



HAL
open science

Switching Games of Backward Stochastic Differential Equations

Ying Hu, Shanjian Tang

► **To cite this version:**

Ying Hu, Shanjian Tang. Switching Games of Backward Stochastic Differential Equations. 2008.
hal-00287645v1

HAL Id: hal-00287645

<https://hal.science/hal-00287645v1>

Preprint submitted on 12 Jun 2008 (v1), last revised 25 Nov 2013 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Switching Games of Backward Stochastic Differential Equations

Ying Hu* and Shanjian Tang†

June 12, 2008

Abstract

In this paper, we study the switching game of one-dimensional backward stochastic differential equations (BSDEs). This gives rise to a new type of multi-dimensional obliquely reflected BSDEs, which is a system of BSDEs reflected on the boundary of a special unbounded convex domain along an oblique direction. The existence of the adapted solution is obtained by the penalization method, the monotone convergence, and the a priori estimations. The uniqueness is obtained by a verification method (the first component of any adapted solution is shown to be the vector value of a switching problem for Reflected BSDEs). Finally, we show the existence of both the value and the saddle point for the switching game. More specifically, we prove that the value process of the switching game is given by the first component of the solution of the multi-dimensional obliquely reflected BSDEs and the saddle point can also be constructed using the latter.

Key Words. Backward stochastic differential equations, oblique reflection, switching game, saddle point

Abbreviated title. Switching games and obliquely reflected BSDEs

AMS Subject Classifications. 60H10, 93E20

*IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France. Part of this work was completed when this author was visiting Laboratory of Mathematics for Nonlinear Sciences, Fudan University, whose hospitality is greatly appreciated. *E-mail:* Ying.Hu@univ-rennes1.fr.

†Institute of Mathematics and Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China. This author is supported in part by NSFC Grant #10325101, Basic Research Program of China (973 Program) Grant #2007CB814904, and Chang Jiang Scholars Programme. Part of this work was completed when this author was visiting IRMAR, Université Rennes 1, whose hospitality is greatly appreciated. *E-mail:* sjtang@fudan.edu.cn.

1 Introduction

Let (Ω, \mathcal{F}, P) be a complete probability space, carrying a standard Brownian motion $W = \{W_t\}_{t \geq 0}$ with values in R^d . $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of the Brownian motion W augmented by the P -null sets of \mathcal{F} .

Consider two players I and II , who use their respective switching control processes $a(\cdot)$ and $b(\cdot)$ to control the following BSDE

$$\begin{aligned} U(s) = & \xi + \left(A^{a(\cdot)}(T) - A^{a(\cdot)}(s) \right) - \left(B^{b(\cdot)}(T) - B^{b(\cdot)}(s) \right) \\ & + \int_s^T \psi(r, U(r), V(r), a(r), b(r)) dr - \int_s^T V(r) dW(r). \end{aligned} \quad (1.1)$$

Here, ξ is an m -dimensional random variable measurable with respect to the past of W up to time T . ξ is called the terminal condition and ψ is called the coefficient (also called the generator). $A^{a(\cdot)}(\cdot)$ and $B^{b(\cdot)}(\cdot)$ are the cost processes associated with the switching control processes $a(\cdot)$ and $b(\cdot)$, respectively; they are càdlàg processes. Under suitable conditions, the above BSDE has a unique adapted solution, denoted by $(U^{a(\cdot), b(\cdot)}, V^{a(\cdot), b(\cdot)})$.

Player I chooses the switching control $a(\cdot)$ from a given finite set Λ to minimize the cost

$$J(a(\cdot), b(\cdot)) = U^{a(\cdot), b(\cdot)}(0), \quad (1.2)$$

and each of his instantaneous switchings from one scheme $i \in \Lambda$ to another different scheme $i' \in \Lambda$ incurs a positive cost which will be specified by the function $k(i, i')$, $(i, i') \in \Lambda \times \Lambda$. While Player II chooses the switching control $b(\cdot)$ from a given finite set Π to maximize the cost $J(a(\cdot), b(\cdot))$, and each of his instantaneous switchings from one scheme $j \in \Pi$ to another different scheme $j' \in \Pi$ incurs a positive cost which will be specified by the function $l(j, j')$, $(j, j') \in \Pi \times \Pi$. We are interested in the existence and the construction of the value process as well as the saddle point.

The solution of the above-stated switching game will appeal to the following new type of reflected backward stochastic differential equation (RBSDE for short) with oblique reflection: for $(i, j) \in \Lambda \times \Pi$ and $t \in [0, T]$,

$$\left\{ \begin{array}{l} Y_{ij}(t) = \xi_{ij} + \int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j) ds \\ \quad - \int_t^T dK_{ij}(s) + \int_t^T dL_{ij}(s) - \int_t^T Z_{ij}(s) dW(s), \\ Y_{ij}(t) \leq \min_{i' \neq i} \{Y_{i'j}(t) + k(i, i')\}, \\ Y_{ij}(t) \geq \max_{j' \neq j} \{Y_{ij'}(t) - l(j, j')\}, \\ \int_0^T \left(Y_{ij}(s) - \min_{i' \neq i} \{Y_{i'j}(s) + k(i, i')\} \right) dK_{ij}(s) = 0, \\ \int_0^T \left(Y_{ij}(s) - \max_{j' \neq j} \{Y_{ij'}(s) - l(j, j')\} \right) dL_{ij}(s) = 0. \end{array} \right. \quad (1.3)$$

Here, the unknowns are the processes $\{Y(t)\}_{t \in [0, T]}$, $\{Z(t)\}_{t \in [0, T]}$, $\{K(t)\}_{t \in [0, T]}$, and $\{L(t)\}_{t \in [0, T]}$, which are required to be adapted with respect to the natural completed

filtration of the Brownian motion W . Moreover, K and L are componentwisely increasing processes. The last two relations in (1.3) are called the upper and lower minimal boundary conditions.

One-dimensional RBSDEs were first studied by El Karoui et al. [7] in the case of one obstacle, and then by Cvitanic and Karatzas [3] in the case of two obstacles. In both papers, it is recognized that one-dimensional reflected BSDEs, with one obstacle and with two obstacles, are generalizations of optimal stopping and Dynkin games, respectively. Nowadays, the literature on one-dimensional reflected BSDEs is very rich. The reader is referred to Hamadène and Hassani [9] and Peng and Xu [17], among others, for the one-dimensional reflected BSDEs with two obstacles.

Multidimensional RBSDE was studied by Gegout-Petit and Pardoux [8], but their BSDE is reflected on the boundary of a convex domain along the inward normal direction, and their method depends heavily on the properties of this inward normal reflection (see (1)-(3) in [8]). We note that in a very special case (e.g., ψ is independent of z), Ramasubramanian [18] studied a BSDE in an orthant with oblique reflection. Multi-dimensional BSDEs reflected along an oblique direction rather than a normal direction, still remains to be open in general, even in a convex domain, let alone in a nonconvex domain. Note that there are some papers dealing with SDEs with oblique reflection (see, e.g. [14] and [5]).

Recently, the authors [13] studied the optimal switching problem for one-dimensional BSDEs, and the associated following type of obliquely reflected multi-dimensional BSDEs: for $i \in \Lambda$,

$$\left\{ \begin{array}{l} Y_i(t) = \xi_i + \int_t^T \psi(s, Y_i(s), Z_i(s), i) ds \\ \quad - \int_t^T dK_i(s) - \int_t^T Z_i(s) dW(s), \\ Y_i(s) \leq \min_{i' \neq i} \{Y_{i'}(s) + k(i, i')\}, \\ \int_0^T \left(Y_i(s) - \min_{i' \neq i} \{Y_{i'}(s) + k(i, i')\} \right) dK_i(s) = 0. \end{array} \right. \quad (1.4)$$

It should be added that an incomplete and less general form of RBSDE (1.4) (where the minimal condition of (1.4) is missing and the generator ψ does not depend on (y, z)) is suggested by [2]. But they did not discuss the existence and uniqueness of solution, which is considered to be difficult. See Remark 3.1 in [2]. Lately, Hamadène and Zhang [11] also studied RBSDE (1.4) in a more general form.

More recently, Tang and Zhong [22] discussed the mixed switching and stopping problem for one-dimensional BSDEs, and obtained the existence and uniqueness result for the the associated following type of obliquely reflected multi-dimensional BSDEs:

for $i \in \Lambda$ and $t \in [0, T]$,

$$\left\{ \begin{array}{l} Y_i(t) = \xi_i + \int_t^T \psi(s, Y_i(s), Z_i(s), i) ds \\ \quad - \int_t^T dK_i(s) + \int_t^T dL_i(s) - \int_t^T Z_i(s) dW(s), \\ Y_i(s) \leq \min_{i' \neq i} \{Y_{i'}(s) + k(i, i')\}, \quad Y_i(t) \geq S(t), \\ \int_0^T \left(Y_i(s) - \min_{i' \neq i} \{Y_{i'}(s) + k(i, i')\} \right) dK_i(s) = 0, \\ \int_0^T (Y_i(t) - S(t)) dL_i(t) = 0. \end{array} \right. \quad (1.5)$$

Here, S is a previously given $\{\mathcal{F}_t, 0 \leq t \leq T\}$ -adapted process with some suitable regularity.

RBSDE (1.3) arises from the switching game for BSDEs, and its form is more complicated than that of RBSDE (1.4), which arises from the optimal switching problem for BSDEs. For each fixed $j \in \Pi$, if we do not impose the following constraint:

$$Y_{ij}(t) \geq \max_{j' \neq j} \{Y_{ij'}(t) - l(j, j')\}, \quad t \in [0, T], \quad (1.6)$$

and its related boundary condition:

$$\int_0^T \left(Y_{ij}(s) - \max_{j' \neq j} \{Y_{ij'}(s) - l(j, j')\} \right) dL_{ij}(s) = 0, \quad (1.7)$$

then we can take $L \equiv 0$, and RBSDE (1.3) is reduced to RBSDE (1.4).

RBSDE (1.3) evolves in the closure \overline{Q} of domain Q :

$$\begin{aligned} Q := \left\{ (y_{ij}) \in \mathbb{R}^{m_1 \times m_2} : y_{ij} < y_{i'j} + k(i, i') \right. \\ \quad \text{for any } i, i' \in \Lambda \text{ such that } i' \neq i \text{ and } j \in \Pi; \\ \quad \quad \quad y_{ij} > y_{ij'} - l(j, j') \\ \quad \quad \quad \left. \text{for any } j, j' \in \Pi \text{ such that } j' \neq j \text{ and } i \in \Lambda \right\}, \end{aligned} \quad (1.8)$$

which is convex and unbounded. The boundary ∂Q of domain Q consists of the boundaries ∂D_{ij}^- and ∂D_{ij}^+ , $(i, j) \in \Lambda \times \Pi$, with

$$D_{ij}^- := \{(y_{ij}) \in \mathbb{R}^m : y_{ij} < y_{i'j} + k(i, i'), \text{ for any } i' \in \Lambda \text{ such that } i' \neq i\}$$

and

$$D_{ij}^+ := \{(y_{ij}) \in \mathbb{R}^m : y_{ij} > y_{ij'} - l(j, j'), \text{ for any } j' \in \Pi \text{ such that } j' \neq j\}$$

for $(i, j) \in \Lambda \times \Pi$. That is,

$$\partial Q = \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} \left(\partial D_{ij}^- \cup \partial D_{ij}^+ \right).$$

In the interior of \overline{Q} , each equation in (1.3) is independent of others. On the boundary, say ∂D_{ij}^- (resp. ∂D_{ij}^+), the (i, j) -th equation is switched to another one (i', j) (resp.

(i, j')), and the solution is reflected along the oblique direction $-e_{ij}$ (resp. e_{ij}), which is the negative (resp. positive) direction of the (i, j) -th coordinate axis.

The existence and uniqueness of solution for RBSDE (1.3) constitutes a main contribution of this paper. We prove the existence by a penalization method, whereas the uniqueness is obtained by a verification method: first we introduce an optimal switching problem for multi-dimensional RBSDEs of form (1.4), then we prove that the first component Y of any adapted solution (Y, Z, K, L) of RBSDE (1.3) is the (vector) value for the optimal switching problem.

Solution of RBSDE (1.3) presents new difficulties when one follows either our previous work [13] using the penalization method, or Hamadène and Zhang [11] using a Picard approximation. In fact, even for the proof of the existence, we have to use the representation for obliquely reflected BSDEs—an extended version of our previous representation result proved in [13]. Moreover, we have to impose, for the proof of the existence, the additional technical condition that the generator ψ is uniformly bounded.

There exist different methods in the literature for the study of switching control and game problems. For the classical method of quasi-variational inequalities, the reader is referred to the book of Bensoussan and Lions [1]. See Tang and Yong [21] and Tang and Hou [20] and the references therein for the theory of variational inequalities and the dynamic programming for optimal stochastic switching control and switching games. But these works are restricted to the Markovian case. Recently, using the method of Snell envelope (see, e.g. El Karoui [6]) combined with the theory of scalar valued RBSDEs, Hamadene and Jeanblanc [10] studied the switching problem with two modes (i.e., $m = 2$) in the non-Markovian context. Djehiche, Hamadene and Popier [4] generalized their result to the above switching problem with multi modes. The BSDE approach, firstly developed in Hu and Tang [13] for optimal stochastic switching and taking the advantage of modern theory and techniques of BSDEs, permits to state and solve these problems in a general non-Markovian framework. This paper is devoted to the development of the BSDE approach for stochastic switching games.

The paper is organized as follows: in Section 2, we introduce some notation and formulate the switching game for one-dimensional BSDEs. We prove the existence of solution by a penalization method in Section 3, whereas in Section 4 we study the uniqueness. The last section is devoted to the proof of the existence of the value process and the construction of the saddle point for our switching game.

2 Notations, and Formulation of our switching game

2.1 Notations

Let us fix a nonnegative real number $T > 0$. First of all, $W = \{W_t\}_{t \geq 0}$ is a standard Brownian motion with values in R^d defined on some complete probability space (Ω, \mathcal{F}, P) . $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of the Brownian motion W augmented by

the P -null sets of \mathcal{F} . All the measurability notions will refer to this filtration. In particular, the sigma-field of predictable subsets of $[0, T] \times \Omega$ is denoted by \mathcal{P} .

We denote by $S^2(R^{m_1 \times m_2})$ or simply by S^2 the set of $R^{m_1 \times m_2}$ -valued, adapted and càdlàg processes $\{Y(t)\}_{t \in [0, T]}$ such that

$$\|Y\|_{S^2} := E \left[\sup_{t \in [0, T]} |Y(t)|^2 \right]^{1/2} < +\infty.$$

$(S^2, \|\cdot\|_{S^2})$ is then a Banach space.

We denote by $M^2((R^{m_1 \times m_2})^d)$ or simply by M^2 the set of (equivalent classes of) predictable processes $\{Z(t)\}_{t \in [0, T]}$ with values in $(R^{m_1 \times m_2})^d$ such that

$$\|Z\|_{M^2} := E \left[\int_0^T |Z(s)|^2 ds \right]^{1/2} < +\infty.$$

M^2 is then a Banach space endowed with this norm.

We define also

$$N^2(R^{m_1 \times m_2}) : = \{K = (K_{ij}) \in S^2 : \text{for any } (i, j) \in \Lambda \times \Pi, K_{ij}(0) = 0, \\ \text{and } K_{ij}(\cdot) \text{ is increasing } \},$$

which is abbreviated as N^2 . $(N^2, \|\cdot\|_{S^2})$ is then a Banach space.

2.2 Hypotheses

Consider now the RBSDE (1.3). The generator ψ is a random function $\psi : [0, T] \times \Omega \times R \times R^d \times \Lambda \times \Pi \rightarrow R$ whose component $\psi(\cdot, i, j)$ is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(R) \otimes \mathcal{B}(R^d)$ for each pair $(i, j) \in \Lambda \times \Pi$, and the terminal condition ξ is simply an $R^{m_1 \times m_2}$ -valued \mathcal{F}_T -measurable random variable. The cost functions k and l for two players are defined on $\Lambda \times \Lambda$ and $\Pi \times \Pi$, respectively; their values are both positive.

We assume the following Lipschitz condition on the generator.

Hypothesis 2.1. (i) The generator $\psi(\cdot, 0, 0) := (\psi(\cdot, 0, 0, i, j))_{i \in \Lambda, j \in \Pi} \in M^2$.

(ii) There exists a constant $C > 0$ such that, for each $(t, y, y', z, z', i, j) \in [0, T] \times R \times R \times R^d \times R^d \times \Lambda \times \Pi$,

$$|\psi(t, y, z, i, j) - \psi(t, y', z', i, j)| \leq C(|y - y'| + |z - z'|), \quad a.s.$$

We make the following assumption on the cost functions k and l of both players, which is standard in the literature.

Hypothesis 2.2. (i) For $i \in \Lambda$, $k(i, i) = 0$. For $(i, i') \in \Lambda \times \Lambda$ such that $i \neq i'$, $k(i, i') > 0$.

(ii) For $j \in \Pi$, $l(j, j) = 0$. For $(j, j') \in \Pi \times \Pi$ such that $j \neq j'$, $l(j, j') > 0$.

(iii) For any $(i, i', i'') \in \Lambda \times \Lambda \times \Lambda$ such that $i \neq i'$ and $i' \neq i''$,

$$k(i, i') + k(i', i'') > k(i, i'').$$

(iv) For any $(j, j', j'') \in \Pi \times \Pi \times \Pi$ such that $j \neq j'$ and $j' \neq j''$,

$$l(j, j') + l(j', j'') > l(j, j'').$$

2.3 Statement of our switching game

Let $\{\theta_j\}_{j=0}^\infty$ be an increasing sequence of stopping times with values in $[0, T]$ and $\forall j$, α_j is an \mathcal{F}_{θ_j} -measurable random variable with values in Λ , and χ is the indicator function. We assume moreover that there exists an integer-valued random variable $N(\cdot)$ such that $\theta_N = T$ P -a.s. and $N \in L^2(\mathcal{F}_T)$. Then we define the admissible switching strategy for player I as follows:

$$a(s) = \alpha_0 \chi_{\{\theta_0\}}(s) + \sum_{j=1}^N \alpha_{j-1} \chi_{(\theta_{j-1}, \theta_j]}(s). \quad (2.1)$$

We denote by \mathcal{A} the set of all these admissible switching strategies for Player I , and by \mathcal{A}^i the subset of \mathcal{A} consisting of admissible switching strategies starting from the scheme $i \in \Lambda$. In the same way, we denote by \mathcal{A}_t the set of all the admissible strategies for Player I , starting at the time t (or equivalently $\theta_0 = t$), and by \mathcal{A}_t^i the subset of \mathcal{A}_t consisting of admissible switching strategies starting at time t from the scheme $i \in \Lambda$.

For any $a(\cdot) \in \mathcal{A}$, we define the associated (cost) process $A^{a(\cdot)}$ as follows:

$$A^{a(\cdot)}(s) = \sum_{j=1}^{N-1} k(\alpha_{j-1}, \alpha_j) \chi_{[\theta_j, T]}(s). \quad (2.2)$$

Obviously, $A^{a(\cdot)}(\cdot)$ is a càdlàg process.

In an identical way, we define the admissible switching strategy $b(\cdot)$ for Player II , and introduce the notations $\mathcal{B}, \mathcal{B}^j, \mathcal{B}_t, \mathcal{B}_t^j$ for the scheme $j \in \Pi$ for Player II , as well as $\mathcal{B}^{b(\cdot)}$ for $b(\cdot) \in \mathcal{B}$.

Now we are in position to introduce the switched BSDEs for both players.

$$\begin{aligned} U(s) &= \xi_{a(T)b(T)} + \left(A^{a(\cdot)}(T) - A^{a(\cdot)}(s) \right) - \left(B^{b(\cdot)}(T) - B^{b(\cdot)}(s) \right) \\ &\quad + \int_s^T \psi(r, U(r), V(r), a(r), b(r)) dr - \int_s^T V(r) dW(r). \end{aligned} \quad (2.3)$$

This is a (slightly) generalized BSDE: it is equivalent to the following standard BSDE:

$$\begin{aligned} \bar{U}(s) &= \xi_{a(T)b(T)} + A^{a(\cdot)}(T) - B^{b(\cdot)}(T) \\ &\quad + \int_s^T \psi(r, \bar{U}(r) - A^{a(\cdot)}(r) + B^{b(\cdot)}(r), \bar{V}(r), a(r)) dr \\ &\quad - \int_s^T \bar{V}(r) dW(r), \end{aligned} \quad (2.4)$$

via the simple change of variable:

$$\bar{U}(s) = U(s) + A^{a(\cdot)}(s) - B^{b(\cdot)}(s), \quad \bar{V}(s) = V(s).$$

Hence, BSDE (2.3) has a solution in $S^2 \times M^2$. We denote by $(U^{a(\cdot), b(\cdot)}, V^{a(\cdot), b(\cdot)})$ this solution. We note that U is only a càdlàg process.

The switching game problem with the initial scheme $(i, j) \in \Lambda \times \Pi$ is stated as follows: Player I aims to minimize $U^{a(\cdot), b(\cdot)}(t)$ over $a(\cdot) \in \mathcal{A}_t^i$, while Player II aims to maximize $U^{a(\cdot), b(\cdot)}(t)$ over $b(\cdot) \in \mathcal{B}_t^j$.

3 Existence

In this section, we state and prove our existence result for RBSDE (1.3).

We need the following additional technical assumption.

Hypothesis 3.1. *Assume that the generator ψ is uniformly bounded with respect to all its arguments.*

we shall use $\left| \psi \right|_{\infty}$ to denote the least upper bound of $|\psi|$.

Definition 3.1. *A solution to RBSDE (1.3) is defined to be a set $(Y, Z, K, L) = \{Y(t), Z(t), K(t), L(t)\}_{t \in [0, T]}$ of predictable processes with values in $(\mathbb{R}^{m_1 \times m_2})^{1+d+1+1}$ such that P -a.s., $t \mapsto Y(t)$, $t \mapsto K(t)$, and $t \mapsto L(t)$ are continuous, $t \mapsto Z(t)$ belongs to $L^2(0, T)$, $t \mapsto \psi(t, Y_i(t), Z_i(t), i)$ belongs to $L^1(0, T)$ and P -a.s., RBSDE (1.3) holds.*

We are now in position to state the existence result.

Theorem 3.1. *Let Hypotheses 2.1, 2.2 and 3.1 be satisfied. Assume that*

$$\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$$

takes values in \bar{Q} . Then RBSDE (1.3) has a solution (Y, Z, K, L) in $S^2 \times M^2 \times (N^2)^2$.

We shall use a penalization method to construct a solution to RBSDE (1.3). We observe (as mentioned in the introduction) that RBSDE (1.3) consists of the m_2 systems of m_1 -dimensional obliquely reflected BSDEs of the form like (1.4):

$$\left\{ \begin{array}{l} Y_{ij}(t) = \xi_{ij} + \int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j) ds \\ \quad - \int_t^T dK_{ij}(s) + \int_t^T dL_{ij}(s) - \int_t^T Z_{ij}(s) dW(s), \\ Y_{ij}(t) \leq \min_{i' \neq i} \{Y_{i'j}(t) + k(i, i')\}, \\ \int_0^T \left(Y_{ij}(s) - \min_{i' \neq i} \{Y_{i'j}(s) + k(i, i')\} \right) dK_{ij}(s) = 0; \quad i \in \Lambda, \end{array} \right. \quad (3.1)$$

with the unknown processes being

$$(Y_{ij}, Z_{ij}, K_{ij}; i = 1, 2, \dots, m_1)$$

(the process $(L_{1j}, \dots, L_{m_1j})$ is taken to be previously given) for $j = 1, 2, \dots, m_2$. These m_2 systems have been well studied by Hu and Tang [13]. In RBSDE (1.3), they are coupled together by the processes $(L_{1j}, \dots, L_{m_1j})$ through the constraint

$$Y_{ij}(t) \geq \max_{j' \neq j} \{Y_{ij'}(t) - l(j, j')\}, \quad (i, j) \in \Lambda \times \Pi \quad (3.2)$$

and the minimal boundary condition:

$$\int_0^T \left(Y_{ij}(s) - \max_{j' \neq j} \{Y_{ij'}(s) - l(j, j')\} \right) dL_{ij}(s) = 0, \quad (i, j) \in \Lambda \times \Pi. \quad (3.3)$$

Therefore, it is natural to consider the following penalized system of RBSDEs (the unknown processes are $(Y_{ij}, Z_{ij}, K_{ij}; i \in \Lambda, j \in \Pi)$):

$$\left\{ \begin{array}{l} Y_{ij}(t) = \xi_{ij} + \int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j) ds \\ \quad + n \sum_{j'=1}^{m_2} \int_t^T (Y_{ij}(s) - Y_{ij'}(s) + l(j, j'))^- ds \\ \quad - \int_t^T dK_{ij}(s) - \int_t^T Z_{ij}(s) dW(s); \\ Y_{ij}(t) \leq \min_{i' \neq i} \{Y_{i'j}(t) + k(i, i')\}; \\ \int_0^T \left(Y_{ij}(s) - \min_{i' \neq i} \{Y_{i'j}(s) + k(i, i')\} \right) dK_{ij}(s) = 0; \quad (i, j) \in \Lambda \times \Pi. \end{array} \right. \quad (3.4)$$

Note that when $j' = j$, we have, in view of Hypothesis 2.2 (ii),

$$(Y_{ij}(s) - Y_{ij'}(s) + l(j, j'))^- = 0. \quad (3.5)$$

A striking difference between RBSDE (3.4) and the RBSDE which is studied by Hu and Tang [13], lies in the fact that the i -th set of unknown variables

$$(y_i, z_i, k_i) \in R \times R^d \times R$$

in the latter is replaced in the former with

$$((Y_{ij})_{j \in \Pi}, (Z_{ij})_{j \in \Pi}, (K_{ij})_{j \in \Pi}) \in R^{m_2} \times (R^{m_2})^d \times R^{m_2}.$$

By slightly adapting the relevant arguments in our previous work [13], we can show the following theorem.

Theorem 3.2. *For any integer n , RBSDE (3.4) has a unique solution (Y^n, Z^n, K^n) in the space $S^2 \times M^2 \times N^2$. Furthermore, we have the following representation:*

$$Y_{ij}^n(t) = \operatorname{ess\,inf}_{a(\cdot) \in \mathcal{A}_t^i} U_j^{a(\cdot), n}(t), \quad (t, i, j) \in [0, T] \times \Lambda \times \Pi. \quad (3.6)$$

Here, for any $a(\cdot) \in \mathcal{A}_t^i$, $(U^{a(\cdot), n}, V^{a(\cdot), n})$ is the unique solution to the following BSDE:

$$\begin{aligned} U_j(s) &= \xi_{a(T)j} + [A^{a(\cdot)}(T) - A^{a(\cdot)}(s)] + \int_s^T \psi(r, U_j, V_j, a(r), j) dr \\ &\quad + n \sum_{j'=1}^{m_2} \int_s^T (U_j - U_{j'} + l(j, j'))^- dr - \int_s^T V_j(r) dW(r), \\ &\quad j \in \Pi, \quad s \in [t, T]. \end{aligned} \quad (3.7)$$

Intuitively, as n tends to $+\infty$, we expect that the sequence of solutions

$$\{(Y^n, Z^n, K^n)\}_{n=1}^\infty$$

together with the penalty term

$$L_{ij}^n(t) := n \sum_{j'=1}^{m_2} \int_0^t (Y_{ij}(s) - Y_{ij'}(s) + l(j, j'))^- ds, \quad (t, i, j) \in [0, T] \times \Lambda \times \Pi$$

will have a limit (Y, Z, K, L) , which solves RBSDE (1.3).

For this purpose, it is crucial to prove that the penalty term is bounded in some suitable sense. Then we are naturally led to compute

$$(Y_{ij}(t) - Y_{ij'}(t) + l(j, j'))^-,$$

using Itô-Tanaka's formula, as done in Hu and Tang [13]. However, in our present situation, the additional term K^n appears in RBSDE (3.4), which gives rise to a serious difficulty to derive the bound of L^n in the preceding procedure. In what follows, we shall use the representation result for Y^n in the preceding theorem to get around the difficulty.

We have the following lemma

Lemma 3.1. *For $j, j' \in \Pi, a(\cdot) \in \mathcal{A}_t^i$, we have*

$$n \left(U_j^{a(\cdot), n}(s) - U_{j'}^{a(\cdot), n}(s) + l(j, j') \right)^- \leq 2 \left| \psi \right|_\infty, \quad s \in [t, T]. \quad (3.8)$$

Proof. We suppress the superscripts $(a(\cdot), n)$ of $U_j^{a(\cdot), n}$ and $U_{j'}^{a(\cdot), n}$ for simplicity. The process $\bar{U}_{jj'}(s) := U_j(s) - U_{j'}(s) + l(j, j')$, $s \in [t, T]$ satisfies the following BSDE:

$$\begin{aligned} & \bar{U}_{jj'}(s) \\ = & \bar{U}_{jj'}(T) + \int_s^T [\psi(r, U_j(r), V_j(r), a(r), j) - \psi(r, U_{j'}(r), V_{j'}(r), a(r), j')] dr \\ & + n \sum_{j''=1}^{m_2} \int_s^T \bar{U}_{jj''}(r)^- dr - n \sum_{j''=1}^{m_2} \int_s^T \bar{U}_{j'j''}(r)^- dr \\ & - \int_s^T (V_j(r) - V_{j'}(r)) dW(r), \quad s \in [t, T]. \end{aligned} \quad (3.9)$$

Applying Tanaka's formula (see, e.g. [19]), we have

$$\begin{aligned} & \bar{U}_{jj'}(s)^- + n \sum_{j''=1}^{m_2} \int_s^T \chi_{\mathcal{L}_{jj''}^-}(r) \bar{U}_{jj''}(r)^- dr \\ & - n \sum_{j''=1}^{m_2} \int_s^T \chi_{\mathcal{L}_{j'j''}^-}(r) \bar{U}_{j'j''}(r)^- dr + \frac{1}{2} \int_s^T d\hat{L}_{jj'}(r) \\ = & \int_s^T \chi_{\mathcal{L}_{jj'}^-}(r) [\psi(r, U_j(r), V_j(r), a(r), j) - \psi(r, U_{j'}(r), V_{j'}(r), a(r), j')] dr \\ & - \int_s^T \chi_{\mathcal{L}_{jj'}^-}(r) (V_j(r) - V_{j'}(r)) dW(r), \quad s \in [t, T] \end{aligned} \quad (3.10)$$

where

$$\mathcal{L}_{jj'}^- := \{(s, \omega) \in [t, T] \times \Omega : \bar{U}_{jj'}(s) < 0\}, \quad (3.11)$$

and $\widehat{L}_{jj'}$ is the local time of the process $\bar{U}_{jj'}$ at 0. We have

$$\begin{aligned}
& \bar{U}_{jj'}(s)^- + n \int_s^T \bar{U}_{jj'}(r)^- dr + \frac{1}{2} \int_s^T d\widehat{L}_{jj'}(r) \\
= & \int_s^T \mathcal{L}_{jj'}^-(r) [\psi(r, U_j(r), V_j(r), a(r), j) - \psi(r, U_{j'}(r), V_{j'}(r), a(r), j')] dr \\
& - \int_s^T \mathcal{L}_{jj'}^-(r) (V_j(r) - V_{j'}(r)) dW(r) \\
& + n \int_s^T (\mathcal{L}_{jj'}^-(r))^- (\bar{U}_{jj'}(r))^- dr \\
& + n \sum_{j'' \neq j, j'' \neq j'} \int_s^T \mathcal{L}_{jj'}^-(r) [\bar{U}_{jj''}(r)^- - \bar{U}_{jj''}(r)^-] dr.
\end{aligned} \tag{3.12}$$

We claim that the integrands of the integrals in the last two terms of (3.12) are all less than or equal to zero. In fact, since

$$\{y \in R^m : y_j - y_{j'} + l(j, j') < 0\} \cap \{y \in R^m : y_{j'} - y_j + l(j', j) < 0\} = \emptyset,$$

due to the fact that

$$l(j, j') + l(j', j) > l(j, j) = 0,$$

we have

$$\mathcal{L}_{jj'}^-, \bar{U}_{jj'}^- = 0, \quad j, j' \in \Pi. \tag{3.13}$$

Secondly, for $j, j', j'' \in \Pi$, taking into consideration both Hypothesis 2.2 (ii), i.e.,

$$l(j, j') + l(j', j'') > l(j, j''),$$

and the elementary inequality that $x_1^- - x_2^- \leq (x_1 - x_2)^-$, for any two real numbers x_1 and x_2 , we have

$$\begin{aligned}
& \mathcal{L}_{jj'}^-(\bar{U}_{jj''}^- - \bar{U}_{jj''}^-) \\
\leq & \mathcal{L}_{jj'}^-(\bar{U}_{jj''}^- - \bar{U}_{jj''}^-) = \mathcal{L}_{jj'}^-(U_{j'} - U_j + l(j', j'') - l(j, j''))^-.
\end{aligned} \tag{3.14}$$

The last equality holds in the last relations, since

$$\{y \in R^m : y_j - y_{j'} + l(j, j') < 0\} \cap \{y \in R^m : y_{j'} - y_j + l(j', j'') - l(j, j'') < 0\} = \emptyset.$$

Concluding the above, we have

$$\begin{aligned}
& \bar{U}_{jj'}(s)^- + n \int_s^T \bar{U}_{jj'}(r)^- dr + \frac{1}{2} \int_s^T d\widehat{L}_{jj'}(r) \\
= & \int_s^T \mathcal{L}_{jj'}^-(r) [\psi(r, U_j(r), V_j(r), a(r), j) - \psi(r, U_{j'}(r), V_{j'}(r), a(r), j')] dr \\
& - \int_s^T \mathcal{L}_{jj'}^-(r) (V_j(r) - V_{j'}(r)) dW(r).
\end{aligned} \tag{3.15}$$

Using Itô's formula, we have

$$\begin{aligned}
& \exp(-ns)\bar{U}_{jj'}(s)^- + \frac{1}{2} \int_s^T \exp(-nr) d\widehat{L}_{jj'}(r) \\
&= \int_s^T \exp(-nr) \mathcal{L}_{jj'}^-(r) [\psi(r, U_j(r), V_j(r), a(r), j) - \psi(r, U_{j'}(r), V_{j'}(r), a(r), j')] dr \\
& \quad - \int_s^T \mathcal{L}_{jj'}^-(r) \exp(-nr) (V_j(r) - V_{j'}(r)) dW(r).
\end{aligned} \tag{3.16}$$

Taking the conditional expectation with respect to \mathcal{F}_s on the both sides of the last equality, in view of Hypothese 3.1, we have

$$\begin{aligned}
& \exp(-ns) (U_j(s) - U_{j'}(s) + l(j, j'))^+ \\
& \leq 2 \int_s^T \exp(-nr) \left| \psi \right|_{\infty} dr \\
& \leq 2n^{-1} \left| \psi \right|_{\infty} [\exp(-ns) - \exp(-nT)] \\
& \leq 2n^{-1} \left| \psi \right|_{\infty} \exp(-ns).
\end{aligned} \tag{3.17}$$

This ends the proof. \square

We have the following estimates for the L^∞ bound of Y_{ij}^n .

Lemma 3.2. *The sequence $\{Y_{ij}^n(t)\}_{n=1}^\infty$ is decreasing. Moreover,*

$$-E[|\xi||\mathcal{F}_t] - |\psi|_{\infty} T \leq Y_{ij}^n(t) \leq E[|\xi||\mathcal{F}_t] + 3|\psi|_{\infty} T; \quad E\left[\sup_t |Y_{ij}^n(t)|^2\right] \leq c. \tag{3.18}$$

Proof. In view of the comparison result for multi-dimensional RBSDEs of Tang and Zhong [22] (which is a natural generalization to RBSDEs of the comparison theorem for multi-dimensional BSDEs), we see that the sequence $\{Y_{ij}^n(t)\}_{n=1}^\infty$ is decreasing.

We have the following two facts:

- (1) $U_j^{a(\cdot), n}(s) \geq -E[|\xi||\mathcal{F}_t] - |\psi|_{\infty} T$.
- (2) Take $\bar{a}(\cdot) \equiv i$. We have

$$U_j^{\bar{a}(\cdot), n}(s) \leq E[|\xi||\mathcal{F}_t] + |\psi|_{\infty} T + 2|\psi|_{\infty} T. \tag{3.19}$$

In view of the representation formula in Theorem 3.2, we conclude the proof. \square

Therefore, from the previous representation result, we have

Lemma 3.3. *We have*

$$n (Y_{ij}^n(t) - Y_{i'j'}^n(t) + l(j, j'))^- \leq 2 \left| \psi \right|_{\infty}. \tag{3.20}$$

Proof. We have

$$Y_{ij}^n(t) - Y_{i'j'}^n(t) + l(j, j') \leq \operatorname{ess\,sup}_{a(\cdot) \in \mathcal{A}_t^i} \left(U_j^{a(\cdot), n}(t) - U_{j'}^{a(\cdot), n}(t) + l(j, j') \right). \tag{3.21}$$

Hence,

$$\begin{aligned} & (Y_{ij}^n(t) - Y_{ij'}^n(t) + l(j, j'))^- \\ \leq & \operatorname{ess\,sup}_{a(\cdot) \in \mathcal{A}_t^i} \left(U_j^{a(\cdot), n}(t) - U_{j'}^{a(\cdot), n}(t) + l(j, j') \right)^- \leq 2n^{-1} |\psi|_\infty. \end{aligned} \quad (3.22)$$

□

We have

Lemma 3.4. *The pair of processes (Z_{ij}^n, K_{ij}^n) are uniformly bounded in $M^2 \times N^2$ for $(i, j) \in \Lambda \times \Pi$.*

Proof. From the RBSDE for Y_{ij}^n , using Itô's formula and Lemmas 3.2 and 3.3, we have

$$\begin{aligned} & E|Y_{ij}^n(0)|^2 + E \int_0^T |Z_{ij}^n(s)|^2 ds \\ \leq & E|Y_{ij}^n(T)|^2 + E \int_0^T |Y_{ij}^n(s)| n (Y_{ij}^n(s) - Y_{ij'}^n(s) + l(j, j'))^- ds \\ & + E \int_0^T |Y_{ij}^n(s)| \cdot |\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), a(s), j) - \psi(s, Y_{ij'}^n(s), Z_{ij'}^n(s), a(s), j')| ds \\ & + E \int_0^T |Y_{ij}^n(s)| dK_{ij}^n(s) \\ \leq & E|Y_{ij}^n(T)|^2 + 3|\psi|_\infty E \int_0^T |Y_{ij}^n(s)| ds + E \left[\sup_t |Y_{ij}^n(t)|^2 K_{ij}^n(T) \right] \\ \leq & C_\epsilon + \epsilon E[(K_{ij}^n(T))^2] \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} & E[(K_{ij}^n(T))^2] \\ \leq & CE|Y_{ij}^n(T)|^2 + CE|Y_{ij}^n(0)|^2 + CE \int_0^T n \left[(Y_{ij}^n(s) - Y_{ij'}^n(s) + l(j, j'))^- \right]^2 ds \\ & + CE \int_0^T |\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), a(s), j) - \psi(s, Y_{ij'}^n(s), Z_{ij'}^n(s), a(s), j')|^2 ds \\ & + CE \int_0^T |Z_{ij}^n(s)|^2 ds \\ \leq & c + CE \int_0^T |Z_{ij}^n(s)|^2 ds. \end{aligned} \quad (3.24)$$

They together conclude the proof. □

Define

$$\beta_{ij}^n(s) := n \sum_{j'=1}^{m_2} (Y_{ij}^n(s) - Y_{ij'}^n(s) + l(j, j'))^-. \quad (3.25)$$

Then,

$$L_{ij}^n(t) = \int_0^t \beta_{ij}^n(s) ds. \quad (3.26)$$

We have

Lemma 3.5. For $(i, j) \in \Lambda \times \Pi$ and integer n , there is a uniformly bounded process α_{ij}^n such that K_{ij}^n has the following form:

$$K_{ij}^n(t) = \int_0^t \alpha_{ij}^n(s) ds, \quad t \in [0, T]. \quad (3.27)$$

The two matrix-valued processes

$$\{\alpha^n\}_{n=1}^\infty := \{(\alpha_{ij}^n; i \in \Lambda, j \in \Pi)\}_{n=1}^\infty$$

and

$$\{\beta^n\}_{n=1}^\infty := \{(\beta_{ij}^n; i \in \Lambda, j \in \Pi)\}_{n=1}^\infty$$

are bounded in M^2 .

Proof. Fix the integer n . Consider the following penalized BSDEs:

$$\begin{aligned} Y_{ij}^{n,m}(t) &= \xi_{ij} + \int_t^T [\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), i, j) + \beta_{ij}^n(s)] ds \\ &\quad - m \sum_{i'=1}^{m_1} \int_t^T \left(Y_{ij}^{n,m} - Y_{i'j}^{n,m} - k(i, i') \right)^+ ds \\ &\quad - \int_t^T Z_{ij}^{n,m}(s) dW(s), \end{aligned} \quad (3.28)$$

with $(i, j) \in \Lambda \times \Pi$ and $t \in [0, T]$. It has a unique solution, denoted by $(Y_{ij}^{n,m}, Z_{ij}^{n,m})$.

Proceeding identically as in Lemma 3.2, we can prove that

$$\alpha_{ij}^{n,m} := m \sum_{i'=1}^{m_1} \left(Y_{ij}^{n,m} - Y_{i'j}^{n,m} - k(i, i') \right)^+ \leq c. \quad (3.29)$$

Therefore, $\{\alpha_{ij}^{n,m}\}_{m=1}^\infty$ has a weak limit in $\mathcal{L}_{\mathcal{F}}^2(0, T)$, denoted by α_{ij}^n . Then α_{ij}^n is also uniformly bounded by the same constant c .

Define

$$K_{ij}^{n,m}(t) := \int_0^t \alpha_{ij}^{n,m}(s) ds, \quad t \in [0, T]. \quad (3.30)$$

From Hu and Tang [13], we have

$$\begin{aligned} \lim_{m \rightarrow \infty} Y_{ij}^{n,m}(t) &= Y_{ij}^n(t), \\ \lim_{m \rightarrow \infty} Z_{ij}^{n,m}(t) &= Z_{ij}^n(t), \\ \lim_{m \rightarrow \infty} K_{ij}^{n,m}(t) &= \int_0^t \alpha_{ij}^n(s) ds = K_{ij}^n(t). \end{aligned} \quad (3.31)$$

□

We have

Lemma 3.6. The sequence $\{Y^n, Z^n\}$ has a strong limit (Y, Z) in $S^2 \times M^2$. The two sequences $\{\alpha^n\}$ and $\{\beta^n\}$ have subsequences which converge to α and β weakly in M^2 , respectively.

Proof. Note that Y_{ij}^n is decreasing in n . In view of Lemma 3.2, using Lebesgue dominant convergence theorem, we can show the strong convergence of $\{Y^n\}$ in the space M^2 . Note that (Y^n, Z^n) solves the following BSDE:

$$\begin{aligned} Y_{ij}^n(t) &= \xi_{ij} + \int_t^T (\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), i, j) + \beta_{ij}^n(s) - \alpha_{ij}^n(s)) ds \\ &\quad - \int_t^T Z_{ij}^n(s) dW(s), \quad (i, j) \in \Lambda \times \Pi, \end{aligned} \quad (3.32)$$

with $\{\alpha^n\}$ and $\{\beta^n\}$ being bounded in M^2 .

We now prove the strong convergence of Z^n . Using Itô's formula, we have

$$\begin{aligned} &|Y_{ij}^n(0) - Y_{ij}^m(0)|^2 + E \int_0^T |Z_{ij}^n(s) - Z_{ij}^m(s)|^2 ds \\ &= 2E \int_0^T (Y_{ij}^n(s) - Y_{ij}^m(s)) (\psi(s, Y_{ij}^n(s), Z_{ij}^n(s), i, j) - \alpha_{ij}^n(s) + \beta_{ij}^n(s)) ds \\ &\quad - 2E \int_0^T (Y_{ij}^n(s) - Y_{ij}^m(s)) (\psi(s, Y_{ij}^m(s), Z_{ij}^m(s), i, j) - \alpha_{ij}^m(s) + \beta_{ij}^m(s)) ds. \end{aligned} \quad (3.33)$$

Using Cauchy-Schwartz inequality and Hypothesis 3.1, we have

$$\lim_{n, m \rightarrow \infty} E \int_0^T |Z^n - Z^m|^2 ds = 0. \quad (3.34)$$

Then, it is standard to show the strong convergence of $\{Y^n\}$ in the space S^2 .

Since (see Lemma 3.5) we have

$$\left| \alpha^n \right|_{M^2} \leq c, \quad \left| \beta^n \right|_{M^2} \leq c, \quad (3.35)$$

the last assertion of the lemma is obvious. \square

Proof of Theorem 3.1 : Define for $(i, j, t) \in \Lambda \times \Pi \times [0, T]$,

$$K_{ij}(t) := \int_0^t \alpha_{ij}(s) ds, \quad L_{ij}(t) := \int_0^t \beta_{ij}(s) ds \quad (3.36)$$

and

$$K := (K_{ij}), \quad L := (L_{ij}). \quad (3.37)$$

We shall show that (Y, Z, K, L) solves RBSDE (1.3).

First, fixing $t \in [0, T]$ and letting $n \rightarrow \infty$ in BSDE (3.32), we take the weak limit in $L^2(\mathcal{F}_T)$. Then, we see that (Y, Z, K, L) solves the following BSDE:

$$\begin{aligned} Y_{ij}(t) &= \xi_{ij} + \int_t^T \psi(s, Y_{ij}(s), Z_{ij}(s), i, j) ds - \int_t^T dK_{ij}(s) \\ &\quad + \int_t^T dL_{ij}(s) - \int_t^T Z_{ij}(s) dW(s), \quad (i, j) \in \Lambda \times \Pi. \end{aligned} \quad (3.38)$$

Now we check out the boundary conditions. From Hu and Tang [13], we have

$$E \int_0^T \left(Y_{ij}^n(s) - \min_{i' \neq i} \{Y_{i'j}^n(s) + k(i, i')\} \right)^- \alpha_{ij}^n(s) ds = 0. \quad (3.39)$$

Setting $n \rightarrow \infty$, we have

$$E \int_0^T \left(Y_{ij}(s) - \min_{i' \neq i} \{Y_{i'j}(s) + k(i, i')\} \right)^- \alpha_{ij}(s) ds = 0. \quad (3.40)$$

That is,

$$E \int_0^T \left(Y_{ij}(s) - \min_{i' \neq i} \{Y_{i'j}(s) + k(i, i')\} \right)^- dK_{ij}(s) = 0. \quad (3.41)$$

On the other hand, from the construction, we have

$$E \int_0^T \left(Y_{ij}^n(s) - \max_{j' \neq j} \{Y_{ij'}^n(s) - l(j, j')\} \right)^+ \beta_{ij}^n(s) ds = 0. \quad (3.42)$$

Setting $n \rightarrow \infty$, we have

$$E \int_0^T \left(Y_{ij}(s) - \max_{j' \neq j} \{Y_{ij'}(s) - l(j, j')\} \right)^+ \beta_{ij}(s) ds = 0. \quad (3.43)$$

That is,

$$E \int_0^T \left(Y_{ij}(s) - \max_{j' \neq j} \{Y_{ij'}(s) - l(j, j')\} \right)^+ dL_{ij}(s) = 0. \quad (3.44)$$

The proof is then complete. \square

4 Uniqueness

In this section, we prove the uniqueness by a verification method. Let (Y, Z, K, L) be a solution in the space $S^2 \times M^2 \times N^2 \times N^2$ to RBSDE (1.3). We shall prove that \tilde{Y} is in fact the (vector) value for an optimal switching problem of RBSDEs. For this purpose, we introduce the following optimal switching problem for RBSDEs.

For $a(\cdot) \in \mathcal{A}_t^i$, we denote by $(U_j^{a(\cdot)}, V_j^{a(\cdot)}, K_j^{a(\cdot)}; j \in \Pi)$ the unique solution of the following RBSDEs:

$$\begin{aligned} U_j^{a(\cdot)}(s) &= \xi_{a(T)j} + \left(A^{a(\cdot)}(T) - A^{a(\cdot)}(s) \right) \\ &\quad + \int_s^T \psi(r, U_j^{a(\cdot)}(r), V_j^{a(\cdot)}(r), a(r), j) dr \\ &\quad + \int_s^T dL_j(r) - \int_s^T V_j^{a(\cdot)}(r) dW(r), \quad s \in [t, T]; \\ &\int_0^T \left(U_j^{a(\cdot)} - \max_{j' \neq j} \{U_{j'}^{a(\cdot)} - l(j, j')\} \right) dL_j(r) = 0; \quad j \in \Pi. \end{aligned} \quad (4.1)$$

We have the following representation for the first component of the adapted solution to RBSDE (1.3), which immediately implies the uniqueness of the adapted solution to RBSDE (1.3).

Theorem 4.1. *Suppose that Hypotheses 2.1 and 2.2 hold. Assume that the random variable*

$$\xi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$$

takes values in \bar{Q} . Let (Y, Z, K, L) be a solution in the space $S^2 \times M^2 \times N^2 \times N^2$ to RBSDE (1.3). Then we have the representation:

$$Y_{ij}(t) = \operatorname{ess\,inf}_{a(\cdot) \in \mathcal{A}_t^i} U_j^{a(\cdot)}(t), \quad (i, j, t) \in \Lambda \times \Pi \times [0, T]. \quad (4.2)$$

Proof. Assume $t = 0$ without loss of generality. Otherwise, it suffices to consider the admissible switching strategies starting at time t .

For the following $a(\cdot) \in \mathcal{A}_0^i$:

$$a(s) = i\chi_0(s) + \sum_{p=1}^N \alpha_{p-1} \chi_{[\theta_{p-1}, \theta_p]}(s), \quad s \in [0, T], \quad (4.3)$$

we define for $s \in [0, T]$ and $j \in \Pi$,

$$\tilde{Y}_j^{a(\cdot)}(s) := \sum_{p=1}^N Y_{\alpha_{p-1}, j}(s) \chi_{[\theta_{p-1}, \theta_p]}(s) + \xi_{a(T)} \chi_{\{T\}}(s), \quad (4.4)$$

$$\tilde{Z}_j^{a(\cdot)}(s) := \sum_{p=1}^N Z_{\alpha_{p-1}, j}(s) \chi_{[\theta_{p-1}, \theta_p]}(s), \quad (4.5)$$

$$\tilde{K}_j^{a(\cdot)}(s) := \sum_{p=1}^N \int_{\theta_{p-1} \wedge s}^{\theta_p \wedge s} dK_{\alpha_{p-1}, j}(r), \quad (4.6)$$

$$\tilde{L}_j^{a(\cdot)}(s) := \sum_{p=1}^N \int_{\theta_{p-1} \wedge s}^{\theta_p \wedge s} dL_{\alpha_{p-1}, j}(r). \quad (4.7)$$

Noting that $\tilde{Y}_j^{a(\cdot)}(\cdot)$ is a càdlàg process with jump $Y_{\alpha_p, j}(\theta_p) - Y_{\alpha_{p-1}, j}(\theta_p)$ at θ_p , $p = 1, \dots, N-1$, we deduce that

$$\begin{aligned} & \tilde{Y}_j^{a(\cdot)}(s) - \tilde{Y}_j^{a(\cdot)}(0) \\ &= \sum_{p=1}^N \int_{\theta_{p-1} \wedge s}^{\theta_p \wedge s} [-\psi(r, Y_{\alpha_{p-1}, j}(r), Z_{\alpha_{p-1}, j}(r), \alpha_{p-1}, j) dr + Z_{\alpha_{p-1}, j}(r) dW(r) \\ & \quad + dK_{\alpha_{p-1}, j}(r) - dL_{\alpha_{p-1}, j}(r)] \\ & \quad + \sum_{q=1}^{N-1} [Y_{\alpha_q, j}(\theta_q) - Y_{\alpha_{q-1}, j}(\theta_q)] \chi_{[\theta_q, T]}(s) \\ &= \sum_{p=1}^N \int_{\theta_{p-1} \wedge s}^{\theta_p \wedge s} [-\psi(r, \tilde{Y}_{\alpha_{p-1}, j}(r), \tilde{Z}_{\alpha_{p-1}, j}(r), \alpha_{p-1}, j) dr + \tilde{Z}_{\alpha_{p-1}, j}(r) dW(r) \\ & \quad + d\tilde{K}_{\alpha_{p-1}, j}(r) - d\tilde{L}_{\alpha_{p-1}, j}(r)] \\ & \quad + \tilde{A}_j^{a(\cdot)}(s) - A^{a(\cdot)}(s), \end{aligned}$$

where

$$\tilde{A}_j^{a(\cdot)}(s) = \sum_{p=1}^{N-1} [Y_{\alpha_p, j}(\theta_p) + k(\alpha_{p-1}, \alpha_p) - Y_{\alpha_{p-1}, j}(\theta_p)] \chi_{[\theta_p, T]}(s), \quad s \in [0, T], \quad (4.8)$$

and it is an increasing process due to the fact that

$$Y(t) \in \bar{Q}, \quad \forall t \in [0, T].$$

Consequently, we conclude that $(\tilde{Y}_j^{a(\cdot)}, \tilde{Z}_j^{a(\cdot)}, \tilde{L}_j^{a(\cdot)}; j \in \Pi)$ is a solution of the following RBSDE:

$$\begin{aligned} & \tilde{Y}_j^{a(\cdot)}(s) \\ = & \xi_{a(T)j} - [(\tilde{K}_j^{a(\cdot)}(T) + \tilde{A}_j^{a(\cdot)}(T)) - (\tilde{K}_j^{a(\cdot)}(s) + \tilde{A}_j^{a(\cdot)}(s))] \\ & + A^{a(\cdot)}(T) - A^{a(\cdot)}(s) + \tilde{L}_j^{a(\cdot)}(T) - \tilde{L}_j^{a(\cdot)}(s) \\ & + \int_s^T \psi(r, \tilde{Y}_j^{a(\cdot)}(r), \tilde{Z}_j^{a(\cdot)}(r), a(r), j) dr - \int_s^T \tilde{Z}_j^{a(\cdot)}(r) dW(r), \quad j \in \Pi \end{aligned} \quad (4.9)$$

with the following boundary condition:

$$\int_0^T \left(\tilde{Y}_j^{a(\cdot)}(r) - \max_{j' \neq j} \left\{ \tilde{Y}_{j'}^{a(\cdot)}(r) - l(j, j') \right\} \right) d\tilde{L}_j^{a(\cdot)}(r) = 0, \quad j \in \Pi. \quad (4.10)$$

Since both $\tilde{K}^{a(\cdot)}$ and $\tilde{A}^{a(\cdot)}$ are increasing càdlàg processes, from the comparison theorem of Tang and Zhong [22] for multi-dimensional RBSDEs, we conclude that

$$\tilde{Y}_j^{a(\cdot)}(0) \leq U_j^{a(\cdot)}(0), \quad j \in \Pi, a(\cdot) \in \mathcal{A}_0^i$$

which implies that

$$Y_{ij}(0) \leq \operatorname{ess\,inf}_{a(\cdot) \in \mathcal{A}_0^i} U_j^{a(\cdot)}(0), \quad j \in \Pi.$$

On the other hand, we set $\theta_0^* = 0, \alpha_0^* = i$. We define the sequence $\{\theta_p^*, \alpha_p^*\}_{p=1}^\infty$ in an inductive way as follows:

$$\theta_p^* := \inf\{s \geq \theta_{p-1}^* : Y_{\alpha_{p-1}^*, j}(s) = \min_{p' \neq \alpha_{j-1}^*} \{Y_{p', j}(s) + k(\alpha_{p-1}^*, p')\} \wedge T, \quad (4.11)$$

and α_p^* is an $\mathcal{F}_{\theta_p^*}$ -measurable random variable such that

$$Y_{\alpha_{p-1}^*, j}(\theta_p^*) = Y_{\alpha_p^*, j}(\theta_p^*) + k(\alpha_{p-1}^*, \alpha_p^*),$$

with $p = 1, 2, \dots$. We define

$$a^*(s) := i\chi_{\{0\}}(s) + \sum_{p=1}^N \alpha_{p-1}^* \chi_{(\theta_{p-1}^*, \theta_p^*]}(s), \quad s \in [0, T]. \quad (4.12)$$

Then, we have

$$\tilde{K}_j^{a^*(\cdot)} \equiv 0, \quad \tilde{A}_j^{a^*(\cdot)} \equiv 0. \quad (4.13)$$

Moreover, we have

$$\left(\tilde{Y}_j^{a^*(\cdot)}, \tilde{Z}_j^{a^*(\cdot)}, \tilde{L}_j^{a^*(\cdot)}; j \in \Pi \right) \in S^2 \times M^2 \times N^2, \quad (4.14)$$

and it satisfies the following RBSDE:

$$\begin{aligned} \tilde{Y}_j^{a^*(\cdot)}(s) &= \xi_{a^*(T)j} + A^{a^*(\cdot)}(T) - A^{a^*(\cdot)}(s) + \tilde{L}_j^{a^*(\cdot)}(T) - \tilde{L}_j^{a^*(\cdot)}(s) \\ &\quad + \int_s^T \psi(r, \tilde{Y}_j^{a^*(\cdot)}(r), \tilde{Z}_j^{a^*(\cdot)}(r), a^*(r), j) dr \end{aligned} \quad (4.15)$$

$$- \int_s^T \tilde{Z}_j^{a^*(\cdot)}(r) dW(r), \quad j \in \Pi. \quad (4.16)$$

Now, it is standard to show from RBSDE (4.15) that $A^{a^*(\cdot)}(T) \in L^2(\mathcal{F}_T)$. Then, following the same arguments of Hu and Tang [13], we see that P -a.s. ω , there exists an integer $N(\omega) \in L^2(\mathcal{F}_T)$ such that $\theta_N^* = T$. The switching strategy $a^*(\cdot)$ is admissible, i.e., $a^*(\cdot) \in \mathcal{A}_0^i$. Moreover,

$$Y_i(0) = U^{a^*(\cdot)}(0).$$

The proof is complete. \square

5 Existence of the value and the saddle point.

We denote by $(Y_{ij}, Z_{ij}, K_{ij}, L_{ij}; i \in \Lambda, j \in \Pi)$ the unique solution of RBSDEs (1.3). We define $(a^*(\cdot), b^*(\cdot))$ as follows.

We define

$$\theta_0^* := 0, \tau_0^* := 0; \alpha_0^* := i, \beta_0^* := j. \quad (5.1)$$

We define stopping times θ_p^*, τ_p^* , Λ -valued random variable α_p^* , and Π -valued random variable β_p^* for $p = 1, 2, \dots$, in the following inductive manner:

$$\begin{aligned} \theta_p^* &:= \inf \left\{ s \geq \theta_{p-1}^* \wedge \tau_{p-1}^* : Y_{\alpha_{p-1}^*, \beta_{p-1}^*}(s) = \min_{i' \neq i} \{ Y_{i', \beta_{p-1}^*}(s) + k(\alpha_{p-1}^*, i') \} \right\} \wedge T, \\ \tau_p^* &:= \inf \left\{ s \geq \theta_{p-1}^* \wedge \tau_{p-1}^* : Y_{\alpha_{p-1}^*, \beta_{p-1}^*}(s) = \max_{j' \neq j} \{ Y_{\alpha_{p-1}^*, j'}(s) - l(\beta_{p-1}^*, j') \} \right\} \wedge T. \end{aligned} \quad (5.2)$$

Then we choose α_p^* such that $\alpha_p^* := \alpha_{p-1}^*$ if $\theta_p^* > \tau_p^*$ and otherwise it is taken from Λ to satisfy the following identity:

$$Y_{\alpha_{p-1}^*, \beta_{p-1}^*}(\theta_p^*) = Y_{\alpha_p^*, \beta_{p-1}^*}(\theta_p^*) + k(\alpha_{p-1}^*, \alpha_p^*). \quad (5.3)$$

Obviously, α_p^* is $\mathcal{F}_{\theta_p^* \wedge \tau_p^*}$ -measurable. We also choose β_p^* such that $\beta_p^* := \beta_{p-1}^*$ if $\tau_p^* \geq \theta_p^*$ and otherwise it is taken from Π to satisfy the following identity:

$$Y_{\alpha_{p-1}^*, \beta_{p-1}^*}(\tau_p^*) = Y_{\alpha_{p-1}^*, \beta_p^*}(\tau_p^*) - l(\beta_{p-1}^*, \beta_p^*). \quad (5.4)$$

Obviously, β_p^* is $\mathcal{F}_{\theta_p^* \wedge \tau_p^*}$ -measurable.

Definition 5.1. A sequence $\{i_p, j_p\}_{p=0}^N$ where $i_p \in \Lambda$ and $j_p \in \Pi$ for $p = 0, 1, \dots, N$ is called a loop if

(i) either $i_{p-1} = i_p$ or $j_{p-1} = j_p$ for each $p = 1, \dots, N-1$

and

(ii) $i_N = i_0$ and $j_N = j_0$.

A loop who contains no other loop (except itself) is called a primary loop.

Define for $i, p \in \Lambda, j, q \in \Pi$, and either $i = p$ or $j = q$, the following set:

$$F_{ip}^{jq} := \left\{ y = (y_{i'p'}) \in R^{m_1 \times m_2} : \begin{aligned} & y_{ij} = y_{pj} + k(i, p) \text{ if } j = q; \\ & \text{or } y_{ij} = y_{iq} - l(j, q) \text{ if } i = p \end{aligned} \right\}. \quad (5.5)$$

In this section, we need the following additional assumption, which is standard in the literature of switching games (see, e.g., Tang and Hou [20, Hypothesis 4, page 924]).

Hypothesis 5.1. *For any loop $\{i_p, j_p\}_{p=0}^N$ with $N \leq m_1 m_2$, we have*

$$\sum_{p=0}^{N-1} k(i_p, i_{p+1}) - \sum_{p=0}^{N-1} l(j_p, j_{p+1}) \neq 0. \quad (5.6)$$

The assumption of no loop of zero cost implies that

$$\inf_{y(i_p j_p) \in F_{i_{p-1} j_{p-1}}^{j_p i_p} \cap \{y: |y| \leq Y^*(T, \omega)\} \cap \bar{Q}} \left(\sum_{p=1}^N |y(i_p j_p) - y(i_{p-1} j_{p-1})|^2 \right)^{1/2} > 0 \quad (5.7)$$

for any loop $\{i_p, j_p\}_{p=0}^N$ and *a.s.* ω . This infimum will be called the length of the loop, relative to $Y(\omega)$.

Since there are only a finite number of primary loops, the least one among all these primary loops' lengths relative to $Y(\omega)$ is strictly positive *a.s.*, which will be denoted by $c(\omega)$.

On the other hand, as Y satisfies (1.3), it is easy to check that

$$E \left[\sum_{p=1}^{\infty} |Y(\theta_p \wedge \tau_p) - Y(\theta_{p-1} \wedge \tau_{p-1})|^2 \right] < \infty. \quad (5.8)$$

As a consequence, there exists $N(\omega)$ such that $\theta_N \wedge \tau_N = T$. Otherwise, there would be infinite number of loops, whose length is almost surely not less than $c(\omega)$, and therefore it is contradictory to the last inequality.

Theorem 5.1. *Let Hypotheses 2.1, 2.2, and 5.1 be satisfied. Let*

$$(Y_{ij}, Z_{ij}, K_{ij}, L_{ij}; i \in \Lambda, j \in \Pi)$$

be a solution of RBSDEs (1.3) in $S^2 \times M^2 \times N^2 \times N^2$. Then, $(Y_{ij}; i \in \Lambda, j \in \Pi)$ is the value process for our switching game, and the switching strategy $a^(\cdot) := (\theta_p^* \wedge \tau_p^*, \alpha_p^*)$ for Player I and the switching strategy $b^*(\cdot) := (\theta_p^* \wedge \tau_p^*, \beta_p^*)$ for Player II is a saddle point of the switching game.*

Proof. We assume $t = 0$ without loss of generality. Otherwise, it suffices to consider the admissible switching strategies starting at time t .

For each pair $(a(\cdot), b(\cdot)) \in \mathcal{A}_0^i \times \mathcal{B}_0^j$, the following BSDE

$$\begin{aligned} U(s) &= \xi + A^{a(\cdot)}(T) - A^{a(\cdot)}(s) - \left(B^{b(\cdot)}(T) - B^{b(\cdot)}(s) \right) \\ &\quad + \int_s^T \psi(r, U(r), V(r), a(r), b(r)) dr - \int_s^T V(r) dW(r) \end{aligned} \quad (5.9)$$

has a unique solution, denoted by $(U^{a(\cdot), b(\cdot)}, V^{a(\cdot), b(\cdot)}) \in S^2 \times M^2$.

We shall show the following

$$U_0^{a^*(\cdot), b(\cdot)} \leq U_0^{a^*(\cdot), b^*(\cdot)} \leq U_0^{a(\cdot), b^*(\cdot)}. \quad (5.10)$$

For the following

$$\begin{aligned} a(s) &= i\chi_0(s) + \sum_{p=1}^N \alpha_{p-1} \chi_{(\theta_{p-1}, \theta_p]}, \\ b(s) &= j\chi_0(s) + \sum_{q=1}^N \beta_{q-1} \chi_{(\tau_{q-1}, \tau_q]}, \end{aligned} \quad (5.11)$$

there is a refinement $\{\theta'_j\}$ of both sequences of stopping times $\{\theta_j\}$ and $\{\tau_j\}$ such that

$$\begin{aligned} a(s) &= i\chi_0(s) + \sum_{p=1}^N \alpha'_{p-1} \chi_{(\theta'_{p-1}, \theta'_p]}, \\ b(s) &= j\chi_0(s) + \sum_{p=1}^N \beta'_{p-1} \chi_{(\theta'_{p-1}, \theta'_p]}. \end{aligned} \quad (5.12)$$

Then we define

$$Y^{a(\cdot), b(\cdot)}(s) = \sum_{p=1}^N Y_{\alpha'_{p-1}, \beta'_{p-1}}(s) \chi_{[\theta'_{p-1}, \theta'_p)}(s) + \xi_{a(T)b(T)} \chi_{\{T\}}(s), \quad (5.13)$$

$$Z^{a(\cdot), b(\cdot)}(s) = \sum_{p=1}^N Z_{\alpha'_{p-1}, \beta'_{p-1}}(s) \chi_{[\theta'_{p-1}, \theta'_p)}(s), \quad (5.14)$$

$$K^{a(\cdot), b(\cdot)}(s) = \sum_{p=1}^N \int_{\theta'_{p-1} \wedge s}^{\theta'_p \wedge s} dK_{\alpha'_{p-1}, \beta'_{p-1}}(r), \quad (5.15)$$

$$L^{a(\cdot), b(\cdot)}(s) = \sum_{p=1}^N \int_{\theta'_{p-1} \wedge s}^{\theta'_p \wedge s} dL_{\alpha'_{p-1}, \beta'_{p-1}}(r). \quad (5.16)$$

Noting that $Y^{a(\cdot), b(\cdot)}(\cdot)$ is a càdlàg process with jump $Y_{\alpha'_p, \beta'_p}(\theta'_p) - Y_{\alpha'_{p-1}, \beta'_{p-1}}(\theta'_p)$ at

$\theta'_p, p = 1, \dots, N-1$, we deduce that

$$\begin{aligned}
& Y^{a(\cdot), b(\cdot)}(s) - Y^{a(\cdot), b(\cdot)}(0) \\
&= - \sum_{p=1}^N \int_{\theta'_{p-1} \wedge s}^{\theta'_p \wedge s} \psi(r, Y_{\alpha'_{p-1}, \beta'_{p-1}}(r), Z_{\alpha'_{p-1}, \beta'_{p-1}}(r), \alpha'_{p-1}, \beta'_{p-1}) dr \\
&\quad + \sum_{p=1}^N \int_{\theta'_{p-1} \wedge s}^{\theta'_p \wedge s} Z_{\alpha'_{p-1}, \beta'_{p-1}}(r) dW(r) \\
&\quad + \sum_{p=1}^N \int_{\theta'_{p-1} \wedge s}^{\theta'_p \wedge s} \left[dK_{\alpha'_{p-1}, \beta'_{p-1}}(r) - dL_{\alpha'_{p-1}, \beta'_{p-1}}(r) \right] \\
&\quad + \sum_{q=1}^{N-1} [Y_{\alpha'_q, \beta'_q}(\theta'_q) - Y_{\alpha'_{q-1}, \beta'_{q-1}}(\theta'_q)] \chi_{[\theta'_q, T]}(s) \\
&= - \sum_{p=1}^N \int_{\theta'_{p-1} \wedge s}^{\theta'_p \wedge s} \psi(r, Y_{\alpha'_{p-1}, \beta'_{p-1}}(r), Z_{\alpha'_{p-1}, \beta'_{p-1}}(r), \alpha'_{p-1}, \beta'_{p-1}) dr \\
&\quad + \sum_{p=1}^N \int_{\theta'_{p-1} \wedge s}^{\theta'_p \wedge s} Z_{\alpha'_{p-1}, \beta'_{p-1}}(r) dW(r) \\
&\quad + \sum_{p=1}^N \int_{\theta'_{p-1} \wedge s}^{\theta'_p \wedge s} \left[dK_{\alpha'_{p-1}, \beta'_{p-1}}(r) - dL_{\alpha'_{p-1}, \beta'_{p-1}}(r) \right] \\
&\quad + \tilde{A}^{a(\cdot), b(\cdot)}(s) - A^{a(\cdot)}(s) - \left(\tilde{B}^{a(\cdot), b(\cdot)}(s) - B^{b(\cdot)}(s) \right),
\end{aligned}$$

where

$$\tilde{A}^{a(\cdot), b(\cdot)}(s) = \sum_{q=1}^{N-1} \chi_{a(s) \neq a(s+)} \left[Y_{\alpha'_q, \beta'_q}(\theta'_q) + k(\alpha'_{q-1}, \alpha'_q) - Y_{\alpha'_{q-1}, \beta'_{q-1}}(\theta'_q) \right] \chi_{[\theta'_q, T]}(s), \tag{5.17}$$

and

$$\begin{aligned}
& \tilde{B}^{a(\cdot), b(\cdot)}(s) \\
&= \sum_{p=1}^{N-1} \chi_{b(s) \neq b(s+), a(s) = a(s+)} \left[Y_{\alpha'_p, \beta'_p}(\theta'_p) - Y_{\alpha'_{p-1}, \beta'_{p-1}}(\theta'_p) + l(\beta'_{p-1}, \beta'_p) \right] \chi_{[\theta'_p, T]}(s).
\end{aligned} \tag{5.18}$$

They are increasing processes, due to the fact that

$$Y(t) \in \bar{Q}, \quad \forall t \in [0, T].$$

We have

$$\tilde{A}^{a^*(\cdot), b(\cdot)} = 0, \quad K^{a^*(\cdot), b(\cdot)} = 0, \quad \forall b(\cdot) \in \mathcal{B}_0^j. \tag{5.19}$$

Consequently, we conclude that $(Y^{a^*(\cdot),b(\cdot)}, Z^{a^*(\cdot),b(\cdot)})$ solves the following BSDE:

$$\begin{aligned}
Y(s) &= \xi_{a^*(T)b(T)} + \left[A^{a^*(\cdot)}(T) - A^{a^*(\cdot)}(s) \right] \\
&\quad + \left[\tilde{B}^{a^*(\cdot),b(\cdot)}(T) - \tilde{B}^{a^*(\cdot),b(\cdot)}(s) \right] - \left[B^{b(\cdot)}(T) - B^{b(\cdot)}(s) \right] \\
&\quad + \int_s^T \psi(r, Y(r), Z(r), a^*(r), b(r)) dr \\
&\quad + \int_s^T dL^{a^*(\cdot),b(\cdot)}(r) - \int_s^T Z(r) dW(r). \tag{5.20}
\end{aligned}$$

First we have

$$A^{a^*(\cdot)}(T) \in L^2(\mathcal{F}_T), \quad B^{b^*(\cdot)}(T) \in L^2(\mathcal{F}_T), \tag{5.21}$$

which together with Hypothesis 2.2 imply that $a^*(\cdot) \in \mathcal{A}_0^i$ and $b^*(\cdot) \in \mathcal{B}_0^j$. We only show the first inclusion, and the second one is shown in a symmetrical way.

We define $\bar{b}(\cdot) \equiv j$. Then, we have

$$\tilde{B}^{a^*(\cdot),\bar{b}(\cdot)} \equiv 0, \quad B^{\bar{b}(\cdot)} \equiv 0. \tag{5.22}$$

Putting the two equalities into RBSDEs (5.20), we see that $(Y^{a^*(\cdot),\bar{b}(\cdot)}, Z^{a^*(\cdot),\bar{b}(\cdot)})$ satisfies the following BSDE:

$$\begin{aligned}
Y^{a^*(\cdot),\bar{b}(\cdot)}(s) &= \xi_{a^*(T)j} + \left[A^{a^*(\cdot)}(T) - A^{a^*(\cdot)}(s) \right] \\
&\quad + \int_s^T \psi(r, Y^{a^*(\cdot),\bar{b}(\cdot)}(r), Z^{a^*(\cdot),\bar{b}(\cdot)}(r), a^*(r), j) dr \\
&\quad + \int_s^T dL^{a^*(\cdot),\bar{b}(\cdot)}(r) - \int_s^T Z^{a^*(\cdot),\bar{b}(\cdot)}(r) dW(r), \quad s \in [0, T]. \tag{5.23}
\end{aligned}$$

Since

$$Y^{a^*(\cdot),\bar{b}(\cdot)} \in S^2, \quad Z^{a^*(\cdot),\bar{b}(\cdot)} \in M^2, \quad L^{a^*(\cdot),\bar{b}(\cdot)} \in N^2, \tag{5.24}$$

in view of Hypothesis 2.1, it is standard to derive from equality (5.23) for $s = 0$ that $A^{a^*(\cdot)}(T) \in L^2(\mathcal{F}_T)$.

On the one hand, since $\tilde{B}^{a^*(\cdot),b(\cdot)}$ and $L^{a^*(\cdot),b(\cdot)}$ are increasing processes, we have from the comparison theorem for BSDEs that

$$Y^{a^*(\cdot),b(\cdot)}(s) \geq U^{a^*(\cdot),b(\cdot)}(s). \tag{5.25}$$

Letting $s = 0$, we have

$$Y_{ij}(0) \geq U^{a^*(\cdot),b(\cdot)}(0). \tag{5.26}$$

On the other hand, since

$$\tilde{B}^{a^*(\cdot),b^*(\cdot)} = 0, \quad L^{a^*(\cdot),b^*(\cdot)} = 0, \tag{5.27}$$

we have

$$Y^{a^*(\cdot),b^*(\cdot)}(s) = U^{a^*(\cdot),b^*(\cdot)}(s). \tag{5.28}$$

Letting $s = 0$, we have

$$Y_{ij}(0) = U^{a^*(\cdot),b^*(\cdot)}(0). \tag{5.29}$$

The proof is complete. \square

References

- [1] A. Bensoussan and J. L. Lions, Impulse Control and Quasivariational Inequalities. *Gauthier-Villars, Montrouge*, 1984.
- [2] R. Carmona and M. Ludkovski, Optimal switching with applications to energy tolling agreements. Preprint.
- [3] Cvitanic, J. and Karatzas, I., Backward SDEs with reflection and Dynkin games. *Annals of Probability* 24 (4), 2024–2056.
- [4] B. Djehiche, S. Hamadène and A. Popier, The multi-state starting and stopping problem. arXiv: 0707.2663v1 [math.PR] (with the new title: a finite horizon optimal multiple switching problem), July 18, 2007.
- [5] P. Dupuis and H. Ishii, SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.* 21 (1993), 554–580.
- [6] N. El Karoui, Les aspects probabilistes du contrôle stochastique. *Ninth Saint Flour Probability Summer School – 1979 (Saint Flour, 1979)*, pp. 73–238, Lecture Notes in Math., 876, Springer, Berlin, 1981.
- [7] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng and M. C. Quenez, Reflected solutions of backward SDE’s, and related obstacle problems for PDE’s. *Ann. Probab.* 25 (1997), 702–737.
- [8] A. Gegout-Petit and E. Pardoux, Equations différentielles stochastiques rétrogrades réfléchies dans un convexe. *Stochastics Stochastic Rep.* 57 (1996), 111–128.
- [9] S. Hamadène and M. Hassani, BSDEs with two reflecting barriers: the general result. *Probab. Theory Relat. Fields* 132 (2005), 237–264.
- [10] S. Hamadène and M. Jeanblanc, *On the starting and stopping problem: application in reversible investments*. *Math. Oper. Res.* 32 (2007), 182–192.
- [11] S. Hamadène and J. Zhang, The starting and stopping problem under Knightian uncertainty and related systems of reflected BSDEs. arXiv:0710.0908v1 [math.PR], October 4, 2007.
- [12] Y. Hu and S. Peng, On the comparison theorem for multi-dimensional BSDEs. *C. R. Math. Acad. Sci. Paris* 343 (2006), 135–140.
- [13] Y. Hu and S. Tang, Multi-dimensional backward stochastic differential equations with oblique reflection and optimal switching. Arxiv:0706.4365v2 [math.PR], July 4, 2007.
- [14] P. L. Lions and A. S. Sznitman, Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* 38 (1984), 511–537.
- [15] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* 14 (1990), 55–61.

- [16] E. Pardoux and S. Peng, Backward SDEs and quasilinear PDEs. *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, pp. 200–217, Lecture Notes in Control Inform. Sci., 176, Springer, Berlin, 1992.
- [17] S. Peng and M. Xu, The smallest g -supermartingale and reflected BSDE with single and double L^2 obstacles, *Ann. I. H. Poincaré. Probability* 41 (2005), 605–630.
- [18] S. Ramasubramanian, Reflected backward stochastic differential equations in an orthant. *Proc. Indian Acad. Sci. Math. Sci.* 112 (2002), 347–360.
- [19] D. Revuz and M. Yor, Continuous Martingales and Brownian Motion. Third Edition. *Springer, Berlin*, 1999.
- [20] S. Tang and S. Hou, Switching games of stochastic differential systems, *SIAM J. Control and Optimization* 46 (2007), 900-929.
- [21] S. Tang and J. Yong, Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach. *Stochastics Stochastics Rep.* 45 (1993), 145–176.
- [22] S. Tang and W. Zhong, Multi-dimensional reflected BSDEs with both fixed and inter-connected reflecting barriers, preprint, 2007.