}^+ + \mu(d, T^{+\frac{1}{2}}) \tag{6.12}$$

where

$$\mathcal{I}^+ = \int_0^{+\infty} e^{-\frac{t}{d}} \tanh\big(\xi + \frac{d}{2} + \frac{it}{2\pi}\big) d\,t - \int_0^{+\infty} e^{-\frac{t}{d}} \tanh\big(\xi - \frac{d}{2} + \frac{it}{2\pi}\big) d\,t$$

If we substitute $t + \pi i\, d$ in the first part, $t - \pi i\, d$ in the second part we obtain

$$\mathcal{I}^+ = -\int_{-\pi i d}^{+\infty - \pi i d} e^{-\frac{t}{d}} \tanh\big(\xi + \frac{it}{2\pi}\big) d\,t + \int_{\pi i d}^{+\infty + \pi i d} e^{-\frac{t}{d}} \tanh\big(\xi - \frac{it}{\pi}\big) d\,t \quad .$$

Now, we apply Cauchy's theorem

$$\begin{aligned} \mathcal{I}^+ &= -\int_0^{+\infty} e^{-\frac{t}{d}} \Big(\tanh\big(\xi + \frac{it}{2\pi}\big) - \tanh\big(\xi - \frac{it}{2\pi}\big)\Big) d\,t \\ &+ \int_0^{-\pi i d} e^{-\frac{t}{d}} \tanh\big(\xi + \frac{it}{2\pi}\big) d\,t - \int_0^{\pi i d} e^{-\frac{t}{d}} \tanh\big(\xi - \frac{it}{2\pi}\big) d\,t \quad . \end{aligned}$$

Substituting $t = -i\, ds$ in the second part, $t = i\, ds$ in the third part, we obtain

$$\begin{aligned} \mathcal{I}^+ &= \int_0^{+\infty} e^{-\frac{t}{d}} \Big(\tanh\big(\xi + \frac{it}{2\pi}\big) - \tanh\big(\xi - \frac{it}{2\pi}\big)\Big) d\,t \\ &- 2i\, d \int_0^{\pi} \cos(s) \tanh\big(\xi + d\frac{s}{2\pi}\big) ds. \end{aligned}$$

With (6.12) this implies for $0 < u \leq 1$

$$\mathcal{H}_0(d, T^+) = -\mathcal{H}_0(d, u) + \mu_0^+(d, u) \tag{6.13}$$

where

$$\mu_0^+(d, u) = \frac{d}{2\pi} \int_0^{\pi} \cos(s) \tanh\big(\xi + d\frac{s}{2\pi}\big) ds + \mu(d, T^{+\frac{1}{2}}).$$

By real analytic continuation, this formula is valid for $-1 < u \leq 1$. We use the same method for $\mathcal{H}_0(d, T^{-\frac{1}{2}}), \mathcal{H}_1(d, T^{\pm\frac{1}{2}})$ and obtain for $-1 < u \leq 1$

$$\begin{aligned} \mathcal{H}_0(d, T^{-\frac{1}{2}}) &= -\mathcal{H}_0(d, u) + \mu_0^-(d, u) \\ \mathcal{H}_1(d, T^{+\frac{1}{2}}) &= -\mathcal{H}_1(d, u) + \mu_1^+(d, u) \\ \mathcal{H}_1(d, T^{-\frac{1}{2}}) &= -\mathcal{H}_1(d, u) + \mu_1^-(d, u) \end{aligned} \tag{6.14}$$

where

$$\begin{aligned} \mu_0^-(d, u) &= \frac{d}{2\pi} \int_0^{\pi} \cos(s) \tanh\big(\xi - \frac{d\, s}{2\pi}\big) d\, s + \mu_0(d, T^{-\frac{1}{2}}) \\ \mu_1^+(d, u) &= -\frac{d}{4\pi^2} \int_0^{\pi} \sin(s) \tanh\big(\xi + \frac{d\, s}{2\pi}\big) d\, s + \mu_1(d, T^{+\frac{1}{2}}) \\ \mu_1^-(d, u) &= -\frac{d}{4\pi^2} \int_0^{\pi} \sin(s) \tanh\big(\xi - \frac{d\, s}{2\pi}\big) d\, s + \mu_1(d, T^{-\frac{1}{2}}) \\ \mu_1(d, u) &= \frac{1}{4\pi^2} \tau_1(u) d - \frac{1}{8\pi^4} \tau_3(u) d^3 + \frac{3}{8\pi^6} \tau_5(u) d^5 - \frac{45}{16\pi^8} \tau_7(u) d^7 \end{aligned}$$

(2)- Using the definition of the operator $\mathcal{S}$ in (2.13) and (1) of this Lemma, the result is immediate.

(3)- The proof of (3) is similar to that of (2).

(4)- For $k=1$, we differentiate (6.11) with respect to d. Because

$$\frac{\partial T^{\pm\frac{1}{2}}}{\partial d} = \pm\frac{1}{2}\Big(1-(T^{\pm\frac{1}{2}})^2\Big),$$

then

$$\mathcal{H}_2(d,u) \pm \frac{d}{2}\Big(1-(T^{\pm\frac{1}{2}})^2\Big)\frac{\partial \mathcal{H}_1}{\partial u}(d,T^{\pm\frac{1}{2}}) = -\mathcal{H}_2(d,u) + d\mu_1'^{\pm}(d,u)$$

implies

$$\begin{aligned}
\mathcal{S}\big(\mathcal{H}_2\big) &= -\frac{d}{2}\mathcal{C}\Big((1-u^2)\frac{\partial \mathcal{H}_1}{\partial u}\Big) + d\big(\mu_1'^{+}(d,u) - \mu_1'^{-}(d,u)\big) \\
&= \frac{d}{2}(1-u^2)\frac{\partial \mathcal{H}_1}{\partial u} + \mu_2(d,u) \\
&= \frac{1}{2}\mathcal{H}_0(d,u) + \mu_2(d,u)
\end{aligned}$$

where $\mu_2(d,u)$ is analytic function beginings with d^{10}.

(5)- Using (2.15) and (2), (3) of previous lemma , we obtain

$$\begin{aligned}
\mathcal{S}\Big(\frac{u}{\tau_2}\mathcal{H}_0\Big) &= \mathcal{S}\Big(\frac{u}{\tau_2}\Big)\mathcal{C}\big(\mathcal{H}_0\big) + \mathcal{C}\Big(\frac{u}{\tau_2}\Big)\mathcal{S}\big(\mathcal{H}_0\big) \\
&= -\frac{1}{2}(u^2+1)\sinh(d)\frac{\mathcal{H}_0}{\tau_2(u)} + \mu_4(d,u),
\end{aligned}$$

where the function $\mu_4(d,u)$ is analytic, beginnings with d^{11}. This completes proof of the Lemma.

In the sequel we consider $u_0 \in]-1,0]$.

Proposition 6.2. *We have*

1. *Uniformly for $u_0 \le u \le 1$,*

$$\begin{aligned}
\mathcal{H}_0(d,u) &\sim \sum_{\substack{n=2 \\ n\ even}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u) d^n \ \ as\ d \searrow 0 \\
\mathcal{H}_1(d,u) &\sim \sum_{\substack{n=3 \\ n\ odd}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^{n+1} \tau_n(u) d^n \ \ as\ d \searrow 0 \\
\mathcal{H}_2(d,u) &\sim \sum_{\substack{n=3 \\ n\ odd}}^{\infty} n\,!\Big(\frac{i}{2\pi}\Big)^{n+1} \tau_n(u) d^n \ \ as\ d \searrow 0 \qquad (6.15)
\end{aligned}$$

2. *For* $i = 1..3$, $\left|\frac{\partial \mathcal{H}_i}{\partial u}(d,u)\right| \leq Kd$ *for* $u_0 < u \leq 1 \quad (d > 0)$

Proof. The proof of this proposition is similar to that of [11].

With the aim of applying the results of [10], we consider $S_{n-2}(u) = R_n(u)$, where $S_n(u)$ is the remainder term in (5.38). Then $R_n(u), n$ is a sequence of polynomials of degree at most n and

$$\|R_n\|_n = O\Big((n-5)\,!(2\pi)^{-1}\log(n)\Big)$$

Lemma 6.3. *[10] If we define*

$$\begin{aligned} r(t,u): &= \sum_{n=10}^{\infty} R_n(u)\frac{t^{n-1}}{(n-1)\,!} \qquad (t \in \mathbb{C}, |t| \leq \pi^2, u_0 \leq u \leq 1) \\ r(t,u): &= r\big(\pi^2,u\big) + \big(t-\pi^2\big)\frac{\partial r}{\partial t}\big(\pi^2,u\big) \quad (t > \pi^2, u_0 \leq u \leq 1) \\ \mathcal{R}(d,u): &= \int_0^{\infty} e^{-\frac{t}{d}}\, r(t,u)dt \end{aligned}$$

then

1. *r is continuously differentiable function on the set B of all (t,u) such that u satisfies $u_0 \leq u \leq 1$ and t is a complex number and satisfies $|t| \leq \pi^2$ or $t > \pi^2$. The restriction of r to $u_0 \leq u \leq 1, |t| \leq \pi^2$ is twice continuously differentiable. for fixed $u_0 \leq u \leq 1$ the function $r(t,u)$ is analytic in $|t| < \pi^2$*

2. *$\mathcal{R}(d,u)$ is continuous, partially differentiable with respect to u, has continuous partial derivative and*

$$\mathcal{R}(d,u) \sim \sum_{n=10}^{\infty} R_n(u)d^n \ \text{ as } d \searrow 0 \tag{6.16}$$

3. $\big|\mathcal{R}(d,u)\big| \leq Kd^3$, $\left|\frac{\partial \mathcal{R}}{\partial u}(d,u)\right| \leq Kd^3$ *for* $u_0 \leq u \leq 1 \quad (d > 0)$

The importance of our definition of $\mathcal{R}$ lies in a certain compatibility with insertion of the functions T^+, T^- for u. First let

$$\begin{aligned} \sum_{n=10}^{\infty} R_n^+(u)d^n &= \sum_{n=11}^{\infty} R_n(T^+)d^n \\ \sum_{n=10}^{\infty} R_n^-(u)d^n &= \sum_{n=11}^{\infty} R_n(T^-)d^n \end{aligned}$$

We obtain a new sequences $R_n^+(u), R_n^-(u)$ of polynomials of degree at most n. This follows from the relation

$$p\big(T^+(d,u)\big) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k p(u) d^k \quad . \tag{6.17}$$

Theorem 4.2 implies

$$\begin{aligned}\left\|R_n^+(u)\right\|_n &= O\Big((n-5)\,!(2\pi)^{-n}\log(n)\Big)\\ \left\|R_n^-(u)\right\|_n &= O\Big((n-5)\,!(2\pi)^{-n}\log(n)\Big)\end{aligned}$$

Therefore we can use the previous lemma for $R_n^+(u),\ R_n^-(u)$ and obtain functions $\mathcal{R}^+(d,u),\ \mathcal{R}^-(d,u)$.

Theorem 6.4. *There is a positive constant K independent of d,u such that*

$$\begin{aligned}\left|\mathcal{R}^+(d,u)-\mathcal{R}(d,T^+)\right| &\le Kd^3e^{-\frac{\pi^2}{d}}, \quad \textit{for}\,(d>0,u_0<u\le 1),\\ \left|\mathcal{R}^-(d,u)-\mathcal{R}(d,T^-)\right| &\le Kd^3e^{-\frac{\pi^2}{d}} \quad \textit{for}\,(d>0,u_0<u\le 1).\end{aligned}$$

Proof. The proof is exactly the one of [10] .

Definition 6.5. *Let $\mathcal{D}(d,u)$ be a function defined for $0<d<d_0$ and $u_0<u<1$. We say that $\mathcal{D}(d,u)$ has property G if*

$$\mathcal{D}(d,u)=\int_0^\infty e^{-\frac{t}{d}}\,q(t,u)dt \quad (0<d<d_0,\ u_0<u<1)$$

is the Laplace transform of some function $q(t,u)$ that has the following properties

1. *$q(t,u)$ is defined if $u_0<u<1$ and either t is complex and $|t|<\pi^2$ or t is real and $t\ge 0$,*
2. *$q(t,u)$ is analytic in $|t|<\pi^2$ for $u_0<u<1$,*
3. *$q(t,u)$ restricted to $0\le t<\pi^2$ or $t\ge\pi^2$ is continuous and the $\lim_{t\to\pi^2}q(t,u)$ exists for every $u_0<u<1$,*
4. *there is a positve constant K such that*

$$|q(d,u)|\le Ke^{Kt} \quad \textit{for} \quad t\ge 0,(0<d<d_0,\ u_0<u<1)$$

Lemma 6.6. *For $u_0<u\le 1$, we have*

1. *If $\mathcal{H}_i(d,u), i=0,1,2$ are the functions of (6.9) then*

$$d^2\mathcal{H}_k(d,u)=(1-u^2)\tilde{\mathcal{H}}_k(d,u)+\mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big), \quad k=0,1$$

and

$$d^3\mathcal{H}_2(d,u)=(1-u^2)\tilde{\mathcal{H}}_2(d,u)+\mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big),$$

where $\tilde{\mathcal{H}}_i(d,u), i=1,2,3$ have property G.

2. *Let k be a psitive integer. If $\mathcal{D}_1, \mathcal{D}_2$ have property G and their first terms in the Taylor development at $d = 0$, begin with d^k then*

$$\mathcal{D}_1(d,u)\mathcal{D}_2(d,u) = d^k\mathcal{D}(d,u) + \mathcal{O}\Big(d^k e^{-\frac{\pi^2}{d}}\Big),$$

where $\mathcal{D}(d,u)$ has property G

3. *Any function $\mathcal{D}(d,u)$ analytic in a neighborhood of $d = 0$ has property G if $\mathcal{D}(0,u) = 0$ for all u,*

4. *If $\mathcal{R}(d,u)$ is defined by lemma 6.3 then $\dfrac{1}{d^2}\mathcal{R}(d,u)$ has property G*

5. *If $\mathcal{D}_1, \mathcal{D}_2$ have property G then so do $\mathcal{D}_1 + \mathcal{D}_2$, $\mathcal{D}_1 - \mathcal{D}_2$ and $\mathcal{D}_1 \cdot \mathcal{D}_2$*

6. *If $\mathcal{D}(d,u)$ has property G then*

$$\big|\mathcal{D}(d,u)\big| \le Kd \quad \big(0 < d < \frac{1}{K}\big) \tag{6.18}$$

with some constant $K > 0$ independent of u.

Proof

1. For $i = 0$,

   (i)-If $u > 0$, we have

$$d^2\mathcal{H}_0 = (1-u^2)\int_0^\infty e^{-\frac{t}{d}} g_2(t,u)dt \tag{6.19}$$

   where

$$g_2(t,u) = \frac{1}{(1-u^2)}\int_0^t\int_0^\tau h_0(s,u)ds\,d\tau$$

   $g_2(t,u)$ has a logarithmic singularity at $t_k(s) = (2k+1)\pi^2 \pm d\dfrac{2\pi\, s}{\varepsilon}i$ for $(k \ge 0, s > 0)$. it is analytic in $|t| < \pi^2$ and $\lim_{t\to\pi^2} g_2(t,u)$ exists.

   If we put

$$\tilde{\mathcal{H}}_0(d,u) = \int_0^\infty e^{-\frac{t}{d}}\tilde{g}_2(t,u)d\,t$$

   where

$$\tilde{g}_2(t,u) = \begin{cases} g_2(t,u), & \text{if } t \le \pi^2 \\ g_2(\pi^2,u), & \text{if } t \ge \pi^2 \end{cases}$$

   then $\tilde{\mathcal{H}}_0(d,u)$ has property G and

$$d^2\mathcal{H}_0(d,u) = (1-u^2)\tilde{\mathcal{H}}_0(d,u) + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big).$$

(ii)- For $-u_0 < u < 1$, where $0 < u_0 < 1$,, we have

$$\begin{aligned}\mathcal{H}_0(d,u) &= \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} h_0(t,u)dt + 2\pi i \sum_{k\geq 0} Res\Big(e^{-\frac{t}{d}} h_0(t,u), t_k(s)\Big) \\ &= \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} h_0(t,u)dt + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big)\end{aligned}$$

where $\dfrac{\pi}{2} < \varphi < \dfrac{\pi}{4}$. For $0 < u < u_0$, this formula coincides with the formula

$$\int_0^{\infty} e^{-\frac{t}{d}} h_0(t,u)dt$$

and extends it by real analytic continuation for $-u_0 < u < 0$.
This implis

$$d^2\mathcal{H}_0(d,u) = (1-u^2)\int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} g_2(t,u)dt + \mathcal{O}\Big((1-u^2)d^2e^{-\frac{\pi^2}{d}}\Big)$$

we obtain

$$\begin{aligned}d^2\mathcal{H}_0(d,u) &- (1-u^2)\tilde{\mathcal{H}}_0(d,u) = (1-u^2)\int_\Gamma e^{-\frac{t}{d}} g_2(t,u)d\Gamma \\ &+ \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big),\end{aligned}$$

where Γ is the path following the real line from infinity to $(\pi^2, 0)$, then along the vertical line from $(\pi^2, 0)$ to $\Big(\pi^2, \pi^2\tan(\varphi)\Big)$ and finally along the line $y = \tan(\varphi)x$ from $\Big(\pi^2, \pi^2\tan(\varphi)\Big)$ to infinity.
Since $g_2(., u)$ is bounded on Γ, then

$$\Big|d^2\mathcal{H}_0(d,u) - (1-u^2)\tilde{\mathcal{H}}_0(d,u)\Big| \leq K(1-u^2)e^{-\frac{\pi^2}{d}},$$

where K is positive constant. Finally

$$d^2\mathcal{H}_0(d,u) = (1-u^2)\tilde{\mathcal{H}}_0(d,u) + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big) \quad \text{for} \quad (0 < u_0 < u \leq 1)$$

For $i = 1$ we can use the same method.
For $i = 2$, we use the same method with

$$d^3\mathcal{H}_2 = (1-u^2)\int_0^{\infty} e^{-\frac{t}{d}} g_2(t,u)dt \tag{6.20}$$

where

$$g_2(t,u) = \frac{1}{(1-u^2)}\int_0^t \int_0^\sigma \int_0^\tau h_2(s,u)ds\, d\sigma\, d\tau.$$

2. We assume that $\mathcal{D}_1(d,u), \mathcal{D}_2(d,u)$ have property G and their first terms in the Taylor development at $d=0$ begin with d^k. Then

$$\mathcal{D}_1 = \int_0^\infty e^{-\frac{t}{d}} f(t,u) d\, t$$
$$\mathcal{D}_2 = \int_0^\infty e^{-\frac{t}{d}} g(t,u) d\, t$$

where $f(t,u), g(t,u)$ are analytic in $|t| < \pi^2$ and $f(t,u) = O(t^{k-1}), g(t,u) = O(t^{k-1})$.

$$\mathcal{D}_1(d,u)\mathcal{D}_2(d,u) = \int_0^\infty e^{-\frac{t}{d}} (f * g)(t,u) d\, t \tag{6.21}$$

Since

$$\begin{aligned} h(t,u) &= (f*g)(t,u) = \int_0^s f(t,u) g(t-s,u) ds \\ &= \int_0^t f(t-s,u) g(s,u) ds \\ &= \int_{t/2}^t f(s,u) g(t-s,u) ds + \int_{t/2}^t f(t-s,u) g(s,u) ds. \end{aligned}$$

For $t < \pi^2$, the function $h(t,u)$ is k times differentiable with respect to t and

$$\begin{aligned} h'(t,u) &= f(t,u)g(0,u) + f(0,u)g(t,u) - f(\frac{t}{2},u) g(\frac{t}{2},u) \\ &+ \int_{t/2}^t f(s,u) g'(t-s,u) ds + \int_{t/2}^t f'(t-s,u) g(s,u) ds \\ &= \int_{t/2}^t f(s) g'(t-s) ds + \int_{t/2}^t f'(t-s,u) g(s,u) ds \\ &- f(\frac{t}{2},u) g(\frac{t}{2},u) \\ h^{(k)}(t,u) &= \int_{t/2}^t f(s) g^{(k-1)}(t-s) ds + \int_{t/2}^t f^{(k-1)}(t-s,u) g(s,u) ds \\ &- \sum_{n=0}^{k-1} f^{(n)}(\frac{t}{2},u) g^{(k-1-n)}(\frac{t}{2},u) \end{aligned}$$

Observe that $h^{(k)}(t,u)$ is contninuous on $[0, 2\pi^2[$, it is analytic for $|t| < \pi^2$. If we put

$$\tilde{h}(t,u) = \begin{cases} h(t,u), & \text{if } t < \pi^2 \\ (t-\pi^2)^k. & \text{if } t \geq \pi^2 \end{cases}$$

then

$$\int_0^\infty e^{-\frac{t}{d}} \tilde{h}^{(k)}(t,u) d\, t,$$

has property G and

$$\begin{aligned}(\mathcal{D}_1\cdot\mathcal{D}_2)(d,u) &= \int_0^\infty e^{-\frac{t}{d}}h(t,u)dt=\int_0^\infty e^{-\frac{t}{d}}\tilde{h}(t,u)dt+\mathcal{O}\Big(d^k e^{-\frac{\pi^2}{d}}\Big)\\ &= d^k\int_0^\infty e^{-\frac{t}{d}}\tilde{h}^{(k)}(t,u)dt+\mathcal{O}\Big(d^k e^{-\frac{\pi^2}{d}}\Big)\\ &= d^k\mathcal{D}(d,u)+\mathcal{O}\Big(d^k e^{-\frac{\pi^2}{d}}\Big)\end{aligned}$$

where $\mathcal{D}(d,u)$ has property G.

For $0\le u\le 1$, the proof of (3), (4), (5) and (6) is exactly the one of [10]. This proof is valid for $u_0<u\le 1$.

Now we have a formal solution of the equation (2.2) and an asymptotic estimate for its coefficients. With the results on the functions in the beginning of this section we have enough information to be able to give a precise function which satisfies the equation (2.2) with an exponentially small error as $d\to 0$.

In Theorem 2.1 we found that (2.2) has a uniquely determined formal power series solution

$$A(d,u) = U(d,u)+Q(d,u)G(d,u) \tag{6.22}$$

where U,Q are defined in (5.2), (5.6) and G is given by (5.38). This suggest that we put

$$\begin{aligned}\mathcal{G}(d,u) &= \Big(\frac{\alpha}{d^2}+(\beta+\frac{\alpha}{3})\Big)\frac{u\,\mathcal{H}_0(d,u)}{\tau_2(u)}-\big(\beta d+\frac{\alpha}{d}\big)\mathcal{H}_2(d,u)\\ &+ \delta d\mathcal{H}_1(d,u)+d^{-2}\mathcal{R}(d,u)\\ \mathcal{A}(d,u) &= U(d,u)+Q(d,u)\mathcal{G}(d,u)\end{aligned} \tag{6.23}$$

for $d>0, u_0<u\le 1$, where $\mathcal{H}_i(d,u), i=0..2$ are defined in (6.9) and $\mathcal{R}(d,u)$ is the function corresponding to $R_n,\ n=8.10..$ according to lemma 6.3. Using (1) of proposition 6.2 and (2) of lemma 6.3, we obtain

$$\mathcal{G}(d,u)\sim G(d,u)\quad \text{as } d\searrow 0 \text{ for every } u_0<u\le 1. \tag{6.24}$$

Consequently

$$\mathcal{A}(d,u)\ \sim\ A(d,u)\quad \text{as } d\searrow 0 \text{ for every } u_0<u\le 1. \tag{6.25}$$

Theorem 6.7. *The function $\mathcal{G}(d,u)$ satisfies (5.10) except for an exponentially small error. More precisely*

$$\Big|V_1\cdot\mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big)-W_2\cdot\mathcal{S}^2(\mathcal{G})-W_3\mathcal{SC}(\mathcal{G})-Q_1W_1\mathcal{G}-Q_1f_2\Big(d,u,\mathcal{G}\Big)\Big|\le Kd^3(1-u^2)e^{-\frac{\pi^2}{d}},$$

uniformly for $(u_0<u<1, 0<d<d_0,)$, where K is a constant independent of d and u.

In the proof of this Theorem the functions $\mathcal{D}_i(d,u), i = 1.2..$ have property G.
Proof. We set

$$\mathcal{F}(d,u) = V_1 \cdot \mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) - W_2 \cdot \mathcal{S}^2(\mathcal{G}) - W_3\mathcal{SC}(\mathcal{G}) - Q_1W_1\mathcal{G} - Q_1 f_2\Big(d,u,\mathcal{G}\Big). \quad (6.26)$$

Using (2), (4), (5) of Lemma 6.1, and (6.23), we obtain

$$\mathcal{S}(\mathcal{G}) = -\Big(\alpha_1 \sinh(d)\frac{1}{\tau_2} - (\frac{\alpha_1}{2}\sinh(d) - \frac{\beta_1}{2})\Big)\mathcal{H}_0 + (1-u^2)d^8\mathcal{D}_1(d,u) + d^{-2}\mathcal{S}(\mathcal{R})$$

where $\mathcal{D}_1(d,u)$ has property G and

$$\begin{aligned} \alpha_1 : &= \frac{\alpha}{d^2} + (\beta + \frac{\alpha}{3}) \\ \beta_1 : &= \frac{\alpha}{d} + \beta d \end{aligned}$$

This with lemma 6.1 imply

$$\begin{aligned} V_1\mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) &= d^5\mathcal{D}_2(d,u)\mathcal{H}_0 + (1-u^2)d^{11}\mathcal{D}_3(d,u) + d^{-2}V_1\mathcal{S}(Q_1)\mathcal{CS}(\mathcal{R}) \\ &+ d^{-2}V_1\mathcal{C}(Q_1)\mathcal{S}^2(\mathcal{R}), \end{aligned}$$

where $\mathcal{D}_2(d,u), \mathcal{D}_3(d,u)$ have property G. Because

$$\begin{aligned} \mathcal{C}(Q_1) &= (1-u^2)d^2\big(1+\mathcal{O}(d^2)\big), \\ \mathcal{S}(Q_1) &= (1-u^2)d^3\big(u+\mathcal{O}(d^2)\big), \\ V_1 &= 1+\mathcal{O}(d^2) \end{aligned}$$

it is sufficient to apply Theorem 6.4, (4) of Lemma 6.6 for $\mathcal{R}$ and (1) of Lemma 6.6 for $\mathcal{H}_0$ we obtain

$$V_1\mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) = d^5\mathcal{D}_2(d,u)\mathcal{H}_0 + (1-u^2)d^2\mathcal{D}_4(d,u) + \mathcal{O}\Big((1-u^2)d^3e^{-\frac{\pi^2}{d}}\Big)$$

where $\mathcal{D}_4(d,u)$ has property G. With (1) of lemma 6.6 this implies

$$V_1\mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) = (1-u^2)d^2\mathcal{D}_5(d,u) + \mathcal{O}\Big((1-u^2)d^3e^{-\frac{\pi^2}{d}}\Big) \quad (6.27)$$

Using the same method for the terms $W_2 \cdot \mathcal{S}^2(\mathcal{G})$, $W_3\mathcal{SC}(\mathcal{G})$ and $Q_1W_1\mathcal{G}$, we obtain

$$\begin{aligned} W_2 \cdot \mathcal{S}^2(\mathcal{G}) &= (1-u^2)d^{10}\mathcal{D}_6(d,u) + \mathcal{O}\Big((1-u^2)d^{10}e^{-\frac{\pi^2}{d}}\Big), \\ W_3\mathcal{SC}(\mathcal{G}) &= (1-u^2)d^9\mathcal{D}_7(d,u) + \mathcal{O}\Big((1-u^2)d^{10}e^{-\frac{\pi^2}{d}}\Big), \\ Q_1W_1\mathcal{G} &= (1-u^2)d^8\mathcal{D}_8(d,u) + \mathcal{O}\Big((1-u^2)d^{10}e^{-\frac{\pi^2}{d}}\Big). \quad (6.28) \end{aligned}$$

To study the term $Q_1 f_2\Big(d,u,\mathcal{G}\Big)$, we first treat $\mathcal{G}^2$ and $\mathcal{G}^3$.

$$\begin{aligned}\mathcal{G}^2(d,u) &= \alpha_1^2\frac{u^2}{\tau_2^2}\mathcal{H}_0^2+\beta_1^2\mathcal{H}_2^2+\delta^2d^2\mathcal{H}_1^2+d^{-4}\mathcal{R}^2-2\alpha_1\beta_1\frac{u}{\tau_2}\mathcal{H}_0\mathcal{H}_2+2\alpha_1\delta\frac{u}{\tau_2}\mathcal{H}_0\mathcal{H}_1\\ &+ 2d^{-2}\alpha_1\frac{u}{\tau_2}\mathcal{H}_0\mathcal{R}-2\beta_1\delta d\mathcal{H}_1\mathcal{H}_2-2d^{-2}\beta_1\mathcal{R}\mathcal{H}_2+2d^{-1}\delta\mathcal{R}\mathcal{H}_1 \qquad (6.29)\end{aligned}$$

Using (1), (2) and (4) of the lemme 6.6 we obtain

$$\mathcal{G}(d,u)^2=d^2\mathcal{D}_9(d,u)+\mathcal{O}\Big(d^2e^{-\frac{\pi^2}{d}}\Big) \qquad (6.30)$$

With (2) of lemma 6.6 this implies

$$\mathcal{G}(d,u)^3=\mathcal{G}(d,u)\mathcal{G}(d,u)^2=d^8\mathcal{D}_{10}(d,u)+\mathcal{O}\Big(d^8e^{-\frac{\pi^2}{d}}\Big). \qquad (6.31)$$

We can rewrite

$$f_2\big(d,u,\mathcal{G}\big)=y_0(d,u)+y_1(d,u)\mathcal{G}^2f_{1,1}\big(d,u,\mathcal{G}^2\big)+y_2(d,u)d^2\mathcal{G}^3f_{1,2}\big(d,u,\mathcal{G}^2\big) \qquad (6.32)$$

where $f_{1,1}, f_{1,2}$ and $y_i, i=0,1,2$ are analytic .

Lemma 6.8. *For $i=1,2$,*

$$f_{1,i}(d,u,\mathcal{G}^2)=\int_0^\infty e^{-\frac{t}{d}}f_i(t,u)dt+\mathcal{O}\Big(d^2e^{-\frac{\pi^2}{d}}\Big)\quad for(d>0,u_0<u\leq 1) \qquad (6.33)$$

where $f_i(.,u)$ is analytic in $|t|<\pi^2$ and continuous in $[0,\pi^2[$ and $[\pi^2,\infty[$

Proof. Using (6.30), we can rewrite

$$\mathcal{G}^2(d,u)=\int_0^\infty e^{-\frac{t}{d}}g(t,u)dt+\mathcal{O}\Big(d^2e^{-\frac{\pi^2}{d}}\Big)$$

where $g(.,u)$ is analytic in $|t|<\pi^2$, it is continuous in $[0,\pi^2[$ and $[\pi^2,\infty[$, twice differentiable in $|t|<\pi^2$. We obtain

$$f_{1,i}(d,u,\mathcal{G}^2)=\sum_{n=1}^{\infty}f_{n,i}(d,u)\mathcal{G}^{2n}$$

where

$$f_{n,i}(d,u)=\int_0^\infty e^{-\frac{t}{d}}\varphi_{n,i}(t,u)dt$$

and $\varphi_{n,i}(t,u)$ are entire functions.

Using the proof of theorem 5.1 from [1], we find

$$f_{1,i}(d,u,\mathcal{G}^2)=\int_0^\infty e^{-\frac{t}{d}}f_i(t,u)dt+\mathcal{O}\Big(d^2e^{-\frac{\pi^2}{d}}\Big) \qquad (6.34)$$

where the series

$$\sum_{n=1}^{\infty}(\varphi_{n,i} * g^{*n})(t,u)\Big), \quad g^{*n} = g * ... * g, n \text{ times}$$

is uniformly convergent to a function $f_i(t,u)$ analytic in $|t| < \pi^2$ and continuous in $[0,\pi^2[$ and $[\pi^2,\infty[$, satisfies also

$$|f_i(t,u)| \leq K\exp(Kt) \quad \text{for} \;\; t \geq 0, u_0 < u \leq 1.$$

Then

$$f_{1,i}(d,u,\mathcal{G}^2) = \mathcal{Y}_i(d,u) + \mathcal{O}\Big(d^2 e^{-\frac{\pi^2}{d}}\Big) \quad \text{for}(d > 0, u_0 < u \leq 1),$$

where $\mathcal{Y}_i(d,u), i = 1,2$ have property G.

This lemma with (6.32) and (2) of lemma 6.6 imply

$$Q_1 f_2\Big(d,u,\mathcal{G}\Big) = d^6(1-u^2)\mathcal{D}_{10}(d,u) + \mathcal{O}\Big(d^6(1-u^2)e^{-\frac{\pi^2}{d}}\Big) \tag{6.35}$$

Combining (6.27), (6.28) and (6.35), we find

$$\mathcal{F}(d,u) \;\; = \;\; d^2(1-u^2)\mathcal{D}(d,u) + \mathcal{K}(d,u)$$

where $\mathcal{D}(d,u)$ has property G and

$$\big|\mathcal{K}(d,u)\big| \leq Kd^3(1-u^2)de^{\frac{-\pi^2}{d}} \quad \text{for} \quad (0 < d < d_0,\, u_0 < u < 1). \tag{6.36}$$

Hence

$$\mathcal{F}(d,u) = (1-u^2)d^2\int_0^{\infty} e^{-\frac{t}{d}}q(t,u)dt + \mathcal{K}(d,u)$$

where $q(t,u)$ is analytic in $|t| < \pi^2$, it is continuous on $[0,\pi^2[$ and $]\pi^2,\infty[$, has a limit as $t \to \pi^2$ for every $u_0 < u < 1$ and satisfies

$$|q(t,u)| \leq K\, e^{Kt}, \tag{6.37}$$

with a constant K independent of u. If $q(t,u) = \sum_{n=0}^{\infty} q_n(u)t^n$ is the power series of $q(t,u)$ near $t = 0$, Watson's lemma with (6.36) imply

$$\mathcal{F}(d,u) \sim \sum_{n=0}^{\infty} n!(1-u^2)q_n(u)d^{n+3} \quad \text{as } d \searrow 0 \text{ for every } \; u_0 < u < 1 \;\; .$$

On the other hand because of its definition

$$\begin{aligned}\mathcal{F}(d,u) \;\; &\sim \;\; V_1 \cdot \mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) - W_2 \cdot \mathcal{S}^2(\mathcal{G}) - W_3\mathcal{S}\mathcal{C}(\mathcal{G}) \\ &- \;\; Q_1W_1\mathcal{G} - Q_1 f_2\Big(d,u,\mathcal{G}\Big) = 0 + 0d + \cdots \qquad ,\end{aligned}$$

since the formal series G satisfies (5.10). This means that all $q_n \equiv 0$. Thus we obtain for $u_0 < u < 1$ with (6.37)

$$\begin{aligned}(1-u^2)d^2\int_0^\infty e^{-\frac{t}{d}}|q(t,u)|dt &\leq (1-u^2)d^2\int_{\pi^2}^\infty e^{-\frac{t}{d}}Ke^{Kt}dt \\ &\leq K(1-u^2)d^3e^{-\frac{\pi^2}{d}} \quad (0<d<d_0)\end{aligned}$$

and thus

$$\big|\mathcal{F}(d,u)\big| \leq K(1-u^2)d^3e^{-\frac{\pi^2}{d}} \quad (0<d<d_0)\ .$$

We have proved that

$$\begin{aligned}\Big|V_1\cdot\mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) &- W_2\cdot\mathcal{S}^2(\mathcal{G}) - W_3\mathcal{SC}(\mathcal{G}) - Q_1W_1\mathcal{G} - Q_1f_2\Big(d,u,\mathcal{G}\Big)\Big| \\ &\leq K(1-u^2)d^3e^{-\frac{\pi^2}{d}},\end{aligned}$$

for $0<d<d_0$, $u_0<u<1$, i.e. $\mathcal{Q}(d,u)$ is a quasi-solution of (5.10) on this interval. This implies that the function $\mathcal{A}(d,u)$ defined in (6.23) is a quasi-solution of (2.2), more precisely

$$\begin{aligned}\Big|\sqrt{\frac{1-(T^+)^2}{1-u^2}}\mathcal{A}(d,T^+) &+ \sqrt{\frac{1-(T^-)^2}{1-u^2}}\mathcal{A}(d,T^-) - 2\mathcal{A}(d,u) - f\big(\varepsilon,\mathcal{A}(d,u)\big)\Big| \\ &\leq Kde^{-\frac{\pi^2}{d}},\end{aligned} \tag{6.38}$$

7 Distance Between Points of Manifolds

Clearly, if $q_\varepsilon(t)$ is an exact solution of the difference equation (1.1), then $(q_\varepsilon(t),p_\varepsilon(t))$, where $p_\varepsilon(t)=\frac{1}{\varepsilon}\big(q_\varepsilon(t)-q_\varepsilon(t-\varepsilon)\big)$, is an exact solution of the the system (1.2). In the introduction, we have mentioned that the stable manifold W_s^- of this system at $A=(0,0)$ is parametrized by $t\to(q_\varepsilon^-(t),p_\varepsilon^-(t))$ and the unstable manifold W_u^+ of (1.2) at $B=(2\pi,0)$ is parametrized by $t\to(q_\varepsilon^+(t),p_\varepsilon^+(t))$, where $(q_\varepsilon^-(t),p_\varepsilon^-(t))$ is an exact solution of (1.2) and $q_\varepsilon^+(t)=2\pi-q_\varepsilon^-(-t), p_\varepsilon^+(t)=p_\varepsilon^-(-t+\varepsilon)$.

In the previous section, we have constructed a quasi-solution $\mathcal{A}(d,u)$ for equation (2.2), i.e. it satisfies this equation with an exponentially small error. We denote by $\tilde{W}_s$, $\tilde{W}_u$ the manifolds close to W_s^- respectively W_u^+ parametrized by $t\mapsto\big(\xi_-(t),\varphi_-(t)\big)$ respectively $t\mapsto\big(\xi_+(t),\varphi_+(t)\big)$, where $\xi_-(t)=\sqrt{1-u(t)^2}\mathcal{A}\big(d,u(t)\big)+q_{0d}(t)$ and $\xi_+(t)=2\pi-\xi_-(-t)$, $\varphi_-(t)=\dfrac{1}{\varepsilon}\big(\xi_-(t)-\xi_-(t-\varepsilon)\big)$, $\varphi_+(t)=\dfrac{1}{\varepsilon}\big(\xi_+(t)-\xi_+(t-\varepsilon)\big)$. Here and in the sequel, we often omit to indicate the dependence with respect to ε for the sake of simplicity of notation.

We will first show that the vertical distance between some point (q_1,p_1) of the stable manifold W_s^- and the manifold $\tilde{W}_s$ is exponentially small. For this purpose,

we consider the sequence $Z_n = (q_n, p_n)$ on the stable manifold W_s^-, defined by

$$Z_{n+1} = \begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \phi \begin{pmatrix} q_n \\ p_n \end{pmatrix} = \begin{pmatrix} q_n + \varepsilon p_{n+1} \\ p_n + \varepsilon \sin(q_n) \end{pmatrix}, \text{ for } n = 1, 2, 3, ... \tag{7.1}$$

There is a sequence t_n such that

$$\xi_-(t_n) = q_n = \sqrt{1 - u(t_n)^2}\, \mathcal{A}\big(d, u(t_n)\big) + q_{0d}(t_n). \tag{7.2}$$

The vertical projection of the point (q_n, p_n) on the manifold $\tilde{W}_s$ is the point (q_n, b_n), where $b_n = \varphi_-(t_n) = g(q_n) := \varphi_-\big(\xi_-^{-1}(q_n)\big)$ for $n = 1, 2, 3, ...$. We denote by $\triangle_\varepsilon(n)$ the vertical distance between the point (q_n, p_n) and the manifold $\tilde{W}_s$. Then

$$\triangle_\varepsilon(n) = p_n - b_n. \tag{7.3}$$

For the ϕ-image of the point (q_n, b_n) on $\tilde{W}_s$, we find

$$\begin{aligned} \begin{pmatrix} \tilde{q}_{n+1} \\ \tilde{b}_{n+1} \end{pmatrix} &= \phi \begin{pmatrix} q_n \\ b_n \end{pmatrix} = \begin{pmatrix} q_n + \varepsilon \tilde{b}_{n+1} \\ b_n + \varepsilon \sin(q_n) \end{pmatrix}, \\ &= \begin{pmatrix} \xi_-(t_n) + \varepsilon \tilde{b}_{n+1} \\ \varepsilon^{-1}\big(\xi_-(t_n) - \xi_-(t_n - \varepsilon)\big) + \varepsilon \sin\big(\xi_-(t_n)\big) \end{pmatrix} \\ &= \begin{pmatrix} 2\xi_-(t_n) - \xi_-(t_n - \varepsilon) + \varepsilon^2 \sin\big(\xi_-(t_n)\big) \\ \varepsilon^{-1}\big(\xi_-(t_n) - \xi_-(t_n - \varepsilon)\big) + \varepsilon \sin\big(\xi_-(t_n)\big) \end{pmatrix} \end{aligned} \tag{7.4}$$

As the function ξ_- satisfies equation (1.1) except for an exponentially small error because of (6.38) and (7.2), we have

$$\begin{aligned} \tilde{q}_{n+1} &= \xi_-(t_n + \varepsilon) + e_{n+1}(\varepsilon) \\ \tilde{b}_{n+1} &= \varphi_-(t_n + \varepsilon) + \frac{1}{\varepsilon} e_{n+1}(\varepsilon), \end{aligned} \tag{7.5}$$

where $|e_{n+1}(\varepsilon)| \le K\varepsilon \exp(-\dfrac{\pi^2}{\varepsilon})$ with some K independent of ε and n. Since $g\Big(\xi_-(t_n + \varepsilon)\Big) = \varphi_-(t_n + \varepsilon)$, we have

$$\begin{aligned} \tilde{b}_{n+1} &= g\Big(\xi_-(t_n + \varepsilon)\Big) + \frac{1}{\varepsilon} e_{n+1}(\varepsilon) \\ &= g\Big(\tilde{q}_{n+1} - e_{n+1}(\varepsilon)\Big) + \frac{1}{\varepsilon} e_{n+1}(\varepsilon) \end{aligned} \tag{7.6}$$

On the other hand, using (7.3) and the definition of ϕ, we obtain

$$\begin{aligned} \begin{pmatrix} \tilde{q}_{n+1} \\ \tilde{b}_{n+1} \end{pmatrix} &= \phi \begin{pmatrix} q_n \\ p_n - \triangle_\varepsilon(n) \end{pmatrix} = \begin{pmatrix} q_n + \varepsilon p_n + \varepsilon^2 \sin(q_n) - \varepsilon \triangle_\varepsilon(n) \\ p_n + \varepsilon \sin(q_n) - \triangle_\varepsilon(n) \end{pmatrix}, \\ &= \begin{pmatrix} q_{n+1} - \varepsilon \triangle_\varepsilon(n) \\ p_{n+1} - \triangle_\varepsilon(n) \end{pmatrix}. \end{aligned} \tag{7.7}$$

With (7.3) and (7.6) this implies

$$\begin{aligned}\triangle_\varepsilon(n+1) &= p_{n+1} - b_{n+1} = p_{n+1} - g(q_{n+1}) \\ &= \triangle_\varepsilon(n) + \tilde{b}_{n+1} - g\Big(\tilde{q}_{n+1} + \varepsilon\triangle_\varepsilon(n)\Big) \\ &= \triangle_\varepsilon(n) + g\Big(\tilde{q}_{n+1} - e_{n+1}(\varepsilon)\Big) - g\Big(\tilde{q}_{n+1} + \varepsilon\triangle_\varepsilon(n)\Big) + \frac{1}{\varepsilon}e_{n+1}(\varepsilon).\end{aligned}$$

Using Taylor expansion we obtain

$$\triangle_\varepsilon(n+1) = \big(1 - \varepsilon g'(\theta_{n+1})\big)\triangle_\varepsilon(n) + \frac{1}{\varepsilon}\big(1 - \varepsilon g'(\theta_{n+1})\big)e_{n+1}(\varepsilon) \qquad (7.8)$$

where $\tilde{q}_{n+1} - e_{n+1}(\varepsilon) < \theta_{n+1} < \tilde{q}_{n+1} + \varepsilon\triangle_\varepsilon(n)$.

Now g is ε-close to the curve $p = -2\sin(q/2)$, hence

$$g'(\theta_{n+1}) = -\cos\Big(\frac{\theta_{n+1}}{2}\Big) + O(d).$$

Thus given any positive $\mu < \pi$, there is a positive constant c such that for all $q_1 \le \mu$, all n and sufficiently small d,

$$1 - \varepsilon g'(\theta_{n+1}) \ge 1 + \varepsilon c \quad .$$

It is now convenient to write (7.8) in the form

$$\triangle_\varepsilon(n) = \big(1 - \varepsilon g'(\theta_{n+1})\big)^{-1}\triangle_\varepsilon(n+1) - \frac{1}{\varepsilon}e_{n+1}(\varepsilon)$$

As $\triangle_\varepsilon(n) \to 0$ as $n \to \infty$, this implies that that there is a positive constant K such that

$$\big|\triangle_\varepsilon(n)\big| \le Ke^{-\frac{\pi^2}{\varepsilon}}\sum_{k=0}^{\infty}(1+\varepsilon c)^{-k}$$

Consequently

$$\triangle_\varepsilon(n) = O\Big(\frac{1}{\varepsilon}\exp\Big(-\frac{\pi^2}{\varepsilon}\Big)\Big). \qquad (7.9)$$

In particular

$$\mathrm{dist}_v((q_1,p_1),\tilde{W}_s) = O\Big(\frac{1}{\varepsilon}\exp\Big(-\frac{\pi^2}{\varepsilon}\Big)\Big), \qquad (7.10)$$

where (q_1,p_1) is any point on the stable manifold W_s^-, provided $q_1 \le \mu < \pi$; here dist_v denotes the vertical distance.

The estimate (7.10) can be extended to any $\mu < 2\pi$ and a starting point (q_1,p_1) with $q_1 \le \mu$ in the following way. The relation (7.8) remains valid, only now we just have the existence of some constant $c > 0$ such that $1 - \varepsilon g'(\theta_{n+1}) \ge 1 - \varepsilon c$ for all n. As system (1.2) can be regarded as a one-step numerical method for the

system (1.3) of differential equations and the starting point is at a distance $\mathcal{O}(\varepsilon)$ of its solution $(q_0(t), q_0'(t))$, results on the convergence of one-step methods can be applied and yield that $q_k = q_0(t_1 + (k-1)\varepsilon) + \mathcal{O}(\varepsilon)$, where $q_0(t_1) = q_1$, uniformly for integer k, $1 \le k \le L/\varepsilon$, where L is any positive constant. We choose L such $q_0(t_1 + L) \le \mu/2 < \pi$. Repeated application of (7.8) now gives

$$|\triangle_\varepsilon(n) - \triangle_\varepsilon(n + [L/\varepsilon])| \le K e^{-\frac{\pi^2}{\varepsilon}} \sum_{k=0}^{[L/\varepsilon]} (1 - \varepsilon c)^{-k} \le \frac{K e^{Lc}}{c\varepsilon} e^{-\frac{\pi^2}{\varepsilon}} \quad .$$

To the quantity $\triangle_\varepsilon(n + [L/\varepsilon])$, inequality (7.9) can be applied, because $q_{n+[L/\varepsilon]} \le \mu/2 < \pi$. Thus (7.9) and hence also (7.10) remain valid also uniformly for $0 < q_1 \le \mu$, provided $\mu < 2\pi$.

The analogous reasoning applies to the vertical distance of a point $(\tilde{q}_1, \tilde{p}_1)$ on the unstable manifold W_u^+ from the manifold $\tilde{W}_u$ and yields

$$\mathrm{dist}_v((\tilde{q}_1, \tilde{p}_1), \tilde{W}_u) = O\Big(\frac{1}{\varepsilon} \exp\Big(-\frac{\pi^2}{\varepsilon}\Big)\Big). \tag{7.11}$$

Another method to obtain (7.11) consists in using (7.10) and symmetry.

Now we will estimate the vertical distance between the two manifolds $\tilde{W}_s$ and $\tilde{W}_u$. As the quasi-solution $\mathcal{A}(d,u)$ is defined for $-1 < u =: \tanh(t) < 1$, we can define

$$\begin{aligned} \mathcal{A}^+(d,u) &= -\mathcal{A}(d,-u), \\ \xi_+(t) &= \sqrt{1-u(t)^2}\,\mathcal{A}^+\big(d,u(t)\big) + 2\pi - q_{0d}(-t) = 2\pi - \xi_-(-t), \end{aligned} \tag{7.12}$$

$$D_\varepsilon(t) = \xi_+(t) - \xi_-(t) \quad \text{for } -\frac{4}{3} < t < \frac{4}{3} \quad . \tag{7.13}$$

Using (7.12) and the definition of $\xi_-(t)$ we find

$$\begin{aligned} D_\varepsilon(t) &= -\sqrt{1-u(t)^2}\Big(\mathcal{A}\big(d,u(t)\big) + \mathcal{A}\big(d,-u(t)\big)\Big) - q_{0d}(t) - q_{0d}(-t) + 2\pi \\ &= -\sqrt{1-u(t)^2}\Big(\mathcal{A}\big(d,u(t)\big) + \mathcal{A}\big(d,-u(t)\big)\Big) \end{aligned} \tag{7.14}$$

With (5.5) and (6.23) this implies

$$\begin{aligned} D_\varepsilon(t) = -\sqrt{1-u(t)^2}\,Q(d,u)\Bigg[&\alpha_1 \frac{u}{\tau_2}\Big(\mathcal{H}_0(d,u) - \mathcal{H}_0(d,-u)\Big) - \\ &\beta_1\Big(\mathcal{H}_2(d,u) + \mathcal{H}_2(d,-u)\Big) + \delta d\Big(\mathcal{H}_1(d,u) + \mathcal{H}_1(d,-u)\Big)\Bigg] \end{aligned}$$

where $Q(d,u)$ is defined in (5.6), α_1, β_1 are defind in (6.27) and $\mathcal{H}_i(d,u)$ are defined in (6.9) and can be continued analytically to $-1 < u \le 1$ as in (6.10).

Using the fact that the functions $u\,h_0(s,u), h_1(s,u), h_2(s,u)$ in (6.9) are odd, we can apply the residue theorem and obtain for $-1 < u < 1$ that

$$\begin{aligned}\mathcal{H}_i(d,u) + \mathcal{H}_i(d,-u) &= \sum_{Im(s_k(t))<0} 2\pi i \,\mathrm{Res}\Big(e^{-\frac{s}{d}} h_i(s,u), s_k(t)\Big) \\ &\quad - \sum_{Im(s_k(t))>0} 2\pi i \,\mathrm{Res}\Big(e^{-\frac{s}{d}} h_i(s,u), s_k(t)\Big), \quad i = 0,1,2,\end{aligned}$$

where $s_k(t) = \pi^2 \pm \frac{2d\pi\, t}{\varepsilon}\, i + 2k\pi^2$ for $k \geq 0$. We obtain

$$\begin{aligned}\mathrm{Res}\Big(e^{-\frac{s}{d}} h_0(s,u), s_k(t)\Big) &= \frac{1}{2} e^{-\frac{(k+1)\pi^2}{d}} e^{\mp \frac{2\pi t i}{\varepsilon}} \\ \mathrm{Res}\Big(e^{-\frac{s}{d}} h_1(s,u), s_k(t)\Big) &= \pm \frac{i}{4\pi} e^{-\frac{(k+1)\pi^2}{d}} e^{\mp \frac{2\pi t i}{\varepsilon}} \\ \mathrm{Res}\Big(e^{-\frac{s}{d}} h_2(s,u), s_k(t)\Big) &= \pm \frac{i}{4\varepsilon d}\big(\pi\varepsilon i \mp 2dt\big) e^{-\frac{(k+1)\pi^2}{d}} e^{\mp \frac{2\pi t i}{\varepsilon}}\end{aligned}$$

and hence

$$D_\varepsilon(t) = \frac{1}{d^2}\phi_1(t,\varepsilon) \exp\Big(-\frac{\pi^2}{d}\Big) + O\Big(e^{-\frac{\pi^2}{d}}\Big). \tag{7.15}$$

where

$$\begin{aligned}\phi_1(t,\varepsilon) &= 2\pi\alpha\Big[\sinh\Big(\frac{dt}{\varepsilon}\Big) + t/\cosh\Big(\frac{dt}{\varepsilon}\Big)\Big]\sin\Big(\frac{2\pi t}{\varepsilon}\Big) \\ &+ \Big[\frac{\pi\alpha}{\cosh\Big(\frac{dt}{\varepsilon}\Big)}\Big]\cos\Big(\frac{2\pi t}{\varepsilon}\Big)\end{aligned} \tag{7.16}$$

As a consequence of (7.15) and (7.12), we obtain immediately that

$$\xi^{\pm}(0) = \pi + \mathcal{O}(\varepsilon^{-2} e^{-\pi^2/\varepsilon}) \ . \tag{7.17}$$

Now, let us take a point $(\xi_+(t), \varphi_+(t))$ on the manifold $\tilde{W}_u$. We suppose that the point $(\xi_-(t_1), \varphi_-(t_1))$ is its vertical projection on the manifold $\tilde{W}_s$. We will evaluate the vertical distance between these two points

$$\mathrm{dist}_v(t) = \varphi_+(t) - \varphi_-(t_1) = \varphi_+(t) - g\big(\xi_+(t)\big), \tag{7.18}$$

where $g(x) = \varphi_-\Big(\xi_-^{-1}(x)\Big)$. Thus by (7.13)

$$\mathrm{dist}_v(t) = \varphi_+(t) - g\big(\xi_-(t) + D_\varepsilon(t)\big). \tag{7.19}$$

Using Taylor expansion, we find

$$\mathrm{dist}_v(t) = \varphi_+(t) - g\big(\xi_-(t)\big) - D_\varepsilon(t) g'\big(\eta(t)\big), \tag{7.20}$$

where $\xi_-(t) < \eta(t) < \xi_-(t) + D_\varepsilon(t)$. Here

$$\eta(t) = q_{0d}(t) + O(d), \quad \text{hence} \tag{7.21}$$

$$g'\big(\eta(t)\big) = -\cos\Big(\frac{\eta(t)}{2}\Big) + O(d) = -\tanh\Big(\frac{dt}{\varepsilon}\Big) + O(d). \tag{7.22}$$

As $g\big(\xi_-(t)\big) = \varphi_-(t)$, this yields

$$\begin{aligned} \mathrm{dist}_v(t) &= \varphi_+(t) - \varphi_-(t) - D_\varepsilon(t) g'_-\big(\eta(t)\big), \\ &= \frac{1}{\varepsilon}\big(\xi_+(t) - \xi_+(t-\varepsilon)\big) - \frac{1}{\varepsilon}\big(\xi_-(t) - \xi_-(t-\varepsilon)\big) - D_\varepsilon(t) g'_-\big(\eta(t)\big), \\ &= \frac{1}{\varepsilon}\big(D_\varepsilon(t) - D_\varepsilon(t-\varepsilon)\big) - D_\varepsilon(t) g'\big(\eta(t)\big). \end{aligned} \tag{7.23}$$

Now formula (7.15) applies and we obtain

$$D_\varepsilon(t-\varepsilon) = D_\varepsilon(t) + \frac{1}{d}\phi_2(t,\varepsilon)\exp\Big(-\frac{\pi^2}{d}\Big) + O\Big(e^{-\frac{\pi^2}{d}}\Big) \tag{7.24}$$

where

$$\begin{aligned} \phi_2(t,\varepsilon) &= -2\pi\alpha\Big[\frac{\cosh\Big(\frac{dt}{\varepsilon}\Big)^2 + 1}{\cosh\Big(\frac{dt}{\varepsilon}\Big)} - t\frac{\tanh\Big(\frac{dt}{\varepsilon}\Big)}{\cosh\Big(\frac{dt}{\varepsilon}\Big)}\Big]\sin\Big(\frac{2\pi t}{\varepsilon}\Big) \\ &+ \pi\alpha\Big[\frac{\tanh\Big(\frac{dt}{\varepsilon}\Big)}{\cosh\Big(\frac{dt}{\varepsilon}\Big)}\Big]\cos\Big(\frac{2\pi t}{\varepsilon}\Big) \quad . \end{aligned} \tag{7.25}$$

With (7.23) and (7.22) this implies

$$\mathrm{dist}_v(t) = \frac{1}{d^2}\Big[-\phi_2(t,\varepsilon) + \tanh\Big(\frac{dt}{\varepsilon}\Big)\phi_1(t,\varepsilon)\Big]e^{-\frac{\pi^2}{d}} + O\Big(\frac{1}{d}e^{-\frac{\pi^2}{d}}\Big) \quad . \tag{7.26}$$

Consequently, for $-\dfrac{4}{3} \le t \le \dfrac{4}{3}$

$$\mathrm{dist}_v(t) = \frac{4\pi\alpha}{\varepsilon^2}\cosh(t)\sin\Big(\frac{2\pi t}{\varepsilon}\Big)e^{-\frac{\pi^2}{\varepsilon}} + O\Big(\frac{1}{\varepsilon}e^{-\frac{\pi^2}{\varepsilon}}\Big) \quad \text{as } \varepsilon \searrow 0. \tag{7.27}$$

Combining this result with (7.10) and (7.11), we finally obtain our main result theorem 1.1, because $\xi_\pm(t) = q_0(t) + \mathcal{O}(\varepsilon^2)$ uniformly with respect to t on any finite interval.

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