$C^{1+\alpha}$-REGULARITY FOR TWO-DIMENSIONAL ALMOST-MINIMAL SETS IN ^{n}

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$C^{1+\alpha}$-REGULARITY FOR TWO-DIMENSIONAL ALMOST-MINIMAL SETS IN $\mathbb{R}^n$

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Résumé. On donne une nouvelle démonstration et une généralisation partielle du résultat de Jean Taylor [Ta] qui dit que les ensembles presque-minimaux de dimension 2 dans $\mathbb{R}^3$ (au sens d'Almgren) sont localement $C^{1+\alpha}$-equivalents à des cônes minimaux. La démonstration est plutôt élémentaire, mais utilise quand même un résultat local de séparation de [D3] et une généralisation [DDT] du théorème du disque topologique de Reifenberg. L'idée directrice est encore que si X est le cône sur un arc de petit graphe lipschitzien sur la sphère, mais n'est pas contenu dans un disque, on peut utiliser un graphe de fonction harmonique pour déformer X et diminuer substantiellement sa surface. Le résultat de séparation local est utilisé pour se ramener à des unions de cônes sur des arcs de graphes Lipschitziens. Une bonne partie de la démonstration se généralise à des ensembles de dimension 2 dans $\mathbb{R}^n$, mais dans ce cadre notre résultat final de régularité sur E dépend éventuellement de la liste des cônes minimaux obtenus par explosion à partir de E en un point.

Abstract. We give a new proof and a partial generalization of Jean Taylor's result [Ta] that says that Almgren almost-minimal sets of dimension 2 in $\mathbb{R}^3$ are locally $C^{1+\alpha}$-equivalent to minimal cones. The proof is rather elementary, but uses a local separation result proved in [D3] and an extension of Reifenberg's parameterization theorem [DDT]. The key idea is still that if X is the cone over an arc of small Lipschitz graph in the unit sphere, but X is not contained in a disk, we can use the graph of a harmonic function to deform X and diminish substantially its area. The local separation result is used to reduce to unions of cones over arcs of Lipschitz graphs. A good part of the proof extends to minimal sets of dimension 2 in $\mathbb{R}^n$, but in this setting our final regularity result on E may depend on the list of minimal cones obtained as blow-up limits of E at a point.

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1. Introduction

This paper can be seen as a continuation of [D3], and an attempt to give a slightly different proof of results of J. Taylor [Ta] and extend some of them to higher codimensions. In [D3] we described various sufficient conditions for an almost-minimal set E of dimension 2 in $\mathbb{R}^n$ to be locally equivalent to a minimal cone through a biHölder mapping. Here we are interested in getting a more precise (typically, $C^{1,\alpha}$) local equivalence with a minimal cone. A more precise argument seems to be needed to get these, because instead of proving that some quantities that measure the closeness to minimal cones stay small

(which may be obtained by compactness and limiting arguments), we shall need to show that they decay at some definite speed.

Let us try to describe the scheme of the proof. As the reader will see, the ideas are basically the same as in [Ta], even though we shall try to make the argument as modular as possible and avoid currents. On the other hand, the proof will sometimes become more complicated because we want some of it to apply to two-dimensional sets in $\mathbb{R}^n$.

Let E be a reduced almost minimal set (see Definition 1.10 below), fix $x \in E$, and set

$$\theta(r) = r^{-2} H^2(E \cap B(x, r)) \tag{1.1}$$

for r small. Here and throughout the paper, H^2 denotes the two-dimensional Hausdorff measure. It is important that if the gauge function in the definition of almost-minimality is small enough, then $\theta(r)$ is almost nondecreasing, to the point that

$$d(x) = \lim_{t \to 0^+} \theta(t) \tag{1.2}$$

exists. Then we can also show that all the blow-up limits of E at x are minimal cones with constant density $d(x)$.

For the local biHölder equivalence of E to a minimal cone, we do not need so much more than this, but for the $C^{1,\alpha}$ equivalence, we shall also need to know that the distance in $B(x, r)$ from E to a minimal cone goes to 0 at a definite speed. We shall introduce the density excess

$$f(r) = \theta(r) - d(x) = \theta(r) - \lim_{t \to 0^+} \theta(t) \tag{1.3}$$

and prove that under suitable conditions, it tends to 0 like a power of r. See Theorem 4.5 for a differential inequality, and Lemma 5.11 for the main example of ensuing decay. Then we shall also show that $f(r)$ controls the Hausdorff distance from E to minimal cones in small balls. [See Theorem 11.4 for the main estimate, and Theorem 12.8 and Proposition 12.28 for consequences when the gauge function is less than Cr^b. It will then be possible to deduce the $C^{1,\alpha}$ equivalence from a variant [DDT] of Reifenberg's topological disk theorem [R1]. [See Theorem 1.15 and Corollary 12.25.]

The main ingredient in the proof of the fact that $\theta(r)$ is almost nondecreasing is a comparison of E with the cone (centered at x) over the set $E \cap \partial B(x, r)$, which gives an estimate for $H^2(E \cap B(x, r))$ in terms of $H^1(E \cap \partial B(x, r))$. Similarly, the key decay estimate for f will rely on a comparison with a more elaborate competitor with the same trace on $\partial B(x, r)$.

More precisely, we shall first compare E with the cone over $E \cap \partial B(x, r)$, but we shall also need to show that if $E \cap \partial B(x, r)$ is not close to $C \cap \partial B(x, r)$ for a minimal cone C, then we can also improve on the cone over $E \cap \partial B(x, r)$ and save a substantial amount of area.

In a way, the central point of the argument is the following observation. Let $\mathcal{C}_T$ denote a conical sector in the plane, with aperture $T \in (0, \pi)$, and let $F : \mathcal{C}_T \to \mathbb{R}^{n-2}$ be a Lipschitz function with small norm, which is homogeneous of degree 1. Thus the graph

Γ of F is a cone in $\mathbb{R}^n$. Also suppose that $F(x) = 0$ on the two half lines that form the boundary of $\mathcal{C}_T$. The observation is that if $F \neq 0$, it is possible to replace $\Gamma \cap B(0,1)$ with another surface Γ' with the same boundary, and whose area is substantially smaller. If the Lipschitz constant is small enough, the main term in the computation of the area of Γ is the energy $\int |\nabla F|^2$, and a simple Fourier series computation shows that when F is homogeneous of degree 1, it is so far from being harmonic in $\mathcal{C}_T$ that replacing it with the harmonic extension of its values on $\partial(\mathcal{C}_T \cap B(0, 9/10))$ will produce a Γ' with substantially smaller area. Notice that by taking $T < \pi$, we exclude the unpleasant case when Γ is a plane but $F \neq 0$. See Section 8 for the detailed construction.

A good part of our argument will consist in trying to reduce to this simple situation. First we shall construct a net g of simple curves in $E \cap \partial B(x, r)$ (in Section 6). This will involve having some control on the topology of E, and this is the main place where we shall use some results from [D3]. Then we shall evaluate how much area we can win when we compare with the cone over $E \cap \partial B(x, r)$, then with the cone over the net g, then with the cone over the simplified net ρ obtained from g by replacing the simple curves with arcs of great circles. Finally, we shall also need to evaluate whether we can improve on ρ when its measure is different from $2rd(x)$. At this point in the argument, and when $n > 3$, we apparently need to add an extra assumption (the full length property of Definition 4.10) on the minimal cones that we use to approximate E, or the blow-up limits of E at x.

If we compare with J. Taylor's approach, maybe the main difference in the organization of the proof is that here we set aside (in [D3]) the topological information that we need to construct a competitor. In ambient dimension 3, this amounts to a separation property; in higher dimensions, we shall use more of the biHölder parameterization provided by [D3]. We use this information only once in the proof of the decay estimate, to simplify the construction of Γ and the competitors. We also keep the final geometric part (the construction of a parameterization of E once we know that it stays close to minimal cones) mostly separate from the main estimate, and we import it from [DDT].

We shall try not to use too much geometric measure theory, but it is not clear that this attempt will be entirely successful. First of all, some of it is hidden in the rectifiability results for almost-minimal sets from [Al] and [DS] and the stability of these sets under limits [D1], that were used extensively in [D3]. We also use a generalization [DDT] of Reifenberg's parameterization theorem to situations where E looks like cone of type $\mathbb{Y}$ or $\mathbb{T}$, both in [D3] to get a topological description of E locally, and more marginally here to say that estimates on density and Hausdorff distances to minimal cones can be translated into $C^{1,\alpha}$ estimates. Finally, we could not resist using the co-area theorem a few times.

One of the main motivations for this work was to understand the techniques of [Ta] and then apply them to a similar question for minimizers of the Mumford-Shah functional in $\mathbb{R}^3$; see [Le], where apparently more than the mere statement of the regularity result in [Ta] was needed.

We also would like to know to which extent the $C^{1,\alpha}$ regularity result of [Ta] extends to two-dimensional sets in $\mathbb{R}^n$. Here we shall only prove the $C^{1,\alpha}$ regularity of E near x under some assumption on the blow-up limits of E at x, and many questions will remain unaddressed, in particular because we don't have a list of minimal cones.

Let us be more specific now and try to state our main result. We shall work with

almost-minimal (rather than minimal) sets in a domain. The extra generality is probably quite useful, and will not be very costly. Our definition of almost-minimality will involve a gauge function, i.e., a nondecreasing function $h : (0, +\infty) \to [0, +\infty]$ such that $\lim_{\delta \to 0} h(\delta) = 0$. In this definition, we are given an open set $U \subset \mathbb{R}^n$ and a relatively closed subset E of U, with locally finite Hausdorff measure, i.e., such that

$$H^2(E \cap B) < +\infty \quad \text{for every compact ball } B \subset U. \tag{1.4}$$

We want to compare E with competitors $F = \varphi_1(E)$, where $\{\varphi_t\}_{0 \le t \le 1}$, is a one-parameter family of continuous functions $\varphi_t : U \to U$, with the following properties:

$$\varphi_0(x) = x \quad \text{for } x \in U, \tag{1.5}$$

$$\text{the function } (t, x) \to \varphi_t(x), \text{ from } [0, 1] \times U \text{ to } U, \text{ is continuous,} \tag{1.6}$$

$$\varphi_1 \text{ is Lipschitz} \tag{1.7}$$

(but we never care about the Lipschitz constant in (1.7)) and, if we set

$$W_t = \{x \in U \, ; \, \varphi_t(x) \neq x\} \quad \text{and } \widehat{W} = \bigcup_{t \in [0,1]} \Big[W_t \cup \varphi_t(W_t) \Big], \tag{1.8}$$

then

$$\widehat{W} \text{ is relatively compact in } U \quad \text{and} \quad \operatorname{diam}(\widehat{W}) < \delta. \tag{1.9}$$

Definition 1.10. *We say that the closed set E in U is an almost-minimal set (of dimension 2) in U, with gauge function h, if (1.4) holds and if*

$$H^2(E \setminus F) \le H^2(F \setminus E) + h(\delta)\, \delta^2 \tag{1.11}$$

for each $\delta > 0$ and each family $\{\varphi_t\}_{0 \le t \le 1}$ such that (1.5)-(1.9) hold, and were we set $F = \varphi_1(E)$. We say that E is reduced when

$$H^2(E \cap B(x, r)) > 0 \quad \text{for } x \in E \text{ and } r > 0. \tag{1.12}$$

This definition is of course inspired by Almgren's definition of "restricted sets" [Al], but we use a slightly different accounting. We could use other definitions of almost-minimal sets; for instance, we could have replaced (1.11) with

$$H^2(E \cap W_1) \le H^2(\varphi_1(E \cap W_1)) + h(\delta)\delta^2, \tag{1.13}$$

and this would have yielded exactly the same class of almost-minimal sets with gauge h (see Proposition 4.10 in [D3]). Also see Definition 4.1 in [D3] for a slightly smaller class of almost-minimal sets, with a definition closer to Almgren's.

Denote by E^* the closed support of the restriction of H^2 to E, i.e., set

$$E^* = \big\{x \in E\,;\, H^2(E \cap B(x,r)) > 0 \text{ for every } r > 0\big\}. \tag{1.14}$$

Thus E is reduced when $E = E^*$. It is fairly easy to see that E^* is closed in U, $H^2(E \backslash E^*) = 0$, and E^* is almost-minimal with gauge h when E is almost-minimal with gauge h (see Remark 2.14 in [D3]). Thus we may restrict our attention to reduced almost-minimal sets, as the other almost-minimal sets are obtained from the reduced ones by adding sets of vanishing Hausdorff measure.

Even when we do not say this explicitly, the almost minimal sets and minimal cones in this paper will be assumed to be reduced.

Even when E is a reduced minimal set, it can have singularities, but the list of singularities is fairly small, especially when $n = 3$.

First consider minimal cones, i.e., minimal sets that are also cones. It was proved by E. Lamarle [La], A. Heppes [He], and J. Tayor [Ta], that there are exactly three types of (nonempty) reduced minimal cones in $\mathbb{R}^3$: the planes, the sets of type $\mathbb{Y}$ obtained as unions of three half planes with a common boundary L and that make 120° angles along L, and sets of type $\mathbb{T}$ which are cones T over the union of the edges of a regular tetrahedron, and centered at the center of the tetrahedron. See Figures 1.1 and 1.2.

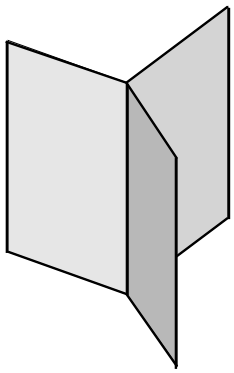

Figure 1.1. A cone of type $\mathbb{Y}$

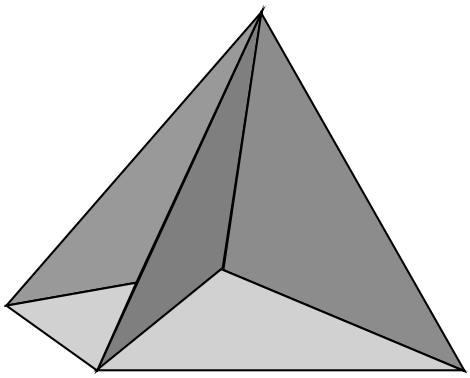

Figure 1.2. A cone of type $\mathbb{T}$

When $n > 3$, the precise list of reduced minimal cones (centered at the origin) is not known, but they have the same structure as in $\mathbb{R}^3$. The set $K = E \cap \partial B(0,1)$ is composed of a finite collection of great circles and arcs of great circles, which can only meet by sets of three and with 120° angles. See the beginning of Section 2 for a more precise description. We still have the 2-planes (which correspond to a single circle) and the sets of type $\mathbb{Y}$ (where K is the union of three half circles that meet at two antipodal points), and we call "sets of type $\mathbb{T}$" all the other nonempty reduced minimal cones. But for $n > 3$ the list of sets of type $\mathbb{T}$ is larger, and not well investigated.

It turns out that if E is a reduced almost-minimal set with a small enough gauge function h, then for $x \in E \cap U$ we have a reasonably good local description of E near x.

In particular, it is shown in [D3] (after [Ta] when $n = 3$) that for r small, E coincides near $B(x, r)$ with the image of a reduced minimal cone by a biHölder homeomorphism of space.

Our main statement here is roughly the same, except that we require h to be slightly smaller, add an assumption on the list of blow-up limits of E at x when $n > 3$, but obtain a better regularity.

Theorem 1.15. *Let E be a reduced almost-minimal set in the open set $U \subset \mathbb{R}^n$, with a gauge function h such that $h(r) \leq Cr^\alpha$ for r small enough (and some choice of $C \geq 0$ and $\alpha > 0$). Let $x \in E$ be given. If $n > 3$, suppose in addition that some blow-up limit of E at x is a full length minimal cone (See Definition 2.10). Then there is a unique blow-up limit X of E at x, and $x + X$ is tangent to E at x. In addition, there is a radius $r_0 > 0$ such that, for $0 < r < r_0$, there is a $C^{1,\beta}$ diffeomorphism $\Phi : B(0, 2r) \to \Phi(B(0, 2r))$, such that $\Phi(0) = x$ and $|\Phi(y) - x - y| \leq 10^{-2}r$ for $y \in B(0, 2r)$, and $E \cap B(x, r) = \Phi(X) \cap B(x, r)$.*

Jean Taylor's result from [Ta] corresponds to $n = 3$.

See Definition 3.11 for the definition of blow-up limits, and Definition 2.10 for full length minimal cones. Let us just say here that every (reduced) minimal cone in $\mathbb{R}^3$ is full length. This is also the case of the explicitly known minimal cones in higher dimension. Thus we do not know whether every minimal cone is full length. See Section 14.

Roughly speaking, the full length condition requires that if one deforms $X \cap \partial B(0, 1)$ into a nearby net of arcs of great circles which is strictly longer, then the cone over this deformation is far from being a minimal cone.

In the statement, $\beta > 0$ depends only on n, α, and the full length constants for X if $n > 3$. The $C^{1,\beta}$ diffeomorphism conclusion means that $D\Phi$ and its inverse are Hölder-continuous with exponent β.

Unfortunately, we have no lower bound on r, even when $n = 3$ and x is a point of type $\mathbb{T}$. [We could perhaps have hoped for lower bounds on r that depend only on h and the distances from x to ∂U and to the points of E of "higher types".]

Theorem 1.15 can be generalized in two ways. First, we can use gauge functions h that do not decrease as fast as a power, but we shall always assume that

$$h_1(r) = \int_0^r h(2t)\frac{dt}{t} \text{ is finite for } r \text{ small enough,} \tag{1.16}$$

because we shall rely on a near monotonicity- and a parameterization result from [D3] where this is used.

If we assume a suitably strong Dini condition (somewhat stronger than (1.16)), we can proceed as in Theorem 1.15, and get a decay for the density excess $f(r)$ and the numbers $\beta_Z(x, r)$ that measure the good approximation of E by minimal cones which is sufficient to give the uniqueness of the tangent cone X and a local C^1 equivalence to X.

If we only assume (1.16), and this time assume that every blow-up limit of E at x has the full length property, we still get some decay of $f(r)$ and the $\beta_Z(x, r)$, but not enough to imply the C^1 equivalence of E to a minimal cone or even the uniqueness of the tangent cone. See Section 13 for a little more detail.

We can also replace the full length condition with a weaker one, but this will lead to a slow decay of f that does not yield C^1 estimates. See Section 13 again.

The plan for the rest of the paper is the following. Section 2 is devoted to a rapid description of the minimal cones, and the definition of the full length property.

In Section 3 we define the density $\theta(r)$ and the density excess $f(r)$, record simple consequences of the almost monotonicity of density for almost minimal sets, and say a few words about blow-up limits.

The main decay estimate for f will come from a differential inequality that follows from a comparison inequality. This comparison inequality is stated in Section 4 (Theorem 4.5), where we also check that two technical assumptions are satisfied almost-everywhere. The proof takes up most of Sections 6-10.

In Section 5 we explain why Theorem 4.5 leads to a differential inequality and a decay estimate for f.

The proof of Theorem 4.5 really starts in Section 6, and is performed on a single ball $B = B(x, r)$; the long-term goal is to construct a nice competitor for E in B, and thus get an upper bound on $H^2(E \cap B)$. In Section 6 we construct a net g of simple curves $g_{j,k}$ in ∂B, that looks like the intersection of ∂B with a minimal cone (see Lemma 6.11). The construction uses some separating properties of E.

In Section 7 we approximate the simple curves $g_{j,k}$ with Lipschitz curves $\Gamma_{j,k}$; this will be useful because it is easier to find retractions on a neighborhood of the cone over $\cup\Gamma_{j,k}$. Most of the section consists in checking that we diminish the length proportionally to the amount of $g_{j,k}$ that we replace with $\Gamma_{j,k}$. The main ingredients are simple one-dimensional computations and the Hardy-Littlewood maximal function.

Section 8 explains how to replace the cone over a small Lipschitz curve like the $\Gamma_{j,k}$ with a Lipschitz surface with the same trace on ∂B, but with smaller area. The idea is to use the approximation of the minimal surface equation by the Laplace equation and observe that the cone is the graph of a function which is homogeneous of degree 1, whence reasonably far from harmonic if it is not almost affine. The estimate relies on simple Fourier series computations.

A first competitor is defined in Section 9. The main point is to map most of $E \cap B$ to the finite union of Lipschitz surfaces constructed in Section 8, without losing too much near ∂B. Most of the area estimates for Theorem 4.5 are done at the end of the section.

Section 10 contains the final epiperimetry argument, which concerns the possible improvement of E when it is equal, or very close to the cone over a union ρ of geodesics (but yet is different from the approximating minimal cone X). This is the place where the geometry of X plays a role, and where the full length property is used.

In Section 11, we switch to a slightly different subject, and show that when $f(r)$ is very small, $E \cap B(0, C^{-1}r)$ is well approximated by a minimal cone (but yet we lose a power in the estimate). See Theorem 11.4 and Corollary 11.85 for the main statements. The proof uses the same competitor as in Section 9, and a slightly unpleasant construction of transverse curves, which we use to control the way in which the good approximating minimal cone that we get on almost every concentric sphere depends on the sphere.

Section 12 contains the consequence of the decay estimate (Theorem 4.5) and the approximation results (Theorem 11.4 and Corollary 11.85), when $h(r) \leq Cr^b$, in terms

of better approximation by cones and local C^1-equivalence to a minimal cone. See Theorem 12.8, Proposition 12.28, and Corollary 12.25. Theorem 1.15 is a special case of this.

Section 13 contains some technical improvements, concerning better estimates for Proposition 12.28 and Corollary 12.25, larger gauge functions (sufficiently large powers of $\mathrm{Log}(C/r)$), and weaker variants of the full length property.

Finally the full length property is discussed again in Section 14, where it is established for minimal cones in $\mathbb{R}^3$ and a few others.

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2. Full length minimal cones.

We start with a rough description of the reduced minimal cones of dimension 2 in $\mathbb{R}^n$, that we take from [D3]. We start with simple definitions.

For us a cone centered at the origin will be a positive cone, i.e., a set X such that $\lambda x \in X$ when $x \in X$ and $\lambda \geq 0$. In general, a cone will just be a translation of a cone centered at the origin.

A minimal set (in $\mathbb{R}^n$) is the same thing as an almost minimal set in $U = \mathbb{R}^n$ with the gauge function $h = 0$. That is, E is closed, $H^2(E \cap B) < +\infty$ for every ball B, and

$$H^2(E \setminus f(E)) \leq H^2(f(E) \setminus E) \tag{2.1}$$

for every Lipschitz function f such that $f(x) = x$ out of some compact set. The reader should maybe be warned that the literature contains many other definitions of "minimal sets" or soap films.

By minimal cone, we simply mean a cone which is also a minimal set in $\mathbb{R}^n$. We shall always assume that it is reduced, even if we don't say it explicitly.

Observe that the 2-planes, and the sets of type $\mathbb{Y}$ (three half 2-planes with a common boundary L, and that make 120° angles along L) are still minimal in any ambient dimension $n \geq 3$. All the other minimal cones will be called cones of type $\mathbb{T}$.

The simplest example of a cone of type $\mathbb{T}$ is obtained by taking a regular tetrahedron in some affine 3-space, and then letting T be the cone over the union of the edges centered at the center of the tetrahedron. We get a minimal set, even if $n > 3$ (because competitors could always be projected back to $\mathbb{R}^3$).

When $n = 3$, Lamarle [La], Heppes [He], and Taylor [Ta] showed that these are the only cones of type $\mathbb{T}$ that we can get. When $n > 3$, the union of two orthogonal 2-dimensional planes is a minimal cone, and possibly this stays true if the two planes are almost orthogonal. Also, the product of two one-dimensional sets Y lying in orthogonal two-planes is likely to be minimal. And there may be lots of other, much wilder examples, but the truth is that the author does not know any.

At least we know the following rough description of minimal cones. Let E be a (reduced) minimal cone of dimension 2 in $\mathbb{R}^n$, and suppose for convenience that X is centered at the origin. Set $K = E \cap \partial B(0, 1)$. Then K is a finite union of great circles and arcs of great circles $\mathfrak{C}_j$, $j \in J$. The $\mathfrak{C}_j$ can only meet when they are arcs of great circles,

and only by sets of 3 and at a common endpoint. Then they make angles of 120° at that point. The arcs of circles have no free endpoints either (each endpoint is the end of exactly three arcs). In addition, there is a small constant $\eta_0 > 0$, that depends only on n, such that

$$\text{the length of each arc } \mathfrak{C}_j \text{ is at least } 10\eta_0 \tag{2.2}$$

and

$$\begin{aligned} &\text{if } i, j \in J,\ i \neq j, \text{ and } x \in \mathfrak{C}_i \text{ is such that } \operatorname{dist}(x, \mathfrak{C}_j) \leq \eta_0, \text{ then } \mathfrak{C}_i \text{ and } \mathfrak{C}_j \\ &\text{are both arcs of circles, and they have a common endpoint in } \overline{B}(x, \operatorname{dist}(x, \mathfrak{C}_j)). \end{aligned} \tag{2.3}$$

Next set

$$d(E) = H^2(E \cap B(0,1)) = \frac{1}{2} H^1(K) = \frac{1}{2} \sum_{j \in J} \text{length}(C_j), \tag{2.4}$$

where the last equalities come from the fact that $E \cap B(0,1)$ is the essentially disjoint union of the angular sectors over the arcs $\mathfrak{C}_j$. Recall from Lemma 14.12 in [D3] that there is a constant $d_T > \frac{3\pi}{2}$ such that $d(E) \geq d_T$ when E is a minimal cone of type $\mathbb{T}$.

It will be easier to manipulate K after we cut some of the $\mathfrak{C}_j$ into two or three shorter sub-arcs. For each $\mathfrak{C}_j$ whose length is more than $\frac{9\pi}{10}$, say, we cut $\mathfrak{C}_j$ into two or three essentially disjoint sub-arcs $\mathfrak{C}_{j,k}$ with roughly the same length. For the other arcs $\mathfrak{C}_j$, those whose length is at most $\frac{9\pi}{10}$, we leave them as they are, i.e., decompose them into a single arc $\mathfrak{C}_{j,k}$. Altogether, we now have a decomposition of K into essentially disjoint arcs of circles $\mathfrak{C}_{j,k}$, $(j,k) \in \widetilde{J}$, such that

$$10\eta_0 \leq \text{length}(\mathfrak{C}_{j,k}) \leq \frac{9\pi}{10} \quad \text{for } (j,k) \in \widetilde{J}. \tag{2.5}$$

The $\mathfrak{C}_{j,k}$ will be easier to use, because they are geodesics of ∂B that are determined by their endpoints in a reasonably stable way.

Let us also distinguish two types of vertices. We denote by V the collection of all the endpoints of the various $\mathfrak{C}_{j,k}$. Then let V_0 be the set of vertices $x \in V$ that lie at the end of exactly three arcs $\mathfrak{C}_{j,k}$. Those are the vertices that were already present in our initial decomposition of K into $\mathfrak{C}_j$. Finally set $V_1 = V \setminus V_0$. Those are the endpoints that we just added, and each one lies exactly in two arcs $\mathfrak{C}_{j,k}$. Notice that because of (2.3), if (j,k) and $(j',k') \in \widetilde{J}$ are different pairs, then

$$\operatorname{dist}(\mathfrak{C}_{j,k}, \mathfrak{C}_{j',k'}) \geq \eta_0 \ \text{ or else } \ \mathfrak{C}_{j,k} \text{ and } \mathfrak{C}_{j',k'} \text{ have a common endpoint } x \in V. \tag{2.6}$$

We shall sometimes refer to this construction as the standard decomposition of K. The precise choice of our decomposition of the long $\mathfrak{C}_j$ into two or three $\mathfrak{C}_{j,k}$ does not really matter, but we could make it unique by cutting $\mathfrak{C}_j$ into as few equal pieces as possible.

Next we consider deformations of K, obtained as follows. Let $\eta_1 < 10^{-1}\eta_0$ be given, and denote by $\Phi(\eta_1)$ the set of functions $\varphi : V \to \partial B(0,1)$ such that

$$|\varphi(x) - x| \le \eta_1 \quad \text{for } x \in V. \tag{2.7}$$

Let $\varphi \in \Phi(\eta_1)$ and $(j,k) \in \widetilde{J}$ be given. Denote by a and b the extremities of $\mathfrak{C}_{j,k}$. We denote by $\varphi_*(\mathfrak{C}_{j,k})$ the geodesic of $\partial B(0,1)$ that goes from $\varphi(a)$ to $\varphi(b)$, where the uniqueness comes from (2.5) and (2.7). Finally set

$$\varphi_*(K) = \bigcup_{(j,k)\in\widetilde{J}} \varphi_*(\mathfrak{C}_{j,k}) \tag{2.8}$$

Thus we just deform K as a net of geodesics, by moving its vertices. Finally denote by $\varphi_*(X)$ the cone over $\varphi_*(K)$.

Notice that by (2.5), (2.6), and (2.7), the geodesics $\varphi_*(\mathfrak{C}_{j,k})$ all have lengths larger than $9\eta_0$, and only meet at their vertices $\varphi(x), x \in V$, with angles that are still close to $2\pi/3$ when $x \in V_0$ and to π when $x \in V_1$. In particular,

$$H^1(\varphi_*(K)) = \sum_{(j,k)\in\widetilde{J}} \text{length}(\varphi_*(\mathfrak{C}_{j,k})). \tag{2.9}$$

Definition 2.10. *Let X be a (reduced) minimal cone centered at the origin. We say that X is a full length minimal cone when there is a standard decomposition of K as above, an $\eta_1 < \eta_0/10$, and a constant $C_1 \ge 1$, such that if $\varphi \in \Phi(\eta_1)$ is such that*

$$H^1(\varphi_*(K)) > H^1(K), \tag{2.11}$$

then there is a deformation $\widetilde{X}$ of $\varphi_*(X)$ in $B(0,1)$ such that

$$H^2(\widetilde{X} \cap B(0,1)) \le H^2(\varphi_*(X) \cap B(0,1)) - C_1^{-1}[H^1(\varphi_*(K)) - H^1(K)]. \tag{2.12}$$

By deformation of $\varphi_*(X)$ in $B(0,1)$, we mean a set of the form $\widetilde{X} = f(\varphi_*(X))$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and such that $f(x) = x$ for $x \in \mathbb{R}^n \setminus B(0,1)$ and $f(B(0,1)) \subset B(0,1)$.

So we only require that when a small deformation $\varphi_*(K)$ is longer than K, it cannot be associated with a minimal cone, and we require this with uniform "elliptic" estimates.

Here we allowed ourselves to choose any standard decomposition of K, but probably the required property does not depend on this choice.

Naturally, when X is not centered at the origin, we say that it is a full length minimal cone when it is the translation of a full length minimal cone centered at the origin.

We shall see in Section 14 that the minimal cones in $\mathbb{R}^3$ all have full length. For instance, if X is a plane, K is a great circle, and $H^1(\varphi_*(K)) > 2\pi$, it turns out that we can find a vertex $x \in V_1$ such that the angle of the two geodesics $\varphi_*(\mathfrak{C}_{j,k})$ that leave from

x is different from π by at least $C^{-1}[H^1(\varphi_*(K)) - 2\pi]^{1/2}$, and then it is easy to find $\widetilde{X}$ as above. The situation for cones of type $\mathbb{Y}$ or $\mathbb{T}$ is similar: the length excess of $\varphi_*(K)$ gives a lower bound on the wrongness of the angle at some vertex.

It is a priori possible that every minimal cone is full length, but we really lack examples to discuss this seriously. Notice however that if X is a full length minimal cone, it cannot be embedded in a one-parameter family of minimal cones X_t centered at the origin such that $\frac{\partial}{\partial t} H^2(X_t \cap B(0,1)) \neq 0$. See Section 14 for a slightly longer discussion of these issues.

Remark 2.13. Some weaker version of Theorem 4.5 below will work with a weaker (more degenerate) version of the full length property, where we replace $[H^1(\varphi_*(K)) - H^1(K)]$ in (2.12) with $[H^1(\varphi_*(K)) - H^1(K)]^N$. See 13.32.

Remark 2.14. We shall not use the fact that our cones are minimal cones, but only the description given at the beginning of this section (that is, the description of K as a union of great circles and arcs of great circles, all the way up to (2.3). If E is a cone that satisfies all these properties, we shall call it a minimal-looking cone. This comes with an additional constant, the small constant η_0 in (2.2) and (2.3). Note that some minimal-looking cones are not minimal, for instance the cone over the union of the edges of a cube in $\mathbb{R}^3$.

Essentially all the results proved in this paper have versions for minimal-looking cones. See the end of Section 13 for a slightly more detailed account.

3. Simple facts about density and limits.

Let E be a reduced almost minimal set in $U \subset \mathbb{R}^n$, with gauge function h. We shall always assume that

$$h_1(r) = \int_0^r h(2t)\frac{dt}{t} < +\infty \quad \text{for } r \text{ small}, \tag{3.1}$$

as in (1.16), in particular because (3.1) seems to be needed for the following result of almost-monotonicity for the density. There exist constants $\lambda \geq 1$ and $\varepsilon_0 > 0$, that depend only on the ambient dimension n, such that the following holds. Fix $x \in E$, and set

$$\theta(r) = r^{-2} H^2(E \cap B(x,r)) \tag{3.2}$$

as above. Let $R > 0$ be such that $B(x,R) \subset U$, $h_1(R) < +\infty$, and $h(R) \leq \varepsilon_0$. Then

$$\theta(r)\, e^{\lambda h_1(r)} \text{ is a nondecreasing function of } r \in (0,R). \tag{3.3}$$

See for instance Proposition 5.24 in [D3]. Our main estimate will be an improvement of (3.3) when h is small and E is close to a full length minimal cone. Before we come to that, let us mention some simple consequences of (3.3).

Notice that (3.1)-(3.3) imply the existence of the limit density

$$d(x) = \lim_{r \to 0} \theta(r) \tag{3.4}$$

for every $x \in E$. Then we can introduce the density excess

$$f(r) = \theta(r) - d(x) = \theta(r) - \lim_{t \to 0^+} \theta(t) \tag{3.5}$$

as in (1.3). Let us check that

$$f(r) \le f(s) + Ch_1(s) \quad \text{for } 0 < r < s < R \tag{3.6}$$

when $B(x, 2R) \subset U$ and $h_1(2R) \le \varepsilon_0/2$. Indeed, (3.1) yields $h(R) \le \varepsilon_0$, so we can apply (3.3) and get that

$$\theta(r) \le \theta(r)\, e^{\lambda h_1(r)} \le \theta(s)\, e^{\lambda h_1(s)} \le (1 + Ch_1(s))\, \theta(s) \le \theta(s) + CCh_1(s) \tag{3.7}$$

because $\theta(s) \le C$ by the local Ahlfors-regularity of E (see Lemma 2.15 in [D3] in this context, but the property comes from [DS]); this is why we required that $B(x, 2R) \subset U$. Then (3.6) follows by subtracting $d(x)$ from both sides.

When we let r tend to 0 in (3.7), we get that

$$d(0) \le \theta(s) + Ch_1(s), \quad \text{or equivalently} \quad f(s) \ge -Ch_1(s), \quad \text{for} \quad 0 < s < R. \tag{3.8}$$

Let us also say a few words about blow-up limits. For the definition of convergence, it is convenient to set

$$\begin{aligned} d_{x,r}(E, F) = r^{-1} \sup \big\{ \operatorname{dist}(y, E)\, ;\, y \in F \cap B(x, r) \big\} \\ + r^{-1} \sup \big\{ \operatorname{dist}(y, F)\, ;\, y \in E \cap B(x, r) \big\} \end{aligned} \tag{3.9}$$

when E and F are nonempty closed sets in a domain U that contains $B(x, r)$. By convention, $\sup \big\{ \operatorname{dist}(y, E)\, ;\, y \in F \cap B(x, r) \big\} = 0$ when $F \cap B(x, r)$ is empty.

Definition 3.10. *Let $\{E_k\}$ be a sequence of subsets of $\mathbb{R}^n$, and let F be a closed set in the open set $U \subset \mathbb{R}^n$. We say that $\{E_k\}$ converges to E in U when $\lim_{k \to +\infty} d_{x,r}(E_k, F) = 0$ for every ball $B(x, r)$ such that $\overline{B}(x, r) \subset U$.*

The definition would perhaps look more classical if we replaced the balls $B(x, r)$ with compact subsets of U, but the result is the same because every compact subset of U is contained in a finite union of balls $B(x, r)$ such that $\overline{B}(x, r) \subset U$.

Definition 3.11. *A blow-up limit of E at x is a closed set F that can be obtained as limit in $\mathbb{R}^n$ of a sequence $\{r_k^{-1}(E - x)\}$, where $\{r_k\}$ is a sequence of radii such that $\lim_{k \to +\infty} r_k = 0$.*

Since every ball in $\mathbb{R}^n$ is contained in a $B(0, \rho)$, this means that $\lim_{k \to +\infty} d_{0,\rho}(E_k, F) = 0$ for every $\rho > 0$, where we set $E_k = r_k^{-1}(E - x)$, or equivalently that

$$\lim_{k \to +\infty} d_{x, \rho r_k}(E, x + r_k F) = 0 \quad \text{for every } \rho > 0. \tag{3.12}$$

The fact that the set E_k naturally lives in $U_k = r_k^{-1}(U - x)$, which may be smaller than $\mathbb{R}^n$, does not matter, because U_k eventually contains every $\overline{B}(0, \rho)$.

Observe that E always has at least one blow-up limit at x. Indeed, by a standard compactness result on the Hausdorff distance (see for instance Section 34 of [D2]),

$$\begin{gathered}\text{every sequence } \{r_k\} \text{ which tends to 0 has a subsequence} \\ \text{for which the } r_k^{-1}(E - x) \text{ converge to some limit } X.\end{gathered} \tag{3.13}$$

In addition, the blow-up limits are simpler, because of the following.

Proposition 3.14. *If X is a blow-up limit of E at x, then X is a reduced minimal cone, with density*

$$H^2(X \cap B(0,1)) = \lim_{r \to 0} r^{-2} H^2(E \cap B(x,r)) = d(x). \tag{3.15}$$

This is Proposition 7.31 in [D3], but let us rapidly say how it works. The main point is a result of [D1], which says that the Hausdorff measure is lowersemicontinuous along sequences of quasiminimal sets. Then limits of reduced almost-minimal sets with a given gauge function g are reduced almost-minimal sets with the same gauge function g.

Here we get the minimality of X because $\{r_k\}$ tends to 0, so we can take $g(t) = h(rt)$ for arbitrarily small values of r. Then

$$H^2(X \cap B(0,\rho)) = \lim_{r \to 0} r^{-2} H^2(E \cap B(x, r\rho)) = \rho^2 d(x) \tag{3.16}$$

by (3.12) and because the Hausdorff measure goes to the limit well; see Proposition 7.31 in [D3] again. So (3.15) holds, but also the density of X is constant, and X is a cone by Theorem 6.2 in [D3]. □

When $n = 3$, there are only three types of (nonempty) reduced minimal cones: the planes, the sets of type $\mathbb{Y}$, and the cones over tetrahedron edges like T in Figure 1.2. So Proposition 3.14 says that $d(x)$ can only take the values π, $3\pi/2$, or $3\text{Argcos}(-1/3) \approx 1.82 \cdot \pi$, depending on the common type of all the blow-up limits of E at x.

When $n > 3$, there are more possibilities. If $d(x) = \pi$, every blow-up limit of E at x is a plane and we say that x is a P-point. If $d(x) = 3\pi/2$, every blow-up limit is a set of type $\mathbb{Y}$, and x is called a Y-point. Otherwise, Lemma 14.12 in [D3] says that $d(x) \geq d_T$ for some $d_T > 3\pi/2$, and we say that x is a T-point. We do not know much more in this case, there may even be many non mutually isometric blow-up limits of E at x (but they would all have the same density $d(x)$). See [D3], Section 14 and the beginning of Section 16, for a little more detail.

4. The main comparison statement.

Here we want to state the main technical result of the paper. Let E be a reduced almost minimal set in $U \subset \mathbb{R}^n$, with a gauge function h that satisfies (3.1), and let $x \in E$ and $r > 0$ be given.

We shall make the following assumptions on r, which will allow us to construct competitors and then prove a differential inequality for f. First we assume that for some small $\varepsilon > 0$,

$$B(x, 100r) \subset U, \quad f(50r) = \theta(50r) - d(x) \leq \varepsilon, \quad h_1(60r) = \int_0^{60r} h(2t)\frac{dt}{t} \leq \varepsilon, \tag{4.1}$$

and there is minimal cone X centered at the origin, such that

$$d_{x,100r}(E, x + X) \leq \varepsilon. \tag{4.2}$$

We shall also use the standard decomposition of $K = X \cap \partial B(0,1)$ into arcs of great circles $\mathfrak{C}_{j,k}$, and the corresponding set of vertices V, as described in Section 2.

Our second assumption is a little more technical, but we shall see later that it is satisfied for almost every r. We require that for every continuous nonnegative function f on $\mathbb{R}^n$,

$$\lim_{\rho \to 0} \rho^{-1} \int_{t \in (r-\rho, r)} \int_{E \cap \partial B(0,t)} f(z)\, dH^1(z)\, dt = \int_{E \cap \partial B(0,r)} f(z)\, dH^1(z) \tag{4.3}$$

and that

$$\sup_{\rho > 0} \ \rho^{-1} \int_{t \in (r-\rho, r)} H^1(E \cap \partial B(0,t))\, dt < +\infty \tag{4.4}$$

(the finiteness at r of the corresponding maximal function).

Theorem 4.5. *Let $\eta_1 < \eta_0/10$ be given. If $\varepsilon > 0$ is small enough, depending on η_1, we have the following result. Let E be a reduced almost minimal set in $U \subset \mathbb{R}^n$, with the gauge function h, and let $x \in E$ and $r > 0$ be such that (4.1)-(4.4) hold. Then there is a mapping $\varphi : V \to \partial B(0,1)$ such that (2.7) holds,*

$$r H^1(\varphi_*(K)) \leq H^1(E \cap \partial B(x,r)) \tag{4.6}$$

and

$$\begin{aligned} H^2(E \cap B(x,r)) \leq \frac{r}{2} H^1(E \cap \partial B(x,r)) \\ - 10^{-5} r \left[H^1(E \cap \partial B(x,r)) - r H^1(\varphi_*(K))\right] + 4r^2 h(2r). \end{aligned} \tag{4.7}$$

If in addition X is a full length minimal cone with constants C_1 and η_1, and such that $H^2(X \cap B(0,1)) \leq d(x)$, then

$$\begin{aligned} H^2(E \cap B(x,r)) \leq \frac{r}{2} H^1(E \cap \partial B(x,r)) \\ - \alpha r \left[H^1(E \cap \partial B(x,r)) - 2d(x)r\right] + 4r^2 h(2r), \end{aligned} \tag{4.8}$$

where the small $\alpha > 0$ depends on C_1.

In (4.7) and (4.8), the term with square brackets corresponds to an improvement over the estimate that leads to the monotonicity property (3.3). More properties of $\varphi_*(K)$ and intermediate objects will show up along the proof.

The proof of Theorem 4.5 will keep us busy for quite some time, but let us already try to say how it will go. Of course (4.7) and (4.8) will be obtained by comparing E with a competitor, what will be long is the construction of the competitor.

By homogeneity, we shall immediately reduce to the case when $x = 0$ and $r = 1$. We shall also assume that $H^1(E \cap \partial B(0,1))$ is not much larger than $2d(0)$, because otherwise (4.7) and (4.8) hold more or less trivially. First we shall find the points $\varphi(y)$, $y \in V$, and at the same time a net of simple curves $g_{j,k} \subset E \cap \partial B(0,1)$ that connect them like the $\mathfrak{C}_{j,k}$ do. That is, $g_{j,k}$ goes from $\varphi(y)$ to $\varphi(z)$, where y and z denote the endpoints of $\mathfrak{C}_{j,k}$.

For instance, if X is a vertical cone of type $\mathbb{Y}$, the $g_{j,k}$ are six curves that connect two points near the poles to three points near the equator. The construction will be done in Section 6, and the proof will rely on separation properties that come from the biHölder description of E that was obtained in [D3].

In Section 7 we shall replace the $g_{j,k}$ with Lipschitz graphs with small constants and the same endpoints, the $\Gamma_{j,k}$. The construction is a simple maximal function argument, but we shall need to estimate $H^1(\Gamma_{j,k} \setminus g_{j,k})$ in terms of how much length we win in the process.

The reason why we prefer the $\Gamma_{j,k}$ is that it is easier to modify the cone over a Lipschitz graph and diminish its area. The idea is to approach the minimal surface equation with the laplacian, and observe that a homogeneous function is rarely harmonic. We shall improve the cone over each $\Gamma_{j,k}$ separately, and then glue the result to get a good competitor for the cone over $\cup_{j,k}\Gamma_{j,k}$. This will save a substantial amount of area, unless the initial $\Gamma_{j,k}$ were already quite close to geodesics.

We do this modification in Section 8, and in Section 9 combine the three previous sections to get a competitor for E and prove (4.7).

We also make sure that our competitor has a tip near the origin that coincides with the cone over $\varphi_*(K)$, so if X is a full length minimal cone and $H^1(\varphi_*(K)) > 2d(x)$, we can use Definition 2.10 to improve again our competitor near the origin, and prove (4.8). So the proof of Theorem 4.5 will be completed at the end of Section 10.

Let us now explain why some of our assumptions can be obtained easily. We start with (4.1) and (4.2).

Lemma 4.9. *Let E be a reduced almost minimal set in $U \subset \mathbb{R}^n$, with a gauge function h such that (3.1) holds, and let $x \in E$ be given. For each $\varepsilon > 0$, we can find $r_0 > 0$ such that for $0 < r < r_0$, (4.1) and (4.2) hold for some cone X which is a blow-up limit of E at x.*

Here X is allowed to depend on r. Notice that $H^2(X \cap B(0,1)) = d(x)$ (by (3.15)). The full length property is not automatic, even though in the context of Theorem 1.15, it will be enough to use this extra assumption once.

Let x and ε be given. The fact that (4.1) holds for r small follows directly from (3.1) and (3.5). Suppose for a moment that there are arbitrarily small radii r such that (4.2) holds for no blow-up limit X. Let $\{r_k\}$ be a sequence of such radii that tends to 0. By (3.13), we can replace $\{r_k\}$ with a subsequence for which the $r_k^{-1}(E-x)$ converge to some limit X. Then X is a blow-up limit of E at x, and by (3.12) $d_{x,100r_k}(E, x+X)$ tend to 0. This contradiction completes the proof of Lemma 4.9. □

Let us also see why when X is a plane or a cone with the density $d(x)$, we may even dispense with the density assumption in (4.1).

Lemma 4.10. *For each $\delta > 0$, we can find $\varepsilon > 0$ such that if E is a reduced almost minimal set in $U \subset \mathbb{R}^n$, $B(x, 2r) \subset U$, $h(3r) \leq \varepsilon$, and (4.2) holds for some minimal cone X centered at the origin and such that $H^2(X \cap B(0,1)) \leq d(x)$, then $f(r) \leq \delta$.*

Indeed with these assumptions (verified with $\varepsilon/2$), Lemma 16.43 in [D3], applied with $F = x + X$, implies that

$$\begin{aligned} H^2(E \cap B(x,r)) &\leq H^2(F \cap B(x,(1+\delta)r)) + \delta r^2 \\ &= H^2(X \cap B(0,(1+\delta)r)) + \delta r^2 \leq \left[(1+\delta)^2 d(x) + \delta\right] r^2, \end{aligned} \tag{4.11}$$

hence $\theta(x,r) = r^{-2}H^2(E \cap B(x,r)) \leq d(x) + C\delta$ and $f(r) \leq C\delta$. The additional constant C is unimportant, and Lemma 4.10 follows. □

When (4.2) holds for some plane X, the condition that $H^2(X \cap B(0,1)) \leq d(x)$ is automatically satisfied, because π is the smallest possible density. This is not the case when X is a cone of type $\mathbb{Y}$, but at least, if ε and $h_1(r)$ are small enough, lemma 16.24 in [D3] gives a point of type Y near x, to which we can apply Lemma 4.10. We don't know this when X is of type $\mathbb{T}$.

We now turn to (4.3) and (4.4).

Lemma 4.12. *If E is a reduced almost minimal set in $U \subset \mathbb{R}^n$ and $B(x, r_0) \subset U$, (4.3) and (4.4) hold for almost every $r \in (0, r_0)$.*

We just need to know that $B(x, r_0) \subset U$ to make sure that E is closed in $B(x, r_0)$ and $H^2(E \cap B(x, r_1)) < +\infty$ for $r_1 < r_0$. Now we just need to check that (4.3) and (4.4) hold for almost every $r \in (0, r_1)$, and take a countable union.

We want to apply the coarea formula, and for this we need to know that E is rectifiable. This is verified in Section 2 of [D3] (see below (2.24) there). The reader should not worry that rectifiability is "only" proved for generalized quasiminimal sets in [D3]; the point of introducing generalized quasiminimal sets was specifically to make sure that our almost-minimal sets would automatically be generalized quasiminimal sets. Compare Definitions 2.10 and 4.3 in [D3] (or read the few lines just above Lemma 4.7 there), and also see Proposition 4.10 there if you want to use Definition 4.8 in [D3].

So we are allowed to apply the coarea formula (Theorem 3.2.22 in [Fe]) on the rectifiable set $E \cap B(0, r_1)$. We apply it to the C^1 mapping $g : z \to |z|$, and integrate against a bounded nonnegative Borel-measurable function f. The level sets are

the $g^{-1}(t) = E \cap \partial B(0,t)$, $0 < t < r_1$, and we get that

$$\int_{E\cap B(0,r_1)} f(z)\, J_g(z)\, dH^2(z) = \int_{t\in(0,r_1)} \Big\{ \int_{E\cap\partial B(0,t)} f(z)\, dH^1(z) \Big\}\, dt, \tag{4.13}$$

where J_g is the appropriate 1-jacobian (or gradient size) of g on $E \cap B(0,r_1)$, and the fact that the inside integral $\int_{E\cap\partial B(0,t)} f(z)\, dH^1(z)$ is a measurable function of t is part of the theorem.

For the sake of Lemma 4.12, we just need to know that $J_g(z) \le 1$, which is essentially obvious because g is 1-Lipschitz, but later on we shall need more information that we give now.

Since E is rectifiable, it has an approximate tangent plane $P(x)$ at x for H^2-almost every $x \in E$. Incidentally, we even know here that $P(x)$ is a true tangent plane (but this won't really matter), because E is locally Ahlfors-regular of dimension 2. See for instance Lemma 2.15 in [D3] for the local Ahlfors regularity and Exercise 41.21 in [D2] for the true tangent planes. We only need to compute $J_g(x)$ at points x where $P(x)$ exists. Notice also that, essentially by definition, $J_g(x)$ is almost-everywhere the same as if E were equal to $P(x)$. Thus

$$J_g(x) = \cos\alpha(x) \quad \text{almost-everywhere on } E, \tag{4.14}$$

where $\alpha(x) \in [0, \pi/2]$ denotes the angle of the radius $[0,x]$ with the tangent plane to E at x (that is, the smallest angle with a vector of the plane). When $n = 3$, we could also define $\alpha(x)$ by saying that $\pi/2 - \alpha(x)$ is the non oriented angle of $[0,x]$ with a unit normal to $P(x)$.

Return to the proof of Lemma 4.12. Set $\widetilde{f}(t) = \int_{E\cap\partial B(0,t)} f(z)\, dH^1(z)$ for $t \in (0, r_1)$. This is the inside integral in the right-hand side of (4.13), we already know that it is measurable, and we also get that $\widetilde{f}$ is integrable, because (4.13) says that

$$\int_{t\in(0,r)} \widetilde{f}(t)\, dt = \int_{E\cap B(0,r_1)} f(z)\, J_h(z)\, dH^2(z) \le ||f||_\infty\, H^2(E\cap B(0,r_1)) < +\infty. \tag{4.15}$$

Almost every $r \in (0, r_1)$ is a Lebesgue density point for $\widetilde{f}$, which means that

$$\lim_{\rho\to 0}\ \frac{1}{2\rho} \int_{(r-\rho,r+\rho)} |\widetilde{f}(t) - \widetilde{f}(r)|\, dt = 0. \tag{4.16}$$

Observe that (4.3) (for this single f) follows from (4.16) and the triangle inequality, but we shall still need some uniformity to control all the continuous functions at the same time. Let us take care of (4.4) in the mean time.

Take $f = 1$, and denote by b the corresponding function $\widetilde{f}$. Thus $b(t) = H^1(E \cap \partial B(0,t))$. As before, b is integrable on $(0, r_1)$. Denote by b^* its (non centered) Hardy-Littlewood maximal function; thus

$$b^*(r) = \sup\Big\{ \frac{1}{|I|} \int_I b(t)dt\,;\, I \subset (0,r_1) \text{ is an interval that contains } r \Big\}. \tag{4.17}$$

for $r \in (0, r_1)$. The Hardy-Littlewood maximal theorem (on page 5 of [St]) says that $b^*(r) < +\infty$ for almost-every $r \in (0, r_0)$. Obviously (4.4) holds when $b^*(r) < +\infty$, so we can safely return to (4.3).

Now let $C(\overline{B}(0, r_1))$ denote the set of continuous functions on $\overline{B}(0, r_1)$, and select a countable family $\{f_i\}$, $i \in \mathbb{N}$, in $C(\overline{B}(0, r_1))$, which is dense in the *sup* norm. We know that for almost every $r \in (0, r_1)$, (4.17) holds, and so does (4.16) for every $\widetilde{f}_i$.

Fix $r \in (0, r_1)$ like this. We know that (4.4) holds for r, and we just need to check that (4.3), or even (4.16), holds for any given f. Set $a(\rho) = \int_{(r-\rho, r+\rho)} |\widetilde{f}(t) - \widetilde{f}(r)|\, dt$ for $\rho > 0$ small (so that $(r-\rho, r+\rho) \subset (0, r_1)$). We just need to show that $\lim_{\rho \to 0} \rho^{-1} a(\rho) = 0$.

Let $\varepsilon > 0$ be given. Let i be such that $||f - f_i||_\infty \leq \varepsilon$. Then

$$\begin{aligned} a(\rho) &\leq \int_{(r-\rho, r+\rho)} \Big[|\widetilde{f}(t) - \widetilde{f}_i(t)| + |\widetilde{f}_i(t) - \widetilde{f}_i(r)| + |\widetilde{f}_i(r) - \widetilde{f}(r)| \Big]\, dt \\ &\leq \int_{(r-\rho, r+\rho)} \big[|\widetilde{f}_i(t) - \widetilde{f}_i(r)| + 2||f - f_i||_\infty\, H^1(E \cap \partial B(0,t)) \big]\, dt \qquad (4.18) \\ &\leq 2\varepsilon\rho + 2\varepsilon \int_{(r-\rho, r+\rho)} H^1(E \cap \partial B(0,t))\, dt \end{aligned}$$

for ρ small enough, by (4.16) for $\widetilde{f}_i$ and because $||f - f_i||_\infty \leq \varepsilon$. Since

$$\int_{(r-\rho, r+\rho)} H^1(E \cap \partial B(0,r))\, dt = \int_{(r-\rho, r+\rho)} b(t) \leq 2\rho b^*(r) \qquad (4.19)$$

by definition of b and b^*, (4.18) shows that $a(\rho) \leq [2 + 4b^*(r)]\, \varepsilon\rho$ for ρ small enough. Thus $a(\rho)/\rho$ tends to 0 and (4.16) holds for f, as needed.

This completes our proof of Lemma 4.12. □

5. Differential inequalities and decay for the density excess f

In this section we show how the estimate (4.8) in the conclusion of Theorem 4.5 can be used to derive a differential inequality on f, which itself leads to a decay estimate for f. Once we have all this, we shall use f to control the geometry of E, but this will not happen before Sections 11 and 12.

Here and in the rest of the paper, E is a reduced two-dimensional almost-minimal set in the open set $U \subset \mathbb{R}^n$, with a gauge function h such that (3.1) holds.

Let us first discuss how to recover functions like $\theta(r)$ from their derivative almost-everywhere. From now on, we shall work with $x = 0$ to simplify our notation.

Lemma 5.1. *Suppose $0 \in U$, and set $D = \operatorname{dist}(0, \mathbb{R}^n \setminus U)$ and $v(r) = H^2(E \cap B(0,r))$ for $0 < r < D$. Let μ denote the image by the radial projection $z \to |z|$ of the restriction of H^2 to E; thus μ is a positive Radon measure on $[0, D)$, and $v(r) = \mu\big([0,r)\big) = \int_{[0,r)} d\mu$ for $0 < r < D$. Let $b(r)$ be a positive C^1 function on $(0, D)$. Then*

$$b(y)v(y) - b(x)v(x) = \int_{[x,y)} b(r)\, d\mu(r) + \int_{[x,y)} b'(r)v(r)\, dr \quad \text{for } 0 < x < y < D. \qquad (5.2)$$

In addition, v is differentiable almost-everywhere and

$$b(y)v(y) - b(x)v(x) \geq \int_x^y [b(r)v'(r) + b'(r)v(r)]\, dr \quad \text{for } 0 < x < y < D. \tag{5.3}$$

In order to prove (5.2), compute $I = \int_{x \leq s \leq t < y} b'(s) ds d\mu(t)$ in two different ways. If we integrate in s first, we get that $I = \int_{[x,y)} [b(t) - b(x)] d\mu(t)$, while integrating in t first yields $I = \int_{[x,y)} b'(s)(v(y) - v(s)) ds$. Then

$$\begin{aligned} \int_{[x,y)} b(r) d\mu(r) + \int_{[x,y)} b'(r)v(r) dr &= I + b(x) \int_{[x,y)} d\mu(t) - I + v(y) \int_{[x,y)} b'(s) ds \\ &= b(x)[v(y) - v(x)] + v(y)[b(y) - b(x)] = v(y)b(y) - v(x)b(x), \end{aligned} \tag{5.4}$$

so (5.2) holds. The fact that $v'(r)$ exists for almost every $r < D$ is a standard fact about nondecreasing functions, and so is the fact that the absolutely continuous part of $d\mu$ is $v'(r)dr$. In particular, $v'(r)dr \leq d\mu$ and (5.3) follows from (5.2) because b is nonnegative. Lemma 5.1 follows. □

Let us immediately record what this means in terms of $\theta(r) = r^{-2} H^2(E \cap B(0,r)) = r^{-2} v(r)$.

Lemma 5.5. *Still assume that $0 \in U$ and set $D = \operatorname{dist}(0, \mathbb{R}^3 \setminus U)$. The function θ is differentiable almost-everywhere on $(0, D)$, with*

$$\theta'(r) = r^{-2} v'(r) - 2r^{-3} v(r) = r^{-2} v'(r) - 2r^{-1} \theta(r) \quad \text{almost-everywhere,} \tag{5.6}$$

and

$$\theta(y) - \theta(x) \geq \int_x^y \theta'(r)\, dr \quad \text{for } 0 < x < y < D. \tag{5.7}$$

Indeed it is easy to see that if v is differentiable at r, then θ is differentiable at r too, with the formula in (5.6). Then (5.7) is just the same as (5.3). □

Let us also check that (with the notation of Lemma 5.1)

$$v'(r) \geq H^1(E \cap \partial B(0,r)) \text{ for almost every } r \in (0, D). \tag{5.8}$$

Pick any $r_1 \in (0, D)$ and apply the co-area formula (4.13). [We are in the same situation as in Lemma 4.12, where the assumption was that $r_0 \leq D$, except for the fact that now $x = 0$.] Take for f the characteristic function of $A = B(0,r) \setminus B(0, r - \rho)$, with $0 < \rho \leq r < r_1$. We get that

$$\begin{aligned} \int_{E \cap A} J_g(z)\, dH^2(z) &= \int_{t \in (0, r_1)} \left\{ \int_{E \cap \partial B(0,t)} f(z)\, dH^1(z) \right\} dt \\ &= \int_{t \in [r-\rho, r)} H^1(E \cap \partial B(0,t))\, dt \end{aligned} \tag{5.9}$$

and, since $J_g(z) \le 1$ in (4.13), we get that

$$v(r) - v(r-\rho) = H^2(E \cap A) \ge \int_{t \in [r-\rho, r)} H^1(E \cap \partial B(0,t))\, dt. \tag{5.10}$$

Now fix r, divide (5.10) by ρ, and let ρ tend to 0. The left-hand side tends to $v'(r)$ if $v'(r)$ exist. The right-hand side tends to $H^1(E \cap \partial B(0,t))$ when r is a Lebesgue point for the function $t \to H^1(E \cap \partial B(0,t))$. This function is locally integrable by (5.9) (recall that its measurability is part of the co-area formula), so the Lebesgue density theorem says that almost every $r \in (0, r_1)$ is a Lebesgue point, and (5.10) yields (5.8).

Let us now see how to deduce a differential inequality and a decay estimate from the conclusion of Theorem 4.5.

Lemma 5.11. *Let $0 < x < y < D = \operatorname{dist}(0, \mathbb{R}^3 \setminus U)$ be given, and suppose that for some $\alpha \in (0, 1/3)$, (4.8) holds for almost every $r \in (x, y)$. Then $f(r) = \theta(r) - d(0)$ is differentiable almost-everywhere on (x, y), with*

$$rf'(r) \ge \frac{4\alpha}{1-2\alpha} f(r) - 24h(2r) \quad \text{for almost every } r \in (x,y), \tag{5.12}$$

and then

$$f(x) \le (x/y)^a f(y) + Cx^a \int_x^y r^{-a-1} h(2r) dr, \tag{5.13}$$

where we set $a = \frac{4\alpha}{1-2\alpha}$.

Indeed, we already know from Lemma 5.5 that f is differentiable almost-everywhere on $(0, D)$, and (5.6) says that

$$f'(r) = \theta'(r) = r^{-2} v'(r) - 2r^{-1}\theta(r) \ge r^{-2} H^1(E \cap \partial B(0,r)) - 2r^{-1}\theta(r) \tag{5.14}$$

almost-everywhere, by (5.8).

Set $X = \dfrac{1}{2r} H^1(E \cap \partial B(0,r))$. Thus $rf'(r) \ge 2X - 2\theta(r)$. But (4.8) (divided by r^2) says that $(1-2\alpha)X \ge \theta(r) - 2\alpha d(0) - 4h(2r)$. Hence

$$\begin{aligned} rf'(r) &\ge 2X - 2\theta(r) \ge \frac{2}{1-2\alpha}\big[\theta(r) - 2\alpha d(0) - 4h(2r)\big] - 2\theta(r) \\ &\ge \frac{4\alpha}{1-2\alpha}\big[\theta(r) - d(0)\big] - \frac{8h(2r)}{1-2\alpha} = \frac{4\alpha f(r)}{1-2\alpha} - \frac{8h(2r)}{1-2\alpha}, \end{aligned} \tag{5.15}$$

which yields (5.12) because $\alpha < 1/3$.

Now let us integrate (5.12) to prove (5.13). Set $g(r) = r^{-a} f(r)$; then

$$g'(r) = r^{-a} f'(r) - a r^{-a-1} f(r) = r^{-a-1}[rf'(r) - af(r)] \ge -24 r^{-a-1} h(2r) \tag{5.16}$$

almost-everywhere on (x, y), by (5.12). We claim that

$$g(y) - g(x) \geq \int_x^y g'(r)dr \geq -24 \int_x^y r^{-a-1} h(2r)dr. \tag{5.17}$$

By (5.16), we just need to worry about the first inequality. Notice that $g(r) = r^{-a} f(r) = r^{-a}\theta(r) - r^{-a}d(0)$. The analogue for $-r^{-a}d(0)$ of the first inequality in (5.17) is trivial, and for $r^{-a}\theta(r) = r^{-a-2}v(y)$, it is just equivalent to (5.3) with $b(r) = r^{-a-2}$. So the first part of (5.17) follows from Lemma 5.1, and (5.17) holds. Now

$$f(x) = x^a g(x) \leq x^a g(y) + 24x^a \int_x^y r^{-a-1} h(2r)dr \tag{5.18}$$

by (5.17); (5.13) and Lemma 5.10 follow. □

The next two examples are here to explain why (5.13) is really a decay estimate. The computation for the first one will be used in Section 13; Example 5.21 is just here to give an idea of what happens with weaker decay conditions.

Example 5.19. Let us check that

$$f(x) \leq x^a y^{-a} f(y) + C' x^b \tag{5.20}$$

when x and $y > x$ are as in Lemma 5.11 and $h(r) \leq Cr^b$ for some $b \in (0, a)$. (Even if h is much smaller, we shall not get better bounds than $f(x) \leq Cx^a$ anyway.) Indeed $\int_x^y r^{-a-1} h(2r)dr \leq C2^b \int_x^y r^{b-a-1}dr = C2^b(a-b)^{-1}(x^{-a+b} - y^{-a+b}) \leq C2^b(a-b)^{-1}x^{-a+b}$, hence (5.20) follows from (5.13).

Example 5.21. Suppose that $h(r) \leq C[\log(A/r)]^{-b}$ for some constants $A, b > 0$. Then

$$f(x) \leq (x/y)^a f(y) + C(x/A)^{a/2} + C\big[\log\big(\frac{A}{2x}\big)\big]^{-b} \leq C_{A,y}\big[\log\big(\frac{A}{2x}\big)\big]^{-b} \tag{5.22}$$

when $x < y < A/3$ satisfy the assumptions of Lemma 5.11. [However, we should remember that we have to assume (3.1) to prove (5.13), so we cannot really take $h(r) = C[\log(A/r)]^{-b}$ with $b \leq 1$.]

Indeed, (5.13) yields $f(x) \leq (x/y)^a f(y) + Cx^a \int_x^y r^{-a-1}[\log(\frac{A}{2r})]^{-b}dr$. We cut the domain of integration (x, y) into two region, I_1 where $\log(\frac{A}{2r}) \geq \frac{1}{2}\log(\frac{A}{2x})$ and $I_2 = (x, y) \setminus I_1$. Then

$$\int_{I_1} r^{-a-1}\big[\log\big(\frac{A}{2r}\big)\big]^{-b}dr \leq \big[\frac{1}{2}\log\big(\frac{A}{2x}\big)\big]^{-b} \int_{I_1} r^{-a-1}dr \leq Cx^{-a}\big[\log\big(\frac{A}{2x}\big)\big]^{-b}. \tag{5.23}$$

On I_2 we have that $\frac{A}{2r} \leq \big(\frac{A}{2x}\big)^{1/2}$, hence $r \geq (Ax/2)^{1/2}$. Thus

$$\begin{aligned} \int_{I_2} r^{-a-1}\big[\log\big(\frac{A}{2r}\big)\big]^{-b}dr &\leq \big[\log\big(\frac{A}{2y}\big)\big]^{-b} \int_{I_2} r^{-a-1}dr \\ &\leq C\int_{I_2} r^{-a-1}dr \leq C(Ax/2)^{-a/2}, \end{aligned} \tag{5.24}$$

We sum (5.23) and (5.24), multiply by Cx^a, add $(x/y)^a f(y)$ and get (5.22).

Even if h goes very slowly to 0, (5.13) still gives a definite decay, with the same sort of argument as above. For instance, we can pick any $z \in [x, y]$, and observe that

$$\begin{aligned} \int_x^y r^{-a-1} h(2r) dr &\le \int_x^z r^{-a-1} h(2z) dr + \int_z^y r^{-a-1} h(2y) dr \\ &\le a^{-1} x^{-a} h(2z) + a^{-1} z^{-a} h(2y), \end{aligned} \tag{5.25}$$

which leads to

$$f(x) \le x^a y^{-a} f(y) + Ch(2z) + Cx^a z^{-a} h(2y). \tag{5.26}$$

One can then optimize the choice of z (depending on x), but anyway it is easy to force the right-hand side of (5.26) to tend to 0. But again, remember that we use (3.1) to prove (5.13).

It is amusing that (5.13) leads to some definite decay for f in all cases, even when (3.1) fails. But this probably does not mean that we could use a variant of the more delicate method here to get the results of [D3] under weaker assumptions on h.

We end the section with a discussion of a weaker form of Lemma 5.11 when (4.8) is replaced with the weaker inequality (5.27). This part will only needed in Section 13.

In the good cases, which include all the cases when $n = 3$ and y is small enough, we shall be able to prove (4.8). In some other cases, for instance if the tangent cones to E only satisfy a weaker full length condition, we may be a little less lucky but still get that

$$\theta(r) \le \frac{1}{2r} H^1(E \cap \partial B(0, r)) - \alpha \Big[\frac{1}{2r} H^1(E \cap \partial B(0, r)) - d(0) \Big]_+^N + Ch(2r) \tag{5.27}$$

for some $N > 1$. Here the $+$ means that we take the positive part (so $A_+^N = 0$ for $A \le 0$). Let us record the estimates that this would yield; we shall discuss the weaker full length conditions in Section 13.

Lemma 5.28. *Let $0 < x < y < D = \operatorname{dist}(0, \mathbb{R}^3 \backslash U)$ be given, and suppose that (5.27) holds for some $\alpha \in (0, 1/2)$, some $N > 1$, and almost-every $r \in (x, y)$. Then f is differentiable almost-everywhere on (x, y), and*

$$r f'(r) \ge 2\alpha \, f(r)_+^N - Ch(2r) \quad \text{for almost every } r \in (x, y). \tag{5.29}$$

As before, the differentiability of f almost everywhere comes from Lemma 5.5. Set $X = \frac{1}{2r} H^1(E \cap \partial B(0, r))$ as above; thus (5.14) still says that $r f'(r) \ge 2X - 2\theta(r)$, and (5.27) says that

$$\theta(r) - X \le -\alpha \big[X - d(0) \big]_+^N + Ch(2r) \tag{5.30}$$

We first prove (5.29) when $\theta(r) \le X$. Then

$$\begin{aligned} rf'(r) &\ge 2X - 2\theta(r) \ge 2\alpha\big[X - d(0)\big]_+^N - 2Ch(2r) \\ &\ge 2\alpha\big[\theta(r) - d(0)\big]_+^N - 2Ch(2r) = 2\alpha f(r)_+^N - 2Ch(2r) \end{aligned} \tag{5.31}$$

by definition of $f(r)$. So (5.29) holds in this case.

If $\theta(r) > X$, (5.30) says that $\theta(r) - X \le Ch(2r)$. Then $\big[X - d(0)\big]_+^N \ge \big[\theta(r) - d(0)\big]_+^N - Ch(2r)$ and the proof of (5.31) yields $rf'(r) \ge 2\alpha f(r)_+^N - Ch(2r)$ again. This proves (5.29) and Lemma 5.28. □

Remark 5.32. As before, (5.29) and Lemma 5.4 yield a decay estimate, but which is not nearly as good as in the situation of Lemma 5.11. For instance, we claim that if (5.27) holds for almost-every $r \in (0, y)$ for some choice of $y < D$, $\alpha \in (0, 1/2)$, and $N > 1$, and if

$$h(r) \le C[\mathrm{Log}(\frac{1}{r})]^{-\frac{N}{N-1}} \quad \text{for } r \text{ small}, \tag{5.33}$$

then

$$f(x) \le C_1 \Big[\mathrm{Log}\Big(\frac{2y}{x}\Big)\Big]^{-\frac{1}{N-1}} \quad \text{for } 0 < x < y, \tag{5.34}$$

for some C_1 that also depends on y and $f(y)$. Set

$$\varphi(r) = C_1\big[\mathrm{Log}(\frac{2y}{r})\big]^{-\frac{1}{N-1}} \quad \text{for } r < y. \tag{5.35}$$

We choose φ of this form because

$$r\varphi'(r) = \frac{C_1}{N-1}\big[\mathrm{Log}(\frac{2y}{r})\big]^{-\frac{N}{N-1}} = \frac{C_1^{1-N}}{N-1}\,\varphi(x)^N, \tag{5.36}$$

which is similar to (5.29).

If C_1 is large enough, $\varphi(y) > f(y)$. Then suppose that $f(r) \ge \varphi(r)$ for some $r < y$, and denote by x the supremum of such r. Notice that $f(x) \ge \varphi(x)$ because φ is continuous, $f(r) = \theta(r) - d(0)$, and $r^2\theta(r)$ is nondecreasing. Hence $x < y$. For $\delta > 0$ small,

$$\begin{aligned} \theta(x+\delta) - \theta(x) &\ge \int_x^{x+\delta} \theta'(r)\,dr \ge \int_x^{x+\delta} \big[2\alpha\, f(r)_+^N - Ch(2r)\big]\,\frac{dr}{r} \\ &\ge \int_x^{x+\delta} \Big\{2\alpha\, f(r)_+^N - C[\mathrm{Log}(\frac{2y}{r})]^{-\frac{N}{N-1}}\Big\}\,\frac{dr}{r} \\ &\ge 2\alpha\delta x^{-1} f(x_+)_+^N - C\delta x^{-1}[\mathrm{Log}(\frac{2y}{x})]^{-\frac{N}{N-1}} + o(\delta) \end{aligned} \tag{5.37}$$

by Lemma 5.5, because $\theta' = f'$, by (5.29), and by (5.33). Here again, C may depend on y. At the same time,

$$\theta(x+\delta) - \theta(x) = f(x+\delta) - f(x) \le \varphi(x+\delta) - \varphi(x) = \delta\varphi'(x) + o(\delta) \tag{5.38}$$

because $f(x+\delta) < \varphi(x+\delta)$ and $f(x) \geq \varphi(x)$, which yields

$$2\alpha f(x_+)_+^N - C[\mathrm{Log}(\frac{2y}{x})]^{-\frac{N}{N-1}} \leq x\varphi'(x) = \frac{C_1}{N-1}\big[\mathrm{Log}\big(\frac{2y}{x}\big)\big]^{-\frac{N}{N-1}} \tag{5.39}$$

by (5.36). If C_1 is larger than $2(N-1)C$, this yields $2\alpha f(x_+)_+^N \leq \frac{C_1}{2(N-1)}\big[\mathrm{Log}\big(\frac{2y}{x}\big)\big]^{-\frac{N}{N-1}}$. In addition, $\varphi(x) \leq f(x) \leq f(x_+)$ by definition of x and because $\theta(r)$ is nondecreasing and the other functions are continuous. Thus $2\alpha\varphi(x)^N \leq \frac{C_1}{2(N-1)}\big[\mathrm{Log}\big(\frac{2y}{x}\big)\big]^{-\frac{N}{N-1}}$. This contradicts (5.35) if C_1 is large enough. So there is no x as above, $f(r) < \varphi(r)$ for $r < y$, and (5.34) follows.

Even if we take h much smaller than in (5.33), the differential inequality in (5.29) will not give a much better decay than the one in (5.34). For instance, φ in (5.35) solves (5.29) with $h = 0$ when C_1 is small , and it does not decay much near 0.

And unfortunately (5.34) does not give enough control to allow us to prove that E is locally C^1-equivalent to a minimal cone. See Section 13 for a rapid discussion of the weaker full length conditions.

6. Separation properties and the construction of a net of curves

The next few sections will be devoted to the proof of Theorem 4.5. So we fix a reduced almost minimal set E in the open set $U \subset \mathbb{R}^n$, with gauge function h, and we let $x \in E$ and $r > 0$ be such that (4.1) and (4.2) hold. [We shall not use (4.3) and (4.4) before Section 9.]

Since our statement is invariant under translations and dilations, we can assume that $x = 0$ and $r = 1$, which will simplify the notation.

Our ultimate goal is to construct a nice competitor for E in $B(0,1)$, which in particular will be better than the cone over $E \cap \partial B(0,1)$ unless $E \cap \partial B(0,1)$ is already very nice. In this section we concentrate on finding a net of curves $g_{j,k} \subset E \cap \partial B(0,1)$; see Lemma 6.11.

We already have a constant $\varepsilon > 0$ in play (in (4.2)), and our construction will use another small parameter $\tau > 0$, also to be chosen later. Although we shall not always repeat this, all the estimates below hold only when τ is small enough, and then ε is small enough, depending on τ.

We set $B = B(0,1)$ and $\partial B = \partial B(0,1)$ to save notation. Observe before we start that

$$d(0) - C\varepsilon \leq \theta(r) \leq d(0) + C\varepsilon \quad \text{for } 0 < r \leq 50, \tag{6.1}$$

by (4.1), (3.6), and (3.8). The next lemma will allow us to restrict our attention to the case when

$$H^1(E \cap \partial B) \leq 2(1+\tau)\, d(0). \tag{6.2}$$

Lemma 6.3. *If (6.2) fails, the conclusions of Theorem 4.5 hold, and in particular (4.8) holds for every $\alpha \in (0, 1/3)$.*

We start with (4.8), which is the most important. If (6.2) fails,

$$
\begin{aligned}
H^2(E\cap B(x,r)) - \frac{r}{2}H^1(E\cap\partial B(x,r)) + \alpha r\Big[H^1(E\cap\partial B(x,r)) - 2d(x)r\Big]& \\
= \theta(1) - \frac{1-2\alpha}{2}\,H^1(E\cap\partial B) - 2\alpha d(0)& \\
\le \big[d(0)+C\varepsilon\big] - (1-2\alpha)(1+\tau)\,d(0) - 2\alpha d(0)& \\
= C\varepsilon - (1-2\alpha)\,\tau\,d(0) < 0&
\end{aligned}
\tag{6.4}
$$

because $x = 0$ and $r = 1$, by (6.1), and if ε is small enough; (4.8) follows.

Let us also check (4.6) and (4.7). We take $\varphi(x) = x$ for $x \in V$. Recall that for Theorem 4.5, $K = X \cap \partial B$, where X still denotes the minimal cone in (4.2), so we get that $\varphi_*(K) = K = X \cap \partial B$. We need to control $D = H^2(X \cap B(0,1))$, so we apply Lemma 16.43 in [D3], with $E = X$, $F = E$, $x = 0$, $r = 1$, and $\delta = \tau/3$. The assumptions are satisfied by (4.1) and (4.2), and we get that

$$
\begin{aligned}
D = H^2(X\cap B(0,1)) &\le H^2(E\cap B(0,1+\delta)) + \delta \\
&= (1+\delta)^2\theta(0,1+\delta) + \delta \le (1+3\delta)d(0) = (1+\tau)\,d(0)
\end{aligned}
\tag{6.5}
$$

by (6.1), because $d(0) \ge \pi$, and if ε is small enough. The same lemma, with E and X exchanged, also yields

$$
\theta(1) = H^2(E\cap B(0,1)) \le H^2(X\cap B(0,1+\delta)) + \delta = (1+\delta)^2 D + \delta \tag{6.6}
$$

hence

$$
D \ge (1+\delta)^{-2}\theta(1) - \delta \ge (1+\delta)^{-2}[d(0) - C\varepsilon] - \delta \ge (1-3\delta)\,d(0) = (1-\tau)\,d(0), \tag{6.7}
$$

again by (6.1) and because $d(0) \ge \pi$. Observe that (6.5) and (6.7) do not rely on the failure of (6.2). Then (4.6) holds because

$$
H^1(\varphi_*(K)) = H^1(K) = 2D \le 2(1+\tau)\,d(0) \le H^1(E\cap\partial B) \tag{6.8}
$$

because X is the cone over K, by (6.5) and because (6.2) fails. Next (4.7) holds because

$$
\begin{aligned}
H^2(E\cap B(x,r)) - \frac{r}{2}H^1(E\cap\partial B(x,r)) + 10^{-5}r\Big[H^1(E\cap\partial B(x,r)) - rH^1(\varphi_*(K))\Big]& \\
= \theta(1) - \frac{1-2\cdot 10^{-5}}{2}\,H^1(E\cap\partial B) - 2\cdot 10^{-5}D& \\
\le \big[d(0)+C\varepsilon\big] - (1-2\cdot 10^{-5})(1+\tau)\,d(0) - 2\cdot 10^{-5}D& \\
= C\varepsilon - \tau d(0) + 2\cdot 10^{-5}(1+\tau)\,d(0) - 2\cdot 10^{-5}D& \\
\le C\varepsilon - \tau d(0) + 2\cdot 10^{-5}d(0)[(1+\tau) - (1-\tau)] < 0&
\end{aligned}
\tag{6.9}
$$

(as for (6.4)) because $x = 0$ and $r = 1$, by (6.8) and (6.1), because (6.2) fails, and by (6.7). This completes the proof of Lemma 6.3. □

So we shall now assume that (6.2) holds. Recall from (4.2) that we have a minimal cone X centered at the origin, such that

$$d_{0,100}(E, X) \leq \varepsilon. \tag{6.10}$$

We shall again use a standard decomposition of $K = X \cap \partial B$ into geodesic arcs $\mathfrak{C}_{j,k}$, $(j,k) \in \widetilde{J}$, as described in Section 2. We still denote by $V = V_0 \cup V_1$ the collection of endpoints of the $\mathfrak{C}_{j,k}$, with V_0 corresponding to the original vertices (those who belong to three $\mathfrak{C}_{j,k}$) and V_1 to the added vertices (where only two $\mathfrak{C}_{j,k}$ end, and make 180° angles). We want to draw curves $g_{j,k}$ in $E \cap \partial B$ with the same structure, as in the following lemma.

Lemma 6.11. *We can find points $\varphi(y)$, $y \in V$, and simple arcs $g_{j,k}$ in $E \cap \partial B$, so that*

(6.12) *if y and z denote the endpoints of $\mathfrak{C}_{j,k}$, the endpoints of $g_{j,k}$ are $\varphi(y)$ and $\varphi(z)$,*

(6.13) *the arcs $g_{j,k}$ are disjoint, except for their endpoints,*

$$|\varphi(y) - y| \leq C\tau \quad \text{for } y \in V, \tag{6.14}$$

and

$$\operatorname{dist}(z, \mathfrak{C}_{j,k}) \leq C\tau \text{ for } z \in g_{j,k}\,. \tag{6.15}$$

The proof will rely on a local description of E as a biHölder image of a minimal cone, which we take from [D3], plus a little bit of topology.

Let us apply Lemma 16.19 in [D3] if 0 is a point of type P, Lemma 16.25 in [D3] if 0 is a Y-point, and Lemma 16.56 in [D3] if it is a T-point. Each time we take for τ the same constant as in (6.2). [There is nothing subtle here, we are just saving some notation, and anyway we shall take τ small.] By (4.1), the lemma applies if ε is small enough. We get that B is a biHölder ball for E, of the same type as 0, and with the constant τ. By Definition 15.10 in [D3], this means that there is a reduced minimal cone X centered at the origin, and a biHölder mapping $f : B(0,2) \to \mathbb{R}^n$, with the following properties:

$$|f(y) - y| \leq \tau \quad \text{for } y \in B(0,2), \tag{6.16}$$

$$(1-\tau)\,|y-z|^{1+\tau} \leq |f(y) - f(z)| \leq (1+\tau)\,|y-z|^{1-\tau} \quad \text{for } y, z \in B(0,2), \tag{6.17}$$

$$B(0, 2-\tau) \subset f(B(0,2)), \tag{6.18}$$

and

$$E \cap B(0, 2-\tau) \subset f(X \cap B(0,2)) \subset E. \tag{6.19}$$

In addition, the proof in [D3] says that we can take the same cone X as in (6.10); the point is we can use $Z(0,3) = X$ when we apply Corollary 15.11 in [D3], and the proof of [DTT] allows us to take $X = Z(0,3)$ above.

In fact we shall only be interested in the restriction of f to $X \cap B(0,2)$, and in particular we do not need to know (6.18) (or (6.17) when y or z lies out of X). The precise biHölder estimate will not be used either, but we need to know that f is a homeomorphism. We shall use f to find a set $G_1 \subset E \cap \partial B$ with a good separation property; in the special case of dimension $n = 3$, and in particular if we used the "Mumford–Shah" definition of almost minimal sets, we could obtain the separation property more easily. See Remark 6.46.

First fix an arc $\mathfrak{C}_{j,k}$, denote by x and x' the two endpoints of $\mathfrak{C}_{j,k}$, and set

$$\mathfrak{C}' = \mathfrak{C}'_{j,k} = \big\{ z \in \mathfrak{C}_{j,k} \, ; \, \mathrm{dist}(z, \{x, x'\}) \geq 5\tau \big\} \tag{6.20}$$

(a slightly smaller arc). If τ is smaller than $\eta_0/10$, (2.2) says that $\mathfrak{C}'$ is not empty. Denote by H the intersection of $\overline{B}(0, 3/2)$ with the cone over $\mathfrak{C}'$. Thus H is a sector in a 2-disk. Then set

$$F = H \cap f^{-1}(E \cap \partial B) = \big\{ x \in H \, ; \, |f(x)| = 1 \big\}, \tag{6.21}$$

because $f(H) \subset E$ since $H \subset X$ and by (6.19). Notice that

$$\mathrm{dist}(z, \mathfrak{C}') = ||z| - 1| \leq \tau \text{ for } z \in F, \tag{6.22}$$

just because $|f(z)| = 1$ and by (6.16). Also, F is a level set of the restriction of $|f|$ to H, so it separates 0 from $\frac{3}{2}\mathfrak{C}'$ in H, just because (6.16) says that $|f(0)| < 1$ and that $|f(y)| > 1$ for $y \in \frac{3}{2}\mathfrak{C}'$.

Now we use some topology. The set H is compact, connected, locally connected, and simply connected. The compact set $F \subset H$ separates the two connected pieces $\{0\}$ and $\frac{3}{2}\mathfrak{C}'$. By 52.III.1 on page 335 of [Ku], there is a compact connected set $F_1 \subset F$ such that

$$F_1 \text{ separates 0 from } \tfrac{3}{2}\mathfrak{C}' \text{ in } H. \tag{6.23}$$

The author wishes to thank A. Ancona for telling him about this separation theorem, and is happy to share the statement and reference. We shall use it this way for the moment, but in fact, since H is a planar domain (a topological disk) we could manage with a less subtle theorem, for instance Theorem 14.3 on p. 123 in [Ne]. We shall say a few words about this later, in Remark 6.45.

For each $y \in \mathfrak{C}'$, set $I_y = \big\{ ty \, ; \, 1/2 \leq t \leq 3/2 \big\}$. If I_y did not meet F_1, we could use it to connect 0 (or equivalently $\frac{1}{2}\mathfrak{C}'$) to $\frac{3}{2}\mathfrak{C}'$ in $H \setminus F_1$. So we can find $y^\sharp \in F_1 \cap I_y$. Observe that $|y^\sharp - y| \leq ||y^\sharp| - 1| \leq \tau$ by (6.22), hence $|f(y^\sharp) - y| \leq |f(y^\sharp) - y^\sharp| + |y^\sharp - y| \leq 2\tau$ by (6.16).

Set $G_{j,k} = f(F_1)$. Thus $G_{j,k}$ is a connected subset of $E \cap \partial B$ (by (6.19) and (6.21)),

$$\mathrm{dist}(y, G_{j,k}) \leq 2\tau \text{ for } y \in \mathfrak{C}' \tag{6.24}$$

(because $f(y^\sharp) \in G_{j,k}$), and

$$\operatorname{dist}(z, \mathfrak{C}') \le 2\tau \text{ for } z \in G_{j,k}\,, \tag{6.25}$$

because $z = f(y)$ for some $y \in F_1 \subset F$, then $|y' - y| \le \tau$ for some $y' \in \mathfrak{C}'$, by (6.22), and then $\operatorname{dist}(z, \mathfrak{C}') \le |f(y) - y'| \le |f(y) - y| + |y - y'| \le 2\tau$.

When we apply (6.24) to the endpoints of $\mathfrak{C}'$ and recall that $\mathfrak{C}' = \mathfrak{C}'_{j,k}$ is a geodesic which is barely shorter than $\mathfrak{C}_{j,k}$, we get that

$$H^1(G_{j,k}) \ge \operatorname{length}(\mathfrak{C}_{j,k}) - 15\tau \tag{6.26}$$

because $G_{j,k}$ is connected. By the definition (6.20) and (2.3) in particular (and if τ is small compared to η_0), the various $\mathfrak{C}'_{j,k}$ lie at distances greater than 10τ from each other, and then (6.25) says that the $G_{j,k}$ are disjoint.

Recall that $X \cap \partial B(0,1)$ is the disjoint union of the $\mathfrak{C}_{j,k}$, so

$$\sum_{j,k} \operatorname{length}(\mathfrak{C}_{j,k}) = H^1(X \cap \partial B(0,1)) = 2H^2(X \cap B(0,1)) \tag{6.27}$$

because X is a cone. Also recall from (6.5) and (6.7) that

$$|H^2(X \cap B(0,1)) - d(0)| \le \tau d(0) \tag{6.28}$$

if ε is small enough, depending on τ. For each fixed pair $(j,k) \in \widetilde{J}$,

$$\begin{aligned} H^1(G_{j,k}) &= \sum_{(j',k') \in \widetilde{J}} H^1(G_{j',k'}) - \sum_{(j',k') \neq (j,k)} H^1(G_{j',k'}) \\ &= H^1\Big(\bigcup_{(j',k') \in \widetilde{J}} G_{j',k'} \Big) - \sum_{(j',k') \neq (j,k)} H^1(G_{j',k'}) \\ &\le H^1(E \cap \partial B) - \sum_{(j',k') \neq (j,k)} \big[\operatorname{length}(\mathfrak{C}_{j',k'}) - 15\tau \big] \\ &\le 2(1+\tau)\, d(0) - \big[2H^2(X \cap B(0,1)) - \operatorname{length}(\mathfrak{C}_{j,k}) \big] + C\tau \\ &\le \operatorname{length}(\mathfrak{C}_{j,k}) + C_1 \tau \end{aligned} \tag{6.29}$$

because the $G_{j',k'}$ disjoint and contained in $E \cap \partial B$, then by (6.26), (6.2), (6.27), and (6.28). Here C and C_1 depend on $d(0)$ and the number of arcs, but this does not matter, and in addition these constants would be easy to estimate in terms of n alone. Similarly,

$$\begin{aligned} H^1\big([E \cap \partial B] \setminus \bigcup_{j,k} G_{j,k} \big) &= H^1(E \cap \partial B) - \sum_{j,k} H^1(G_{j,k}) \\ &\le 2(1+\tau)\, d(0) - \sum_{j,k} \big[\operatorname{length}(\mathfrak{C}_{j,k}) - 15\tau \big] \\ &\le 2(1+\tau)\, d(0) - 2H^2(X \cap B(0,1)) + C\tau \le C_2 \tau \end{aligned} \tag{6.30}$$

again by (6.26), (6.2), (6.27), and (6.28).

Let us now construct analogues of the $G_{j,k}$ near the vertices $x \in V$. After this, we shall connect the two types of sets and later simplify the net that we get. So let $x \in V$ be given. Set

$$Y(x) = X \cap \partial B(0,1) \cap \overline{B}(x, C_3\tau), \tag{6.31}$$

where $C_3 = 100 + C_1 + C_2$. If τ is small enough, $Y(x)$ is either composed of three small arcs of circle that make 120° angles at x (if $x \in V_0$), or of two small arcs that leave from x in opposite directions (if $x \in V_1$). Then let H denote the intersection of $\overline{B}(0, 3/2)$ with the cone over $Y(x)$, and set $F = H \cap f^{-1}(E \cap \partial B)$ (as in (6.21)).

By the same separation result as above, there is a compact connected set $F_1 \subset F$ that separates 0 from $\frac{3}{2}Y(x)$ in H (see Remark 6.45 below if you prefer to use the separation result in [Ne]). Then set $G(x) = f(F_1)$. As before, $G(x)$ is a connected subset of $E \cap \partial B$,

$$\text{dist}(y, G(x)) \le 2\tau \text{ for } y \in Y(x) \tag{6.32}$$

as in (6.24), and

$$\text{dist}(z, Y(x)) \le 2\tau \text{ for } z \in G(x)\,, \tag{6.33}$$

as in (6.25). In particular, (6.32) says that $G(x)$ has branches that reach out reasonably far from x.

Now we want connect the $G_{j,k}$ to the $G(x)$; we shall do this in little tubes near the extremities of the $\mathfrak{C}_{j,k}$. Let $\mathfrak{C}_{j,k}$ be given, and let $x, x' \in V$ denote its extremities. Set

$$\mathfrak{C}'' = \mathfrak{C}''_{j,k,x} = \big\{y \in \mathfrak{C}_{j,k}\,;\, 10\tau \le |x - y| \le (C_3 - 10)\tau\big\}. \tag{6.34}$$

For $y \in \mathfrak{C}_{j,k}$, denote by $D(y)$ the hyperdisk centered at y, of radius 3τ, and that lies in the hyperplane perpendicular to $\mathfrak{C}_{j,k}$ at y. Let us check that

$$\text{we can find } y \in \mathfrak{C}'' \text{ such that } E \cap \partial B \cap D(y) \text{ has exactly one point.} \tag{6.35}$$

Denote by $T = T_{j,k}$ the union of the $D(y)$, $y \in \mathfrak{C}_{j,k}$. By (6.25) and (6.20), $G_{j,k} \subset T$. By (6.24), $G_{j,k}$ has points within 2τ from the endpoints of $\mathfrak{C}'$, so if we set

$$\mathfrak{C}''' = \big\{z \in \mathfrak{C}_{j,k}\,;\, \text{dist}(z, \{x, x'\}) \ge 10\tau\big\} \subset \mathfrak{C}', \tag{6.36}$$

then $G_{j,k}$ crosses the union of the $D(y)$, $y \in \mathfrak{C}'''$ (by connectedness). Hence, if we define the natural projection $\pi : T \to \mathfrak{C}_{j,k}$ by $\pi(z) = y$ when $z \in D(y)$, we get that

$$\pi(G_{j,k}) \text{ contains } \mathfrak{C}'''. \tag{6.37}$$

Observe that π is $(1 + 10\tau)$-Lipschitz on T (by simple geometry), so (6.37) alone implies that $H^1(G_{j,k}) \ge (1 + 10\tau)^{-1} H^1(\mathfrak{C}''') \ge (1 + 10\tau)^{-1}[\text{length}(\mathfrak{C}_{j,k}) - 21\tau]$. We want to say

that if (6.35) fails, we get a somewhat larger lower bound on $H^1(G_{j,k})$, that contradicts (6.29).

Put the lexicographic order on $\mathbb{R}^n$, and let A denote the set of points $z \in G_{j,k}$ such that $\pi(z) \neq \pi(z')$ for $z' \in G_{j,k}$ strictly smaller than z (thus, z is the first point in $G_{j,k}$ with the projection $\pi(z)$). First, A is a Borel set. Indeed, A is the intersection of the sets A_m, $1 \leq m \leq n$, of points $z \in G_{j,k}$ such that $\pi(z) \neq \pi(z')$ whenever $z' \in G_{j,k}$ is such that $z'_i = z_i$ for $i < m$ and $z'_m < z_m$. And each A_m is itself a countable intersection of a countable union of closed sets $A_{m,p,q}$ where $|\pi(z) - \pi(z')| \geq 2^{-p}$ for $z' \in G_{j,k}$ such that $z'_i = z_i$ for $i < m$ and $z'_m \leq z_m - 2^{-q}$.

Next, $\pi(A)$ contains $\mathfrak{C}'''$, by (6.37) and because $G_{j,k}$ is closed. That is, for each $\xi \in \mathfrak{C}'''$, we can find points $z \in G_{j,k}$ such that $\pi(z) = \xi$, and there is a first one (get the smallest coordinate z_j one j at a time). So

$$\begin{aligned} H^1(A) &\geq (1+10\tau)^{-1} H^1(\pi(A)) \geq (1+10\tau)^{-1} H^1(\mathfrak{C}''') \\ &\geq (1+10\tau)^{-1} [\mathrm{length}(\mathfrak{C}_{j,k}) - 21\tau] \end{aligned} \tag{6.38}$$

(by (6.36)) as before.

Set $A' = G_{j,k} \setminus A$ and $A'' = E \cap \partial B \setminus \bigcup_{(j',k') \in \widetilde{J}} G_{j',k'}$. If (6.35) fails, then for every $y \in \mathfrak{C}''$, we can find at least two points $z \in E \cap \partial B \cap D(y)$. At most one of them lies in A, because π is injective on A; hence the other one lies in $A' \cup A''$, or in some other $G_{j',k'}$, $(j',k') \neq (j,k)$. This last option is impossible, because if $z \in G_{j',k'}$, then $\mathrm{dist}(z, \mathfrak{C}'_{j',k'}) \leq 2\tau$ by (6.25), and z lies out of T (again if τ is small enough compared to η_0, and by (6.20) and (2.3)). So we found $z \in A' \cup A''$ such that $\pi(z) = y$. In other words, $\pi(A' \cup A'') \supset \mathfrak{C}''$, hence

$$\begin{aligned} H^1(A' \cup A'') &\geq (1+10\tau)^{-1} H^1(\pi(A' \cup A'')) \geq (1+10\tau)^{-1} H^1(\mathfrak{C}'') \\ &\geq (1+10\tau)^{-1} [C_3 - 21]\, \tau \geq (78 + C_1 + C_2)\, \tau \end{aligned} \tag{6.39}$$

(because $C_3 = 100 + C_1 + C_2$). Then

$$\begin{aligned} H^1(G_{j,k}) &\geq H^1(A) + H^1(A') \geq H^1(A) + H^1(A' \cup A'') - H^1(A'') \\ &\geq (1+10\tau)^{-1} [\mathrm{length}(\mathfrak{C}_{j,k}) - 21\tau] + (78 + C_1 + C_2)\, \tau - H^1(A'') \\ &\geq (1+10\tau)^{-1} [\mathrm{length}(\mathfrak{C}_{j,k}) - 21\tau] + (78 + C_1 + C_2)\, \tau - C_2 \tau \\ &> \mathrm{length}(\mathfrak{C}_{j,k}) + C_1 \tau \end{aligned} \tag{6.40}$$

by (6.38), (6.39), and (6.30), because $\mathrm{length}(\mathfrak{C}_{j,k}) < \pi$, and if τ is small enough. This contradiction with (6.29) proves (6.35).

For each choice of j, k, and an endpoint x of $\mathfrak{C}_{j,k}$, choose $y \in \mathfrak{C}''$ as in (6.35), and denote by $z_{j,k,x}$ the only point of $E \cap \partial B \cap D(y)$. By the proof of (6.35), we already know that $z_{j,k,x} \in G_{j,k}$. It also lies in $G(x)$, for the same sort of reasons: $G(x)$ is connected and lies close to $Y(x)$ (by (6.33)); by (6.32) it contains points of T on both sides of $D(y)$, so it meets $D(y)$, and then (6.35) says that this happens at $z_{j,k,x}$.

The union of all the $G_{j,k}$ and the $G(x)$ is a subset of $E \cap \partial B$ which is already rather nice, but we shall need to simplify it to get the curves promised in Lemma 6.11. We first fix $x \in V$ and simplify $G(x)$.

Let us assume that $x \in V_0$; the other case will be simpler. Denote by z_1, z_2, and z_3 the three $z_{j,k,x}$ that correspond to the $\mathfrak{C}_{j,k}$ that touch x. Since $G(x)$ is connected, with $H^1(G(x)) \leq H^1(E \cap \partial B) < +\infty$ (by (6.2)), there is a simple arc g_1 in $G(x)$ that goes from z_1 to z_2. See for instance Proposition 30.14 on page 188 of [D2].

For $1 \leq i \leq 3$, z_i lies in the disk $D(y_i)$ for some y_i in the corresponding $\mathfrak{C}_{j,k}$, as in (6.35). Let $T(x)$ denote the connected component of x in $\left\{ z \in \mathbb{R}^n \, ; \, \mathrm{dist}(z, X \cap \partial B) \leq 3\tau \right\} \setminus [D(y_1) \cup D(y_2) \cup D(y_3)]$. Thus $T(x)$ is a little tube centered at x with a Y-shape. Since $G(x) \subset E$, (6.10) says that $G(x)$ can only meet $\partial T(x)$ on the $D(y_i)$, hence precisely at the points z_i (by (6.35)). Now g_1 is simple and goes from z_1 to z_2, so its interior does not meet z_1 and z_2, and it can only meet z_3 at a single point. Recall also that $g_1 \subset G(x)$, which is contained in a slightly longer tube (as in (6.33)), so g_1 cannot lie outside of $T(x)$ and connect z_1 to z_2. Thus g_1 lies in the interior or $T(x)$, except for its endpoints z_1 and z_2, and perhaps z_3 that could be touched once.

If x lies in V_1 instead, we construct g_1 as before, except that $T(x)$ looks like a tube and there is no z_3, and we stop here.

Return to the case when $x \in V_0$. As before, there is a simple arc g_2' in $T(x)$ that goes from z_3 to z_1. This arc meets g_1 (maybe immediately, if $z_3 \in g_1$, and maybe also at z_1 or z_2; this does not matter). Let $\varphi(x)$ denote the first point of g_1 that we meet when we run along g_2', and denote by g_2 the arc of g_2' between z_3 and $\varphi(x)$. Our substitute for $G(x)$ will be $g(x) = g_1 \cup g_2$, which we see as the union of three disjoint simple arcs in $T(x)$ that connect $\varphi(x)$ to the three z_j. [We don't care if some of these arcs are reduced to one point.]

Now we define the $g_{j,k}$. Denote by x_1 and x_2 the two extremities of $\mathfrak{C}_{j,k}$, and set $z_i = z_{j,k,x_i}$ for $i = 1, 2$. Also denote by y_i the point of $\mathfrak{C}''$ such that z_i is the only point of $E \cap \partial B \cap D(y_i)$.

Since $G_{j,k}$ connects z_1 to z_2 and $H^1(G_{j,k}) < +\infty$, we can find a simple arc g in $G_{j,k}$, that goes from z_1 to z_1. Notice that $g \subset G_{j,k} \subset T$, where T is as above (below (6.35)). Let T' denote the component of $T \setminus [D(y_1) \cup D(y_2)]$ that lies between these two disks; (6.10) says that g can only meet $\partial T'$ on the disk $D(y_i)$, hence at the z_i. As before, g is simple, so it only meets $\partial T'$ twice, at z_1 and z_1, and it is otherwise contained in the interior of T' (because we know from (6.25) that it is contained in T, and otherwise it could not connect z_1 to z_2).

Our path $g_{j,k}$ is obtained by following the arc of $g(x_1)$ between $\varphi(x_1)$ and z_1, then g, and then the arc of $g(x_2)$ from z_2 to $\varphi(x_2)$.

Observe that the $T(x)$, $x \in V$, are disjoint, that $T(x)$ only meets the tube T' associated to $\mathfrak{C}_{j,k}$ when x is one of the extremities of $\mathfrak{C}_{j,k}$ (and then the intersection is contained in the disk $D(y_{j,k,x})$ that contains $z_{j,k,x}$), and that the various T' are also disjoint. Then the arcs $g_{j,k}$ are simple, and $g_{j,k}$ only meets $g_{j',k'}$ when $\mathfrak{C}_{j,k}$ and $\mathfrak{C}_{j',k'}$ have a common endpoint x, in which case $g_{j,k} \cap g_{j',k'} = \{\varphi(x)\}$.

We are now ready to prove Lemma 6.11. We already defined the $\varphi(x)$, $x \in V$, and the curves $g_{j,k}$. We have (6.12) by construction, we just checked (6.13), and (6.14) holds

because $\varphi(x) \in g(x) \subset T(x)$, which itself is contained in $B(x, (C_3+3)\tau)$ (because the y_i lie in $\mathfrak{C}''$, see (6.34), (6.35), and the definition of $T(x)$ somewhat below (6.40)).

So we just need to check (6.15). Recall that each $g_{j,k}$ is composed of an arc $g \subset G_{j,k}$, which stays within 2τ from $\mathfrak{C}_{j,k}$ by (6.25), an arc of $g(x_1)$ which is contained in $T(x_1) \subset B(x_1, (C_3+3)\tau)$, and an arc of $g(x_2) \subset T(x_2) \subset B(x_2, (C_3+3)\tau)$. Thus $g_{j,k}$ stays within $(C_3+3)\tau)$ from $\mathfrak{C}_{j,k}$, as needed. This completes our proof of Lemma 6.11. □

Notice that

$$H^1(g_{j,k}) \geq \text{length}(\mathfrak{C}_{j,k}) - C\tau \tag{6.41}$$

because $\mathfrak{C}_{j,k}$ is a geodesic with length at most $\frac{9\pi}{10}$ (by (2.5)), and because by (6.14) its endpoints are within $C\tau$ from the endpoints of $g_{j,k}$. On the other hand,

$$\sum_{j,k} H^1(g_{j,k}) = H^1\big(\bigcup_{j,k} g_{j,k}\big) \leq H^1(E \cap \partial B) \leq 2(1+\tau)\, d(0) \tag{6.42}$$

because the $g_{j,k}$ are disjoint and by (6.2), so

$$H^1(g_{j,k}) \leq \text{length}(\mathfrak{C}_{j,k}) + C_4\tau \tag{6.43}$$

by the proof of (6.29), and then

$$H^1\big([E \cap \partial B] \setminus \bigcup_{j,k} g_{j,k}\big) \leq C_5\tau \tag{6.44}$$

by the proof of (6.30).

We end this section with two remarks on the construction of the connected sets $G_{j,k}$ and $G(x)$.

Remark 6.45. We could also use Theorem 14.3 in [Ne], instead of the stronger separation result from [Ku], to construct the $G_{j,k}$ and analogues of the $G(x)$.

We start with $G_{j,k}$. We still work in the intersection H of $\overline{B}(0, 3/2)$ with the cone over $\mathfrak{C}' = \mathfrak{C}'_{j,k}$ (see near (6.21)), and need to know that if the compact set $F = H \cap f^{-1}(E \cap \partial B)$ separates 0 from $\frac{3}{2}\mathfrak{C}'$ in H, then there is a compact connected set $F_1 \subset F$ that still separates 0 from $\frac{3}{2}\mathfrak{C}'$ in H (as in (6.23)).

Theorem 14.3 in [Ne] gives this, but it is stated when H is the two-dimensional sphere. We just need a small argument to reduce to that case. Let $\widetilde{H}$ be a second copy of H, with the copy $\widetilde{F}$ of F in it. Glue $\widetilde{H}$ to H, by identifying every point of ∂H (two arc segments from the origin to the extremities of $\frac{3}{2}\mathfrak{C}'$, plus $\frac{3}{2}\mathfrak{C}'$ itself) to its copy in $\widetilde{H}$. This gives a sphere $S = H \cup \widetilde{H}$ and a compact set $\widehat{F} = F \cup \widetilde{F} \subset S$.

Let us check that $\widehat{F}$ separates 0 from $\frac{3}{2}\mathfrak{C}'$ in S. If not, there is a curve $\zeta \subset S \setminus \widehat{F}$ that goes from 0 to $\frac{3}{2}\mathfrak{C}'$. If $\zeta \subset H$, we get a contradiction immediately with the fact that F separates 0 from $\frac{3}{2}\mathfrak{C}'$ in H. Otherwise, let $\widehat{\zeta}$ denote the union of ζ and its symmetric image

$\widetilde{\zeta}$ (obtained by exchanging the copies H and $\widetilde{H}$); then $\widehat{\zeta}$ is connected because ζ meets $\widetilde{\zeta}$ at the origin, and $\widehat{\zeta}$ does not meet $\widehat{F}$ because $\widehat{F}$ is symmetric.

Set $\zeta' = \widehat{\zeta} \cap H$. It is connected too, because if O_1 and O_2 are disjoint open sets in H that cover ζ', then the symmetric extensions $\widehat{O}_1$ and $\widehat{O}_2$ are disjoint, open in S, and cover $\widehat{\zeta}$, a contradiction. Finally, ζ' does not meet F because $\widehat{\zeta}$ does not meet $\widehat{F}$. Since ζ' contains 0 and meets $\frac{3}{2}\mathfrak{C}'$, we get a contradiction. So $\widehat{F}$ separates 0 from $\frac{3}{2}\mathfrak{C}'$ in S.

By Theorem 14.3 in [Ne], we get a connected piece $F' \subset \widehat{F}$ that separates 0 from $\frac{3}{2}\mathfrak{C}'$ in S. Its closure is still connected and contained in $\widehat{F}$ (because $\widehat{F}$ is compact), so we can assume that F' is compact.

Denote by $\widetilde{F}'$ the symmetric copy of F', and set $\widehat{F}' = F' \cup \widetilde{F}'$. Then $\widehat{F}'$ is also contained in $\widehat{F}$ (because $\widehat{F}$ is symmetric), and it also separates 0 from $\frac{3}{2}\mathfrak{C}'$ in S. In addition F' meets ∂H, because otherwise it would not separate 0 from $\frac{3}{2}\mathfrak{C}'$ in S, so F' meets $\widetilde{F}'$, and $\widehat{F}'$ is connected.

Set $F_1 = \widehat{F}' \cap H$. This set is contained in F (because $\widehat{F}' \subset \widehat{F}$, and it is connected too; the argument is the same as for ζ' above. It separates 0 from $\frac{3}{2}\mathfrak{C}'$ in H because F' separates in S, so it satisfies all the desired properties.

Now we construct substitutes for the $G(x)$, $x \in V$. If $x \in V_1$, the set $Y(x) = X \cap \partial B(0,1) \cap \overline{B}(x, C_3\tau)$ in (6.31) is composed of two short arcs of great circles that leave from x in opposite directions, the piece of cone H is a again a simple topological disk, and we can use Theorem 14.3 in [Ne] as above.

When $x \in V_0$, $Y(x)$ has a fork, and we shall proceed differently. Each branch of $Y(x)$ comes from a $\mathfrak{C}_{j,k}$, and we use (6.35) to find a point y in the branch such the disk $D(y)$ only meets $E \cap \partial B$ once, at a point $z_{j,k,x}$ of $G_{j,k}$. This way, we get three points $y_{j,k,x}$, one in each branch of $Y(x)$, which we decide to call y_1, y_2, and y_3. Also call z_i the only point of $E \cap \partial B \cap D(y_i)$. All we need to do is find a connected set $G(x) \subset T(x)$ that connects the three z_i, because after this we notice that since $D(y_i)$ only meet $E \cap \partial B$ once, each z_i lies in the corresponding $G_{j,k}$, and we can proceed as before.

First we want to connect z_1 to z_2. Let Y_i denote the arc of $Y(x)$ that goes through y_i, and let H be the intersection of $\overline{B}(0, 3/2)$ with the cone over $Y_1 \cup Y_2$. Then set $F = H \cap f^{-1}(E \cap \partial B)$, and notice that F separates 0 from $\frac{3}{2}Y$ in H, again because it is a level set. Since H is a simple topological disk, we can apply Theorem 14.3 in [Ne] as above, and find a compact connected set $F_1 \subset F$ that separates 0 from $\frac{3}{2}Y$ in H. Denote by y_1' and y_2' the endpoints of Y_1; notice that F_1 contains a point in $I_1 = [0, y_1']$, because otherwise we could use I_1 to connect 0 to $\frac{3}{2}Y$. Similarly, F_1 meets $I_2 = [0, y_2']$.

Now set $G = f(F_1)$; G is compact, connected, and contained in $E \cap \partial B \cap T(x)$. It contains points in $f(I_1)$ and $f(I_2)$, so it crosses the two disks $D(y_i)$, and hence contains z_1 and z_2. Similarly, there is a compact connected set $G' \subset E \cap \partial B$ that contains z_1 and z_3, and we can take $G(x) = G \cup G'$. Thus we can use Theorem 14.3 in [Ne] instead of 52.III.1 in [Ku].

Remark 6.46. When $n = 3$, we can get the connected sets $G_{j,k}$ and $G(x)$ (and then proceed as above) with a little less information than the full parameterization f of (6.16)-(6.19).

Let X be as in (6.10), and denote by W_i, $1 \le i \le 2$, 3, or 4 (depending on the type of X) the connected components of $\overline{B} \setminus X_{100\varepsilon}$, where $X_{100\varepsilon} = \{y \in \mathbb{R}^3 \, ; \, \mathrm{dist}(y, X) \le 100\varepsilon\}$ denotes a closed 100ε-neighborhood of X. We know from (6.10) that the W_i don't meet E. Now assume that

(6.47) the $W_i \cap \partial B$ lie in different connected components of $E \cap \partial B$.

[This is obviously the case if we know that E separates the W_i in $\overline{B}$, and the existence of the parameterization f in (6.16)-(6.19) gives this information.]

Then we can use a slightly different separation argument (also based on Theorem 14.3 in [Ne]) to produce the connected sets $G_{j,k}$ and $G(x)$. This time we work directly in slices of ∂B and find connected pieces of $E \cap \partial B$ that separate, instead of using f^{-1} and working on a level set in a piece of cone.

For instance, when we need to construct $G(x)$ for some $x \in V_0$, we work in the topological disk $H = \partial B \cap B(x, 10^{-1})$, notice that $E \cap H$ separates the three main regions of H from each other, and use the same argument as in Remark 6.45 to find connected sets G and G' in $E \cap H$ that separate them too.

It is a little sad that (still when $n = 3$) the author was not able to find a simple proof of the needed separating property that would not use the parameterisation from [D3], because it is the only place in the argument where we use the geometric part of [D3] and [DDT] heavily. On the other hand, if we want to get all the way to the regularity property stated in Theorem 1.15, this does not make a huge difference, because arguments of the same type (i.e., an extension of the Reifenberg parameterization theorem) are needed there anyway.

Let us also mention that in the special case of "Mumford-Shah" minimal and almost minimal sets (see Section 18 of [D3]), (6.47) is a rather easy consequence of the definitions (see the proof of Theorem 1.9 in [D3], and in particular the proof of (18.31), near the end of Section 18), So at least we get a slightly simpler proof in this case.

7. Construction of a Lipschitz graph and length estimates

The construction of this section will be applied, after a rotation, to each of the curves $g_{j,k}$, $(j, k) \in \widetilde{J}$, that were produced in Lemma 4.11, but the section is fairly independent, and we use slightly different notation.

Here again B is the unit ball in $\mathbb{R}^n$, and we give ourselves a simple rectifiable curve γ in ∂B. We assume that

$$9\eta_0 \le \mathrm{length}(\gamma) \le \frac{10\pi}{11}, \tag{7.1}$$

where $\eta_0 > 0$ is as in (2.2).

The first inequality is mostly here for convenience or normalization. The second one will be needed in some estimates; the point is to make sure that we are far enough from the situation of non uniqueness for geodesics, and incidentally $10\pi/11$ could be be replaced with any constant strictly smaller than π.

Denote by a and b the extremities of γ. We assume that

$$\text{length}(\gamma) \leq \text{dist}_{\partial B}(a, b) + \tau_1, \tag{7.2}$$

where we denote by $\text{dist}_{\partial B}$ the geodesic distance on ∂B, and τ_1 is a positive constant that we can take as small as we want.

We also assume, for simplicity, that a and b lie on the horizontal 2-plane P through the origin, and that

$$\text{dist}(z, P) \leq \tau_1 \quad \text{for } z \in \gamma. \tag{7.3}$$

This does not cost us too much, because (7.1) and (7.2) imply the analogue of (7.3), with τ_1 replaced with a constant τ_1' which tends to 0 with τ_1. We won't need to bother, though, because (7.3) will come for free when γ is one of the $g_{j,k}$.

We want to construct a Lipschitz curve Γ on ∂B, with the same extremities a and b, whose Lipschitz constant is at most η, and which has a big intersection with γ. Here $\eta > 0$ is given in advance, small enough, and we are allowed to choose τ_1 very small, depending on η. Later on, we shall connect this section with the previous one, and we shall be allowed to take τ small, depending on η and τ_1. It will be important to estimate $H^1(\gamma \setminus \Gamma)$, for instance, in terms of

$$\Delta L = \text{length}(\gamma) - \text{dist}_{\partial B}(a, b), \tag{7.4}$$

and not just τ_1 or τ. See (7.30)-(7.32) below for the main properties of Γ. Let us first prove some estimates, and then construct Γ.

Let $z : I \to \partial B$ denote a parameterization of γ by arc-length. Here I is a compact interval, and $|I| = \text{length}(\gamma)$. We write $z(t) = (z_1(t), z_2(t), v(t))$, with $v(t) \in \mathbb{R}^{n-2}$. Notice that $|v(t)| \leq \tau_1$, by (7.3). Then set $w(t) = (1 - |v(t)|^2)^{1/2} = (z_1^2(t) + z_2^2(t))^{1/2} \geq 1 - \tau_1$, and write

$$z(t) = \big(\cos\theta(t) w(t), \sin\theta(t) w(t), v(t)\big) \tag{7.5}$$

for $t \in I$. We take a continuous determination of θ, which is easy because $(z_1(t), z_2(t))$ does not vanish. Notice that w, and then θ, are Lipschitz. Also set

$$l = \text{length}(\gamma) = |I| \quad \text{and} \quad d = \text{dist}_{\partial B}(a, b). \tag{7.6}$$

Thus $\Delta L = l - d$. We shall assume that γ roughly runs counterclockwise on average, so that

$$\int_I \theta'(t) dt = \text{dist}_{\partial B}(a, b) = d\,; \tag{7.7}$$

otherwise, we could always parameterize γ backwards and get the same results.

Lemma 7.8. *If τ_1 is small enough,*

$$\int_I |v'(t)|^2 dt \le 14\,\Delta L. \tag{7.9}$$

First differentiate (7.5) to get that

$$\begin{aligned} z'(t) = \theta'(t)w(t)\big(-\sin\theta(t), \cos\theta(t), 0\big) \\ + w'(t)\big(\cos\theta(t), \sin\theta(t), 0\big) + \big(0, 0, v'(t)\big). \end{aligned} \tag{7.10}$$

The three pieces are orthogonal, so

$$|z'(t)|^2 = \theta'(t)^2 w(t)^2 + |w'(t)|^2 + |v'(t)|^2. \tag{7.11}$$

Recall that $w(t)^2 + |v(t)|^2 = 1$, so $w(t)w'(t) = -\langle v'(t), v(t)\rangle$. Also, z is a parameterization by arc-length, so $|z'(t)| = 1$ almost-everywhere, i.e.,

$$\theta'(t)^2 w^2(t) = 1 - |w'(t)|^2 - |v'(t)|^2 = 1 - \frac{\langle v'(t), v(t)\rangle^2}{w(t)^2} - |v'(t)|^2. \tag{7.12}$$

Set $\lambda = 101/100$ and recall that $|v(t)| \le \tau_1$ by (7.3), so $w^{-2}(t) = [1-|v(t)|^2]^{-1} \le 1+\lambda|v(t)|^2$ if τ_1 is small enough. Hence, ignoring the non-positive middle term in (7.12),

$$\theta'(t)^2 \le \big[1+\lambda|v(t)|^2\big](1-|v'(t)|^2) \le 1+\lambda|v(t)|^2 - |v'(t)|^2 \text{ almost-everywhere on } I. \tag{7.13}$$

We may assume (by translation) that I starts at the origin. Thus $I = [0, l]$, with $l = \text{length}(\gamma)$ as above. Observe that $v(0) = v(l) = 0$ because a and b lie on P. We can extend v to $[-l, l]$ so that it is odd, and then write v as a sum of sines. That is, write

$$v(t) = \sum_{k\ge 1} c_k \sin(\pi k t/l) \tag{7.14}$$

(with vector-valued coefficients c_k when $n > 3$). Then

$$||v||^2_{L^2(I)} = \frac{l}{2}\sum_{k\ge 1} |c_k|^2 \tag{7.15}$$

and

$$||v'||^2_{L^2(I)} = \frac{l}{2}\sum_{k\ge 1}(\pi k/l)^2|c_k|^2 \ge \Big(\frac{\pi}{l}\Big)^2 ||v||^2_{L^2(I)} \ge \frac{121}{100}\,||v||^2_{L^2(I)}\,, \tag{7.16}$$

by (7.1) and (7.6). We now integrate (7.13) and get that

$$\int_I \theta'(t)^2 dt \le |I| + \lambda||v||^2_{L^2(I)} - ||v'||^2_{L^2(I)} \le l - \Big(1 - \frac{100\lambda}{121}\Big)||v'||^2_{L^2(I)} \le l - \frac{1}{7}\,||v'||^2_{L^2(I)}\,. \tag{7.17}$$

Notice that

$$d^2 = \Big\{ \int_I \theta'(t)dt \Big\}^2 \le l \int_I \theta'(t)^2 dt \le l^2 - \frac{l}{7} \int_I |v'(t)|^2 dt \tag{7.18}$$

by (7.7), Cauchy-Schwarz, (7.6), and (7.17). Hence

$$\int_I |v'(t)|^2 dt \le \frac{7}{l}\,(l^2 - d^2) = 7\,\frac{l+d}{l}\,\Delta L \le 14\,\Delta L \tag{7.19}$$

(see below (7.6)). Lemma 7.8 follows. □

We also need some control on θ'. Set

$$f(t) = 1 + 2|v(t)|^2 - \theta'(t). \tag{7.20}$$

Observe that $\theta'(t)^2 \le 1 + 2|v(t)|^2$ almost-everywhere, by (7.13), so $f(t) \ge 0$. On the other hand,

$$\begin{aligned} \int_I f(t)dt &= l + 2\int_I |v(t)|^2 dt - \int_I \theta'(t)dt = l - d + 2\int_I |v(t)|^2 dt \\ &= \Delta L + 2\int_I |v(t)|^2 dt \le \Delta L + 2\int_I |v'(t)|^2 dt \le 30\,\Delta L \end{aligned} \tag{7.21}$$

by (7.7), the line below (7.6), (7.16), and (7.19). So $f(t)$ is often small, and hence $\theta'(t)$ is rarely smaller than $1/2$, say.

We are now ready to construct our η-Lipschitz curve Γ. It will be obtained by modifying z on open intervals where some maximal functions are large. First denote by v^* the (non centered) Hardy-Littlewood maximal function of the derivative of v; thus

$$v^*(t) = \sup \Big\{ \frac{1}{|J|} \int_J |v'(s)|ds\,;\, J \text{ is an interval contained in } I \text{ such that } t \in J \Big\}. \tag{7.22}$$

The Hardy-Littlewood maximal theorem (see for instance [St], p.5) says that $||v^*||_2^2 \le C||v'||_2^2 \le C\Delta L$, by (7.19). Set $Z_1 = \{t \in I\,;\, v^*(t) > \eta/4\}$. Thus

$$|Z_1| \le 16\eta^{-2}||v^*||_2^2 \le C\eta^{-2}\Delta L. \tag{7.23}$$

Similarly set

$$f^*(t) = \sup \Big\{ \frac{1}{|J|} \int_J f(s)\,ds\,;\, J \text{ is an interval contained in } I \text{ such that } t \in J \Big\} \tag{7.24}$$

and $Z_2 = \{t \in I\,;\, f^*(t) > 1/2\}$. Then

$$|Z_2| \le 2||f^*||_{\text{Weak}L^1} \le C||f||_1 \le C\Delta L, \tag{7.25}$$

by the Hardy-Littlewood maximal theorem again and (7.21). Finally set $Z = Z_1 \cup Z_2$; thus

$$|Z| \leq C\eta^{-2}\Delta L. \tag{7.26}$$

It is easy to see (just from the definition of Z_1 and Z_2 with maximal functions) that Z is open in I. We want to keep z as it is on $I \setminus Z$, and on the open intervals that compose Z, replace γ with arcs of geodesics.

First observe that if $t \in I \setminus Z$, $t' \in I$, and J denotes the interval with endpoints t and t', then

$$|v(t') - v(t)| \leq \int_J |v'(u)|\, du \leq |t' - t|\, v^*(t) \leq \eta|t' - t|/4, \tag{7.27}$$

by (7.22) and because $t \notin Z_1$. Similarly, if in addition $t' \geq t$,

$$\begin{aligned} \theta(t') - \theta(t) &= \int_J [f(t) + \theta'(t)]\, dt - \int_J f(t)dt = \int_J [1 + 2v(t)^2]dt - \int_J f(t)dt \\ &\geq (t' - t) - (t' - t)f^*(t) \geq (t' - t)/2 \end{aligned} \tag{7.28}$$

by (7.20), (7.24), and because $t \notin Z_2$. If instead $t' < t$, the same argument yields $\theta(t) - \theta(t') \geq (t - t')/2$. Altogether,

$$\theta(t') - \theta(t) \geq (t' - t)/2 \quad \text{when } 0 \leq t \leq t' \leq l, \text{ and } t \text{ and } t' \text{ do not both lie in } Z. \tag{7.29}$$

So we want to keep the part of γ that corresponds to $I \setminus Z$, and replace $z(Z)$ with arcs of geodesics. Recall that Z is open in I, so it is an at most countable union of disjoint intervals I_j that are open in I. Call a_j and b_j the endpoints of I_j, and denote by ρ_j the arc of geodesic on ∂B that goes from $z(a_j)$ to $z(b_j)$. For each j, we replace the arc of γ between $z(a_j)$ and $z(b_j)$ with ρ_j. This gives a new curve Γ. First observe that

$$\Gamma \text{ has the same endpoints as } \gamma, \tag{7.30}$$

because even if the endpoints of I lie in Z, we keep $z(0)$ and $z(l)$ as endpoints of the corresponding geodesics. Also,

$$H^1(\Gamma \setminus \gamma) \leq H^1(\gamma \setminus \Gamma) \leq C\eta^{-2}\Delta L; \tag{7.31}$$

the first inequality holds because the geodesic ρ_j is never longer than the arc of γ that it replaces, so that $H^1(\Gamma) \leq \text{length}(\Gamma) \leq \text{length}(\gamma) = H^1(\gamma)$ (because γ is simple), while the second inequality comes from (7.26) and the fact that z is 1-Lipschitz. To end this section, we need to check that

$$\Gamma \text{ is a Lipschitz graph with constant } \leq \eta, \tag{7.32}$$

and at the same time we shall explain what we mean by this (see (7.42) or (7.44)). Let us first check that

$$|v(b_j) - v(a_j)| \le \eta(b_j - a_j)/4 \text{ and } \theta(b_j) - \theta(a_j) \ge (b_j - a_j)/2. \tag{7.33}$$

Recall that I_j is a connected component of Z, which is open in I. So $a_j \in I \setminus Z$, except perhaps if $a_j = 0$. Similarly, $b_j \in I \setminus Z$, unless it is the final endpoint of I. Both things cannot happen at the same time, because

$$|b_j - a_j| \le |Z| \le C\eta^{-2}\Delta L \le C\eta^{-2}\tau_1 < |I| \tag{7.34}$$

by (7.26), (7.2), (7.4), (7.1), and if τ_1 is small enough. So a_j or b_j lies in $I \setminus Z$, we can apply (7.27) and (7.29) to them, and we get (7.33).

We want to estimate the average slope of γ between $z(a_j)$ and $z(b_j)$. Denote by π the projection on the horizontal plane. For the next estimate, it is convenient to use complex notations; then $\pi(z(t)) = w(t)e^{i\theta(t)}$ by (7.5), and

$$\begin{aligned} |\pi(z(b_j)) - \pi(z(a_j))| &= \left|w(b_j)e^{i\theta(b_j)} - w(a_j)e^{i\theta(a_j)}\right| \\ &= |e^{i\theta(a_j)}|\,\left|w(b_j)e^{i[\theta(b_j)-\theta(a_j)]} - w(a_j)\right| \\ &= \left|w(b_j)[e^{i[\theta(b_j)-\theta(a_j)]} - 1] + [w(b_j) - w(a_j)]\right| \\ &\ge |w(b_j)|\left|e^{i(\theta(b_j)-\theta(a_j))} - 1\right| - |w(b_j) - w(a_j)|. \end{aligned} \tag{7.35}$$

Recall that $|v(b_j)| \le \tau_1$ by (7.3), so $w(b_j) \ge 1 - \tau_1$. Also, $|b_j - a_j|$ is as small as we want, by (7.34), and $|\theta(b_j) - \theta(a_j)|$ is at most twice larger, for instance because (7.13) says that $|\theta'| \le 2$. Then $|e^{i(\theta(b_j)-\theta(a_j))} - 1| \ge \frac{9}{10}\,|\theta(b_j) - \theta(a_j)| \ge \frac{9}{20}\,(b_j - a_j)$, by (7.33). Finally, $|w(b_j) - w(a_j)| \le |v(b_j) - v(a_j)| \le \eta(b_j - a_j)/4$, because $x \to (1 - x^2)^{1/2}$ is 1-Lipschitz near the origin, and by (7.33) again. Altogether,

$$|\pi(z(b_j)) - \pi(z(a_j))| \ge \frac{9}{20}\,(1 - \tau_1)(b_j - a_j) - \eta(b_j - a_j)/4 \ge \frac{8}{20}\,(b_j - a_j) \tag{7.36}$$

if τ_1 and η are small enough. Then (7.33) yields

$$|v(b_j) - v(a_j)| \le \eta(b_j - a_j)/4 \le \frac{5\eta}{8}\,|\pi(z(b_j)) - \pi(z(a_j))|. \tag{7.37}$$

Recall that $|z(b_j)) - z(a_j)| \le |b_j - a_j|$ is as small as we want, so the variation of the unit tangent vector to the geodesic ρ_j between $z(a_j))$ and $z(b_j))$ is also as small as we want. Then the slope of that tangent is less than $3\eta/4$.

The reader is probably already convinced that (7.32) will easily follow from this, but let us complete the argument brutally. Let us even parameterize Γ with a function $\widetilde{z} : I \to \partial B$, as we did for γ in (7.5). We need to define functions $\widetilde{\theta}$, $\widetilde{v}$, $\widetilde{w}$, and $\widetilde{z}$ on I, so that

$$\widetilde{z}(t) = \big(\cos\widetilde{\theta}(t)\,\widetilde{w}(t), \sin\widetilde{\theta}(t)\,\widetilde{w}(t), \widetilde{v}(t)\big) \tag{7.38}$$

and, since $\widetilde{z}(t) \in \partial B$, $\widetilde{w}(t)^2 + |\widetilde{v}(t)|^2 = 1$.

We keep $\widetilde{\theta}(t) = \theta(t)$, $\widetilde{v}(t) = v(t)$, $\widetilde{w}(t) = w(t)$, and $\widetilde{z}(t) = z(t)$ for $t \in I \setminus Z$. Let I_j be a component of Z, and again denote its extremities by a_j and b_j. We keep $\widetilde{\theta}(a_j) = \theta(a_j)$ and $\widetilde{\theta}(b_j) = \theta(b_j)$, even if a_j or b_j lies in Z, and we define $\widetilde{\theta}$ so that it is affine on I_j. Recall that the geodesic ρ_j from $z(a_j)$ to $z(b_j)$ is short, with a small slope; then, given $\theta \in [\theta(a_j), \theta(b_j)]$, there is a unique point $\xi = \xi(\theta) \in \rho_j$ such that $\pi(\xi) = e^{i\theta}|\pi(\xi)|$. In other words, the fact that $\widetilde{z}(t) \in \rho_j$ and that we want (7.38) with a given $\widetilde{\theta}(t) \in [\theta(a_j), \theta(b_j)]$ determine $\widetilde{z}(t)$, $\widetilde{v}(t)$, and $\widetilde{w}(t)$ uniquely, and these functions are smooth on I_j.

Also recall that because of (7.37), the slope of the tangent vector to ρ_j stays smaller than $3\eta/4$. Then

$$|\widetilde{v}'(t)| \le \frac{3\eta}{4}\,|(\pi \circ \widetilde{z})'(t)| \quad \text{on } I_j. \tag{7.39}$$

But (7.38) yields

$$|(\pi \circ \widetilde{z})'(t)|^2 = \widetilde{\theta}'(t)^2 \widetilde{w}(t)^2 + \widetilde{w}'(t)^2 \le \widetilde{\theta}'(t)^2 \widetilde{w}(t)^2 + |\widetilde{v}'(t)|^2 |\widetilde{v}(t)|^2 \widetilde{w}(t)^{-2} \tag{7.40}$$

by the proof of (7.11), and because $\widetilde{w}(t)\widetilde{w}'(t) = -\langle \widetilde{v}'(t), \widetilde{v}(t)\rangle$ (since $|\widetilde{v}|^2 + |\widetilde{w}|^2 = 1$). By (7.3), the two extremities of ρ_j lie within τ_1 of P, so ρ_j itself lies within $\sqrt{2}\tau_1$ of P, $|\widetilde{v}(t)|^2 \le 2\tau_1^2$, $|\widetilde{w}(t)|^2 \ge 1 - 2\tau_1^2$, and hence

$$\begin{aligned} |\widetilde{v}'(t)|^2 &\le \Big(\frac{3\eta}{4}\Big)^2 |(\pi \circ \widetilde{z})'(t)|^2 \le \Big(\frac{3\eta}{4}\Big)^2 \big[\widetilde{\theta}'(t)^2 \widetilde{w}(t)^2 + |\widetilde{v}'(t)|^2 |\widetilde{v}(t)|^2 \widetilde{w}(t)^{-2}\big] \\ &\le \Big(\frac{3\eta}{4}\Big)^2 \widetilde{\theta}'(t)^2 + \Big(\frac{3\eta}{4}\Big)^2 \frac{2\tau_1^2}{1 - 2\tau_1^2}\,|\widetilde{v}'(t)|^2 \le \Big(\frac{3\eta}{4}\Big)^2 \widetilde{\theta}'(t)^2 + 10^{-2}|\widetilde{v}'(t)|^2 \end{aligned} \tag{7.41}$$

by (7.39) and (7.40), and if η and τ_1 are small enough. This yields

$$|\widetilde{v}'(t)| \le \frac{4\eta}{5}\,\widetilde{\theta}'(t) \quad \text{for } t \in I_j \tag{7.42}$$

(recall that $\widetilde{\theta}'(t) > 0$ by (7.33) and because $\widetilde{\theta}$ is affine on I_j).

We are ready to check that

$$|\widetilde{v}(t') - \widetilde{v}(t)| \le \frac{4\eta}{5}\,[\widetilde{\theta}(t') - \widetilde{\theta}(t)] \quad \text{for } t, t' \in I \text{ such that } t \le t'. \tag{7.43}$$

When t, t' lie in a same I_j, this follows from (7.42). Next suppose that $t \in I_j$ and $t' \in I_k$, with $k \ne j$. Then b_j and a_k both lie in $I \setminus Z$ (because Z is open), and

$$\begin{aligned} |\widetilde{v}(t') - \widetilde{v}(t)| &\le |\widetilde{v}(t') - \widetilde{v}(a_k)| + |\widetilde{v}(a_k) - \widetilde{v}(b_j)| + |\widetilde{v}(b_j) - \widetilde{v}(t)| \\ &\le \frac{4\eta}{5}(\widetilde{\theta}(t') - \widetilde{\theta}(a_k)) + \frac{\eta}{2}\,(\widetilde{\theta}(a_k) - \widetilde{\theta}(b_j)) + \frac{4\eta}{5}(\widetilde{\theta}(b_j) - \widetilde{\theta}(t)) \\ &\le \frac{4\eta}{5}(\widetilde{\theta}(t') - \widetilde{\theta}(t)) \end{aligned} \tag{7.44}$$

by (7.42), (7.27), and (7.29), as needed. The case when t, or t', or both, lie in $I \setminus Z$ is similar.

So (7.43) holds. We can take this, or rather the fact that $\widetilde{v}(t)$ is an η-Lipschitz function of $\widetilde{\theta}(t)$, as a strong definition of (7.32). [That is, the altitude of a point of Γ is an η-Lipschitz function of its projection on $\partial B \cap P$.]

Notice that (7.43) implies the other reasonable definition of (7.32), i.e., that

$$\widetilde{v}(t) \text{ is an } \eta\text{-Lipschitz function of } \pi(\widetilde{z}(t)). \tag{7.45}$$

Indeed $|(\pi \circ \widetilde{z})'(t)| \geq \widetilde{\theta}'(t)\, \widetilde{w}(t) \geq \frac{5}{6}\, \widetilde{\theta}'(t)$ by the first half of (7.40) and because $|\widetilde{w}(t)|^2 \geq 1 - 2\tau_1^2$. Then $\widetilde{\theta}(t)$ is a $\frac{6}{5}$-Lipschitz function of $\pi(\widetilde{z}(t))$, and (7.45) follows from (7.43).

8. Improvement of the cone over a small Lipschitz graph

In this section we consider a Lipschitz graph Γ with small constant contained in the unit sphere, and we try to find a surface in the unit ball, with the same boundary as the cone over Γ, but with a smaller area.

Later on, this construction will be applied to the Lipschitz curves $\Gamma_{j,k}$ that were constructed in the previous section, which themselves will come from the curves $g_{j,k}$ that we found in Section 6. The general idea is to find successive improvements of our almost-minimal set E, but the way we reduce to Lipschitz graphs will be explained later.

So we are given a curve Γ in the unit sphere ∂B. We assume for simplicity that its extremities a and b lie in the horizontal 2-plane. Let us even assume that

$$a = (1, 0, 0) \text{ and } b = (\cos T, \sin T, 0) \text{ for some } T \in [8\eta_0, 10\pi/11], \tag{8.1}$$

where again $\eta_0 > 0$ is as in (2.2). We avoid the case when a and b are nearly antipodal for the same sort of reasons as in Section 7. We also assume that Γ is a Lipschitz graph with constant at most η, by which we mean that we can find an η-Lipschitz function $v : [0, T] \to \mathbb{R}^{n-2}$, with $v(0) = v(T) = 0$, such that if we set

$$w(t) = (1 - |v(t)|^2)^{1/2} \text{ and } z(t) = \big(\cos t\, w(t), \sin t\, w(t), v(t)\big) \text{ for } t \in [0, T], \tag{8.2}$$

z is a parameterization of Γ by $[0, T]$.

Remark 8.3. The curve Γ of Section 7 satisfies these requirements. Indeed, $\text{length}(\Gamma) \leq \text{length}(\gamma) \leq 10\pi/11$ by (7.31) and (7.1); $\text{length}(\Gamma) \geq \text{dist}_{\partial B}(a, b) \geq \text{length}(\gamma) - \tau_1 \geq 9\eta_0 - \tau_1 \geq 8\eta_0$ by (7.2), (7.1), and if τ is small enough; finally the representation (8.2) follows from (7.43) and (7.38): set $\widetilde{\theta}(t) = u$ in (7.38) (this change of variable is all right because $\widetilde{\theta}$ is strictly increasing), and observe that $v(u) = \widetilde{v}(t)$ is an η-Lipschitz function of u, by (7.43).

As in Section 7, we can take η as small as we want, and we shall use this smallness to approximate minimal surfaces with graphs of harmonic functions. We shall try to estimate various gains in terms of $\text{length}(\Gamma) - T$ and the equivalent quantity $||v'||_2^2$.

Define a homogeneous function F on a sector with aperture T by

$$F(r\cos t, r\sin t) = \frac{r\,v(t)}{w(t)} \quad \text{for } r \geq 0 \text{ and } t \in [0,T]. \tag{8.4}$$

Also set

$$D_T = \{(r\cos t, r\sin t)\,;\, r \in (0,1) \text{ and } t \in (0,T)\}, \tag{8.5}$$

and then denote by Σ_F the graph of F over $\overline{D}_T$. Notice that $\Gamma \subset \Sigma_F$, by (8.2) and (8.4).

We want to find a new function G, also defined on $\overline{D}_T$, such that

$$G(r\cos t, r\sin t) = F(r\cos t, r\sin t) \quad \text{for } t \in \{0,T\}, \text{ and for } 9/10 \leq r \leq 1, \tag{8.6}$$

and whose graph Σ_G has a smaller area. [We required that $G(r\cos t, r\sin t) = F(r\cos t, r\sin t)$ for $r \geq 9/10$ to make sure that we do not change anything near Γ.] The natural thing to do is to take the harmonic function with the same boundary values as F, but we shall modify this a little near ∂B (to keep it Lipschitz) and near the center, to allow further modifications. In fact, we shall require that

$$G(r\cos t, r\sin t) = 0 \quad \text{for } 0 \leq r \leq 2\kappa \text{ and } 0 \leq t \leq T, \tag{8.7}$$

where κ is a small positive constant that does not even depend on n.

Lemma 8.8. *We can find a Lipschitz function $G : \overline{D}_T \to \mathbb{R}^{n-2}$ such that (8.6) and (8.7) hold,*

$$|\nabla G(z)| \leq C||v'||_\infty \leq C\eta \quad \textit{for almost-every } z \in D_T, \tag{8.9}$$

and

$$H^2(\Sigma_F) - H^2(\Sigma_G) \geq 10^{-4}\int_0^T |v'(t)|^2 dt \geq 10^{-4}[\text{length}(\Gamma) - T]. \tag{8.10}$$

A good part of the proof is common with the paper [D3], which we shall quote for some of the computations. Set

$$f(t) = F(\cos t, \sin t) = \frac{v(t)}{w(t)} = v(t)(1 - |v(t)|^2)^{-1/2} \quad \text{for } 0 \leq t \leq T. \tag{8.11}$$

Note that $||v||_\infty \leq \eta T$ because v is $C\eta$-Lipschitz and vanishes at 0, hence $w(t) \geq 1 - \eta T$ for $0 \leq t \leq T$, and then

$$f \text{ is } 2||v'||_\infty\text{-Lipschitz, and } f(0) = f(T) = 0. \tag{8.12}$$

A first attempt for G is to take the harmonic extension of f to D_T (with Dirichlet conditions on the boundary), which we define as follows. First we use the fact that $f(0) = f(T) = 0$ to write f as a sum of sines, i.e.,

$$f(t) = \sum_{k \geq 1} \beta_k \sin(\pi k t/T) \tag{8.13}$$

for $0 \leq t \leq T$. (See (13.4) in [D3].) Notice that

$$\frac{\pi^2}{2T} \sum_{k \geq 1} k^2 |\beta_k|^2 = \int_0^T |f'(t)|^2 dt < 4\eta^2 T \tag{8.14}$$

(see (13.5) in [D3] and use (8.12)), so $\sum_{k \geq 1} |\beta_k| < +\infty$ by Cauchy-Schwarz, and the series in (8.13) even converges pointwise. We set

$$G_1(\rho \cos t, \rho \sin t) = \sum_{k \geq 1} \beta_k \rho^{\pi k/T} \sin(\pi k t/T) \tag{8.15}$$

for $t \in \{0, T\}$ and $0 \leq \rho \leq 1$, as in (13.7) in [D3], and where we also get the normal convergence of the series from (8.14). Then $G_1(\cos t, \sin t) = f(t)$, so

$$G_1(z) = F(z) \text{ on } \overline{D}_T \cap \partial B(0, 1), \tag{8.16}$$

and

$$G_1(\rho \cos t, \rho \sin t) = 0 \text{ for } t \in \{0, T\} \text{ and } 0 \leq \rho \leq 1 \tag{8.17}$$

trivially. Direct computations using the expansions in (8.13) and (8.15) and the orthogonality of sines and cosines show that

$$\frac{1}{2} \int_0^T |f'(t)|^2 dt \leq \int_{D_T} |\nabla F|^2 \leq \int_0^T |f'(t)|^2 dt, \tag{8.18}$$

as in [D3], Lemma 13.9, and

$$\int_{D_T} |\nabla G_1|^2 \leq \frac{2T/\pi}{1 + (T/\pi)^2} \int_{D_T} |\nabla F|^2 \leq \frac{220}{221} \int_{D_T} |\nabla F|^2 \tag{8.19}$$

as in (13.19) in [D3], and where the last inequality holds because $T \leq 10\pi/11$ and $\frac{\lambda}{(1+\lambda^2)}$ is an increasing function of $\lambda \in [0, 1]$.

Incidentally, it is fortunate for us that even when $n > 3$, what we need later is really $|\nabla G_1|^2 = \left|\frac{\partial G_1}{\partial x}\right|^2 + \left|\frac{\partial G_1}{\partial y}\right|^2$, which is easier to compute, rather than for instance the square of the operator norm of the differential DG_1 (acting from $\mathbb{R}^2$ to $\mathbb{R}^{n-2}$), which may be different. The reader may check that in polar coordinates, one may also compute $|\nabla G_1|^2$ as $\left|\frac{\partial G_1}{\partial r}\right|^2 + \left|\frac{1}{r}\frac{\partial G_1}{\partial \theta}\right|^2$.

We unfortunately need some additional control on ∇G_1. The radial and tangential derivatives of G_1 at $z = (\rho\cos t, \rho\sin t)$ are

$$\frac{\partial G_1}{\partial \rho}(\rho\cos t, \rho\sin t) = \sum_{k\geq 1} \beta_k \, \frac{\pi k}{T\rho} \, \rho^{k\pi/T} \sin(\pi k t/T) \tag{8.20}$$

and

$$\frac{1}{\rho}\frac{\partial G_1}{\partial t}(\rho\cos t, \rho\sin t) = \sum_{k\geq 1} \beta_k \, \frac{\pi k}{T\rho} \, \rho^{k\pi/T} \cos(\pi k t/T), \tag{8.21}$$

which yields

$$\begin{aligned} \int_0^T |\nabla G_1(\rho\cos t, \rho\sin t)|^2 dt &= \int_0^T \Big|\frac{\partial G_1}{\partial \rho}\Big|^2 dt + \int_0^T \Big|\frac{1}{\rho}\frac{\partial G_1}{\partial t}\Big|^2 dt \\ &= T\sum_{k\geq 1} \Big|\beta_k \, \frac{\pi k}{T\rho}\Big|^2 \rho^{2k\pi/T} = \frac{\pi^2}{T}\sum_{k\geq 1} k^2 |\beta_k|^2 \, \rho^{-2} \, \rho^{2k\pi/T} \end{aligned} \tag{8.22}$$

as in [D3], (13.13) and its analogue for the tangential derivative. Since $T < \pi$, $\rho^{-2}\,\rho^{2k\pi/T} \leq 1$ for $\rho \leq 1$ and we immediately get that

$$\int_0^T |\nabla G_1(\rho\cos t, \rho\sin t)|^2 dt \leq \frac{\pi^2}{T}\sum_{k\geq 1} k^2|\beta_k|^2 = 2\int_0^T |f'(t)|^2 dt \tag{8.23}$$

by (8.14). We can also get a pointwise estimate for $\rho < 1$ by adding (8.20) to (8.21) and applying Cauchy-Schwarz. We get that

$$\begin{aligned} |\nabla G_1(\rho\cos t, \rho\sin t)| &\leq \frac{2\pi}{T\rho}\sum_{k\geq 1} k\,|\beta_k|\,\rho^{k\pi/T} \leq C\rho^{-1}\Big\{\sum_{k\geq 1} k^2|\beta_k|^2\Big\}^{1/2}\Big\{\sum_{k\geq 1}\rho^{2k\pi/T}\Big\}^{1/2} \\ &\leq C\rho^{-1}||f'||_2\Big\{\sum_{k\geq 1}\rho^{2k\pi/T}\Big\}^{1/2} = C\rho^{-1}||f'||_2\Big\{\frac{\rho^{2\pi/T}}{1-\rho^{2\pi/T}}\Big\}^{1/2} \\ &\leq C\rho^{\frac{\pi}{T}-1}(1-\rho)^{-1/2}||f'||_2 \leq C(1-\rho)^{-1/2}||f'||_2 \end{aligned} \tag{8.24}$$

by (8.14) and because $T \leq \pi$. The estimate diverges slightly near $\rho = 1$, and unfortunately this is not surprising, because the harmonic extension of a Lipschitz function on the unit disk is sometimes a little less than Lipschitz.

To ameliorate this, we pick a radius $r \in (0,1)$ and use a different function G_2 in the annular region $A = \overline{D}_T \cap \overline{B}(0,1) \setminus B(0,r)$. That is, we keep

$$G_2(z) = G_1(z) \quad \text{for } z \in \overline{D}_T \cap \overline{B}(0,r) \tag{8.25}$$

but take

$$G_2(\rho\cos t, \rho\sin t) = \frac{1-\rho}{1-r}\,G_1(r\cos t, r\sin t) + \frac{\rho - r}{1-r}\,G_1(\cos t, \sin t) \tag{8.26}$$

when $(\rho\cos t, \rho\sin t) \in A$. [We interpolate linearly the values of G_1 on the circles $\partial B(0,r)$ and $\partial B(0,1)$.] Note that the two definitions coincide on $\partial B(0,r)$, and that

$$G_2(z) = G_1(z) = F(z) \quad \text{for } z \in \overline{D}_T \cap \partial B(0,1) \tag{8.27}$$

by (8.16). The radial derivative on A is such that

$$\begin{aligned} \Big|\frac{\partial G_2}{\partial \rho}(\rho\cos t, \rho\sin t)\Big| &= \frac{1}{1-r}\,\big|G_1(\cos t, \sin t) - G_1(r\cos t, r\sin t)\big| \\ &\le \frac{1}{1-r}\int_r^1 |\nabla G_1(x\cos t, x\sin t)|dx \\ &\le C(1-r)^{-1}||f'||_2 \int_r^1 (1-x)^{-1/2}dx \le C(1-r)^{-1/2}||f'||_2 \end{aligned} \tag{8.28}$$

by (8.24). For the derivative in the tangential direction,

$$\begin{aligned} \Big|\frac{1}{\rho}\,\frac{\partial G_2}{\partial t}(\rho\cos t, \rho\sin t)\Big| &\le \frac{r}{\rho}\,\Big|\frac{\partial G_1}{\partial t}(r\cos t, r\sin t)\Big| + \frac{1}{\rho}\,\Big|\frac{\partial G_1}{\partial t}(\cos t, \sin t)\Big| \\ &\le |\nabla G_1(r\cos t, r\sin t)| + 2|f'(t)| \\ &\le C(1-r)^{-1/2}||f'||_2 + 2|f'(t)| \end{aligned} \tag{8.29}$$

because $G_1(\cos t, \sin t) = f(t)$ (by (8.11) and (8.27)) and by (8.24). Altogether,

$$|\nabla G_2(\rho\cos t, \rho\sin t)| \le C(1-r)^{-1/2}||f'||_2 + 2|f'(t)| \le C(1-r)^{-1/2}||v'||_\infty \tag{8.30}$$

on A, because f is $2||v'||_\infty$-Lipschitz (by (8.12)). Recall from (8.25) that $G_2 = G_1$ on $\overline{D}_T \cap \overline{B}(0,r)$; there $|\nabla G_2(\rho\cos t, \rho\sin t)| \le C(1-r)^{-1/2}||f'||_2$ directly by (8.24), and so

$$G_2 \text{ is } C(1-r)^{-1/2}||v'||_\infty\text{-Lipschitz on } \overline{D}_T, \tag{8.31}$$

because G_2 is continuous across $\partial B(0,r)$. We do not mind the factor $(1-r)^{-1/2}$ much here, because we shall soon take $r = 1 - 10^{-6}$.

We also need to estimate $\displaystyle\int_A |\nabla G_2(z)|^2 dz$. First,

$$\begin{aligned} \int_A \Big|\frac{\partial G_2}{\partial \rho}(z)\Big|^2 dz &\le (1-r)^{-2}\int_{\rho=r}^1 \int_{t=0}^T \Big\{\int_{x=r}^1 |\nabla G_1(x\cos t, x\sin t)|dx\Big\}^2 \rho d\rho dt \\ &\le (1-r)^{-1}\int_{t=0}^T \Big\{\int_{x=r}^1 |\nabla G_1(x\cos t, x\sin t)|dx\Big\}^2 dt \\ &\le \int_{t=0}^T \int_{x=r}^1 |\nabla G_1(x\cos t, x\sin t)|^2 dx dt \le 2(1-r)\int_0^T |f'(t)|^2 dt \end{aligned} \tag{8.32}$$

by the first part of (8.28), because the inside integral does not depend on ρ, by Cauchy-Schwarz, Fubini, and (8.23) (with $\rho = x$ and integrated on $(r, 1)$). Similarly,

$$\begin{aligned}\int_A \Big|\frac{1}{\rho}\,\frac{\partial G_2}{\partial t}(z)\Big|^2 dz &\le \int_{\rho=r}^1 \int_{t=0}^T \Big\{|\nabla G_1(r\cos t, r\sin t)| + 2|f'(t)|\Big\}^2 \rho d\rho dt \\ &\le (1-r)\int_{t=0}^T \Big\{|\nabla G_1(r\cos t, r\sin t)| + 2|f'(t)|\Big\}^2 dt \\ &\le 2(1-r)\Big\{\int_{t=0}^T |\nabla G_1(r\cos t, r\sin t)|^2 dt + 4\int_{t=0}^T |f'(t)|^2 dt\Big\} \\ &\le 12(1-r)\int_0^T |f'(t)|^2 dt\end{aligned} \tag{8.33}$$

by (8.29), because the integrand does not depend on ρ, and by (8.23) (with $\rho = r$). Altogether,

$$\int_A |\nabla G_2(z)|^2 dz \le 14(1-r)\int_0^T |f'(t)|^2 dt \le 10^{-4}\int_0^T |f'(t)|^2 dt \tag{8.34}$$

by (8.32) and (8.33), and because we take $r = 1 - 10^{-6}$.

We are now ready to define G. We keep

$$G(z) = F(z) \quad \text{for } z \in \overline{D}_T \setminus B(0, 9/10), \tag{8.35}$$

as suggested by (8.6), and set

$$G(z) = \frac{9}{10}\, G_2\big(\frac{10z}{9}\big) \quad \text{for } z \in \overline{D}_T \cap B(0, 9/10) \setminus B(0, 3\kappa), \tag{8.36}$$

where the small constant κ will be chosen soon. Notice that when $z \in \partial B(0, 9/10)$, $F(z) = \frac{9}{10}\, F\big(\frac{10z}{9}\big) = \frac{9}{10}\, G_1\big(\frac{10z}{9}\big) = \frac{9}{10}\, G_2\big(\frac{10z}{9}\big)$ because F is homogeneous of degree 1 and by (8.27). So our two definitions match on $\partial B(0, 9/10)$. Recall from (8.7) that we need to take

$$G(z) = 0 \quad \text{on } \overline{D}_T \cap B(0, 2\kappa). \tag{8.37}$$

The simplest way to make a continuous transition is to interpolate linearly (as before) and take

$$\begin{aligned}G(\rho\cos t, \rho\sin t) &= \frac{\rho - 2\kappa}{\kappa}\, G(3\kappa\cos t, 3\kappa\sin t) \\ &= \frac{9}{10}\,\frac{\rho - 2\kappa}{\kappa}\, G_1\Big(\frac{30\kappa}{9}\cos t, \frac{30\kappa}{9}\sin t\Big)\end{aligned} \tag{8.38}$$

for $0 \le t \le T$ and $2\kappa \le \rho \le 3\kappa$. [We used (8.36) and (8.25) here.]

This completes our definition of G; let us now check that it satisfies the requirements of Lemma 8.8.

We need to know that $G(\rho\cos t, \rho\sin t) = F(\rho\cos t, \rho\sin t) = 0$ when $t \in \{0, T\}$, to complete our proof of (8.6). But for such t, $F(\rho\cos t, \rho\sin t) = 0$ directly by (8.4) and because $v(0) = v(T) = 0$ (see above (8.2)). Next, $G_1(\rho\cos t, \rho\sin t) = 0$ by (8.17), and then $G_2(\rho\cos t, \rho\sin t) = 0$ by the definition (8.25) or (8.26). Then $G(\rho\cos t, \rho\sin t) = 0$, because it is obtained from F or G_2 by (8.35), (8.36), (8.37) or (8.38).

Since (8.7) holds by (8.37), we turn to the Lipschitz condition (8.9). Since G is continuous across $\partial B(0, 9/10)$, $\partial B(0, 3\kappa)$, and $\partial B(0, 2\kappa)$, we just need to investigate the remaining annular domains separately. Out of $B(0, 9/10)$, $G = F$ by (8.35), so it is enough to check that

$$F \text{ is } 5||v'||_\infty\text{-Lipschitz on } \overline{D}_T. \tag{8.39}$$

Recall that v is η-Lipschitz and vanishes at the origin, so $|v(t)| \le ||v'||_\infty T \le \eta T$, $w(t) \ge 1 - \eta T$ by (8.2), and $|w'(t)| \le 2||v'||_\infty$, again by (8.2). Then (8.4) yields

$$\begin{aligned} |\nabla F(r\cos t, r\sin t)| &\le \Big|\frac{\partial F}{\partial r}(r\cos t, r\sin t)\Big| + \frac{1}{r}\Big|\frac{\partial F}{\partial t}(r\cos t, r\sin t)\Big| \\ &\le \frac{|v(t)|}{w(t)} + \frac{|v'(t)|}{w(t)} + \frac{|v(t)||w'(t)|}{w(t)^2} \le 5||v'||_\infty \end{aligned} \tag{8.40}$$

if η is small enough, and as needed.

Return to G. In $B(0, 9/10) \setminus B(0, 3\kappa)$, we use (8.36) and (8.31). Since $G = 0$ in $B(0, 2\kappa)$, we are just left with $B(0, 3\kappa) \setminus B(0, 2\kappa)$, where (8.38) yields

$$\Big|\frac{1}{\rho}\,\frac{\partial G}{\partial t}(\rho\cos t, \rho\sin t)\Big| \le \frac{3\kappa}{\rho}\,\Big|\nabla G_1\Big(\frac{30\kappa}{9}\cos t, \frac{30\kappa}{9}\sin t\Big)\Big| \le C||f'||_2 \le C||v'||_\infty \tag{8.41}$$

by (8.24) and (8.12), and

$$\Big|\frac{\partial G}{\partial \rho}(\rho\cos t, \rho\sin t)\Big| \le \kappa^{-1}\Big|G_1\Big(\frac{30\kappa}{9}\cos t, \frac{30\kappa}{9}\sin t\Big)\Big| \le C||f'||_2 \le C||v'||_\infty \tag{8.42}$$

this time by (8.17) and (8.24). So (8.9) holds.

We are left with the verification of (8.10). Since $F = G$ out of $B(0, 9/10)$ by (8.35),

$$H^2(\Sigma_F) - H^2(\Sigma_G) = H^2(\Sigma'_F) - H^2(\Sigma'_G), \tag{8.43}$$

where Σ'_F is the graph of the restriction of F to $D' = D_T \cap B(0, 9/10)$, and similarly for Σ'_G. By the area formula,

$$H^2(\Sigma'_G) = \int_{D'} J(z)dz, \tag{8.44}$$

where J is the jacobian of the parameterization $H : z \to (z, G(z))$, which we compute now. Fix $z \in D'$ where G is differentiable, denote by e_1 and e_2 the first two vectors of the canonical basis of $\mathbb{R}^n$ (so e_1 and e_2 form a basis of the horizontal plane P), and set $v_1 = \frac{\partial G}{\partial x}(z)$ and $v_2 = \frac{\partial G}{\partial y}(z)$. Thus, if we set $w_1 = DH(z)(e_1)$ and $w_2 = DH(z)(e_2)$, we get that $w_i = e_i + v_i$ for $i = 1, 2$. Now $J(z) = |w_1||w_2| \sin \alpha$, where $\alpha \in [0, \pi/2]$ is the angle of w_1 with w_2, so

$$\begin{aligned} J(z)^2 &= |w_1|^2|w_2|^2 \sin^2 \alpha = |w_1|^2|w_2|^2[1 - \cos^2 \alpha] = |w_1|^2|w_2|^2 - \langle w_1, w_2\rangle^2 \\ &= (1 + |v_1|^2)(1 + |v_2|^2) - \langle v_1, v_2\rangle^2 = 1 + |v_1|^2 + |v_2|^2 + |v_1|^2|v_2|^2 - \langle v_1, v_2\rangle^2 \\ &\le 1 + |v_1|^2 + |v_2|^2 + |v_1|^2|v_2|^2 \\ &\le 1 + |\nabla G(z)|^2 + |\nabla G(z)|^4 \le 1 + (1 + C\eta)|\nabla G(z)|^2 \end{aligned} \tag{8.45}$$

where the last inequality comes from (8.9). Recall that $(1 + u)^{1/2} \le 1 + \frac{u}{2}$ for $u \ge 0$. Applying this with $u = J(z)^2 - 1$ yields

$$\begin{aligned} H^2(\Sigma'_G) &= \int_{D'} J(z)\, dz = \int_{D'} (1 + u)^{1/2} dz \le \int_{D'} (1 + \frac{u}{2})\, dz \\ &= H^2(D') + \frac{1}{2} \int_{D'} u\, dz \le H^2(D') + \frac{(1 + C\eta)}{2} \int_{D'} |\nabla G(z)|^2 dz \end{aligned} \tag{8.46}$$

by (8.44) and (8.45). Next

$$\begin{aligned} \int_{D'} |\nabla G|^2 &\le \int_{D' \cap B(0,3\kappa)} |\nabla G|^2 + \int_{D' \setminus B(0,3\kappa)} |\nabla G|^2 \\ &\le C \int_{D' \cap B(0,3\kappa)} ||f'||_2^2 + \int_{D' \setminus B(0,3\kappa)} \big|\nabla\big[\frac{9}{10}\, G_2(10z/9)\big]\big|^2 \\ &\le C\kappa^2 ||f'||_2^2 + \frac{81}{100} \int_{D_T} |\nabla G_2|^2 \end{aligned} \tag{8.47}$$

by (8.41), (8.42), (8.36), and a linear change of variable. We choose κ so small that $C\kappa^2 \le 10^{-4}$ in (8.47), and since

$$\int_{D_T} |\nabla G_2|^2 \le \int_{D_T} |\nabla G_1|^2 + \int_A |\nabla G_2|^2 \le \frac{220}{221} \int_{D_T} |\nabla F|^2 + 10^{-4} ||f'||_2^2 \tag{8.48}$$

by (8.25), (8.19) and (8.34), we get that

$$\int_{D'} |\nabla G|^2 \le \frac{81}{100}\, \frac{220}{221} \int_{D_T} |\nabla F|^2 + 2 \cdot 10^{-4}\, ||f'||_2^2\,. \tag{8.49}$$

We also need a lower bound for $H^2(\Sigma'_F)$. This time the area formula yields

$$H^2(\Sigma'_F) = \int_{D'} J_F(z) dz \ge \int_{D'} \left\{1 + |\nabla F(z)|^2\right\}^{1/2} \tag{8.50}$$

where the lower bound comes from the first two lines of (8.45). Recall from (8.39) that F is 5η-Lipschitz on $\overline{D}_T$. Also,

$$(1+u)^{1/2} \geq 1 + \frac{u}{2} - \frac{u^2}{8} \geq 1 + \frac{u}{2} - \frac{25\eta^2 u}{8} \geq 1 + \frac{u}{2}\,(1 - 7\eta^2) \tag{8.51}$$

for $0 \leq u \leq 25\eta^2$; we apply this with $u = |\nabla F(z)|^2$, integrate, and get that

$$H^2(\Sigma'_F) \geq \int_{D'} \left\{1 + |\nabla F|^2\right\}^{1/2} \geq H^2(D') + \frac{1-7\eta^2}{2} \int_{D'} |\nabla F|^2. \tag{8.52}$$

Notice that $\int_{D'} |\nabla F|^2 = \frac{81}{100} \int_{D_T} |\nabla F|^2$ because F is homogeneous of degree 1; then

$$\begin{aligned} H^2(\Sigma_F) - H^2(\Sigma_G) &= H^2(\Sigma'_F) - H^2(\Sigma'_G) \\ &\geq \frac{1-7\eta^2}{2} \int_{D'} |\nabla F|^2 - \frac{1+C\eta}{2} \int_{D'} |\nabla G|^2 \\ &\geq \frac{81}{200} \left[1 - 7\eta^2 - (1+C\eta)\,\frac{220}{221}\right] \int_{D_T} |\nabla F|^2 - (1+C\eta) 10^{-4} ||f'||_2^2 \\ &\geq \frac{81}{200}\,\frac{1}{300} \int_{D_T} |\nabla F|^2 - (1+C\eta) 10^{-4} ||f'||_2^2 \\ &\geq \left[\frac{81}{12 \cdot 10^4} - (1+C\eta) 10^{-4}\right] ||f'||_2^2 \geq 5 \cdot 10^{-4} ||f'||_2^2 \end{aligned} \tag{8.53}$$

by (8.43), (8.52), (8.46), (8.49), if η is small enough, and by (8.18). Since $f = v(1-|v|^2)^{-1/2}$ by (8.11) and $|v| \leq \eta T$ everywhere, we easily get that $||f'||_2^2 \geq \frac{1}{2}||v'||_2^2$, and the first part of (8.10) follows from (8.53). So our proof of Lemma 8.8 will be complete as soon as we check that $\text{length}(\Gamma) - T \leq \int_0^T |v'(t)|^2 dt$

We differentiate in (8.2) or lazily use (7.11) with $\theta' = 1$, and get that $|z'(t)|^2 = w(t)^2 + |w'(t)|^2 + |v'(t)|^2$. Recall that $ww' = -\langle v, v'\rangle$ because $w^2 + |v|^2 = 1$, so $|w'| \leq |v||v'|/w$ and $|z'(t)|^2 \leq 1 + |v'(t)|^2[1 + |v(t)|^2/w(t)^2] \leq 1 + 2|v'(t)|^2$ (recall that $|v(t)| \leq \eta T$ and hence $w(t) \geq 1 - \eta T$). Hence $|z'(t)| \leq 1 + |v'(t)|^2$ (recall that $(1+u)^{1/2} \leq 1 + u/2$ for $u \geq 0$), and

$$\text{length}(\Gamma) = \int_0^T |z'(t)| dt \leq T + \int_0^T |v'(t)|^2 dt, \tag{8.54}$$

as needed. Lemma 8.8 follows. □

9. Retractions near Lipschitz graphs and the construction of a first competitor

We shall now use the previous sections to construct a first competitor for E in $B(0,1)$. We still assume that (6.2) holds (otherwise Lemma 6.3 gives the desired estimates), and we found in Section 6 a collection of curves $g_{j,k}$, $(j,k) \in \widetilde{J}$, with the properties described in Lemma 6.11. Recall in particular that the extremities of $g_{j,k}$ are $\varphi(a)$ and $\varphi(b)$, where a and $b \in V$ are the endpoints of the corresponding $\mathfrak{C}_{j,k}$.

Denote by $D_{j,k}$ the cone over the geodesic that goes from a to b. Thus $D_{j,k}$ is a plane sector bounded by the half lines $[0,a)$ and $[0,b)$. Also denote by $P_{j,k}$ the plane that contains $D_{j,k}$.

Next apply to each arc $g_{j,k}$ the construction of Section 7. More precisely, we should first apply a rotation $\mathcal{R}$ such that $\mathcal{R}(P_{j,k})$ is the horizontal plane P of Section 7, and apply the construction of Section 7 to $\gamma = \mathcal{R}(g_{j,k})$, but, in order to save notation, we shall often pretend that $\mathcal{R}$ is the identity. Let us check that γ satisfies our assumptions (7.1)-(7.3).

First, the geodesic distance in ∂B from $\varphi(a)$ to $\varphi(b)$ is $\text{dist}_{\partial B}(\varphi(a),\varphi(b)) \geq \text{dist}_{\partial B}(a,b) - C\tau \geq 9\eta_0$ by (6.14) and (2.5), so $\text{length}(\gamma) \geq 9\eta_0$. On the other hand, $\text{length}(\gamma) = \text{length}(g_{j,k}) = H^1(g_{j,k}) \leq \text{length}(\mathfrak{C}_{j,k}) + C_4\tau < \frac{10\pi}{11}$ because $g_{j,k}$ is simple, by (6.43), and by (2.5), so (7.1) holds. We also get (7.2), because $\text{length}(\gamma) \leq \text{length}(\mathfrak{C}_{j,k}) + C_4\tau = \text{dist}_{\partial B}(a,b) + C_4\tau \leq \text{dist}_{\partial B}(\varphi(a),\varphi(b)) + C\tau$ by (6.14) and if τ is small enough compared to τ_1. [See the discussion above (7.4) concerning the order in which we choose the constants.] Finally, (7.3) follows from (6.15).

So we can apply the construction of Section 7 to $g_{j,k}$, and we get a curve $\Gamma_{j,k}$ in $\partial B(0,1)$, which is an η-Lipschitz graph over $P_{j,k}$ and has a large intersection with $g_{j,k}$. Denote by $\Gamma^*_{j,k}$ the cone over $\Gamma_{j,k}$. Then $\Gamma^*_{j,k}$ is the graph of a homogeneous Lipschitz function defined on $D_{j,k}$. As was observed in Remark 8.3, we can apply the construction of Section 8 to the curve $\Gamma = \mathcal{R}(\Gamma_{j,k})$, where $\mathcal{R}$ is the same rotation as above, which sends $P_{j,k}$ to P. We get a new surface $\Sigma_{j,k}$, with the properties described near Lemma 8.8. In particular, $\Sigma_{j,k}$ is the graph over $P_{j,k}$ of a $(C\eta)$-Lipschitz function $G_{j,k}$ defined on $D_{j,k}$ (see (8.9)), and since $\Gamma^*_{j,k}$ is the graph of the homogeneous function F in in (8.4) (see the remark below (8.5)),

$$\Sigma_{j,k} \text{ coincides with } \Gamma^*_{j,k} \text{ out of } B(0,19/20) \tag{9.1}$$

by (8.6) and because F and $G_{j,k}$ are both Lipschitz with small constants. Also recall that $\Sigma_{j,k}$ and $\Gamma^*_{j,k}$ are both bounded by the two half lines through $\varphi(a)$ and $\varphi(b)$ that bound $D_{j,k}$ (see (8.6), (8.4), and the line above (8.2)), and that

$$H^2(\Sigma_{j,k} \cap B) \leq H^2(\Gamma^*_{j,k} \cap B) - 10^{-4}[\text{length}(\Gamma_{j,k}) - \text{dist}_{\partial B}(\varphi(a),\varphi(b))], \tag{9.2}$$

by (8.10) and the definition (8.1). Set

$$g = \bigcup_{(j,k)\in\widetilde{J}} g_{j,k}\,,\ \Gamma = \bigcup_{(j,k)\in\widetilde{J}} \Gamma_{j,k}\,,\ \Gamma^* = \bigcup_{(j,k)\in\widetilde{J}} \Gamma^*_{j,k}\,,\ \text{ and } \Sigma = \bigcup_{(j,k)\in\widetilde{J}} \Sigma_{j,k}\,. \tag{9.3}$$

We shall need to define a projection of a neighborhood of Σ onto Σ; which will allow us later to deform $E \cap \overline{B}(0,r)$ onto a subset of Σ.

Lemma 9.4. *There is a neighborhood U of $E \cap \overline{B}$ in $\overline{B}$ and a 50-Lipschitz mapping $p_2 : U \to \Sigma \cap \overline{B}$, such that*

$$p_2(z) = z \quad \textit{for } z \in \Sigma \cap \overline{B} \setminus B(0,1/3) \tag{9.5}$$

and

$$p_2(z) \in \Sigma \cap B(0,1/2) \quad \textit{for } z \in U \cap B(0,1/3). \tag{9.6}$$

We shall start with a first projection on a simpler set R. Set

$$R_{j,k} = \big\{ z \in \mathbb{R}^n \,;\, \operatorname{dist}(z, D_{j,k}) \le 10^{-2} \eta_0 \operatorname{dist}(z, \partial D_{j,k}) \big\} \quad \text{and} \quad R = \bigcup_{(j,k) \in \widetilde{J}} R_{j,k}\,, \tag{9.7}$$

where $\partial D_{j,k}$ is the union of the two half lines that bound $D_{j,k}$ (the half lines through $\varphi(a)$ and $\varphi(b)$, with the notation above), and η_0 is as in Section 2. Thus the $R_{j,k}$ are small conic sectors around the $D_{j,k}$. Observe that

$$\Sigma_{j,k} \subset R_{j,k} \tag{9.8}$$

rather trivially, because $\Sigma_{j,k}$ is a small Lipschitz graph over $D_{j,k}$ bounded by the two half lines of $\partial D_{j,k}$.

We shall first project a small neighborhood of $\Sigma \cap \partial B$ onto $R \cap \partial B$. Set

$$W = \big\{ z \in \partial B \,;\, \operatorname{dist}(z, V) < C_6 \tau \big\}, \tag{9.9}$$

where V is our set of vertices from Section 2, and C_6 will be chosen soon, somewhat larger than the constants in Lemma 6.11.

We shall only need to define the first projection p_0 on $[R \cap \partial B] \cup W$, and on $R \cap \partial B$ we just take $p_0(z) = z$. So we just need to take care of the $B_x = \partial B \cap B(\varphi(x), C_6\tau)$, $x \in V$. Observe that the B_x are far from each other, by (2.5) and (2.6), so we can treat them separately.

Fix $x \in V$. There are two or three arcs $\mathfrak{C}_{j,k}$ that end at x. For all the other arcs $\mathfrak{C}_{j',k'}$, (2.6) says that $\mathfrak{C}_{j',k'}$ lies further than $\eta_0/2$ away from B_x (if τ is small enough), and then (6.14) says that $D_{j',k'}$ still lies further than $\eta_0/3$ away from B_x too. So $R \cap B_x$ is simply composed of the two or three $R_{j,k}$ for which x is an endpoint of $\mathfrak{C}_{j,k}$.

Fix such a pair (j,k). Near B_x, $R_{j,k}$ is a thin conical neighborhood in ∂B of the geodesic $\rho_{j,k}$ that connects $\varphi(x)$ to some other point of $\varphi(V)$, and p_0 will be easier to define first on a larger conical neighborhood $A_{j,k}$ of $\rho_{j,k}$. For $z \in B_x \setminus \{\varphi(x)\}$, denote by $\theta_{j,k}(z) \in [0, \pi]$ the angle of $z - \varphi(x)$ with the tangent at $\varphi(x)$ of $\rho_{j,k}$ (pointing away from $\varphi(x)$). Then set $A_{j,k} = \big\{ z \in B_x \setminus \{\varphi(x)\} \,;\, \theta_{j,k}(z) \le \frac{99}{100}\,\frac{\pi}{3} \big\}$.

We define p_0 on $A_{j,k}$ so that $p_0(z) = z$ when $z \in R_{j,k}$, $p_0(A_{j,k}) \subset B_x \cap R_{j,k}$, and $p_0(z) = \varphi(x)$ when $\theta_{j,k}(z) = \frac{99}{100}\,\frac{\pi}{3}$. For instance, project $z \in A_{j,k}$ onto $R_{j,k}$ along a vector field in ∂B which is roughly rotationally invariant around the tangent line to $\rho_{j,k}$ at $\varphi(x)$ and tangent to $\partial A_{j,k}$ along its conical boundary, as suggested by Figure 9.1.

We can choose p_0 so that it is $\frac{19}{10}$-Lipschitz on $A_{j,k}$, say. Now observe that the two or three $A_{j,k}$ are disjoint, because the $\rho_{j,k}$ make angles larger than $2\pi/3 - C\tau$ at $\varphi(x)$. So we get a clear definition of p_0 on $\cup_{j,k} A_{j,k}$. We take $p_0(z) = \varphi(x)$ on $B_x \setminus \cup_{j,k} A_{j,k}$; then p_0 is continuous across the conical boundaries of the $A_{j,k}$, and we get a $\frac{19}{10}$-Lipschitz map p_0 defined on B_x.

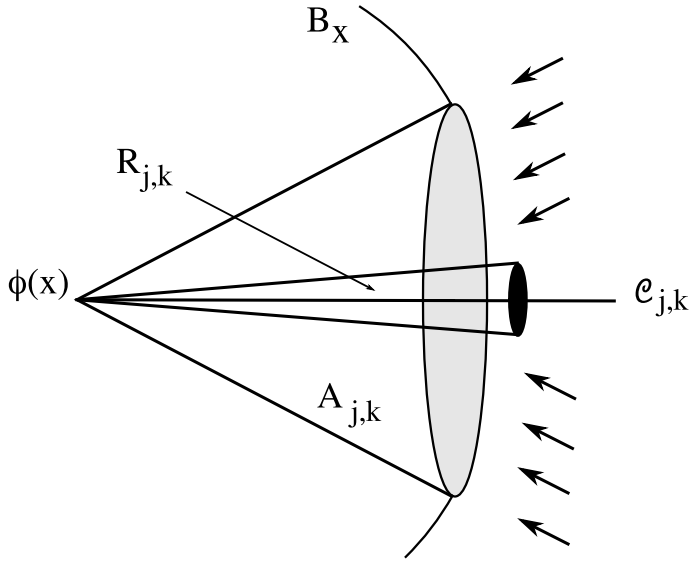

Figure 9.1. Project along the roughly invariant vector field whose values on the vertical plane are suggested by the arrows. This is a picture in ∂B, which we assimilate to $\mathbb{R}^{n-1}$.

Recall that we decided to take $p_0(z) = z$ on R; the two definitions fit on the B_x (because $R \cap A_{j,k} = R_{j,k}$ and R does not meet $B_x \setminus \cup_{j,k} A_{j,k}$), and then

$$p_0 \text{ is 2-Lipschitz on } [R \cap \partial B] \cup W, \tag{9.10}$$

because the B_x are so far from each other and $p_0(z) = z$ on $R \cap B_x$, which is not empty. Record that by construction,

$$p_0(z) = z \text{ for } z \in R \cap \partial B \text{ and } p_0(z) \in R \text{ for } z \in [R \cap \partial B] \cup W. \tag{9.11}$$

The second projection p_1 will be defined on

$$U = \big\{ z \in \overline{B} \,;\, \mathrm{dist}(z, D) \le C_7 \tau \big\}, \tag{9.12}$$

where D is the union of the $D_{j,k}$, and C_7 will depend on η_0, but not on C_6. First set

$$\begin{cases} m(z) = 0 & \text{for } 0 \le |z| \le 10^{-1}\,; \\ m(z) = |z| & \text{for } |z| \ge 2 \cdot 10^{-1}\,; \\ m(z) = 2(|z| - 10^{-1}) & \text{for } 10^{-1} \le |z| \le 2 \cdot 10^{-1} \end{cases} \tag{9.13}$$

(so that m is continuous and piecewise linear) and then

$$p_1(z) = m(z)\, p_0(z/|z|) \text{ for } z \in U. \tag{9.14}$$

Let us check that this is well defined. First, we take $p_1(0) = 0$, which is legitimate because $p_1(z) = 0$ for $z \in B(0, 10^{-1}) \setminus \{0\}$ (since $m(z) = 0$). We also need to check that $p_0(z/|z|)$ is defined when $|z| \ge 10^{-1}$. Set $w = z/|z|$; we want to check that $w \in [R \cap \partial B] \cup W$.

Observe that $\operatorname{dist}(w, D) \le 10C_7\tau$ by (9.12) and because D is a cone, and let $(j,k) \in \widetilde{J}$ be such that $\operatorname{dist}(w, D_{j,k}) \le 10C_7\tau$. If $\operatorname{dist}(w, \partial D_{j,k}) \ge 1000\eta_0^{-1}C_7\tau$, then $w \in R_{j,k}$ by (9.7), and we are happy. Otherwise, recall that $\partial D_{j,k}$ is just the union of two half lines through points $\varphi(x)$, with $x \in V$. So we can find $x \in V$ such that $|w - \varphi(x)| \le 2000\eta_0^{-1}C_7\tau$, and then $x \in B_x \subset W$, by (6.14) and if C_6 is large enough, depending on C_7 and η_0. Thus $w \in [R \cap \partial B] \cup W$ in all cases, and p_1 is well defined.

Let us check that

$$p_1 \text{ is 22-Lipschitz.} \tag{9.15}$$

For $z_1, z_2 \in U \setminus B(0, 10^{-1})$,

$$\begin{aligned} |p_1(z_1) - p_1(z_2)| &\le m(z_1)\,|p_0(z_1/|z_1|) - p_0(z_2/|z_2|)| + |m(z_1) - m(z_2)|\,|p_0(z_2/|z_2|)| \\ &\le 2m(z_1)\Big|\frac{z_1}{|z_1|} - \frac{z_2}{|z_2|}\Big| + |m(z_1) - m(z_2)| \le 22|z_1 - z_2| \end{aligned} \tag{9.16}$$

by (9.10) and because the radial projection $z \to \frac{z}{|z|}$ is 10-Lipschitz on $\overline{B} \setminus B(0, 10^{-1})$. If instead $z_1 \in U \setminus B(0, 10^{-1})$ but $z_2 \in B(0, 10^{-1})$, we get that $|p_1(z_1) - p_1(z_2)| = |p_1(z_1)| \le m(z_1) \le 2|z_1 - z_2|$; the case when $z_1, z_2 \in B(0, 10^{-1})$ is trivial, and (9.15) follows.

Observe that

$$p_1(z) \in R \cap \overline{B} \text{ for } z \in U, \tag{9.17}$$

just by (9.11) and because R is a cone. We shall need to know that the $R_{j,k}$ are essentially disjoint. Let us even check that

$$\operatorname{dist}(z, \bigcup_{(j',k') \neq (j,k)} R_{j',k'}) \ge \operatorname{Min}\Big\{\frac{\eta_0|z|}{10}\,;\, \operatorname{dist}(z, \partial D_{j,k})\Big\} \text{ for } z \in R_{j,k}. \tag{9.18}$$

Since R, the $R_{j,k}$, and $\partial D_{j,k}$ are cones, it is enough to check this when $z \in R_{j,k} \cap \partial B$. Recall from the definition (9.7) that

$$\operatorname{dist}(z, D_{j,k}) \le 10^{-2}\eta_0 \operatorname{dist}(z, \partial D_{j,k}) \le 10^{-2}\eta_0. \tag{9.19}$$

Denote by a and b the extremities of $\mathfrak{C}_{j,k}$. Then $\partial D_{j,k}$ is the union of the two half lines through $\varphi(a)$ and $\varphi(b)$, and $D_{j,k}$ is the cone over the geodesic $\rho_{j,k}$ between $\varphi(a)$ and $\varphi(b)$. Thus, if $\xi \in D_{j,k}$ minimizes the distance to z,

$$\operatorname{dist}(z, \rho_{j,k}) \le \operatorname{dist}(z, \xi/|\xi|) \le |z - \xi| + (1 - |\xi|) \le 2|z - \xi| = 2\operatorname{dist}(z, D_{j,k}) \tag{9.20}$$

because $D_{j,k}$ is the cone over $\rho_{j,k}$ and $|z| = 1$. Now every point of $\rho_{j,k}$ is $C\tau$-close to $\mathfrak{C}_{j,k}$, because the two geodesics $\mathfrak{C}_{j,k}$ and $\rho_{j,k}$ have almost the same extremities, by (6.14), and because (2.5) gives the stability of geodesics. So

$$\begin{aligned} \operatorname{dist}(z, \mathfrak{C}_{j,k}) &\le \operatorname{dist}(z, \rho_{j,k}) + C\tau \le 2\operatorname{dist}(z, D_{j,k}) + C\tau \\ &\le 2 \cdot 10^{-2}\eta_0 + C\tau \le 3 \cdot 10^{-2}\eta_0 \end{aligned} \tag{9.21}$$

by (9.20) and (9.19). Let us prove (9.18) by contradiction, and assume that we can find $(j',k') \neq (j,k)$ and $z' \in R_{j',k'}$ such that

$$|z'-z| < \mathrm{Min}\Big\{\frac{\eta_0}{10}\,;\,\mathrm{dist}(z,\partial D_{j,k})\Big\}. \tag{9.22}$$

We want to get a contradiction. Set $w' = z'/|z'|$, and observe that $\mathrm{dist}(w',\mathfrak{C}_{j',k'}) \leq 3\cdot 10^{-2}\eta_0$ by the proof of (9.21). Then

$$\begin{aligned}\mathrm{dist}(z,\mathfrak{C}_{j',k'}) &\leq |z-z'| + |z'-w'| + \mathrm{dist}(w',\mathfrak{C}_{j',k'})\\ &\leq |z-z'| + (1-|z'|) + 3\cdot 10^{-2}\eta_0\\ &\leq 2|z-z'| + 3\cdot 10^{-2}\eta_0 \leq \eta_0/5\end{aligned} \tag{9.23}$$

because $|z|=1$ and by (9.22). By (2.6), (9.21), and (9.23), $\mathfrak{C}_{j,k}$ and $\mathfrak{C}_{j',k'}$ have a common endpoint x. Since $\mathfrak{C}_{j,k}$ and $\mathfrak{C}_{j',k'}$ are fairly short (by (2.5)) and leave from x with large angles, we also have that z and z' lie in $B(x,\eta_0/2) \subset B(\varphi(x),\eta_0)$. But in $B(\varphi(x),\eta_0)$ the situation is simple, $D_{j,k}$ and $D_{j',k'}$ are two half planes with a common boundary and that make an angle larger than 100°, and $R_{j,k}$ and $R_{j',k'}$ are small conical neighborhoods of $D_{j,k}$ and $D_{j',k'}$; then (9.22) fails, which proves (9.18).

Define $\pi_{j,k}$ on $\overline{B}\cap R_{j,k}$ by the fact that $\pi_{j,k}(z) \in \Sigma_{j,k}$ and $\pi_{j,k}(z) - z \in P_{j,k}^{\perp}$. That is, if we denote by $\pi'_{j,k}$ the orthogonal projection on $P_{j,k}$, $\pi_{j,k}(z)$ is the point of the graph $\Sigma_{j,k}$ such that $\pi'_{j,k}(\pi_{j,k}(z)) = \pi'_{j,k}(z)$. Trivially,

$$\pi_{j,k}(z) = z \quad \text{for } z \in \overline{B}\cap\Sigma_{j,k}. \tag{9.24}$$

Notice that

$$|\pi'_{j,k}(z) - z| = \mathrm{dist}(z,D_{j,k}) \leq 10^{-2}\eta_0\,\mathrm{dist}(z,\partial D_{j,k}) \tag{9.25}$$

for $z \in R_{j,k}$, by (9.7). Next

$$|\pi_{j,k}(z) - \pi'_{j,k}(z)| \leq C\eta\,\mathrm{dist}(\pi'_{j,k}(z),\partial D_{j,k}) \leq 2C\eta\,\mathrm{dist}(z,\partial D_{j,k}) \tag{9.26}$$

because $\Sigma_{j,k}$ is a $C\eta$-Lipschitz graph that contains $\partial D_{j,k}\cap\overline{B}$, and by (9.25). Thus

$$|\pi_{j,k}(z) - z| \leq 2\cdot 10^{-2}\eta_0\,\mathrm{dist}(z,\partial D_{j,k}) \quad \text{for } z \in R_{j,k}\,, \tag{9.27}$$

by (9.25) and (9.26). We want to define π on $\overline{B}\cap R$ by setting

$$\pi(z) = \pi_{j,k}(z) \quad \text{when } z \in \overline{B}\cap R_{j,k} \tag{9.28}$$

so we need to verify that the different definitions match on the intersections. We first check that when $(j',k') \neq (j,k)$,

$$R_{j,k}\cap R_{j',k'}\cap\overline{B} \subset \partial D_{j,k}\cap\partial D_{j',k'}\cap\overline{B} \subset \Sigma\cap\overline{B}. \tag{9.29}$$

Indeed, if $z \in R_{j,k} \cap R_{j',k'} \cap \overline{B}$, (9.18) says that $z \in \partial D_{j,k}$, and by symmetry $z \in \partial D_{j',k'}$ too. The second inclusion follows from (8.6), (8.4), and the line above (8.2).

Now (9.29) says that $\pi_{j,k}(z) = \pi_{j',k'}(z) = z$ on $R_{j,k} \cap R_{j',k'} \cap \overline{B}$, and our various definitions of π match.

We'll need to know that

$$\pi \text{ is 2-Lipschitz on } \overline{B} \cap R. \tag{9.30}$$

To prove (9.30), let $z, z' \in \overline{B} \cap R$ be given, and let (j,k) and (j',k') be such that $z \in R_{j,k}$ and $z' \in R_{j',k'}$. If $(j',k') = (j,k)$, we easily get that $|\pi(z) - \pi(z')| = |\pi_{j,k}(z) - \pi_{j,k}(z')| \leq (1 + C\eta)|z - z'|$, by definition of $\pi_{j,k}$ and because $\Sigma_{j,k}$ is a $C\eta$-Lipschitz graph over $D_{j,k}$. So we may assume that $(j',k') \neq (j,k)$. Then

$$\begin{aligned} |\pi(z) - \pi(z')| &\leq |z - z'| + |\pi_{j,k}(z) - z| + |\pi_{j',k'}(z') - z'| \\ &\leq |z - z'| + 2 \cdot 10^{-2} \eta_0 \left[\operatorname{dist}(z, \partial D_{j,k}) + \operatorname{dist}(z', \partial D_{j',k'}) \right] \end{aligned} \tag{9.31}$$

by (9.27). Without loss of generality, we may assume that $\operatorname{dist}(z, \partial D_{j,k}) \geq \operatorname{dist}(z', \partial D_{j',k'})$, and then (9.31) yields

$$|\pi(z) - \pi(z')| \leq |z - z'| + 4 \cdot 10^{-2} \eta_0 \operatorname{dist}(z, \partial D_{j,k}). \tag{9.32}$$

Recall from (9.18) that

$$|z - z'| \geq \operatorname{Min}\left\{ \frac{\eta_0 |z|}{10} ; \operatorname{dist}(z, \partial D_{j,k}) \right\}. \tag{9.33}$$

If $|z - z'| \geq \operatorname{dist}(z, \partial D_{j,k})$, (9.32) says that $|\pi(z) - \pi(z')| \leq (1 + 4 \cdot 10^{-2} \eta_0)|z - z'|$, and we are happy. Otherwise, $|z - z'| \geq \frac{\eta_0 |z|}{10}$ and (9.32) says that $|\pi(z) - \pi(z')| \leq |z - z'| + 4 \cdot 10^{-2} \eta_0 |z| \leq (1 + 4 \cdot 10^{-1}) |z - z'|$; (9.30) follows.

It could be that π sends some points of $\overline{B} \cap R$ slightly out of $\overline{B}$, so we compose it with the standard retraction r on the sphere, defined by $r(z) = z$ when $z \in \overline{B}$ and $r(z) = z/|z|$ otherwise. Let us check that

$$r \circ \pi(z) \in \Sigma \quad \text{for } z \in \overline{B} \cap R, \tag{9.34}$$

where Σ still denotes the union of the $\Sigma_{j,k}$. Let $z \in \overline{B} \cap R$ be given, and let $(j,k) \in \widetilde{J}$ be such that $z \in R_{j,k}$. If $r \circ \pi(z) = \pi_{j,k}(x)$, we are happy because $\pi_{j,k}(x) \in \Sigma_{j,k}$ by definition of $\pi_{j,k}$. Otherwise, $r \circ \pi_{j,k}(z) \neq \pi_{j,k}(z)$ (by (9.28)), so $\pi_{j,k}(z)$ lies out of $\overline{B}$. Denote by ξ its orthogonal projection on $P_{j,k}$, and observe that $\xi \in \overline{B}$, because it is also the the orthogonal projection of z on $P_{j,k}$, and $z \in \overline{B}$.

Next, $|\pi_{j,k}(x) - \xi| \leq C\eta$ because $\Sigma_{j,k}$ is the graph over $D_{j,k}$ of a $C\eta$-Lipschitz function $G_{j,k}$ which is equal to 0 on the two half lines that bound $D_{j,k}$ (see (8.9) and (8.6)). Hence $|\xi| \geq |\pi_{j,k}(x)| - |\pi_{j,k}(x) - \xi| \geq 1 - C\eta$. But on the region where $9/10 \leq |\xi| \leq 1$, (8.6) and (8.4) say that $G_{j,k}$ is a homogeneous function, so $r \circ \pi_{j,k}(z) \in \Sigma_{j,k}$ because $\pi_{j,k}(z) \in \Sigma_{j,k}$. So (9.34) holds.

We are now ready to prove Lemma 9.4. We take

$$p_2 = r \circ \pi \circ p_1 \tag{9.35}$$

with p_1 defined by (9.14). We know that p_1 is defined on U and take values in $\overline{B} \cap R$ (see (9.14) and (9.17)), so p_2 also is defined on U, by (9.28), and takes value in $\Sigma \cap \overline{B}$, by (9.34). It is 44-Lipschitz because p_1 is 22-Lipschitz by (9.15), π is 2-Lipschitz by (9.30), and r is 1-Lipschitz. We shall check soon that

$$U \text{ is a neighborhood of } E \cap \overline{B} \text{ in } \overline{B}, \tag{9.36}$$

but let us take care of the other properties first. We start with (9.5). Let $z \in \Sigma \cap \overline{B} \setminus B(0, 1/3)$ be given, and let (j, k) be such that $z \in \Sigma_{j,k}$. Then $z \in R_{j,k}$, by the definition (9.7) of $R_{j,k}$, because $\Sigma_{j,k}$ is a $C\eta$-Lipschitz graph over $D_{j,k} \cap \overline{B}$ which contains $\partial D_{j,k} \cap \overline{B}$ (see (8.9) and (8.6)), and if η is small enough. So $p_0(z/|z|) = z/|z|$, by (9.11), and the definition (9.14) says that $p_1(z) = z$, because $m(z) = |z|$ (by (9.13) and because $z \in \overline{B} \setminus B(0, 1/3)$). By (9.28) and (9.24), $\pi(p_1(z)) = \pi(z) = z$, and then $p_2(z) = r(z) = z$ by (9.35) and because $z \in \overline{B}$. So (9.5) holds.

Now we check (9.6). For $z \in U \cap B(0, 1/3)$, $p_1(z) \in R \cap B(0, 1/3)$ because $m(z) \leq |z|$ and p_0 takes values in R (by (9.13) and (9.11)). Then $\pi(p_1(z)) \in \Sigma \cap B(0, 1/2)$ by (9.28), the fact that $\pi_{j,k}$ takes values in $\Sigma_{j,k}$, and (9.27). Finally $p_2(z) = \pi(p_1(z)) \in \Sigma \cap B(0, 1/2)$, by (9.35), and (9.6) follows.

So we are left with (9.36) to prove. Let $x \in E \cap \overline{B}$ be given. Recall from (4.2) that $\text{dist}(x, X) \leq 100\varepsilon$, where X is also the minimal cone which was used in Section 6 to construct the $g_{j,k}$. Thus $\text{dist}(x, |x|\mathfrak{C}_{j,k}) \leq 100\varepsilon$ for some (j, k). Denote by a and b the endpoints of $\mathfrak{C}_{j,k}$; since D contains the geodesic from $\varphi(a)$ to $\varphi(b)$, (6.14) implies that $\text{dist}(x, D) \leq 100\varepsilon + C\tau < C_7\tau$ (if C_7 is large enough), which implies (9.36) (compare with (9.12)). This completes our proof of Lemma 9.4. □

Our next task is to use the mapping p_2 to construct competitors for E in $\overline{B}$.

First let us use the Whitney extension theorem to extend p_2 to $\mathbb{R}^n$ in a Lipschitz way. Since we can compose p_2 with r without changing its values on U (recall that $p_2(U) \subset \overline{B}$), we can do the extension so that $p_2(z) \in \overline{B}$ for all z.

Then let $\xi > 0$ be very small, to be chosen later, define ψ by

$$\begin{cases} \psi(t) = 1 \quad \text{for } 0 \leq t \leq 1 - \xi\,, \\ \psi(t) = 0 \quad \text{for } t \geq 1\,, \\ \psi \quad \text{is affine on } [1 - \xi, 1], \end{cases} \tag{9.37}$$

and set

$$f_1(z) = \psi(|z|)\, p_2(z) + [1 - \psi(|z|)]\, z \quad \text{for } z \in \mathbb{R}^n. \tag{9.38}$$

Observe that f_1 is Lipschitz, and that

$$f_1(z) = z \text{ for } z \in \mathbb{R}^n \setminus B \quad \text{and} \quad f_1(B) \subset \overline{B} \tag{9.39}$$

by (9.38), because $\psi(t) = 0$ for $t \geq 1$, and because $p_2(\overline{B}) \subset \overline{B}$.

Our first competitor for E will be $F_1 = f_1(E)$. Notice that f_1 satisfies the conditions (1.5)-(1.9) that define the admissible deformations, with $\widehat{W} \subset \overline{B}$ and where we set $f_t(z) = tf_1(z) + (1-t)z$ for $t \in [0,1]$ and $z \in \mathbb{R}^n$. So we can apply (1.11), which says that

$$H^2(E \setminus F_1) \leq H^2(F_1 \setminus E) + 4h(2). \tag{9.40}$$

Since E and F_1 coincide out $\overline{B}$, we can add $H^2(E \cap F_1 \cap \overline{B})$ to both sides and get that

$$H^2(E \cap \overline{B}) \leq H^2(F_1 \cap \overline{B}) + 4h(2). \tag{9.41}$$

We shall introduce other competitors later, obtained by deformation of F_1 inside B, so we shall not really use (9.41) directly, but even so it will be useful to estimate $H^2(F_1 \cap \overline{B})$.

Lemma 9.42. *We have that*

$$\limsup_{\xi \to 0_+} H^2(F_1 \cap \overline{B}) \leq H^2(\Sigma \cap B) + 2550 \int_{E \cap \partial B} \operatorname{dist}(z, \Gamma)\, dH^1(z). \tag{9.43}$$

where $\Gamma = \cup_{j,k} \Gamma_{j,k}$ *as in (9.3).*

First observe that

$$f_1(E \cap B(0, 1-\xi)) = p_2(E \cap B(0, 1-\xi)) \subset \Sigma \cap \overline{B} \tag{9.44}$$

by (9.38), because $\psi(t) = 1$ for $t \leq 1 - \xi$, and by Lemma 9.4. Then set

$$A_\xi = B \setminus B(0, 1-\xi). \tag{9.45}$$

If we prove that

$$\limsup_{\xi \to 0_+} H^2(f_1(E \cap A_\xi)) \leq 2550 \int_{E \cap \partial B} \operatorname{dist}(z, \Gamma)\, dH^1(z), \tag{9.46}$$

Lemma 9.42 will follow at once, because $H^1(\Sigma \cap \partial B) < +\infty$ (and hence $H^2(\Sigma \cap \partial B) = 0$) trivially, and also $H^2(f_1(E \cap \partial B)) = 0$, since $H^1(f_1(E \cap \partial B)) < +\infty$ because f_1 is Lipschitz and $H^1(E \cap \partial B) < +\infty$ by (6.2).

We shall use the area formula to compute $H^2(f_1(E \cap A_\xi))$, the coarea formula to estimate the result in terms of an integral on A_ξ, and then (9.46) and the lemma will follow from our additional precautionary assumptions (4.3) and (4.4).

When the ambient dimension is $n = 3$, we could avoid using most of this with a trick. First, we would modify p_2 slightly, so that $p_2(E \cap A_\xi) \subset \partial B$, and then we would also modify (9.38) (by pushing radially on ∂B) so that $f_1(E \cap A_\xi)$ is also contained in ∂B, and in fact in an arbitrary small neighborhood in ∂B of the union of the arcs of geodesics from the points $x \in E \cap \partial B$ to their projections $p_2(x) \in \Gamma$. The area of this small piece of ∂B

around Γ would be easier to estimate, just by covering it with balls centered on $E \cap \partial B \setminus \Gamma$, and (9.46) would follow.

Let us return to the general case and use the area theorem. Recall from Theorem 2.11 of [DS] (also see Section 2 of [D3] for the adaptation to the present setting) that E is rectifiable; the area theorem (Corollary 3.2.20 in [Fe]) says that

$$H^2(f_1(E \cap A_\xi)) \le \int_{E \cap A_\xi} J_{f_1}(z)\, dH^2(z), \tag{9.47}$$

where $J_{f_1}(z)$ denotes the approximate Jacobian of f_1 on E at z.

We need to estimate Jacobians. Denote by $P(z)$ the approximate tangent plane to E at z, which exists for H^2-almost every $z \in E$ (because E is rectifiable). Incidentally, $P(z)$ is even a true tangent plane because E is Ahlfors-regular (see for instance Exercise 41.21 on page 277 of [D2]). Pick an orthonormal basis (v, w) of the vector space parallel to $P(z)$, with v orthogonal to the radial direction $(0, z)$. Denote by $\theta(z) \in [0, \pi/2]$ the (unoriented) angle of $(0, z)$ with w, so that $|\langle w, z\rangle| = |z| \cos\theta(z)$.

We shall compute J_{f_1} at a point $z \in E \cap A_\xi$ where f_1 (or equivalently p_2, since $\psi(|z|)$ is smooth there, and by (9.38)) is differentiable in the direction of $P(z)$. Notice that this is the case almost-everywhere.

In the direction of v, $\psi(|\cdot|)$ has a vanishing differential, and we get that

$$|Df_1(z)(v)| \le \psi(|z|)\, |Dp_2(z)| + [1 - \psi(|z|)] \le 50 \tag{9.48}$$

by (9.38), and because p_2 is 50-Lipschitz by Lemma 9.4.

In the direction of w, we perform a similar computation, but also have a term that comes from the derivative of ψ. The derivative of $\psi(|\cdot|)$ in the direction of w is bounded by $\xi^{-1} \cos\theta(z)$, so we need to add

$$[\xi^{-1} \cos\theta(z)]|p_2(z) - z| \le 51\xi^{-1} \cos\theta(z)\, \mathrm{dist}(z, \Sigma), \tag{9.49}$$

because p_2 is 50-Lipschitz and by (9.5). Thus

$$|Df_1(z)(w)| \le 50 + 51\xi^{-1} \cos\theta(z)\, \mathrm{dist}(z, \Sigma) \tag{9.50}$$

Hence

$$J_{f_1}(z) \le |Df_1(z)(v)|\, |Df_1(z)(w)| \le 2500 + 2550\xi^{-1} \cos\theta(z)\, \mathrm{dist}(z, \Sigma), \tag{9.51}$$

and (9.47) yields

$$\begin{aligned} H^2(f_1(E \cap A_\xi)) &\le \int_{E \cap A_\xi} [2500 + 2550\xi^{-1} \cos\theta(z)\, \mathrm{dist}(z, \Sigma_F)]\, dH^2(z) \\ &= 2500 H^2(E \cap A_\xi) + 2550\xi^{-1} \int_{E \cap A_\xi} \cos\theta(z)\, \mathrm{dist}(z, \Sigma_F)\, dH^2(z). \end{aligned} \tag{9.52}$$

Next let us apply the coarea formula (Theorem 3.2.22 in [Fe]) to the restriction to $E \cap A_\xi$ of the C^1 mapping $h : z \to |z|$, and integrate against the continuous function $z \to \operatorname{dist}(z, \Sigma)$; we get that

$$\begin{aligned} &\int_{E \cap A_\xi} \operatorname{dist}(z, \Sigma) J_h(z)\, dH^2(z) \\ &\qquad = \int_{t \in (1-\xi, 1)} \int_{E \cap \partial B(0,t)} \operatorname{dist}(z, \Sigma)\, dH^1(z)\, dt. \end{aligned} \tag{9.53}$$

Here $J_h(z)$ is a one-dimensional jacobian; it is the largest size of the differential of h applied to a vector of the tangent plane to E at z; if we keep the same basis (v, w) of the vector space parallel to $P(z)$ to compute $J_h(z)$, we get that $J_h(z) = |Dh(z)(w)| = \cos\theta(z)$. We compare to (9.52) and get that

$$\begin{aligned} H^2(f_1(E \cap A_\xi)) &\le 2500 H^2(E \cap A_\xi) \\ &\quad + 2550 \xi^{-1} \int_{t \in (1-\xi, 1)} \int_{E \cap \partial B(0,t)} \operatorname{dist}(z, \Sigma)\, dH^1(z)\, dt. \end{aligned} \tag{9.54}$$

Now we use our assumptions (4.3) and (4.4) (and recall that here $r = 1$); (4.4) implies that $H^2(E \cap A_\xi)$ tends to 0 when ξ tends to 0, and (4.3) (applied to $f(z) = \operatorname{dist}(z, \Sigma)$) says that

$$\lim_{\xi \to 0_+} \xi^{-1} \int_{t \in (1-\xi, 1)} \int_{E \cap \partial B(0,t)} \operatorname{dist}(z, \Sigma)\, dH^1(z)\, dt = \int_{E \cap \partial B} \operatorname{dist}(z, \Sigma)\, dH^1(z). \tag{9.55}$$

So (9.54) implies that

$$\limsup_{\xi \to 0_+} \xi^{-1} H^2(f_1(E \cap A_\xi)) \le 2550 \int_{E \cap \partial B} \operatorname{dist}(z, \Sigma)\, dH^1(z). \tag{9.56}$$

Finally observe that $\Gamma \subset \Sigma$, because (9.1) and (9.3) say that, out of $B(0, 19/20)$, Σ coincides with the cone Γ^* over the union Γ of the curves $\Gamma_{j,k}$ constructed in Section 7. So $\operatorname{dist}(z, \Sigma) \le \operatorname{dist}(z, \Gamma)$ for $z \in E \cap \partial B$, and (9.46) and Lemma 9.42 follow from (9.56). □

We shall end this section with estimates on the right-hand side of (9.43). We start with $\int_{E \cap \partial B} \operatorname{dist}(z, \Gamma)\, dH^1(z)$, so we want to estimate $\operatorname{dist}(z, \Gamma)$ for $z \in E \cap \partial B$, and we shall start with $z \in g = \bigcup_{j,k} g_{j,k}$. Denote by $x_{j,k}$ and $y_{j,k}$ the endpoints of $\mathfrak{C}_{j,k}$ (the order will not matter). The endpoints of $g_{j,k}$ and $\Gamma_{j,k}$ are $\varphi(x_{j,k})$ and $\varphi(y_{j,k})$, by (6.12) and (7.30). We denote by $\rho_{j,k}$ the geodesic in ∂B with the same endpoints (that is, $\varphi(x_{j,k})$ and $\varphi(y_{j,k})$). By (7.31) and the definition (7.4),

$$H^1(\Gamma_{j,k} \setminus g_{j,k}) \le H^1(g_{j,k} \setminus \Gamma_{j,k}) \le C\eta^{-2}[\operatorname{length}(g_{j,k}) - \operatorname{length}(\rho_{j,k})]. \tag{9.57}$$

In addition,

$$\operatorname{length}(g_{j,k}) = H^1(g_{j,k}) \le \operatorname{length}(\mathfrak{C}_{j,k}) + C_4\tau \le \operatorname{length}(\rho_{j,k}) + C\tau \tag{9.58}$$

because $g_{j,k}$ is simple, by (6.43), and because $\mathfrak{C}_{j,k}$ and $\rho_{j,k}$ are geodesics of ∂B, with almost the same endpoints (recall that $\rho_{j,k}$ goes from $\varphi(x_{j,k})$ to $\varphi(y_{j,k})$ and use (6.14)). Thus $H^1(g_{j,k} \setminus \Gamma_{j,k}) \le C\eta^{-2}\tau$, and hence

$$\operatorname{dist}(z, \Gamma_{j,k}) \le C\eta^{-2}\tau \quad \text{for } z \in g_{j,k}, \tag{9.59}$$

because $g_{j,k}$ is connected and has common endpoints with $\Gamma_{j,k}$. Let us also check that

$$\operatorname{dist}(z, \Gamma) \le C\eta^{-2}\tau \quad \text{for } z \in E \cap \partial B. \tag{9.60}$$

Let $z \in E \cap \partial B$ be given. By (4.2), we can find $z_1 \in X$ such that $|z_1 - z| \le 100\varepsilon$. Then $z_2 = z_1/|z_1|$ lies in $X \cap \partial B$ (because X is a cone), and $|z_2 - z| \le 200\varepsilon r$ because $||z_1| - 1| \le 100\varepsilon$. By definitions, z_2 lies in some $\mathfrak{C}_{j,k}$ and it will be enough to prove that $\operatorname{dist}(z_2, \Gamma_{j,k}) \le C\eta^{-2}\tau$, or even, in view of (9.59), that $\operatorname{dist}(z_2, g_{j,k}) \le C\tau$.

By (6.15), $g_{j,k}$ is contained in a little tube of width $C\tau$ around $\mathfrak{C}_{j,k}$. By (6.14), its endpoints lie within $C\tau$ of the endpoints of $\mathfrak{C}_{j,k}$. If z_2 lies within $C\tau$ of an endpoint of $\mathfrak{C}_{j,k}$, we are happy because it lies close to an endpoint of $g_{j,k}$. Otherwise, we know that $g_{j,k}$ crosses the little hyperdisk of radius $C\tau$ perpendicular to $\mathfrak{C}_{j,k}$ at z_2, and we are again happy. This proves that $\operatorname{dist}(z_2, g_{j,k}) \le C\tau$, and (9.60) follows.

Let us now drop the dependence on η from our notation. This does not matter, because we can choose τ and ε very small, depending on η. We obtain that

$$\begin{aligned}
\int_{E\cap\partial B} \operatorname{dist}(z,\Gamma)\, dH^1(z) &\le C\tau H^1(E \cap \partial B \setminus \Gamma) \\
&\le C\tau H^1(E \cap \partial B \setminus g) + C\tau H^1(g \setminus \Gamma) \\
&\le C\tau H^1(E \cap \partial B \setminus g) + C\tau \sum_{j,k} [\operatorname{length}(g_{j,k}) - \operatorname{length}(\rho_{j,k})] \\
&= C\tau H^1(E \cap \partial B \setminus g) + C\tau [H^1(g) - H^1(\rho)],
\end{aligned} \tag{9.61}$$

because $\operatorname{dist}(z, \Gamma) = 0$ on Γ, by (9.60), (9.57), because the $g_{j,k}$ are essentially disjoint, and where we now set

$$\rho = \bigcup_{(j,k)\in\widetilde{J}} \rho_{j,k} \subset \partial B. \tag{9.62}$$

Thus ρ is the union of the geodesics that connect the endpoints of the $g_{j,k}$ and the $\Gamma_{j,k}$, i.e., the $\varphi(x)$, $x \in V$. With the notation of Section 2,

$$\rho = \varphi_*(K), \quad \text{with } K = X \cap \partial B. \tag{9.63}$$

The estimate (9.61) of what we lose in the deformation of E will be enough for our purposes. Let us also look at what we win in (9.43) with the term $H^2(\Sigma \cap B(0,1))$.

Recall from the discussion above (9.3) that $\Sigma = \cup_{j,k} \Sigma_{j,k}$, where $\Sigma_{j,k}$ is the graph of some function $G_{j,k}$ over a sector $\overline{D}_{j,k}$: see above (9.1). The function $G_{j,k}$ was constructed

as in Section 8, starting with the cone $\Gamma^*_{j,k}$ over $\Gamma_{j,k}$, which corresponds to a (homogeneous) function $F_{j,k}$. The various $\Gamma^*_{j,k}$ are essentially disjoint, so (9.2) says that

$$\begin{aligned} H^2(\Sigma \cap B) &\le H^2(\Gamma^* \cap B) - 10^{-4} \sum_{j,k} [\text{length}(\Gamma_{j,k}) - \text{length}(\rho_{j,k})] \\ &\le H^2(\Gamma^* \cap B) - 10^{-4}[H^1(\Gamma) - H^1(\rho)]. \end{aligned} \tag{9.64}$$

Set

$$\delta_1 = H^1(\Gamma) - H^1(\rho) \ge 0 \tag{9.65}$$

(because the $\rho_{j,k}$ are geodesics with the same endpoints as the $\Gamma_{j,k}$),

$$\delta_2 = H^1(g) - H^1(\Gamma) \ge 0 \tag{9.66}$$

(by the first part of (9.57) and because the curves are simple and essentially disjoint), and

$$\delta_3 = H^1(E \cap \partial B) - H^1(g) = H^1(E \cap \partial B \setminus g) \tag{9.67}$$

(because $g \subset E \cap \partial B$): thus

$$H^1(E \cap \partial B) - H^1(\rho) = \delta_3 + \delta_2 + \delta_1. \tag{9.68}$$

We are ready to combine the various estimates. Recall from (9.41) that $H^2(E \cap \overline{B}) \le H^2(F_1 \cap \overline{B}) + 4h(2)$. for every small $\xi > 0$. Then (9.41) and (9.43) say that

$$\begin{aligned} H^2(E \cap \overline{B}) &\le 4h(2) + H^2(\Sigma \cap B) + 2550 \int_{E \cap \partial B} \text{dist}(z, \Gamma)\, dH^1(z) \\ &\le 4h(2) + H^2(\Sigma \cap B) + C\tau H^1(E \cap \partial B \setminus g) + C\tau[H^1(g) - H^1(\rho)] \\ &= 4h(2) + H^2(\Sigma \cap B) + C\tau\delta_3 + C\tau(\delta_1 + \delta_2) \\ &\le 4h(2) + H^2(\Gamma^* \cap B) - 10^{-4}[H^1(\Gamma) - H^1(\rho)] + C\tau(\delta_1 + \delta_2 + \delta_3) \\ &= 4h(2) + \frac{1}{2} H^1(\Gamma) - 10^{-4}\delta_1 + C\tau(\delta_1 + \delta_2 + \delta_3) \\ &= 4h(2) + \frac{1}{2} H^1(E \cap \partial B) - \frac{1}{2}(\delta_2 + \delta_3) - 10^{-4}\delta_1 + C\tau(\delta_1 + \delta_2 + \delta_3) \\ &\le \frac{1}{2} H^1(E \cap \partial B) - 10^{-5}(\delta_1 + \delta_2 + \delta_3) + 4h(2) \\ &= \frac{1}{2} H^1(E \cap \partial B) - 10^{-5}[H^1(E \cap \partial B) - H^1(\rho)] + 4h(2) \end{aligned} \tag{9.69}$$

by (9.61), (9.64), and because Γ^* is the cone over Γ, if τ is small enough, and by (9.68).

Observe that (9.69) is the same thing as (4.7), because we reduced to $x = 0$ and $r = 1$, and by (9.63). We also have (4.6), because here $H^1(\varphi_*(K)) = H^1(\rho) = H^1(E \cap \partial B) - \delta_1 - \delta_2 - \delta_3$ by (9.68). So it will be enough to establish (4.8), and Theorem 4.5 will follow.

Notice that if by chance $H^1(\rho) \leq 2d(0)$, then (9.69) says that

$$H^2(E \cap B) \leq \frac{1}{2} H^1(E \cap \partial B) - 10^{-5}[H^1(E \cap \partial B) - 2d(0)] + 4h(2) \tag{9.70}$$

so (4.8) holds with $\alpha = 10^{-5}$. We cannot expect this to happen automatically, but we shall see in the next section that otherwise, and if X is a full minimal length cone such that $H^2(X \cap B) \leq d(0)$, the cone over ρ is far from being minimal, and we can replace our first competitor F_1 with a better one, improve (9.69), and get (4.8).

10. Our second competitor and the proof of Theorem 4.5

Let us first summarize the situation so far. We started form a reduced almost minimal set E in U and a ball $B(x, r)$ centered on E that satisfies the assumptions (4.1)-(4.4), reduced by translation and dilatation to $x = 0$ and $r = 1$, did some constructions, and eventually obtained a competitor F_1 for E in B, and proved (4.6), (4.7), and (9.69). In the special case when $H^1(\rho) \leq 2d(0)$, (9.69) implies (4.8) and we are finished. Otherwise, when

$$H^1(\rho) > 2d(0), \tag{10.1}$$

we still need to do some work, and this is where we shall use the additional assumptions (about full length minimal cones) in Theorem 4.5. But in the mean time, let us try to continue the construction, improve F_1 neat its tip, and get a slightly stronger variant of (9.69). We do this and state an intermediate result under the current assumptions (4.1)-(4.4), because we shall also use the result in Section 11. We keep the notations of Section 9, and in particular we continue to denote by $\varphi_*(X)$ the cone over ρ, as in (9.63) and near (2.8). The next lemma says that if we can find a deformation $\widetilde{X}$ of $\varphi_*(X)$ in B, with a smaller Hausdorff measure, we can plug it at the tip of F_1, get a better competitor F_2, and improve on (9.69).

Lemma 10.2. *Suppose we can find a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f(x) = x \text{ for } x \in \mathbb{R}^n \setminus B \text{ and } f(B) \subset B, \tag{10.3}$$

and such that if we set $\widetilde{X} = f(\varphi_*(X))$,

$$H^2(\widetilde{X} \cap B) \leq H^2(\varphi_*(X) \cap B) - A \tag{10.4}$$

for some $A > 0$, then

$$H^2(E \cap \overline{B}) \leq \frac{1}{2} H^1(E \cap \partial B) - 10^{-5}[H^1(E \cap \partial B) - H^1(\rho)] - \kappa^2 A + 4h(2). \tag{10.5}$$

We shall not use (10.1) or the full length assumption in the proof. So we want to use f and $\widetilde{X}$ to improve our competitor F_1 from Section 9. Let us first check that

$$F_1 \cap B(0, \kappa) \subset \Sigma \cap B(0, \kappa) = \varphi_*(X) \cap B(0, \kappa). \tag{10.6}$$

For the inclusion, recall that $F_1 = f_1(E)$. Since (9.44) says that $f_1(E \cap B(0, 1-\xi)) \subset \Sigma$, we just need to check that $f_1(z) \notin B(0, 1/2)$ when $z \in E \setminus B(0, 1-\xi)$. This is clear when $z \in E \setminus B$, because then $f_1(z) = z$ by (9.39). Otherwise, $f_1(z)$ lies somewhere between $p_2(z)$ and z, by (9.38) and (9.37), and it is enough to check that $|p_2(z) - z| \leq 1/3$. By Lemma 9.4, p_2 is 50-Lipschitz and $p_2(z) = z$ for $z \in \Sigma \cap \overline{B} \setminus B(0, 1/3)$, so it is enough to check that $\operatorname{dist}(z, \Sigma \cap \overline{B}) \leq 10^{-3}$.

Recall from the beginning of Section 9 that Σ is the union of the $\Sigma_{j,k}$ (by (9.3)), and that $\Sigma_{j,k}$ is the graph over $D_{j,k}$ of some $C\eta$-Lipschitz function $G_{j,k}$ (see above (9.1)). In addition, $G_{j,k}$ vanishes on $\partial D_{j,k} \cap B$ (or equivalently, $\Sigma_{j,k}$ contains $\partial D_{j,k} \cap B$), by (8.6), (8.4), and the line above (8.2). Then every point of $D_{j,k} \cap B$ lies within $C\eta$ from $\Sigma \cap B$.

Now $z \in U$ by (9.36), and hence $\operatorname{dist}(z, D_{j,k}) = \operatorname{dist}(z, D_{j,k} \cap B) \leq C_7\tau$ for some $(j,k) \in \widetilde{J}$, by (9.12) and the line below. Then $\operatorname{dist}(z, \Sigma_{j,k} \cap \overline{B}) \leq C_7\tau + C\eta$, as needed for the inclusion in (10.6).

For the second part of (10.6), recall from (8.7) that $G_{j,k}(z) = 0$ when $|z| \leq 2\kappa$; then

$$\Sigma_{j,k} \cap B(0, \kappa) = D_{j,k} \cap B(0, \kappa) \tag{10.7}$$

(recall that $G_{j,k}$ is $C\eta$-Lipschitz by (8.9)). Denote by a and b the endpoints of $\mathfrak{C}_{j,k}$; then the endpoints of $\Gamma_{j,k}$ are $\varphi(a)$ and $\varphi(b)$ (see below (9.1), for instance), so $D_{j,k}$ coincides in $B(0, \kappa)$ with the cone over the geodesic from $\varphi(a)$ to $\varphi(b)$, which is also denoted by $\varphi_*(\mathfrak{C}_{j,k})$ in (2.8). Since $\varphi_*(X)$ is the cone over $\varphi_*(K) = \bigcup_{j,k} \varphi_*(\mathfrak{C}_{j,k})$, (10.7) implies that $\Sigma \cap B(0, \kappa) = \varphi_*(X) \cap B(0, \kappa)$. This completes our proof of (10.6).

Now we can glue the deformation $\widetilde{X} = f(\varphi_*(X))$ at the tip of F_1. Set

$$f^\kappa(z) = \kappa f(\kappa^{-1} z) \ \text{ for } z \in \mathbb{R}^n, \tag{10.8}$$

and notice that

$$f^\kappa(z) = z \text{ when } |z| \geq \kappa \text{ and } f^\kappa(z) \in B(0, \kappa) \text{ otherwise;} \tag{10.9}$$

then set $f_2 = f^\kappa \circ f_1$. Observe that f_2 is Lipschitz, and f_2 is the endpoint of a one-parameter family of functions that satisfy (1.5)-(1.9) with $\widehat{W} \subset \overline{B}$, just by (10.9) and because f_1 has this property (see below (9.39)). So we may apply (1.11) with $F_2 = f_2(E)$, just as we did for (9.40) and (9.41), and get that

$$H^2(E \setminus F_2) \leq H^2(F_2 \setminus E) + 4h(2) \tag{10.10}$$

and, since E and F_2 coincide out $\overline{B}$,

$$H^2(E \cap \overline{B}) \leq H^2(F_2 \cap \overline{B}) + 4h(2). \tag{10.11}$$

We want to bound the right-hand side of (10.11) slightly better than what we did in Section 9. We do not need to modify our estimate of $H^2(F_1 \cap \overline{B} \setminus B(0, \kappa))$, because (10.9) says that $F_2 = F_1$ out of $B(0, \kappa)$. On $B(0, \kappa)$, we simply used (9.44) to say that

$$H^2(F_1 \cap B(0, \kappa)) \leq H^2(\Sigma \cap B(0, \kappa)) = H^2(\varphi_*(X) \cap B(0, \kappa)) = \kappa^2 H^2(\varphi_*(X) \cap B), \tag{10.12}$$

by (10.6) and because $\varphi_*(X)$ is a cone. Here we say instead that

$$\begin{aligned} F_2 \cap B(0,\kappa) &= f^\kappa(F_1) \cap B(0,\kappa) = f^\kappa(F_1 \cap B(0,\kappa)) \subset f^\kappa(\varphi_*(X) \cap B(0,\kappa)) \\ &= \kappa f(\kappa^{-1}[\varphi_*(X) \cap B(0,\kappa)]) = \kappa f(\varphi_*(X) \cap B) \\ &= \kappa[f(\varphi_*(X)) \cap B] = \kappa[\widetilde{X} \cap B] \end{aligned} \tag{10.13}$$

by (10.6) and (10.9), because $\varphi_*(X)$ is a cone, by (10.3), and by definition of $\widetilde{X}$. This yields

$$H^2(F_2 \cap B(0,\kappa)) \le \kappa^2 H^2(\widetilde{X} \cap B) \le \kappa^2 H^2(\varphi_*(X) \cap B) - A\kappa^2 \tag{10.14}$$

by (10.4). In other words, compared with (10.12), we won $A\kappa^2$. Recall that we obtained in (9.69) that

$$H^2(E \cap \overline{B}) \le \frac{1}{2} H^1(E \cap \partial B) - 10^{-5}[H^1(E \cap \partial B) - H^1(\rho)] + 4h(2); \tag{10.15}$$

then the extra $A\kappa^2$ yields (10.5), and Lemma 10.2 follows. □

Return to the proof of Theorem 4.5. We need to prove (4.8) in the remaining case when (10.1) holds, and now we shall use the extra assumptions in Theorem 4.5, namely, that $H^2(X \cap B) \le d(0)$ and that X is a full length minimal cone with constants C_1 and η_1. Then

$$H^1(K) = H^1(X \cap \partial B) = 2H^2(X \cap B) \le 2d(0) < H^1(\rho) = H^1(\varphi_*(K)) \tag{10.16}$$

because $K = X \cap \partial B$, X is a cone, and by (10.1) and (9.63). Also notice that $\varphi \in \Phi(\eta_1)$ (where $\Phi(\eta_1)$ is as in (2.7)), by (6.14) and if τ is small enough. So Definition 2.10 says that there is a deformation $\widetilde{X}$ of $\varphi_*(X)$ in B such that

$$H^2(\widetilde{X} \cap B) \le H^2(\varphi_*(X) \cap B) - C_1^{-1}[H^1(\varphi_*(K)) - H^1(K)]. \tag{10.17}$$

That is, there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that (10.3) holds and for which $\widetilde{X} = f(\varphi_*(X))$ satisfies (10.17).

Thus we can apply Lemma 10.2 and get (10.5) with

$$A = C_1^{-1}[H^1(\varphi_*(K)) - H^1(K)] = C_1^{-1}[H^1(\rho) - H^1(K)] \tag{10.18}$$

by (9.63). Set $\alpha = C_1^{-1}\kappa^2$; we can safely assume that $\alpha < 10^{-5}$, so (10.5) also holds with 10^{-5} replaced with α, because $H^1(E \cap \partial B) - H^1(\rho) = \delta_1 + \delta_2 + \delta_3 \ge 0$ by (9.68). Thus we get that

$$\begin{aligned} H^2(E \cap \overline{B}) &\le \frac{1}{2} H^1(E \cap \partial B) - \alpha[H^1(E \cap \partial B) - H^1(\rho)] - \kappa^2 A + 4h(2) \\ &= \frac{1}{2} H^1(E \cap \partial B) - \alpha[H^1(E \cap \partial B) - H^1(K)] + 4h(2) \\ &\le \frac{1}{2} H^1(E \cap \partial B) - \alpha[H^1(E \cap \partial B) - 2d(0)] + 4h(2) \end{aligned} \tag{10.19}$$

by (10.5), (10.18), and (10.16). Since here $r = 1$ and $x = 0$, we just established (4.8). This completes our proof of Theorem 4.5. □

To end this section, we now explain how we can construct competitors for cones (to be used in Lemma 10.2 or just to prove that a given cone has the full length property).

We keep the same notation as in Section 2. That is, X is a minimal cone, we set $K = X \cap \partial B$, and we decompose K as a union of arcs $\mathfrak{C}_{j,k}$, $(j,k) \in \widetilde{J}$, that connect vertices $x \in V$. Then we pick $\varphi \in \Phi(\eta_1)$, with $\eta_1 < \eta_0/10$ (as in (2.7)), and define $\varphi_*(K) = \bigcup_{(j,k)\in\widetilde{J}} \varphi_*(\mathfrak{C}_{j,k}) \subset \partial B$ and the cone $\varphi_*(X)$ over $\varphi_*(K)$ as we did near (2.8). The set ρ of (9.62) in Section 9 was a typical example of this. We are interested in finding better competitor for $\varphi_*(X)$ in B, when this is easy to do.

For each vertex $x \in V$, we define a deviation $\alpha_\varphi(x)$ from the ideal angles at $\varphi(x)$ as follows.

First suppose that $x \in V_0$; thus there are three curves $\mathfrak{C}_{j,k}$ which have x as one of their endpoints. Denote by w_1, w_2, and w_3 the unit tangent vectors to the three $\mathfrak{C}_{j,k}$ (pointing away from $\varphi(x)$, and set

$$\alpha_\varphi(x) = |w_1 + w_2 + w_3| \tag{10.20}$$

Of course $\alpha_\varphi(x) = 0$ when the three $\mathfrak{C}_{j,k}$ make 120° angles at $\varphi(x)$; $\alpha_\varphi(x)$ is not exactly equivalent to the maximum deviation from 120° of the mutual angles of the w_j, because when w_1 and w_2 make a 120° angle and $w_1 + w_2 + w_3$ is almost orthogonal to the plane that contains w_1 and w_2, the angle deviation is of the order of $|w_1+w_2+w_3|^2$. A definition with angles would not be precise enough in this case.

When $x \in V_1$, there are only two arcs $\mathfrak{C}_{j,k}$ leaving from x, we denote by w_1 and w_2 their tangent direction at $\varphi(x)$, and by $\theta \in (0, \pi]$ the angle of w_1 and w_2, with the convention that θ is close to π, and we set

$$\alpha_\varphi(x) = \pi - \theta. \tag{10.21}$$

In this case, there is no difficulty, $\alpha_\varphi(x)$ is equivalent to $|w_1 + w_2|$, and we decided to take the "simpler" definition. Finally we set

$$\alpha_+(\varphi) = \sup_{x\in V} \alpha_\varphi(x). \tag{10.22}$$

Since we assumed that $\varphi \in \Phi(\eta_1)$ with η_1 small, we still get that $\alpha_+(\varphi)$ is small (because our description of minimal cones readily gives $\alpha_+(\varphi) = 0$ when φ is the identical map). The next lemma says that when $\alpha_+(\varphi) \neq 0$, we can win $C^{-1}\alpha_+(\varphi)^2$ by deforming $\varphi_*(X)$ in B.

Lemma 10.23. *Let X, $\varphi \in \Phi(\eta_1)$, and $\alpha_+(\varphi)$ be as above. We can find a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that (10.3) holds, and*

$$H^2(\widetilde{X} \cap B) \leq H^2(\varphi_*(X) \cap B) - C^{-1}\alpha_+(\varphi)^2 \tag{10.24}$$

for $\widetilde{X} = f(\varphi_*(X))$. Here C depends only on n and the constant η_0 in Section 2 (associated to the description of the minimal cones).

To prove the lemma, we first select $x \in V$ such that $\alpha_\varphi(x) = \alpha_+(\varphi)$. We we shall first assume that $x \in V_0$ (but the other case is simpler).

Choose coordinates so that the first axis is in the direction of $\varphi(x)$, and hence $\varphi(x) = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Our mapping f will be of the form

$$f(z) = z + \psi(z_1, \rho)\, v, \tag{10.25}$$

where $\rho = (z_2^2 + \cdots + z_n^2)^{1/2}$, and $v = (0, v_2, \cdots, v_n)$ is a small vector perpendicular to the direction of $\varphi(x)$. We choose the nonnegative smooth bump function ψ so that

$$\psi \text{ is supported in } V = [1/4, 1/2] \times [0, 10^{-1}\eta_0], \tag{10.26}$$

where η_0 is as in (2.2) and (2.3),

$$0 \le \psi \le 1 \ \text{ and } \ \partial_2\psi =: \frac{\partial \psi}{\partial r} \le 0 \ \text{ everywhere,} \tag{10.27}$$

$$\psi(z_1, 0) = 1 \text{ for } |z_1 - 1/3| \le 1/10, \tag{10.28}$$

and

$$|\nabla \psi| \le 100\eta_0^{-1} \text{ everywhere.} \tag{10.29}$$

We shall choose v such that $|v| \le 200^{-1}\eta_0$. Then (10.3) holds automatically, and also f is a bijection, for instance because $|Df - Id| < 1/2$ everywhere.

We intend to compute $H^2(f(\varphi_*(X) \cap B))$ with the area formula. We may restrict the computation to $W = \{z \in \mathbb{R}^n\,;\, (z_1, \rho) \in V\}$, because $f(z) = z$ out of W. Let us check that in W, $\varphi_*(X)$ is composed of three faces P_1, P_2, and P_3, which are half-planes that make angles that are close to 120°, and have a common boundary, the half line through $\varphi(x)$. Since $\varphi_*(X)$ is the cone over $\varphi_*(K)$, it is enough to check that in $\partial B \cap B(\varphi(x), 4\eta_0/10)$, $\varphi_*(K)$ is composed of three arcs of great circles that leave from $\varphi(x)$ with angles that are close to 120°. This last follows from the description of X in Section 2 (and in particular (2.2) and (2.3)), and the fact that $\varphi \in \Phi(\eta_1)$ for some $\eta_1 < \eta_0/10$. The vector plane parallel to P_j is spanned by $\varphi(x)$ and the tangent vector w_j at $\varphi(x)$ to the arc of $\varphi_*(K)$ that corresponds to P_j. Thus the three w_j are orthogonal to $\varphi(x)$, and they make with each other angles that are close to 120°.

We shall do the area computation one face at a time, so let us take care of $P_1 \cap W$. Let us chose our coordinates so that $w_1 = (0, 1, 0 \cdots, 0)$. Also set $e_1 = \varphi(x)$ for convenience. We need to compute the differential of f in directions of e_1 and w_1. Notice that $\rho = z_2$ on F_1, so $f(z) = z + \psi(z_1, z_2)\, v$, and the derivatives are $Df(e_1) = e_1 + \partial_1\psi(z_1, z_2)\, v$ and $Df(w_1) = w_1 + \partial_2\psi(z_1, z_2)\, v$.

Next write $v = \beta w_1 + v_3$, where the two first coordinates of v_3 vanish (recall that v is orthogonal to e_1). Then

$$\begin{aligned} Df(e_1)\wedge Df(w_1) &= [e_1 + \partial_1\psi(z_1,z_2)\, v] \wedge [w_1 + \partial_2\psi(z_1,z_2)\, v] \\ &= e_1 \wedge w_1 + [\partial_2\psi(z_1,z_2)\, e_1 - \partial_1\psi(z_1,z_2)\, w_1] \wedge v \\ &= e_1 \wedge w_1 + [\partial_2\psi(z_1,z_2)\, e_1 - \partial_1\psi(z_1,z_2)\, w_1] \wedge [\beta w_1 + v_3] \\ &= [1 + \beta\partial_2\psi(z_1,z_2)]\, e_1 \wedge w_1 + \partial_2\psi(z_1,z_2)\, e_1 \wedge v_3 - \partial_1\psi(z_1,z_2)\, w_1 \wedge v_3 \end{aligned} \tag{10.30}$$

and the jacobian determinant of the restriction of f to F_1 is

$$\begin{aligned} J_1(z) &= |Df(e_1) \wedge Df(w_1)| \\ &= \big\{[1 + \beta\partial_2\psi(z_1,z_2)]^2 + \partial_2\psi(z_1,z_2)^2|v_3|^2 + \partial_1\psi(z_1,z_2)^2|v_3|^2\big\}^{1/2} \\ &\le 1 + \beta\partial_2\psi(z_1,z_2) + C|v|^2 \end{aligned} \tag{10.31}$$

by (10.29) and because $|v|^2 = \beta^2 + |v_3|^2$. Notice also that $\beta = \langle v, w_1\rangle$; hence, when we apply the area formula to compare $H^2(f(P_1 \cap W))$ to $H^2(P_1 \cap W)$, we get that

$$\begin{aligned} H^2(f(P_1 \cap W)) - H^2(P_1 \cap W) &= \int_{P_1\cap W} [J_1(z) - 1]\, dz \\ &\le \int_{P_1\cap W} \big[\beta\partial_2\psi(z_1,\rho) + C|v|^2\big]\, dz \\ &\le \langle v, w_1\rangle \int_{P_1\cap W} \partial_2\psi(z_1,\rho) + C|v|^2 H^2(P_1 \cap W) \\ &\le \langle v, w_1\rangle \int_{P_1\cap W} \partial_2\psi(z_1,\rho) + C|v|^2. \end{aligned} \tag{10.32}$$

We used some coordinates to prove this, but the final result is stated in a more invariant way, and is also valid for the faces P_2 and P_3. The integral $a = -\int_{P_j\cap W} \partial_2\psi(z_1,\rho)$ has the same value for $j = 1, 2, 3$ because of the radial symmetry of ψ, and then

$$\begin{aligned} a &= -\int_{P_j\cap W} \partial_2\psi(z_1,\rho)\, dz \ge -\int_{|z_1-1/3|\le 1/10} \int_0^{10^{-1}\eta_0} \partial_2\psi(z_1,\rho) d\rho dz_1 \\ &= \int_{|z_1-1/3|\le 1/10} [\psi(z_1,0) - \psi(z_1,10^{-1}\eta_0)] dz_1 = \int_{|z_1-1/3|\le 1/10} dz_1 = 1/5 \end{aligned} \tag{10.33}$$

by (10.27), (10.26), and (10.28). The analogue of (10.32) for the face P_j is

$$H^2(f(P_j \cap W)) - H^2(P_j \cap W) \le -a\langle v, w_j\rangle + C|v|^2, \tag{10.34}$$

and when we sum over j we get that

$$\begin{aligned} H^2[f(\varphi_*(X)) \cap B] - H^2[\varphi_*(X) \cap B] &= H^2[f(\varphi_*(X)) \cap W] - H^2[\varphi_*(X) \cap W] \\ &= \sum_j [H^2(f(P_j \cap W)) - H^2(P_j \cap W)] \le -a\,\langle v, w_1 + w_2 + w_3\rangle + C|v|^2 \end{aligned} \tag{10.35}$$

because f is bijective and $f(z) = z$ out of W and by (10.34). Set $s = w_1 + w_2 + w_3$, and choose $v = cs$, where c is a small positive constant. If c is small enough, our constraint that $|v| \le 200^{-1}\eta_0$ is satisfied. Also, $a\langle v, w_1 + w_2 + w_3\rangle = ac|s|^2$, while $C|v|^2 = Cc^2|s|^2 < ac|s|^2/2$ (if $cC \le a/2$), so (10.35) says that

$$\begin{aligned} H^2(f(\varphi_*(X)) \cap B) - H^2(\varphi_*(X) \cap B) &\le -\frac{ac}{2}\,|s|^2 \le -\frac{c}{10}\,|s|^2 \\ &= -\frac{c}{10}\alpha_\varphi(x)^2 = -\frac{c}{10}\alpha_+(\varphi)^2 \end{aligned} \tag{10.36}$$

by (10.20) and our choice of x. Thus (10.24) holds in this case.

Let us now assume that $x \in V_1$. Denote by w_1 and w_2 the unit tangent vectors to the two arcs of $\varphi_*(K)$ that leave from $\varphi(x)$. Thus $\alpha_+(\varphi) = a_\varphi(x) = \pi - \theta$, where $\theta \in (0, \pi]$ denotes the angle between w_1 and w_2 (see (10.21)). As before, θ is close to π because $\varphi \in \Phi(\eta_1)$.

We may now repeat the same proof as above, with the same formula for f, except that now we only have two faces P_1 and P_2 with the same boundary. We do the same computations as before, and we still get that

$$H^2(f(\varphi_*(X)) \cap B) - H^2(\varphi_*(X) \cap B) \le -\frac{c}{10}\,|s|^2, \tag{10.37}$$

but this time with $s = w_1 + w_2$. It is very easy to see that $|s| \ge \alpha_+(\varphi)/2 = (\pi - \theta)/2$, and so (10.24) holds in this case too.

Note that in this simpler case we could also have used the computations of Section 6, applied to a Lipschitz graph composed of two arcs of geodesics, to construct f.

So we proved (10.24) in both cases, and Lemma 10.23 follows. □

11. Approximation of E by cones

So far we focused our energy on proving decay estimates for the density $\theta(r)$. Now we want to show that this density controls the geometric behavior of E. In this section, we show that it controls the Hausdorff distance to minimal cones. In the next one, we shall explain how to deduce $C^{1+\alpha}$ estimates from this.

In this section too we are given a reduced almost-minimal set E in $U \subset \mathbb{R}^3$, with gauge function h, and a point $x \in E$. As before, we set

$$\theta(r) = r^{-2}H^2(E \cap B(x,r)),\ d(x) = \lim_{r\to 0}\theta(r), \text{ and } \ f(r) = \theta(r) - d(x). \tag{11.1}$$

Our main task will be to use the smallness of f to show that E is well approximated by certain types of cones.

Let us be more specific. We start from a minimal cone X, and cut $K = X \cap \partial B(0,1)$ into arcs of circles $\mathfrak{C}_{j,k}$, $(j,k) \in \widetilde{J}$, as we did in Section 2. Thus the $\mathfrak{C}_{j,k}$ are arcs of great circles. The endpoints of these arcs lie in a finite set $V = V_0 \cup V_1$, where V_0 is the initial set of vertices where three arcs $\mathfrak{C}_{j,k}$ meet with 120° angles, and V_1 is the set of added vertices where only two $\mathfrak{C}_{j,k}$ meet, with 180° angles.

We shall approximate E first by perturbations of X by mappings $\varphi \in \Phi(\eta_1)$, with $\eta_1 \le \eta_0/10$ (see near (2.7)), which means that we shall use the sets $\varphi_*(K)$ and $\varphi_*(X)$. Recall that $\varphi_*(K)$ is obtained from K by replacing each arc $\mathfrak{C}_{k,k}$ with endpoints a and $b \in V$ with the geodesic from $\varphi(a)$ to $\varphi(b)$, and that $\varphi_*(X)$ denotes the cone over $\varphi_*(K)$.

We shall denote by $\mathcal{Z}_0(X, \eta_1)$ the collection of cones $\varphi_*(X)$ obtained this way, and by $\mathcal{Z}(X, \eta_1)$ the set of images of cones of $\mathcal{Z}_0(X, \eta_1)$ by translations.

For $Z \in \mathcal{Z}(X, \eta_1)$, we set $\alpha_+(Z) = \alpha_+(\varphi)$, where $\varphi \in \Phi(\eta_1)$ is such that Z is a translation of $\varphi_*(X)$, and $\alpha_+(\varphi)$ is defined in (10.20)-(10.22). Thus $\alpha_+(Z)$ measures the largest difference between the position of the tangent vectors to $\varphi_*(K)$ at a vertex $\varphi(x)$ and the standard position.

We denote by $\mathcal{Z}$ the union of the $\mathcal{Z}(X, \eta_1)$, where X is a minimal cone, and $\eta_1 < \eta_0/10$ (again with η_0 as in (2.2)-(2.6)). Finally we set

$$\beta_Z(x, r) = \inf \big\{ d_{x,r}(E, X) + \alpha_+(Z) \, ; \, X \in \mathcal{Z} \text{ is centered at } x \big\} \tag{11.2}$$

for $x \in E$ and $r > 0$, where

$$\begin{aligned} d_{x,r}(E, Z) = r^{-1} \sup \big\{ \operatorname{dist}(x, Z) \, ; \, x \in E \cap B(x, r) \big\} \\ + r^{-1} \sup \big\{ \operatorname{dist}(x, E) \, ; \, x \in Z \cap B(x, r) \big\}. \end{aligned} \tag{11.3}$$

Theorem 11.4. *There is a positive constant C such that if E is a reduced almost minimal set in U with gauge function h, $x \in E$, and $r_0 > 0$ are such that $B(x, 110r_0) \subset U$, then*

$$\beta_Z(x, r_0) \le C f(110r_0)^{1/3} + C h_1(110r_0)^{1/3}. \tag{11.5}$$

See (1.16) for the definition of $h_1(t)$. The power $1/3$ is probably not optimal; $1/2$ looks more plausible but seems to require more work too. Anyway, if h_1 is small enough, we proved that $f(r)$ decays like a power of r when r tends to 0, so Theorem 11.4 says that the normalized Hausdorff distance $\beta_Z(x, r)$ tends to 0 like a power too. Later on, we shall use this to control the normalized Hausdorff distance to minimal cones, but for the moment we only care about cones in $\mathcal{Z}$.

The proof of Theorem 11.4 will keep us busy for the rest of this section. Let x and r_0 be as in the statement. We can assume that $x = 0$ (by translation invariance), and that

$$f(110r_0) + h_1(110r_0) \le \varepsilon_1, \tag{11.6}$$

with ε_1 as small as we want, because otherwise (11.5) holds with $C = 2\varepsilon_1^{-1/3}$. Let us apply Proposition 7.24 in [D3]; observe that $\theta(110r_0) = d(0) + \varepsilon_1 \le \inf_{0<t<11r_0/10} \theta(t) + C\varepsilon_1$ by (3.8) and (11.6), so we can apply the proposition, and find a minimal cone X centered at 0 such that

$$d_{0,100r_0}(E, X) \le \varepsilon, \tag{11.7}$$

where $\varepsilon > 0$ is any small constant given in advance, and if ε_1 is small enough. We choose ε as in Theorem 4.5 above, which will allow us to apply the construction of Sections 6-10.

Observe that

$$f(r) + h_1(r) \le C\varepsilon_1 \quad \text{for } 0 < r \le 60r_0, \tag{11.8}$$

because h_1 is nondecreasing, and by (3.6). Set $v(r) = H^2(E \cap B(0,r))$, and denote by $\mathcal{R}$ the set of radii $r \in (0, 2r_0)$ such that θ and v are differentiable at r,

$$\theta'(r) = r^{-2}v'(r) - 2r^{-3}v(r) \ \text{ and } \ v'(r) \ge H^1(E \cap \partial B(0,r)), \tag{11.9}$$

and in addition (4.3) and (4.4) hold. Then

$$H^1((0,2r_0) \setminus \mathcal{R}) = 0 \tag{11.10}$$

by Lemma 5.1, Lemma 5.5, (5.8), and Lemma 4.12. We shall get better estimates when we consider radii $r \in \mathcal{R}$ such that $j(r)$ is small, where we set

$$j(r) = rf'(r) + f(r) + h_1(r) = r\theta'(r) + f(r) + h_1(r) + h(2r). \tag{11.11}$$

Our first task will be to fix $r \in \mathcal{R}$ and control the geometry of $E \cap \partial B(0,r)$ in terms of $j(r)$. Observe that

$$H^1(E\cap\partial B(0,r)) \le v'(r) = r^2\theta'(r)+2r^{-1}v(r) = r^2\theta'(r)+2r\theta(r) \le 2rj(r)+2rd(0) \tag{11.12}$$

for $r \in \mathcal{R}$, by (11.9) and because $\theta(r) = f(r) + d(0)$. A first consequence of this is that if we set

$$\mathcal{R}_1 = \big\{r \in \mathcal{R}\,;\, j(r) \le \tau d(0)\big\}; \tag{11.13}$$

then $H^1(E\cap B(0,r)) \le 2(1+\tau)rd(0)$ for $r \in \mathcal{R}_1$. This is the same thing as (6.2), except that here we did not set $r = 1$. Thus we checked that for $r \in \mathcal{R}_1 \cap (r_0/10, 2r_0)$, the analogue when $r \ne 1$ of the assumptions (4.1)-(4.4), and (6.2) are satisfied, that were needed to construct the competitors F_1 and F_2 in Sections 6-10. This allows us to apply Lemma 10.2 to the set $r^{-1}E$, with $A = C^{-1}\alpha_+(\varphi)^2$ coming from Lemma 10.23, and where φ is still as in Lemma 6.11 and the definition of the cone $\rho = \varphi_*(K)$. Recall that Lemma 10.2 itself does not use (10.1) or the full length assumption (see below its statement). Now (10.5) yields

$$\begin{aligned} H^2(r^{-1}E \cap B) \le{}& \frac{1}{2}H^1(r^{-1}E \cap \partial B) \\ &- 10^{-5}[H^1(r^{-1}E \cap \partial B) - H^1(\varphi_*(K))] - \kappa^2 A + 4h(2r) \end{aligned} \tag{11.14}$$

or equivalently

$$\begin{aligned} H^2(E \cap B(0,r)) \le{}& \frac{r}{2}H^1(E \cap \partial B(0,r)) \\ &- 10^{-5}r[H^1(E \cap \partial B(0,r)) - rH^1(\varphi_*(K))] - C^{-1}\alpha_+(\varphi)^2r^2 + 4r^2h(2r), \end{aligned} \tag{11.15}$$

where we drop the dependence on κ from our notation. Define the δ_i, $1 \le i \le 3$, as in (9.65)-(9.67), but for the set $r^{-1}E$. Then $H^1(E \cap \partial B(0,r)) - rH^1(\varphi_*(K)) = r(\delta_1 + \delta_2 + \delta_3)$ by (9.68) and (9.63), and (11.15) says that

$$\begin{aligned} H^2(E \cap B(0,r)) &\le \frac{r}{2} H^1(E \cap \partial B(0,r)) - 10^{-5} r^2 [\delta_1 + \delta_2 + \delta_3] - C^{-1} r^2 \alpha_+(\varphi)^2 + 4r^2 h(2r) \\ &\le r^2 j(r) + r^2 d(0) - 10^{-5} r^2 [\delta_1 + \delta_2 + \delta_3] - C^{-1} r^2 \alpha_+(\varphi)^2 + 4r^2 h(2r) \end{aligned} \tag{11.16}$$

by (11.12). Thus

$$\begin{aligned} 10^{-5}(\delta_1 + \delta_2 + \delta_3) + C^{-1}\alpha_+(\varphi)^2 &\le j(r) + d(0) - r^{-2} H^2(E \cap B(0,r)) \\ &= j(r) + d(0) - \theta(r) + 4h(2r) \end{aligned} \tag{11.17}$$

and, since $\theta(r) \ge d(0) - Ch_1(r)$ by (3.8), we get that

$$\delta_1 + \delta_2 + \delta_3 + \alpha_+(\varphi)^2 \le Cj(r). \tag{11.18}$$

Let us first take care of the union $g = \cup_{j,k} g_{j,k}$, where the $g_{j,k}$ are the simple arcs constructed in in Section 6 for $r^{-1}E$. For the moment we do this for a fixed $r \in \mathcal{R}_1$; later on we shall connect the information coming from different radii, and get control on E itself.

Lemma 11.19. *There is a cone $X(r) \in \mathcal{Z}_0(X, C\tau)$ centered at 0, such that $\alpha_+(Z) \le Cj(r)^{1/2}$ and*

$$\begin{aligned} &\operatorname{dist}(x,g) \le Cj(r)^{1/2} \ \text{ for } x \in X(r) \cap \partial B \ \text{ and} \\ &\qquad \operatorname{dist}(x, X(r) \cap \partial B) \le Cj(r)^{1/2} \ \text{ for } x \in g. \end{aligned} \tag{11.20}$$

Here X is as in (11.7), and τ is chosen as in Section 6, and in particular Lemma 6.12. For the proof, still denote by $\rho_{j,k}$ the geodesic of ∂B with the same endpoints as $g_{j,k}$. Also call ρ the union of the $\rho_{j,k}$, as in (9.62). Obviously $\operatorname{length}(g_{j,k}) \ge \operatorname{length}(\rho_{j,k})$, but

$$\begin{aligned} \sum_j [\operatorname{length}(g_{j,k}) - \operatorname{length}(\rho_{j,k})] &\le H^1(g) - H^1(\rho) = \delta_1 + \delta_2 \\ &\le \delta_1 + \delta_2 + \delta_3 \le Cj(r) \end{aligned} \tag{11.21}$$

because the $g_{j,k}$ are simple and disjoint, by (9.65) and (9.66), because $\delta_3 \ge 0$ by (9.67), and by (11.18).

It follows from (11.21) and a simple geometric argument (or brutally (7.9), the definitions (7.4) and (7.5), the fact that $v(0) = 0$, and Cauchy-Schwarz) that

$$\begin{aligned} &\operatorname{dist}(x,g) \le Cj(r)^{1/2} \ \text{ for } x \in \rho \ \text{ and} \\ &\qquad \operatorname{dist}(x,\rho) \le Cj(r)^{1/2} \ \text{ for } x \in g. \end{aligned} \tag{11.22}$$

Let $X(r)$ be the cone over ρ. Then (11.20) follows from (11.22). Also, $X(r)$ lies in $\mathcal{Z}_0(X, C\tau)$ by Lemma 6.11 and the definition of $\mathcal{Z}_0(X, \tau)$ at the beginning of this section. In addition, $\alpha_+(Z) = \alpha_+(\varphi)$ (see the definition above (11.2)), so $\alpha_+(Z) \leq Cj(r)^{1/2}$ by (11.18). Lemma 11.19 follows. □

Lemma 11.19 gives a good control on g for each $r \in \mathcal{R}_1$. We also control $E \cap \partial B(0,r) \setminus rg$, because

$$H^1(r^{-1}E \cap \partial B \setminus g) = \delta_3 \leq Cj(r) \tag{11.23}$$

by (9.67) and (11.18) (recall that the three δ_j are nonnegative).

Now we want to show that the minimal set $X(r)$ does not depend too much on r. For this we shall need to build transverse curves that are almost radial and meet $rg = rg(r)$ for different values of r. We shall first restrict to $r \in [r_0/3, r_0]$, where r_0 is as in the statement of Theorem 11.4. The construction of transverse curves will take some time.

We keep the minimal set X of (11.7) and the decomposition of $X \cap \partial B(0,1)$ into curves $\mathfrak{C}_{j,k}$, and denote by $\mathfrak{C}'_{j,k}$ the arc of geodesic contained in $\mathfrak{C}_{j,k}$ with the same center as $\mathfrak{C}_{j,k}$, but only half the length. Then let $z \in \mathfrak{C}'_{j,k}$ be given, and denote by P_z the vector hyperplane through z that is perpendicular to $\mathfrak{C}_{j,k}$ at z; we want to construct a curve in $E \cap P_z$, that crosses the annulus $\overline{B}(0,r_0) \setminus B(0,r_0/3)$, but let us start with a connected set. The reader may start looking at Figure 11.1 for a picture.

Lemma 11.24. *There is a compact connected $G_z \subset E \cap P_z$, which is contained in a $C\varepsilon r_0$-neighborhood of the segment $I_z = [r_0 z/5, 3r_0 z/2]$, and that contains points in $\partial B(0, r_0/3)$ and in $\partial B(0, r_0)$.*

We proceed as in Section 6, for the construction of the curves $g_{j,k}$. By (11.7) and (11.6) or (11.8), and if ε_1 and ε are chosen small enough, we can apply Lemma 16.19 or Lemma 6.25 or Lemma 16.56 in [D3], and we get that $B(0,r_0)$ is a biHölder ball for E. [We used $60r_0$ instead of $50r_0$ in the statement above so that we could bound $h(100r_0)$ in terms of $h_1(50r_0)$.] This gives a biHölder mapping $f : B(0,2r_0) \to \mathbb{R}^n$, with the properties (6.16)-(6.19), with a small constant $\tau > 0$ that will be chosen soon, but where we should adapt the estimates to the scale r_0. We can keep the same cone X as in (11.7).

Denote by H_0 the cone over $\mathfrak{C}_{j,k}$, and set $H = H_0 \cap \overline{B}(0, 5r_0/4) \setminus B(0, r_0/4)$. This is a vaguely rectangular plane domain. Denote by L_1 and L_2 the two straight parts of its boundary, obtained as the product by $[r_0/4, 5r_0/4]$ of one of the two endpoints of $\mathfrak{C}_{j,k}$. Note that $f(L_1)$ and $f(L_2)$ lie on different sides of the hyperplane P_z, by (6.16).

Now set $F = H \cap f^{-1}(E \cap P_z) \subset B(0, 2r_0)$; then F separates L_1 from L_2 in H, because if γ is a path from L_1 to L_2 in H, $f(\gamma)$ must cross P_z. By 52.III.1 in [Ku] or the simpler Theorem 14.3 in [Ne] (see Remark 6.45), there is a connected subset of F_1 of F that separates L_1 from L_2 in H. In particular, it meets $\partial B(0, r_0/3)$, because otherwise we could use $H \cap \partial B(0, r_0/3)$ to connect L_1 to L_2 in $H \setminus F_1$. Similarly, F_1 meets $\partial B(0, 4r_0/3)$.

Now set $G_z = f(F_1)$. By definition of F, it is contained in $E \cap P_z$. Obviously, it is connected too. By what we just said and (6.16), it contains points in $\overline{B}(0, r_0/4 + \tau r_0)$ and out of $B(0, 5r_0/4 - \tau r_0)$; by connectedness, it meets $\partial B(0, r_0/3)$ and $\partial B(0, r_0)$ too.

We still need to check that $\text{dist}(y, I_z) \le C\varepsilon r_0$ for $y \in G_z$. Let $x \in F_1$ be such that $f(x) = y$. First observe that $\text{dist}(x, P_z) \le \tau r_0$, by (6.16) and because $y \in P_z$. Since $x \in H$ and P_z is the hyperplane perpendicular to $\mathfrak{C}_{j,k}$ at z, this forces x to be within τr_0 of the line segment $[r_0 z/4, 5r_0 z/4]$. Then $\text{dist}(y, I_z) \le 2\tau r_0$, by (6.16).

Now recall that we took $z \in \mathfrak{C}'_{j,k}$, i.e., far from the extremities of $\mathfrak{C}_{j,k}$; thus, if τ is small enough compared to the constant η_0 in (2.2) and (2.3), the description of X in Section 2 says that, in the region where $\text{dist}(y, I_z) \le 3\tau r_0$, X coincides with the plane H' that contains $\mathfrak{C}_{j,k}$. Since (11.7) says that $\text{dist}(y, X) \le 100\varepsilon r_0$, we get that $\text{dist}(y, H') \le 100\varepsilon r_0$, and then $\text{dist}(y, I_z) \le 100\varepsilon r_0$ because $y \in P_z$. Lemma 11.24 follows. □

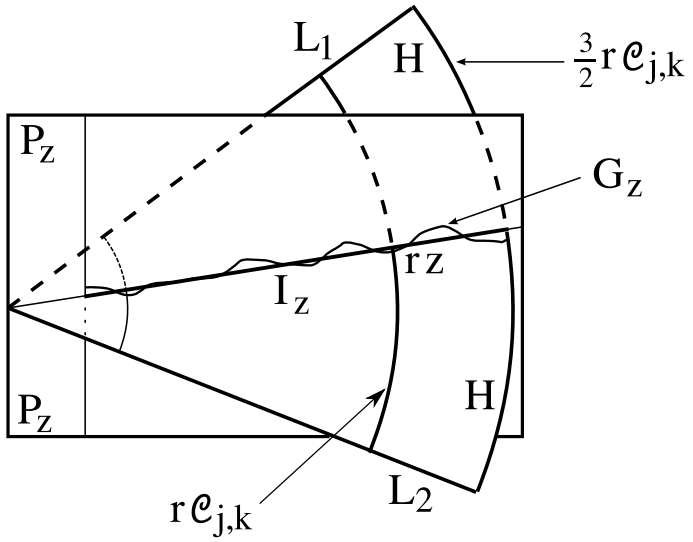

Figure 11.1.

For most choices of $z \in \mathfrak{C}'_{j,k}$, we want to find a nice simple curve in G_z, show that it stays very close to a radius, and at the same time that for many radii r it meets the curve $rg(r)$ studied above (the same one as in Lemma 11.19). This will then make it easy to show that the cones $X(r)$ are close to each other. The acceptable choices of z and r will come from Chebyshev, and before we do this we need to control the averages of certain quantities.

Lemma 11.25. *Set $A = B(0, 5r_0/3) \setminus B(0, r_0/6)$ and, for almost every $x \in E \cap A$, denote by $\alpha(x) \in [0, \pi/2]$ the angle of the radius $[0, x]$ with the tangent plane to E at x. Then*

$$\int_{x \in E \cap A} [1 - \cos \alpha(x)] \, dH^2(x) \le C r_0^2 [f(2r_0) + h_1(2r_0)]. \tag{11.26}$$

We want to prove this with the co-area theorem. Recall that E is rectifiable, and that it even has a true tangent plane $P(x)$ at x for H^2-almost-every $x \in E$. (See the discussion after the statement of Lemma 4.12). So $\alpha(x)$ is defined almost-everywhere, and (4.13) and (4.14), applied with $f = \mathbf{1}_A$, say that

$$\int_{E \cap A} \cos \alpha(x) \, dH^2(x) = \int_{r_0/6 \le r < 5r_0/3} H^1(E \cap \partial B(0, r)) \, dr. \tag{11.27}$$

Let us check that

$$H^1\big((r_0/6, 5r_0/3) \setminus \mathcal{R}_1\big) \le Cr_0[f(2r_0) + h_1(2r_0)], \tag{11.28}$$

where C may depend on τ (through (11.13)), but this does not matter any more because we shall no longer need to modify our choice of τ. Since $\mathcal{R}$ has full measure, the definition (11.13) of $\mathcal{R}_1$ says that $H^1\big((r_0/6, 5r_0/3) \setminus \mathcal{R}_1\big) \le \tau^{-1} d(0)^{-1} \int_{[r_0/6, 5r_0/3]} j(r)dr$, so it is enough to show that

$$\int_{[r_0/6, 5r_0/3]} j(r)dr \le Cr_0[f(2r_0) + h_1(2r_0)]. \tag{11.29}$$

Recall from (11.11) that $j(r) = rf'(r) + f(r) + h_1(r) + h(2r)$. There is no problem with the contribution of $h_1(r) + h(2r)$, because $h_1(r) + h(2r) \le h_1(2r_0)$. Next consider $f(r)$. Notice that for $r \le 5r_0/6$,

$$f(r) \le f(2r_0) + Ch_1(2r_0) \tag{11.30}$$

by (3.6) and because $B(0, r_0) \subset U$, so the contribution of $f(r)$ in (11.29) is under control. We are left with with the contribution of $rf'(r)$. Observe that

$$\begin{aligned} \int_{r_0/6 \le r < 5r_0/3} f'(r)\, dr &= \int_{r_0/6 \le r < 5r_0/3} \theta'(r)\, dr \le \theta(2r_0) - \theta(r_0/6) \\ &= f(2r_0) - f(r_0/6) \le f(2r_0) + Ch_1(r_0/6) \le f(2r_0) + Ch_1(2r_0) \end{aligned} \tag{11.31}$$

by (1.3), (5.7), (3.8), and because h_1 is nondecreasing; (11.29) and (11.28) follow at once.

Let us now focus on $\mathcal{R}_1$. For $r \in \mathcal{R}_1$, we can apply to $r^{-1}E$ the construction (and notation) of the previous sections. Then

$$r^2\theta(r) = H^2(E \cap B(0, r)) \le \frac{r}{2} H^1(E \cap \partial B(0, r)) + 4h(2r)\, r^2 \tag{11.32}$$

by (9.69) and because $H^1(r^{-1}E \cap \partial B) - H^1(\rho) = \delta_1 + \delta_2 + \delta_3 \ge 0$ (see (9.68) and the definitions above it). Equivalently,

$$rH^1(E \cap \partial B(0, r)) \ge 2r^2\theta(r) - 8h(2r)\, r^2. \tag{11.33}$$

But $\theta(r) \ge d(0) - Ch_1(r) \ge d(0) - Ch_1(2r_0)$ by (3.8), and $h(2r) \le h(10r_0/3) \le Ch_1(2r_0) = \int_0^{2r_0} h(2t)\frac{dt}{t}$, because h is nondecreasing and by the definition (1.16) or (3.1) of h_1. Then

$$rH^1(E \cap \partial B(0, r)) \ge 2d(0)\, r^2 - Ch_1(2r_0)\, r^2. \tag{11.34}$$

Thus (11.27) says that

$$\begin{aligned} \int_{E \cap A} \cos\alpha(x)\, dH^2(x) &\ge \int_{r \in \mathcal{R}_1 \cap (r_0/6, 5r_0/3)} H^1(E \cap \partial B(0, r))\, dr \\ &\ge \int_{r \in \mathcal{R}_1 \cap (r_0/6, 5r_0/3)} [2d(0) - Ch_1(2r_0)]\, r dr \\ &\ge d(0)\,\big(\frac{25r_0^2}{9} - \frac{r_0^2}{36}\big) - 2d(0)\, r_0 H^1((r_0/6, 5r_0/3) \setminus \mathcal{R}_1) - Ch_1(2r_0)\, r_0^2 \\ &\ge d(0)\,\big(\frac{25r_0^2}{9} - \frac{r_0^2}{36}\big) - C[f(2r_0) + h_1(2r_0)]\, r_0^2 \end{aligned} \tag{11.35}$$

by (11.28). On the other hand,

$$
\begin{aligned}
H^2(E\cap A) &= \frac{25r_0^2}{9}\,\theta(5r_0/3) - \frac{r_0^2}{36}\,\theta(r_0/3) = \frac{25r_0^2}{9}\,[d(0)+f(5r_0/3)] - \frac{r_0^2}{36}\,\theta(r_0/3)\\
&\le \frac{25r_0^2}{9}\,[d(0)+f(5r_0/3)] - \frac{r_0^2}{36}\,[d(0)-Ch_1(r_0)]\\
&= d(0)\,\big(\frac{25r_0^2}{9} - \frac{r_0^2}{36}\big) + 3r_0^2 f(2r_0) + C\,r_0^2 h_1(2r_0) \qquad (11.36)\\
&\le \int_{E\cap A} \cos\alpha(x)\,dH^2(x) + C[f(2r_0)+h_1(2r_0)]\,r_0^2
\end{aligned}
$$

by (1.1), (3.8), (3.6), and (11.35). This proves (11.26) and Lemma 11.25. □

Return to the compact connected set G_z of Lemma 11.24. Still fix j and k, and for $z\in\mathfrak{C}'_{j,k}$, choose G_z as in Lemma 11.24. Also let r_1 and r_2 be given, with $r_0/3\le r_1<r_2\le r_0$, and pick points $y_1(z)\in G_z\cap\partial B(0,r_1)$ and $y_2(z)\in G_z\cap\partial B(0,r_2)$. Such points exist, because G_z is connected and contains points of $\partial B(0,r_0/3)$ and $\partial B(0,r_0)$. The following lemma says that y_1 and y_2 are often nearly aligned.

Lemma 11.37. *There is a measurable function q defined on $\mathfrak{C}'_{j,k}$, such that*

$$\int_{\mathfrak{C}'_{j,k}} q(z)dH^1(z) \le C[f(2r_0)+h_1(2r_0)]^{1/2} \qquad (11.38)$$

and

$$\left|\frac{y_1}{|y_1|} - \frac{y_2}{|y_2|}\right| \le q(z) \qquad (11.39)$$

for all choices of $z\in\mathfrak{C}'_{j,k}$, r_1, r_2, y_1, and y_2 as above.

Set $S_z=\big\{y\in P_z\,;\,\mathrm{dist}(y,I_z)\le C\varepsilon r_0\big\}$, where P_z, I_z and C are as in Lemma 11.24; thus $G_z\subset S_z$ for all z. Notice also that $S_z\subset A$, where A is as in Lemma 11.25. Let us first check that

$$H^1(G_z)\le H^1(E\cap S_z)<+\infty \text{ for almost every choice of } z\in\mathfrak{C}'_{j,k}. \qquad (11.40)$$

Denote by S the union of the S_z, $z\in\mathfrak{C}'_{j,k}$, and define a Lipschitz mapping $\varphi: E\cap S\to\mathfrak{C}'_{j,k}$ by $\varphi(x)=z$ when $x\in S_z$. We apply the co-area theorem (see 3.2.22 in [Fe]) to φ (or its composition with the inverse of a parameterization of $\mathfrak{C}'_{j,k}$ by arc-length, if you want a mapping to an interval) and get that for any Borel set $T\subset E\cap S$,

$$\int_T J_\varphi(x)\,dH^2(x) = \int_{z\in\mathfrak{C}'_{j,k}} H^1(T\cap\varphi^{-1}(z))\,dH^1(z) = \int_{z\in\mathfrak{C}'_{j,k}} H^1(T\cap S_z)\,dH^1(z), \qquad (11.41)$$

where $J_\varphi(x)$ is the appropriate directional Jacobian. Since φ is $6r_0^{-1}$-Lipschitz on S, $J_\varphi(x) \le 6r_0^{-1}$, and (11.41) with $T = E \cap S$ immediately yields

$$\int_{z \in \mathfrak{C}'_{j,k}} H^1(E \cap S_z)\, dH^1(z) \le 6r_0^{-1} H^2(E \cap S) \le C r_0 \tag{11.42}$$

(by local Ahlfors-regularity or more brutally because $f(60r_0) \le \varepsilon$ by (11.8)). This proves (11.40) (because we already knew its first part). Thus we can safely take $q(z) = +\infty$ when $H^1(E \cap S_z) = +\infty$ and restrict to the other case.

Since G_z is connected and $H^1(G_z) < +\infty$, there is a simple curve $\zeta \subset G_z$ that goes from y_1 to y_2. See for instance Proposition 30.14 in [D2]. Denote by $\xi : I \to K$ a parameterization of ζ by arc-length; such a parameterization exists, because length$(\zeta) \le H^1(G_z) < +\infty$.

Set $v(x) = x/|x|$ for $x \neq 0$. We are interested in the variations of $v \circ \xi$. The differential of v in the unit direction a is $Dv(x) \cdot a = |x|^{-1}\, \widetilde{a}$, where $\widetilde{a}$ is the projection of a on the plane orthogonal to x. For almost every $s \in I$, $\xi'(s)$ exists, and is a unit vector. Then $|(v \circ \xi)'(s)| = |\xi(s)|^{-1} \sin\gamma(s) \le 6r_0^{-1} \sin\gamma(s)$, where $\gamma(s) \in [0, \pi/2]$ is the angle of $\xi'(s)$ with the radius $[0, \xi(s)]$. Hence

$$\left| \frac{y_1}{|y_1|} - \frac{y_2}{|y_2|} \right| = |v(y_1) - v(y_2)| \le 6r_0^{-1} \int_I \sin\gamma(s) ds. \tag{11.43}$$

Next we want to bound the right-hand side of (11.43) by integrals on $E \cap S_z$, and then average over z. It will be more convenient to replace $\sin\gamma(s)$ with quantities that depend only on $x = \xi(s)$.

We need to distinguish different types of points in I. We start with the good set I_1 of points $s \in I$ with the following properties. First, $\xi'(s)$ exists and is a unit vector. Next, E has a (true) tangent 2-plane $P(x)$ at $x = \xi(s)$. Also, $r = |x|$ lies in $\mathcal{R}_1$, which allows us to do the construction of Sections 6-9. Finally, we require the existence of a unit vector v in the direction of $P(x)$, which is orthogonal to $e = x/|x|$, and whose orthogonal projection on P_z has a norm less than 10^{-1}. We shall see soon that this happens often. Let us check that

$$\sin\gamma(s) \le 10\, \alpha(x) \text{ when } s \in I_1, \tag{11.44}$$

where $\alpha(x)$ is the angle of $[0, x]$ with $P(x)$, as in (11.26) and (11.27).

We may assume that $\alpha(x) \le 10^{-1}$, because otherwise (11.44) is trivial. Denote by e_2 any of the two unit vectors perpendicular to P_z. Set $V = P(x) - x$ (the vector plane parallel to $P(x)$). By definition of $\alpha(x)$, we can find a unit vector v_1 in V, with

$$|v_1 - e| \le \alpha(x) \le 10^{-1}. \tag{11.45}$$

Notice that since v_1 is close to e by (11.45), it is nearly orthogonal to v (the unit vector whose existence was required in the definition of E_1). Thus (v_1, v) is a basis of V, and the projection in V on the line through v_1 and parallel to v has a norm smaller than 2.

Recall that $\xi'(s)$ exists and is a unit vector, by definition of I_1. Then $\xi'(s) \in V$, because $\xi(t) \in E$ for all t. So we can write $\xi'(s) = a_1 v_1 + a_2 v$. Recall that the norm of the projection on the line through v_1 parallel to v is less than 2, and hence $|a_1| \leq 2$. We know that $\xi'(s) \cdot e_2 = 0$, because $\xi'(s) \in P_z$ (since the whole curve lies in P_z) and e_2 is orthogonal to P_z. Then $a_1 v_1 \cdot e_2 + a_2 v \cdot e_2 = 0$. In addition, $|v \cdot e_2| \geq 2/3$ because the orthogonal projection of v on P_z has a norm smaller than 10^{-1}, and e_2 spans the orthogonal direction to P_z. Thus

$$\begin{aligned} |a_2| = |a_2 v \cdot e_2|/|v \cdot e_2| &\leq \frac{3}{2} |a_2 v \cdot e_2| = \frac{3}{2} |a_1 v_1 \cdot e_2| \\ &\leq 3|v_1 \cdot e_2| = 3|(v_1 - e) \cdot e_2| \leq 3\alpha(x), \end{aligned} \tag{11.46}$$

because $e \cdot e_2 = 0$ (since $x \in P_z$), and by (11.45). Now recall that $\gamma(s)$ is the angle of $\xi'(s)$ with the direction of $\xi(s) = x$ and $e = x/|x|$. Thus $\sin \gamma(s) = |\pi(\xi'(s))|$, where π denotes the orthogonal projection on $e^\perp$. Now

$$\begin{aligned} \sin \gamma(s) = |\pi(\xi'(s))| = |\pi(a_1 v_1 + a_2 v)| &\leq |a_1||\pi(v_1)| + |a_2| \\ &\leq 2|\pi(v_1)| + 3\alpha(x) = 2|\pi(v_1 - e)| + 3\alpha(x) \leq 5\alpha(x) \end{aligned} \tag{11.47}$$

because $\xi'(s) = a_1 v_1 + a_2 v$ and $|a_1| \leq 2$, by (11.46), and by (11.45). Thus (11.44) holds in all cases.

There is another part of I which will cause no problem; this is the set I_2 of points $s \in I \setminus I_1$ such that $\alpha(\xi(s))$ is defined, and $\alpha(\xi(s)) \geq 10^{-1}$. Notice that (11.44) (trivially) holds for I_2. Hence (11.43) yields

$$\begin{aligned} \left| \frac{y_1}{|y_1|} - \frac{y_2}{|y_2|} \right| &\leq 6 r_0^{-1} \int_I \sin \gamma(s) ds \\ &\leq 6 r_0^{-1} H^1(I \setminus [I_1 \cup I_2]) + 60 r_0^{-1} \int_{I_1 \cup I_2} \alpha(\xi(s)) ds. \end{aligned} \tag{11.48}$$

Now let us chase various exceptional sets. First, the set I_3 where $\xi'(s)$ does not exist, or is not a unit vector, has vanishing measure, so we may drop it.

Next we need to control some exceptional sets in E, before we return to I. Set $W_4 = \{x \in E \cap S\,;\, E \text{ has no tangent plane at } x\}$. Notice that $H^2(W_4) = 0$, because E is rectifiable and locally Ahlfors-regular. See Lemma 2.15 in [D3] and Exercise 41.21 in [D2].

Set $W_5 = \{x \in E \cap S \setminus W_4\,;\, r \notin \mathcal{R}_1 \text{ and } \alpha(x) \leq 10^{-1}\}$. Notice that $\cos \alpha(x) \geq \cos(10^{-1}) \geq 1/2$ on W_5, so

$$\begin{aligned} H^2(W_5) &\leq 2 \int_{W_5} \cos \alpha(x) dH^2(x) = 2 \int_{W_5} J_g(x) dH^2(x) \\ &\leq \int_{[r_0/6, 5r_0/3] \setminus \mathcal{R}_1} H^1(E \cap \partial B(0,r))\, dr \leq \int_{[r_0/6, 5r_0/3] \setminus \mathcal{R}_1} 2r[j(r) + d(0)]\, dr \\ &\leq 2 r_0\, d(0)\, H^1([r_0/6, 5r_0/3] \setminus \mathcal{R}_1) + 2 r_0 \int_{[r_0/6, 5r_0/3]} j(r)\, dr \\ &\leq C r_0^2 [f(2r_0) + h_1(2r_0)], \end{aligned} \tag{11.49}$$

by (4.14) and the analogue of (4.13) for Borel subsets of E, because $r_0/6 \le |x| \le 5r_0/3$ when $x \in S_z$, by the fact that $\mathcal{R}$ has full measure, and by (11.12), (11.28), and (11.29).

Finally let W_6 denote the set of points $x \in E \cap S \setminus W_4$ such that $\alpha(x) \le 10^{-1}$ and $r = |x| \in \mathcal{R}_1$, but we cannot find a unit vector v in the direction V of $P(x)$, as in the definition of I_1. Then

$$\begin{aligned} H^2(W_6) &\le 2\int_{W_6} \cos\alpha(x) dH^2(x) = 2\int_{W_6} J_g(x) dH^2(x) \\ &\le 2\int_{[r_0/6,5r_0/3]\cap\mathcal{R}_1} H^1(W_6 \cap \partial B(0,r))\, dr \end{aligned} \tag{11.50}$$

because $\cos\alpha(x) \ge 1/2$ on W_6, and by (4.14) and (4.13). Let us fix r, and prove that

$$H^1(W_6 \cap \partial B(0,r) \cap \Gamma'_{j,k}) = 0 \tag{11.51}$$

where we set $\Gamma'_{j,k} = r\Gamma_{j,k}$ and $\Gamma_{j,k}$ is the Lipschitz curve in ∂B that was constructed in Section 7 (so that $\Gamma'_{j,k} \subset \partial B(0,r)$). Indeed, for almost every point $x \in W_6 \cap \Gamma'_{j,k}$, $\Gamma'_{j,k}$ has a tangent line L at x, and x is a Lebesgue density point of $E \cap \Gamma'_{j,k}$ (recall that $W_6 \subset E$). Then L is contained in $P(x)$ (which exists because $x \in W_6 \subset E \cap S \setminus W_4$). Let v be a unit vector in the direction of L; we claim that it satisfies the requirements in the definition of I_1 (and then $x \notin W_6$, a contradiction which proves (11.51)). We already know that $v \in V$, because $L \subset P(x)$. Then v is orthogonal to the direction of x, because $\Gamma'_{j,k} \subset \partial B(x,r)$.

We still need to check that v is almost orthogonal to P_z (where $z \in \mathfrak{C}'_{j,k}$ is the one for which $x \in S_z$). Without loss of generality, we can assume that $z = (1,0,\dots 0)$ and P_z is the vertical hyperplane plane $\{x_2 = 0\}$. This also means that our arc of geodesic $\mathfrak{C}_{j,k}$ is contained in the horizontal 2-plane $\{x_3 = \dots = x_n = 0\}$.

By construction, $\Gamma'_{j,k}$ is a small Lipschitz graph over the horizontal 2-plane which contains $\mathfrak{C}_{j,k}$. This means that all the coordinates of v, except perhaps the first two, are smaller than $C\eta$ (see for instance (7.45)). The first coordinate is less than $C\varepsilon$, because v is orthogonal to $e = x/|x|$ and e is very close to $z = (1,0,\dots 0)$, because $x \in S_z$ lies very close to $I_z = [r_0 z/6, 5r_0 z/3]$ (see Lemma 11.24 and the definition of S_z above (11.40)). So v has a small projection on P_z, $x \notin W_6$, and (11.51) holds. Hence (11.50) yields

$$H^2(W_6) \le 2\int_{[r_0/6,5r_0/3]\cap\mathcal{R}_1} H^1(W_6 \cap \partial B(0,r) \setminus \Gamma'_{j,k})\, dr. \tag{11.52}$$

Let $x \in W_6 \cap \partial B(0,r) \setminus \Gamma'_{j,k}$ be given. Then $x \in S = \cup_{z\in\mathfrak{C}'_{j,k}} S_z$, which is contained in the cone over a small neighborhood of $\mathfrak{C}'_{j,k}$. That is, x/r lies in a small neighborhood of $\mathfrak{C}'_{j,k}$. Also recall that $\mathfrak{C}'_{j,k}$ itself lies near the middle of $\mathfrak{C}_{j,k}$, hence reasonably far from the other $\mathfrak{C}_{j',k'}$ (by (2.2) and (2.3)). Then x/r does not lie in any other $\Gamma_{j',k'}$, because $\Gamma_{j',k'}$ stays within $C\eta$ of $\mathfrak{C}_{j',k'}$ (because it is a small Lipschitz graph with endpoints on $g_{j',k'}$, and by (6.15)), and if η is small enough compared to η_0 in (2.2) and (2.3). So $x \notin \Gamma' = \cup_{j',k'} \Gamma'_{j',k'}$.

Recall that $W_6 \subset E$, so

$$\begin{aligned} H^1(W_6 \cap \partial B(0,r) \setminus \Gamma'_{j,k}) &\le H^1(E \cap \partial B(0,r) \setminus \Gamma') = rH^1(r^{-1}E \cap \partial B \setminus \Gamma) \\ &\le rH^1(r^{-1}E \cap \partial B \setminus g) + rH^1(g \setminus \Gamma) = r\delta_3 + r\sum_{j,k} H^1(g_{j,k} \setminus \Gamma) \\ &\le r\delta_3 + r\sum_{j,k} H^1(g_{j,k} \setminus \Gamma_{j,k}) \\ &\le r\delta_3 + Cr\sum_{j,k}[\text{length}(g_{j,k}) - \text{length}(\rho_{j,k})] \le Crj(r), \end{aligned} \tag{11.53}$$

where $g = \cup_{j,k} g_{j,k}$ and the $g_{j,k}$ are the curves of Section 6, by (9.67), because the $g_{j,k}$ are simple and disjoint, by (9.57), and by (11.21). Altogether

$$H^2(W_6) \le C\int_{[r_0/6,5r_0/3]\cap\mathcal{R}_1} rj(r)dr \le Cr_0^2\,[f(2r_0) + h_1(2r_0)] \tag{11.54}$$

by (11.29).

Set $I_4 = I \setminus (I_1 \cup I_2 \cup I_3)$. If $s \in I_4$ and $x = \xi(s)$, $\xi'(s)$ exists and is a unit vector because $s \notin I_3$. If $\xi(s) \notin W_4$, $P(x)$ is defined, $\alpha(x) \le 10^{-1}$ because $s \notin I_2$, and then $\xi(s) \in W_5 \cup W_6$ because otherwise $s \in I_1$. Thus $\xi(I_4) \subset W$, where $W = W_4 \cup W_5 \cup W_6$.

We may now return to (11.48); first observe that

$$\int_{I_1\cup I_2} \alpha(\xi(s))ds \le \int_{E\cap S_z} \alpha(x)dH^1(x) \tag{11.55}$$

because ξ is a parameterization by arc-length of the simple curve ζ, so the direct image of ds on I by ξ is less than or equal to the restriction of H^1 to $E \cap S_z$. Also,

$$H^1(I \setminus [I_1 \cup I_2]) \le H^1(I_3 \cup I_4) = H^1(I_4) \le H^1(S_z \cap W) \tag{11.56}$$

by definition of I_4, because $H^1(I_3) = 0$, because $\zeta(I_4) \subset S_z \cap W$, and again because the image of ds by ξ is no greater than H^1. Thus (11.48) yields

$$\left|\frac{y_1}{|y_1|} - \frac{y_2}{|y_2|}\right| \le 6r_0^{-1}H^1(S_z \cap W) + 60r_0^{-1}\int_{E\cap S_z} \alpha(x)dH^1(x). \tag{11.57}$$

We take $q(z) = 6r_0^{-1}H^1(W \cap S_z) + 60r_0^{-1}\int_{E\cap S_z} \alpha(x)dH^1(x)$; then (11.39) obviously holds, and Lemma 11.37 will follow as soon as we prove (11.38). First

$$\begin{aligned} \int_{z\in\mathfrak{C}'_{j,k}} H^1(W \cap S_z)\,dH^1(z) &= \int_W J_\varphi(x)\,dH^2(x) \\ &\le 6r_0^{-1}H^2(W) \le Cr_0\,[f(r_0) + h_1(2r_0)] \end{aligned} \tag{11.58}$$

by (11.41) (with $T = W$), because $J_\varphi(x) \le 6r_0^{-1}$ almost-everywhere on E, because $H^2(W_4) = 0$, and by (11.49) and (11.54) (recall that $W = W_4 \cup W_5 \cup W_6$). Similarly,

$$\begin{aligned}
&\int_{z\in\mathfrak{C}'_{j,k}} \int_{E\cap S_z} \alpha(x)\, dH^1(x)\, dH^1(z) = \int_{E\cap S} J_\varphi(x)\, \alpha(x)\, dH^2(x) \\
&\le 6r_0^{-1} \int_{E\cap S} \alpha(x)\, dH^2(x) \le 6r_0^{-1} H^2(E\cap S)^{1/2} \Big\{ \int_{E\cap S} \alpha^2(x)\, dH^2(x) \Big\}^{1/2} \\
&\le C \Big\{ \int_{E\cap S} [1-\cos\alpha(x)]\, dH^2(x) \Big\}^{1/2} \le Cr_0[f(2r_0) + h_1(2r_0)]^{1/2}
\end{aligned} \tag{11.59}$$

by the integral version of (11.41), Cauchy-Schwarz, (11.26), and because $S \subset A$. Now $\int_{z\in\mathfrak{C}'_{j,k}} q(z)\, dH^1(z) \le C[f(2r_0) + h_1(2r_0)]^{1/2}$, by (11.58) and (11.59). This proves (11.38), and Lemma 11.37 follows. □

We are now ready to show that the various cones $X(r)$ are close to each other. We shall restrict our attention to $r \in \mathcal{R}_2$, where

$$\mathcal{R}_2 = \big\{ r \in (r_0/3, r_0) \cap \mathcal{R}\,;\, j(r) \le [f(2r_0) + h_1(2r_0)]^{2/3} \big\}. \tag{11.60}$$

Lemma 11.61. *We have that*

$$d_{0,1}(X(r_1), X(r_2)) \le C[f(2r_0) + h_1(2r_0)]^{1/3} \quad \text{for } r_1, r_2 \in \mathcal{R}_2, \tag{11.62}$$

where the local Hausdorff distance function $d_{0,1}$ is as in (11.3).

Let us first get rid of some exceptional sets. Let $r \in \mathcal{R}_2$ be given; observe that $r \in \mathcal{R}_1$ (if ε is small enough; see (11.8) and (11.13)), which allows the construction of competitors the previous sections. Set $E^\sharp = [r^{-1}E \cap \partial B \setminus g] \cup [g \setminus \Gamma]$, where $g = \cup_{j,k} g_{j,k}$ is the union of the arcs of Section 6, and $\Gamma = \cup_{j,k}\Gamma_{j,k}$ is the union of the Lipschitz arcs with the same endpoints constructed in Section 7. Then

$$\begin{aligned}
H^1(E^\sharp) &\le \delta_3 + \sum_{j,k} H^1(g_{j,k} \setminus \Gamma) \le \delta_3 + \sum_{j,k} H^1(g_{j,k} \setminus \Gamma_{j,k}) \\
&\le \delta_3 + C \sum_{j,k} [\text{length}(g_{j,k}) - \text{length}(\rho_{j,k})] \le \delta_3 + C[H^1(g) - H^1(\rho)] \\
&= \delta_3 + C[\delta_1 + \delta_2] \le Cj(r) \le C[f(2r_0) + h_1(2r_0)]^{2/3}
\end{aligned} \tag{11.63}$$

by (9.67), (9.57), (9.62) and the fact that the $\rho_{j,k}$ are disjoint, (9.65) and (9.66), and finally (11.18) and (11.60).

Denote by $Z_{j,k}(r)$ the set of points $z \in \mathfrak{C}'_{j,k}$ such that the set $S_z \subset P_z$ defined above (11.40) meets $rE^\sharp$. Then $Z_{j,k}(r)$ is the image of $rE^\sharp \cap S_z$ by the $6r_0^{-1}$-Lipschitz projection φ from S to $\mathfrak{C}'_{j,k}(r)$ (see below (11.40)), so

$$H^1(Z_{j,k}(r)) \le 6r_0^{-1} H^1(rE^\sharp) \le C[f(2r_0) + h_1(2r_0)]^{2/3}. \tag{11.64}$$

Let us prove that for $z \in \mathfrak{C}'_{j,k} \setminus Z_{j,k}(r)$,

$$E \cap \partial B(0,r) \text{ meets } S_z \text{ exactly once, at a point of } G_z. \tag{11.65}$$

Recall that G_z is the connected subset of $E \cap P_z$ that shows up in Lemma 11.24. It is contained in S_z (see below (11.39)) and meets $\partial B(0,r)$ because it is connected and meets $\partial B(0, r_0/3)$ and $\partial B(0, r_0)$. So $E \cap \partial B(0,r)$ meets S_z at least once (at a point of G_z), and we just need to check that the intersection is reduced to one point. Let us first check that

$$E \cap \partial B(0,r) \cap S_z \subset \Gamma'_{j,k}, \tag{11.66}$$

where again $\Gamma'_{j,k} = r\Gamma_{j,k}$. Let $\xi' \in E \cap \partial B(0,r) \cap S_z$ be given and set $\xi = \xi'/r$. Since $z \in \mathfrak{C}'_{j,k} \setminus Z_{j,k}(r)$, ξ lies out of $E^\sharp$, and hence $\xi \in \Gamma = \cup_{(j',k') \in \widetilde{J}} \Gamma_{j',k'}$. We still need to check that ξ lies in no other $\Gamma_{j',k'}$. Recall that the endpoints of $\Gamma_{j',k'}$ lie in $g_{j',k'}$, which lies very close to $\mathfrak{C}_{j',k'}$ by (6.15); then the whole $\Gamma_{j',k'}$ lies close to $\mathfrak{C}_{j',k'}$, because $\Gamma_{j',k'}$ is a Lipschitz graph with small constant (see the first lines of Section 9, or go to (7.45) and where π is defined above (7.35)), and because we have a good control on the 2-plane that contains the origin and the endpoints of $\Gamma_{j',k'}$ (since the length of $\mathfrak{C}_{j',k'}$ is at most $9\pi/10$ by (2.5)). Now $\xi' \in S_z$, hence ξ lies close to $\mathfrak{C}'_{j,k}$, so ξ lies far from the other $\mathfrak{C}_{j',k'}$, because $\mathfrak{C}'_{j,k}$ lies in the middle of $\mathfrak{C}_{j,k}$ and by (2.2). We checked that $\xi \notin \Gamma_{j',k'}$ for $(j',k') \neq (j,k)$, and so (11.66) holds.

Now $\Gamma'_{j,k}$ is a Lipschitz graph with small constant over the 2-plane that contains its endpoints (by (7.45)), so it cannot cross the vertical hyperplane P_z more than once; (11.65) follows.

Return to Lemma 11.61 and let $r_1, r_2 \in \mathcal{R}_2$ be given. For each choice of $(j,k) \in \widetilde{J}$, we want choose three points z_i in $\mathfrak{C}'_{j,k} \setminus [Z_{j,k}(r_1) \cup Z_{j,k}(r_2)]$. We cut $\mathfrak{C}'_{j,k}$ into three equal intervals I_1, I_2, and I_3, with I_1 in the middle; thus $H^1(I_i) \geq \eta_0$ and $\operatorname{dist}(I_1, I_3) \geq \eta_0$, where η_0 is still as in (2.2). By (11.64) (and (11.8)), we still have that $H^1(I_i \setminus [Z_{j,k}(r_1) \cup Z_{j,k}(r_2)]) \geq \eta_0/2$. We choose $z_i \in I_i \setminus [Z_{j,k}(r_1) \cup Z_{j,k}(r_2)]$ such that

$$q(z_i) \leq \frac{2}{\eta_0} \int_{\mathfrak{C}'_{j,k}} q(z)\, dH^1(z) \leq C[f(2r_0) + h_1(2r_0)]^{1/2}, \tag{11.67}$$

where q is the measurable function of Lemma 11.37 and by (11.38).

Then (11.65), applied to each z_i, gives two points $y_1(z_i) \in G_{z_i} \cap \partial B(0, r_1)$ and $y_2(z_i) \in G_{z_i} \cap \partial B(0, r_2)$. In addition, Lemma 11.37 says that

$$\left| \frac{y_1(z_i)}{|y_1(z_i)|} - \frac{y_2(z_i)}{|y_2(z_i)|} \right| \leq q(z_i) \leq C[f(2r_0) + h_1(2r_0)]^{1/2}. \tag{11.68}$$

By definition of $Z_{j,k}(r_1)$ and $E^\sharp$, the three points $y_1(z_i)/r_1$ lie in the set g associated to r_1 by the construction of Section 6. Then (11.20) says that

$$\operatorname{dist}(y_1(z_i), X(r_1)) \leq Cr_0 j(r_1)^{1/2} \leq Cr_0[f(2r_0) + h_1(2r_0)]^{1/3} \tag{11.69}$$

because $r_1 \in \mathcal{R}_2$ (see (11.60). Similarly, $\operatorname{dist}(y_2(z_i), X(r_2)) \le Cr_0[f(2r_0) + h_1(2r_0)]^{1/3}$.

For $i = 1, 2$, denote by $R^i_{j,k}$ the great circle of ∂B that contains the arc of $X(r_i) \cap \partial B$ which passes near $\mathfrak{C}'_{j,k}$. With the notations of the definition of $\mathcal{Z}_0(X, \eta_1)$ at beginning of this section, $R^i_{j,k}$ is the great circle of ∂B that contains $\varphi_*(\mathfrak{C}_{j,k})$, where $\varphi : V \to \partial B$ is the mapping that determines $X(r_i)$ as an element of $\mathcal{Z}_0(X, C\tau)$ (see Lemma 11.19).

By (11.69), the position of $y_1(z_1)$ and $y_1(z_3)$ determine $R^1_{j,k}$ within an error of at most $C[f(2r_0) + h_1(2r_0)]^{1/3}$. [We do not need the middle point $y_1(z_2)$, but we use the fact that $\operatorname{dist}(I_1, I_3) \ge \eta_0$.] Similarly, the position of $y_2(z_1)$ and $y_2(z_3)$ essentially says where $R^2_{j,k}$ is. Now (11.68) says that the two $R^i_{j,k}$ are $C[f(2r_0)+h_1(2r_0)]^{1/3}$-close to each other. Since this holds for every index (j, k), we get that $X(r_1)$ is $C[f(2r_0)+h_1(2r_0)]^{1/3}$-close to $X(r_2)$ in $B(0, 1)$, as needed for Lemma 11.61. □

Lemma 11.70. *There is a cone $Y \in \mathcal{Z}_0(X, C\tau)$ centered at 0, such that $\alpha_+(Y) \le C[f(2r_0) + h_1(2r_0)]^{1/3}$ and*

$$\operatorname{dist}(y, E) \le Cr_0[f(2r_0) + h_1(2r_0)]^{1/3} \quad \text{for } y \in Y \cap B(0, r_0). \tag{11.71}$$

We want to take $Y = X(r)$ for some $r \in \mathcal{R}_2$, so the first thing to do is check that $\mathcal{R}_2$ is not empty. In fact,

$$\begin{aligned} H^1((r_0/3, r_0) \setminus \mathcal{R}_2) &\le C[f(2r_0) + h_1(2r_0)]^{-2/3} \int_{[r_0/3, r_0]} j(r)\, dr \\ &\le Cr_0[f(2r_0) + h_1(2r_0)]^{1/3}, \end{aligned} \tag{11.72}$$

because $\mathcal{R}$ has full measure (by (11.10)), by the definition (11.60), and by (11.29).

So we can pick $r_1 \in \mathcal{R}_2$ and set $Y = X(r_1)$. Then $\alpha_+(Y) \le Cj(r_1)^{1/2} \le C[f(2r_0) + h_1(2r_0)]^{1/3}$ by Lemma 11.19 and (11.60), and we just need to prove (11.71).

First consider $y \in Y \cap B(0, r_0) \setminus B(0, r_0/3)$. By (11.72), $\mathcal{R}_2$ is $Cr_0[f(2r_0)+h_1(2r_0)]^{1/3}$-dense, so we can find $r \in \mathcal{R}_2$ such that $|r - |y|| \le Cr_0[f(2r_0) + h_1(2r_0)]^{1/3}$.

By Lemma 11.61, we can find $x \in X(r) \cap \partial B$ such that $\left|x - \frac{y}{|y|}\right| \le C[f(2r_0) + h_1(2r_0)]^{1/3}$. Then $x' = rx$ lies in $X(r) \cap \partial B(0, r)$, and $|x' - y| \le r\left|x - \frac{y}{|y|}\right| + \left|\frac{ry}{|y|} - y\right| \le Cr_0[f(2r_0) + h_1(2r_0)]^{1/3}$.

Finally (11.20) gives a point $\xi \in g \subset r^{-1}E \cap \partial B$ such that $|x - \xi| \le j(r)^{1/2} \le C[f(r_0) + h_1(2r_0)]^{1/3}$; thus $\operatorname{dist}(y, E) \le |y - r\xi| \le |y - x'| + r|x - \xi| \le Cr_0[f(2r_0) + h_1(2r_0)]^{1/3}$, as needed.

To complete the proof of Lemma 11.70, we need to control $Y \cap B(0, r_0/3)$ too. We'll do this by constructing new cones Y_l, $l \ge 0$, corresponding to scales 2^l smaller.

By (11.8), $f(r) + h_1(r) \le C\varepsilon_1$ for $0 < r \le 60r_0$, so the analogue of (11.7) holds with r_0 replaced with $2^{-l}r_0$; thus we can apply the same argument as above, and in particular define sets $\mathcal{R}_2(l) \subset (2^{-l}r_0/3, 2^{-l}r_0)$, $l \ge 0$, and, for $r \in \mathcal{R}_2(l)$, a cone $X_l(r)$ as in Lemmas 11.19 and 11.61. We also get a cone Y_l centered at the origin, and such that

$$\begin{aligned} \operatorname{dist}(y, E) \le C2^{-l}r_0[f(2^{1-l}r_0) + h_1(2^{1-l}r_0)]^{1/3} &\le C'2^{-l}r_0[f(2r_0) + h_1(2r_0)]^{1/3} \\ &\text{for } y \in Y_l \cap B(0, 2^{-l}r_0) \setminus B(0, 2^{-l}r_0/3), \end{aligned} \tag{11.73}$$

where the second inequality comes from (3.6). We need to show that

$$d_{0,1}(Y_l, Y_{l+1}) \le C[f(2^{1-l}r_0) + h_1(2^{1-l}r_0)]^{1/3} \le C[f(2r_0) + h_1(2r_0)]^{1/3} \tag{11.74}$$

for $l \ge 0$. Notice that $\mathcal{R}_2(l)$ meets $\mathcal{R}_2(l+1)$, by the analogues of (11.72) for l and $l+1$. Thanks to Lemma 11.61, we may replace Y_{l+1} with any $X_{l+1}(r)$, $\mathcal{R}_2(l+1)$ without really altering (11.74), so we can assume that Y_l and Y_{l+1} were both defined as sets $X_l(r)$ and $X_{l+1}(r)$ for a same $r \in \mathcal{R}_2(l) \cap \mathcal{R}_2(l+1)$. We still need to say why we can even take $X_l(r) = X_{l+1}(r)$, because the reader may be worried that, since the set X_l that satisfies the analogue of (11.7) and is used in the construction of $X_l(r)$ is different from X_{l+1}, it could be impossible to use $X_l(r)$ instead of $X_{l+1}(r)$, in the construction of $X_{l+1}(r)$. This is not the case. The point is that by (11.7) (and maybe at the cost of replacing ε with $\varepsilon/2$, we can safely use $X_l(r)$ in the construction of ρ and $X_{l+1}(r)$ as the cone over ρ (see below (11.22)). We still have that $X_l(r) \in \mathcal{Z}_0(X_{l+1}, C\tau)$ by the same argument as for X_l (and maybe with a slightly larger C), so the rest of the argument goes through essentially unchanged, and we can use X_{l+1} or X_l indifferently.

So we have (11.74), and it is now easy to deduce (11.71) from (11.73). Indeed, let $y \in Y \cap B(0, r_0)$ be given, and let $l \ge 0$ be such that $y \in Y \cap B(0, 2^{-l}r_0) \setminus B(0, 2^{-l}r_0/3)$. If $l = 0$, we can use (11.73) for $l = 0$ and with $Y_0 = Y$. Otherwise, (11.74) says that we can find $y_1 \in Y_1 \cap B(0, 2^{-l}r_0) \setminus B(0, 2^{-l}r_0/3)$, with $|y_1 - y| \le C2^{-l}r_0[f(2r_0) + h_1(2r_0)]^{1/3}$. [Recall that Y and Y_1 are both cones, so $d_{0,2^{-l+1}}(Y_l, Y_{l+1}) = d_{0,1}(Y_l, Y_{l+1})$.] If $l > 1$, we can even iterate this to find $y_l \in Y_l \cap B(0, 2^{-l}r_0) \setminus B(0, 2^{-l}r_0/3)$, with $|y_1 - y| \le Cl2^{-l}r_0[f(2r_0) + h_1(2r_0)]^{1/3}$.

Finally, (11.73) gives a point of E at distance at most $C2^{-l}r_0[f(2r_0) + h_1(2r_0)]^{1/3}$ from y_l, and altogether $\operatorname{dist}(y, E) \le C(l+1)2^{-l}r_0[f(2r_0) + h_1(2r_0)]^{1/3}$. This completes our proof of (11.71), and Lemma 11.70 follows. □

Lemma 11.75. *We also have that*

$$\operatorname{dist}(x, Y) \le Cr_0[f(2r_0) + h_1(2r_0)]^{1/3} \quad \text{for } x \in E \cap B(0, 99r_0/100). \tag{11.76}$$

Observe that (11.76) and (11.71) imply that $d_{0,99r_0/100}(E, Y) \le C[f(2r_0)+h_1(2r_0)]^{1/3}$. Since $\alpha_+(Y) \le C[f(2r_0)+h_1(2r_0)]^{1/3}$ by Lemma 11.70 and $f(2r_0)+h_1(2r_0) \le f(110r_0)+Ch_1(110r_0)]$ by (3.6), this is almost the same thing as (11.5); the only difference is the extra 99/100, but the reader will agree that we could easily have proved Lemma 11.70 with $100r_0/99$, and then the proof of (11.76) below would have yielded (11.5).

So we just need to prove Lemma 11.75, and Theorem 11.4 will follow. Let $x \in E \cap B(0, 99r_0/100)$ be given. We may assume that $x \ne 0$, because the origin lies in Y anyway. Let $l \ge 0$ be such that $x \in B(0, 2^{-l} \cdot 99r_0/100) \setminus B(0, 2^{-l-1} \cdot 99r_0/100)$, We want to show that there is a geometric constant C_1 such that, for $0 < t < C_1^{-1}2^{-l}r_0$, there is a minimal cone $Z(t)$ of type $\mathbb{P}$ or $\mathbb{Y}$, not necessarily centered at x, such that

$$d_{x,t}(E, Z(t)) \le \alpha, \tag{11.77}$$

where the small constant $\alpha > 0$ will be chosen later (depending on the constant η_0 of Section 2. Recall that because of (11.8) and by the proof of (11.7), there is a minimal cone X_l (centered at the origin) such that

$$d_{0,2^{-l}r_0}(E, X_l) \le \varepsilon, \tag{11.78}$$

as in (11.6). We want to use the fact that since x is not too close to the center of X_l, there is a small ball centered at x where X_l coincides with a cone of type $\mathbb{Y}$ or $\mathbb{P}$, and then we can use the results of [D3] to control E in smaller balls centered at x. Later on, we shall compare our results with (11.71), and get that x is close to Y.

We start with the case when the spine of X_l meets $B(x, 2^{-l-k}r_0)$, where the value of $k > 0$ will be decided soon. By (11.78), we can find $z \in E \cap B(x, 2^{-l-k+1}r_0)$, which lies within $\varepsilon 2^{-l}r_0 \leq 2^{-l-k+1}r_0$ of the spine of X_l. If k is large enough, depending on η_0 (in the description of minimal cones in Section 2), X_l coincides with a cone of type $\mathbb{Y}$ in $B(z, 2^{-l-k+11}r_0)$. That is, $B(z, 2^{-l-k+11}r_0)$ is so small compared to its distance to the origin that it only meets the three faces of X_l that we already know about. We translate this cone of type $\mathbb{Y}$ so that its spine contains z, and we get a cone Y' centered at z such that

$$d_{z,2^{-l-k+10}r_0}(E, Y') \leq C\varepsilon. \tag{11.79}$$

If ε is small enough, we can apply Lemma 16.51 in [D3], and we get that $B(z, 2^{-l-k+2}r_0)$ is a biHölder ball of type $\mathbb{Y}$ for E. To be precise, we are not interested in the existence of a biHölder parameterization this time, but only in the fact that for every ball $B(y,t)$ centered on E and contained in $B(z, 2^{-l-k+2}r_0)$, there is a minimal cone $Z(y,t)$, of type $\mathbb{Y}$ or $\mathbb{P}$, such that $d_{y,t}(E, Z(y,t)) \leq \varepsilon'$. In [D3], this property is proved with an ε' which is chosen so small that we can then apply Corollary 15.11 there, and get the biHölder parameterization. But here, we only need to take $\varepsilon' = \alpha$, $y = x$, and $0 < t < 2^{-l-k+1}r_0$ (recall that $z \in E \cap B(x, 2^{-l-k+1}r_0)$). That is, we checked (11.77) in this first case.

Now suppose that the spine of X_l does not meet $B(x, 2^{-l-k}r_0)$. If k is large enough (our final condition on k), (2.3) says that the smaller ball $B(x, 2^{-l-k-1}r_0)$ only meets one face of X_l. That is, X_l coincides with a plane P in $B(x, 2^{-l-k-1}r_0)$. We even know that $\operatorname{dist}(x, P) \leq \varepsilon 2^{-l}r_0$, because $x \in E$ and by (11.78). Translate P so that it passes through x; we get a plane P' such that

$$d_{z,2^{-l-k-2}r_0}(E, P') \leq C\varepsilon. \tag{11.80}$$

If ε is small enough, we can apply Lemma 16.48 in [D3], and we get that $B(x, 2^{-l-k-6}r_0)$ is a biHölder ball of type $\mathbb{P}$ for E. Again, we only need a simple consequence of the proof, namely, the fact that for $t < 2^{-l-k-6}r_0$, there is a plane $P(t)$ such that $d_{x,t}(E, Z(t)) \leq \alpha$. So (11.77) holds in this second case too, and we can take $C_1 = 2^{k+6}$.

Next we want to compare the sets $Z(t)$ to Y_l, where Y_l is the analogue of Y for the smaller ball $B(0, 2^{-l}r_0)$, that shows up in (11.73).

Set $\delta = r_0[f(2r_0) + h_1(2r_0)]^{1/3}$ for convenience, and let us show that

$$Y_l \text{ meets } B(x,t) \text{ for } C_2\delta \leq t \leq 2^{-l}r_0, \tag{11.81}$$

where C_2 will be chosen soon. When $(10C_1)^{-1}2^{-l}r_0 \leq t \leq 2^{-l}r_0$, we first use (11.78) to find $x_1 \in X_l \cap B(x, t/2)$; then we use the fact that by construction, Y_l is $C\tau 2^{-l}r_0$-close to X_l in $B(0, 2^{-l}r_0)$ (recall that Y_l was chosen via Lemma 11.70). This gives $y \in Y_l$ such that $|y - x_1| \leq C\tau 2^{-l}r_0 < t/2$ (if τ is small enough), and then $y \in Y_l \cap B(x,t)$, as needed.

Now suppose that Y_l meets $B(x,t)$ for some $t \le (10C_1)^{-1}2^{-l}r_0$. We want to show that Y_l also meets $B(x,t/2)$ if in addition $t \ge C_2\delta$. By (11.77), we have a cone $Z(10t)$ of type $\mathbb{P}$ or $\mathbb{Y}$, which is $10\alpha rt$-close to E in $B(x,10t)$. Let us check that

$$\operatorname{dist}(y, Z(10t)) \le 20\alpha t \text{ for } y \in Y_l \cap B(x,9t). \tag{11.82}$$

Let $y \in Y_l \cap B(x,9t)$ be given. Observe that $x \in B(0, 2^{-l}\cdot 99r_0/100) \setminus B(0, 2^{-l-1}\cdot 99r_0/100)$ by definition of l, so $y \in B(x,9t) \subset B(0,2^{-l}r_0 \setminus B(0,2^{-l}r_0/3)$ (because $t \le (10C_1)^{-1}2^{-l}r_0$ and $C_1 > 100$), and (11.73) says that we can find $z \in E$ such that $|z-y| \le C2^{-l}\delta$. So $|z-y| \le 10\alpha t$ if $t \ge C_2\delta$ (and C_2 is large enough). Then $z \in E \cap B(x,10t)$, and $\operatorname{dist}(z, Z(10t)) \le 10\alpha t$ by definition of $Z(10t)$; (11.82) follows.

Now Y_l is not exactly a minimal cone, but it lies in $\mathcal{Z}_0(X_l, C\tau)$ (because it was chosen using Lemma 11.70), and X_l is a minimal cone. The fact that in $B(x,9t)$, it is contained in a thin tube around $Z(10t)$ (by (11.82)), which itself is a minimal cone of type $\mathbb{P}$ or $\mathbb{Y}$ and contains points in $B(x,t/100)$ (by (11.77)), implies that Y_l meets $B(x,t/2)$. Indeed, $Z(10t) \cap \partial B(0,|x|) \cap B(x,10t)$ is an arc of geodesic in $\partial B(0,|x|)$ or a fork composed of three arcs of geodesics, and $Y_l \cap \partial B(0,|x|) \cap B(x,9t)$ is also composed of arcs of geodesics, meets $B(x,2t)$, and stays close to $Z(10t) \cap \partial B(0,|x|) \cap B(x,10t)$, so it meets $B(x,t/2)$. So we are able to go from (11.81) for t to (11.81) for $t/2$ when $t \ge C_2\delta$, and this completes the proof of (11.81).

If the range of t in (11.81) is empty, then $\operatorname{dist}(x,Y) \le |x| \le 2^{-l}r_0 \le C_2\delta$ by definition of l; in this case the inequality in (11.76) holds trivially. Otherwise, apply (11.82) to $t = C_2\delta$. This gives a point $y_l \in Y_l$ such that $|y-x| \le C_2\delta$, and we may even assume that $|y| \le 2|x| \le 2^{-l+1}r_0$. Then we apply (11.74) many times, get successive points in the Y_m, $0 \le m \le l$, and at the end a point $y_0 \in Y_0 = Y$ such that $|y_0 - x| \le C_2\delta$. The argument is the same as for the proof of (11.71) once we have (11.74).

This completes our proof of (11.76). Lemma 11.75 follows, and so does Theorem 11.4 (see the comments below the statement of Lemma 11.75). □

Remark 11.83. In fact we could have proved the slightly more precise fact: if h, x, and r are as in the statement of Theorem 11.4, and if $f(110r_0) + h_1(110r_0)$ is small enough, as in (11.6), then

$$\beta_Z(x,r_0) \le Cf(2r_0)^{1/3} + Ch_1(2r_0)^{1/3}. \tag{11.84}$$

That is, we could replace 110 in (11.5) with 2. Of course this looks like a better estimate, but it is unlikely that we shall ever need the improvement, and so we shall stick with the slightly uglier 110. But we shall use the following more specific version of Theorem 11.4.

Corollary 11.85. *For each small enough $\tau > 0$, we can find $\varepsilon > 0$ and $C > 0$ such that if E is a reduced almost minimal set in U with gauge function h, $x \in E$, and $r_0 > 0$ are such that $B(x,110r_0) \subset U$ and we can find a minimal cone X centered at 0 such that $d_{x,100r_0}(E,X) \le \varepsilon$, then we can find a cone $Y \in \mathcal{Z}_0(X,\tau)$ such that*

$$d_{x,r}(E,Y) + \alpha_+(Y) \le C[f(110r_0) + h_1(110r_0)]^{1/3}. \tag{11.86}$$

This is indeed what we proved once we found the minimal cone X in (11.7), except that we just need to replace τ with a slightly smaller constant, to account for the $C\tau$ in Lemma 11.70, for instance. The interest compared Theorem 11.4 is that we say that we can keep the same minimal cone X as a basis for our computations; this may be useful in dimensions $n \geq 4$, for instance if we know that X has the full length property. As before, (11.86) is only interesting when $[f(110r_0) + h_1(110r_0)]^{1/3}$ is smaller than ε. □

12. Joint decay and a proof of Theorem 1.15

In this section we indicate how to deduce Theorem 1.15, or variants, by combining Theorem 4.5, Corollary 11.85, and the simple estimates from Section 5.

As before, E is a reduced almost minimal set in U with gauge function h, we assume that the origin lies in E, and we pick $r_0 > 0$ such that $B(0, 110r_0) \subset U$. In order to simplify the computations, we assume that

$$h(r) \leq C_0 r^b \quad \text{for } 0 < r < 220r_0 \tag{12.1}$$

for some choice of constants $C_0 \geq 0$ and $b \in (0, 1]$. Note that then

$$h_1(r) \leq CC_0 r^b \quad \text{for } 0 < r < 110r_0; \tag{12.2}$$

here and below, we denote by C various constants that depend on n and b, but not on C_0 or ε. We also assume that

$$d_{0,100r_0}(E, X) \leq \varepsilon/500 \tag{12.3}$$

for some cone X centered at 0. Recall that the normalized local Hausdorff distance $d_{x,r}$ is defined in (11.3); here ε is essentially chosen as in the previous sections (and in particular Section 6 and Corollary 11.85), but we shall give a more precise account of quantifiers in the statement of Theorem 12.8. We also assume that

$$H^2(X \cap B(0, 1)) \leq d(0) \tag{12.4}$$

and that

(12.5) X is a full length minimal cone, with constants $\eta_1 \leq \eta_0/10$ and $C_1 \geq 1$.

See Definition 2.10 for full length minimal cones, and recall that η_0 is the small geometric constant from (2.2) and (2.3). We further assume that

$$f(110r_0) + C_0 r_0^b \leq \varepsilon^3, \tag{12.6}$$

where $f(r) = \theta(r) - d(0)$ is as in (3.5).

Our statement will also use the constant $a = \frac{4\alpha}{1-2\alpha} > 0$ introduced in Lemma 5.11, where α is the small positive in constant in (4.8). Thus a depends only on C_1 and the dimension n. In dimension 3, all the minimal cones have the full length property with

a fixed C_1 (see Section 14), so we do not need to mention (12.5) and a is an absolute constant. We choose an exponent b_1 such that

$$0 < b_1 \le b \quad \text{and} \quad b_1 < a. \tag{12.7}$$

Theorem 12.8. *For each choice of $b \in (0,1]$, $C_1 \ge 1$, $\eta_1 \le \eta_0/10$, b_1 as in (12.7), and $\tau > 0$, there are constants $C \ge 0$ and $\underline{\varepsilon} > 0$ such that the following statement holds for $0 < \varepsilon \le \underline{\varepsilon}$. Let E be a reduced almost minimal set in $U \subset \mathbb{R}^n$, with gauge function h, suppose that $0 \in E$, and let $r_0 > 0$ be such that $B(0, 110r_0) \subset U$. Suppose that (12.1) holds for some $C_0 \ge 0$, and that we have (12.6). Finally suppose that (12.3) holds for some minimal cone X such that (12.4) and (12.5) hold. Then there is a minimal cone Y such that*

$$d_{0,r}(E,Y) \le C\big(r/r_0\big)^{b_1/3}\big\{f(110r_0) + C_0 r_0^b\big)\big\}^{b_1/3} \le C\big(r/r_0\big)^{b_1/3}\varepsilon \tag{12.9}$$

for $0 < r \le r_0/2$,

$$Y \in \mathcal{Z}_0(X,\tau), \tag{12.10}$$

and

$$d_{0,1}(Y,X) \le C\varepsilon. \tag{12.11}$$

In addition, Y is the unique blow-up limit of E at 0, and $H^2(Y \cap B) = d(0)$.

The reader will find in [D3] a few examples of sufficient conditions that allow us to apply Theorem 12.8 (and Corollary 12.25 below).

Before we start with the proof of Theorem 12.8, observe that the quantifiers in its statement are compatible with what we did so far. We shall keep the same choices as in the previous sections, except that in two occasions (near (12.18) and (12.20)) we shall again require ε to be small enough.

We want to get Y as the limit of a sequence of cones $Y_l \in Z_0(X,\tau)$, which we shall construct by induction, so that

$$d_{0,2^{-l}r_0}(E,Y_l) + \alpha_+(Y_l) \le C2^{-b_1 l/3}\mu, \tag{12.12}$$

where we set

$$\mu = \big\{f(110r_0) + C_0 r_0^b\big\}^{1/3} \le \varepsilon \tag{12.13}$$

(by (12.6)). Observe that the first cone Y_0 exists, directly by Corollary 11.85, (12.2), (12.3), and (12.6).

Now let $l \ge 1$ be given, assume that we constructed $Y_0, \cdots, Y_{l-1}$, and let us find Y_l. First observe that for $1 \le j < l$ (if $l > 1$; otherwise we do not need the information),

$$\begin{aligned} d_{0,1}(Y_j, Y_{j-1}) = d_{0,2^{-j-1}r_0}(Y_j, Y_{j-1}) &\le 2d_{0,2^{-j}r_0}(Y_j,E) + 2d_{0,2^{-j}r_0}(E,Y_{j-1}) \\ &\le 2d_{0,2^{-j}r_0}(Y_j,E) + 4d_{0,2^{-j+1}r_0}(E,Y_{j-1}) \le 6C2^{-b_1 j/3}\mu \end{aligned} \tag{12.14}$$

because Y_j and Y_{j-1} are cones, by the triangle inequality, and by (12.12). Similarly,

$$d_{0,1}(Y_0, X) = d_{0,r_0/2}(Y_0, X) \leq 2d_{0,r_0}(Y_0, E) + 200d_{0,100r_0}(E, X) \leq 2C\mu + 2\varepsilon/5 \tag{12.15}$$

by (12.12) and (12.3). Let us check that

$$\text{almost-every } r \in (2^{-l}r_0, r_0) \text{ satisfies the assumptions of Theorem 4.5.} \tag{12.16}$$

The first assumption (4.1) is satisfied by (12.6), because h_1 is nondecreasing, and by (3.6). Next (4.3) and (4.4) hold almost-everywhere, by Lemma 4.12. So we just need to prove (4.2), i.e., that

$$d_{0,100r}(E, X) \leq \varepsilon', \tag{12.17}$$

where we denote by ε' the small constant in (4.2) (to avoid confusion), because the additional assumptions on X are satisfied, by (12.4) and (12.5). [This is why we want to keep the same X; incidentally the issue would not arise in dimension 3.]

If $r \geq r_0/500$, (12.17) follows from (12.3). Otherwise, let j denote the largest integer such that $200r \leq 2^{-j}r_0$; then $0 \leq j < l$ because $2^{-l}r_0 < r \leq r_0/500$, and

$$d_{0,100r}(E, X) \leq 2d_{0,200r}(E, Y_j) + 2d_{0,1}(Y_j, X) \leq C'\mu + 4\varepsilon/5 \leq C\varepsilon \tag{12.18}$$

again because Y_j and X are cones, by (12.15), iterations of (12.14), and (12.13). So (12.17) holds if ε is small enough. So (12.16) holds, we can apply Theorem 4.5 to almost every $r \in (2^{-l}r_0, r_0)$, and we get (4.8). This allows us to apply Lemma 5.11, with $y = r_0$ and any $x \in (2^{-l}r_0, r_0)$. We get that

$$\begin{aligned}
f(x) &\leq (x/r_0)^a f(r_0) + Cx^a \int_x^{r_0} r^{-a-1}h(2r)dr \\
&= (x/r_0)^a f(r_0) + CC_0 x^a \int_x^{r_0} r^{b-a-1}dr \\
&\leq (x/r_0)^a f(r_0) + CC_0 x^a \int_x^{r_0} r_0^{b-b_1} r^{b_1-a-1}dr \\
&= (x/r_0)^a f(r_0) + CC_0 r_0^{b-b_1} x^{b_1} \\
&\leq (x/r_0)^a [f(110r_0) + Ch_1(110r_0)] + CC_0 r_0^b (x/r_0)^{b_1} \\
&\leq (x/r_0)^{b_1} [f(110r_0) + CC_0 r_0^b] + CC_0 r_0^b (x/r_0)^{b_1} \leq C(x/r_0)^{b_1}\mu^3
\end{aligned} \tag{12.19}$$

by (5.13), (3.6), because $b_1 < a$, and by (12.2) and (12.13). If $110 \cdot 2^{-l} < 1$, we apply this to $x = 110 \cdot 2^{-l}r_0$, and we get that

$$f(110 \cdot 2^{-l}r_0) \leq C2^{-b_1 l}\mu^3 \leq C2^{-b_1 l}\varepsilon^3 \tag{12.20}$$

(by (12.13)). Otherwise, we simply observe that $f(110 \cdot 2^{-l}r_0) \leq f(110r_0) + Ch_1(110r_0) \leq f(110r_0) + CC_0 r_0^b) \leq C\mu^3$ by (3.6), (12.2), and (12.13). This also yields (12.20) (with a slightly larger constant), because $2^l \leq 110$.

By (12.20) and (12.18) or (12.3), and (for the last time) if ε is small enough, we may apply Corollary 11.85 to the radius $r = 2^{-l}r_0$; we get a cone $Y_l \in \mathcal{Z}_0(X,\tau)$ such that

$$\begin{aligned} d_{0,r}(E,Y_l) + \alpha_+(Y_l) &\le C[f(110r) + h_1(110r)]^{1/3} \le C2^{-b_1 l/3}\mu + C[C_0 r^b]^{1/3} \\ &\le C2^{-b_1 l/3}\mu + C(r/r_0)^{b/3}\mu \le C2^{-b_1 l/3}\mu, \end{aligned} \tag{12.21}$$

by (12.20), (12.13), and because $b_1 \le b$. This is the same as (12.12), and so we can construct the cones Y_l by induction on l.

By (12.14), the Y_l have a limit Y when l tends to $+\infty$, and

$$d_{0,1}(Y,Y_l) \le \sum_{j\ge l} d_{0,1}(Y_j,Y_{j+1}) \le C2^{-b_1 l/3}\mu \tag{12.22}$$

(where we use again the fact that the Y_l are cones to simplify the Hausdorff distance computations). Then

$$d_{0,2^{-l-1}r_0}(E,Y) \le d_{0,2^{-l}r_0}(E,Y_l) + 2d_{0,1}(Y,Y_l) \le C2^{-b_1 l/3}\mu \tag{12.23}$$

by (12.22) and (12.12). For (12.9) we also need the continuous version of (12.23), so let $r \in (0, r_0/2]$ be given, and apply (12.23) to the largest $l \ge 0$ such that $r \le 2^{-l-1}r_0$; we get that

$$\begin{aligned} d_{0,r}(E,Y) &\le 2d_{0,2^{-l-1}r_0}(E,Y) \le C2^{-b_1 l/3}\mu \le C\big(r/r_0\big)^{b_1/3}\mu \\ &= C\big(r/r_0\big)^{b_1/3}\big\{f(110r_0) + C_0 r_0^b\big\}^{1/3} \le C\big(r/r_0\big)^{b_1/3}\varepsilon \end{aligned} \tag{12.24}$$

by (12.13). This is just (12.9).

We already know that each Y_l lies in $\mathcal{Z}_0(X,\tau)$ (see above (12.21)); (12.10) follows by taking limits. Next, (12.11) follows from (12.15), (12.14), and (12.13). By (12.24), $r^{-1}E$ tends to Y when r tends to 0, so Y is the only blow-up limit of E at 0. Then Proposition 3.15 says that Y is a minimal cone, with $H^2(Y \cap B) = d(0)$. This completes our proof of Theorem 12.8.

Corollary 12.25. *For each choice of $n \ge 3$, $b \in (0,1]$, $C_1 \ge 1$, $\eta_1 > 0$ we can find $\beta > 0$ and $\varepsilon_1 > 0$ such that the following holds. Let $U \subset \mathbb{R}^n$ be open and let $E \subset U$ be a reduced almost minimal set in $U \subset \mathbb{R}^n$, with gauge function h. Suppose that $0 \in E$, $r_0 > 0$ is such that $B(0,110r_0) \subset U$, (12.1) holds for some choice of $b \in (0,1]$ and $C_0 \ge 0$,*

$$f(110r_0) + C_0 r_0^b \le \varepsilon_1, \tag{12.26}$$

and

$$d_{0,100r_0}(E,X) \le \varepsilon_1 \tag{12.27}$$

for some full length minimal cone X centered at 0 such that (12.4) and (12.5) hold. Then for $0 < r \le r_0$ there is a $C^{1,\beta}$ diffeomorphism $\Phi : B(0,2r) \to \Phi(B(0,2r))$, such that $\Phi(0) = 0$, $|\Phi(y) - y| \le 10^{-2} r$ for $y \in B(0,2r)$, and $E \cap B(0,r) = \Phi(X) \cap B(0,r)$.

As we shall see later, Theorem 1.15 follows fairly easily from Corollary 12.25. In turn, Corollary 12.25 will be easily deduced from the extension in [DDT] of Reifenberg's topological disk, once we have the following uniform control on the good approximation of E by minimal cones in balls $B(x,r)$ near the origin, which will be proved first.

Proposition 12.28. *Let $\varepsilon_0 > 0$ be given, and then let E and r_0 be as in Corollary 12.25, with the same quantifiers, except that now ε_1 may depend on ε_0 too. Then for $x \in E$ and $r > 0$ such that $x \in E \cap B(0, 10r_0)$ and $0 < r < 10r_0$, we can find a minimal cone $Z(x,r)$, not necessarily centered at x or at the origin, such that*

$$d_{x,r}(E, Z(x,r)) \le (r/r_0)^{\beta} \varepsilon_0. \tag{12.29}$$

We choose the constant b_1 as in (12.7). The minimal cones $Z(x,r)$ will be easier to find when $x = 0$, so we start with this case.

Lemma 12.30. *There is a minimal cone Y such that*

$$d_{0,r}(E, Y) \le (r/r_0)^{b_1/3} \varepsilon_0 \text{ for } 0 \le r \le 100 r_0. \tag{12.31}$$

We want to apply Theorem 12.8. Let $\underline{\varepsilon}$ be as in the statement, and choose $\varepsilon \le \underline{\varepsilon}$ such that $C\varepsilon$ in (12.11) is smaller that $10^{-3}\varepsilon_0$. Next we check the assumptions; (12.1) and (12.4) hold by assumption, (12.3) follows from (12.27) if $\varepsilon_1 < \varepsilon/500$ (and X satisfies (12.4) and (12.5) by assumption), and finally (12.6) follows from (12.26) if $\varepsilon_1 < \varepsilon^3$.

So Theorem 12.8 applies, and we get a minimal cone Y as in its statement. For $0 \le r \le r_0/2$, we apply (12.9) and get that

$$d_{0,r}(E, Y) \le C\big(r/r_0\big)^{b_1/3} \varepsilon \tag{12.32}$$

as needed for (12.31). When $r_0/2 \le r \le 100 r_0$,

$$\begin{aligned} d_{0,r}(E,Y) &\le d_{0,r}(E,X) + d_{0,1}(X,Y) \le (100 r_0/r) d_{0,100r_0}(E,X) + C\varepsilon \\ &\le (100 r_0/r)\varepsilon_1 + 10^{-3}\varepsilon_0 \le (r/r_0)^{b_1/3}\varepsilon_0 \end{aligned} \tag{12.33}$$

because X and Y are cones, by (12.11), (12.27), our choice of ε, and if ε_1 is small enough. Lemma 12.30 follows. Note that in addition we get (12.11) and (12.10), where $\tau > 0$ is any small number given in advance. □

We shall also apply Lemma 12.30 with smaller values of ε_0; this will be helpful to get additional properties, and the only cost will be that we need to take ε_1 even smaller.

Next we want to take care of the small balls centered on the set E_Y of Y-points of $E \cap B(0, 50r_0)$. Recall that E_Y is the set of points $y \in E$ such that every blow-up limit of E at y is a cone of type $\mathbb{Y}$. By Proposition 3.14 and Lemma 14.12 in [D3],

$$E_Y = \big\{ y \in E \,;\, d(y) = 3\pi/2 \big\}. \tag{12.34}$$

Lemma 12.35. *For $y \in E \cap B(0, 50r_0) \setminus \{0\}$ there is a minimal cone $Y(y)$ of type $\mathbb{Y}$, centered at y, such that*

$$d_{y,r}(E, Y(y)) \leq (r/|y|)^{b_1/3}\varepsilon_0 \quad \text{for } 0 \leq r \leq 10^{-2}\eta_0|y|, \tag{12.36}$$

where η_0 is the constant from Section 2.

Let $y \in E_Y \cap B(0, 50r_0)$ be given, set $\delta = 10^{-4}\eta_0|y|$, and let us try to apply Lemma 12.30 to the ball $B(y, \delta)$ (or to be exact, to $E - y$ and $B(0, \delta)$).

We shall prove the analogue of (12.27) with a minimal cone Z of type $\mathbb{Y}$; observe that Z automatically satisfies (12.4), because $d(y) = 3\pi/2$ by (12.34), and (12.5) with some constant that does not even depend on n, by Section 14.

Denote by ε_1' the value of ε_1 in (12.26) and (12.27) that is needed for Lemma 12.30 to hold, with ε_0 replaced with the slightly smaller $[10^{-4}\eta_0]^{b_1/3}\varepsilon_0$, and let us check the assumptions.

Obviously (12.1) holds, and $B(y, 110\delta) \subset B(0, 100r_0) \subset U$. Then we need to check the analogue of (12.26) and (12.27). For (12.27), we need to find a minimal cone Z' of type $\mathbb{Y}$, centered at y, such that

$$d_{y,100\delta}(E, Z') \leq \varepsilon_1'. \tag{12.37}$$

For (12.26), observe that since $d(y) = 3\pi/2$ by (12.34), we need to prove that

$$(110\delta)^{-2}H^2(E \cap B(y, 110\delta)) - \frac{3\pi}{2} + C_0\delta^b \leq \varepsilon_1'. \tag{12.38}$$

In order to obtain both things, we apply Lemma 12.30 to $B(0, r_0)$, but with a smaller constant ε_0' to be chosen soon. The assumptions are still satisfied (if ε_1 is small enough), so we get a minimal cone Y such that (12.31) holds. In particular, taking $r = 2|y|$, we get that

$$d_{0,2|y|}(E, Y) \leq (2|y|/r_0)^{b_1/3}\varepsilon_0' \leq \varepsilon_0'. \tag{12.39}$$

Let us check that there is a minimal cone Z, of type $\mathbb{Y}$ or $\mathbb{P}$, such that

$$Y \cap B(y, 200\delta) = Z \cap B(y, 200\delta). \tag{12.40}$$

Recall from Section 2 that $K' = Y \cap \partial B(0, |y|)$ is composed of great circles or arcs of great circles $\mathfrak{C}_j'$.

Set $D = B(y, \eta_0|y|/10) \cap \partial B(0, |y|)$, and notice that D meets K' because Y is a cone that contains point very near y (by (12.39) and because $y \in E$). Let us first assume that D meets only one circle or arc of circle $\mathfrak{C}_j'$. Denote by Z the plane that contains $\mathfrak{C}_j'$, and let us check (12.40). Observe that D contains no endpoint of $\mathfrak{C}_j'$, because each endpoint meets two other arcs of K'; thus $\mathfrak{C}_j'$ crosses D, and $Z \cap D = \mathfrak{C}_j' \cap D = K' \cap D$. Now Z is a cone, and Y is the cone over K', so Z and Y coincide in $B(y, \eta_0|y|/20)$; (12.40) follows because $200\delta = 2 \cdot 10^{-2}\eta_0|y| < \eta_0|y|/20$.

Now assume that D meets at least two different circles or arcs $\mathfrak{C}'_i$ and $\mathfrak{C}'_j$. Pick $\xi \in \mathfrak{C}'_i$; By (2.3), $\mathfrak{C}'_i$ and $\mathfrak{C}'_j$ have a common endpoint x in $B(\xi, \eta_0|y|/5) \subset B(y, 2\eta_0|y|/5)$. Still by the description in Section 2, there is a third arc $\mathfrak{C}'_k$ that ends at x, and the three arcs leave from x with 120° angles. Denote by Z the minimal cone of type $\mathbb{Y}$ that contains the beginning of these three arcs; we want to show that (12.40) holds for this Z.

First observe that (2.2) says that the lengths of $\mathfrak{C}'_i$, $\mathfrak{C}'_j$, and $\mathfrak{C}'_k$ are at least $10\eta_0|y|$, so these arcs leave D and $K' \cap D$ contains $Z \cap D$. We want to show that $K' \cap D \subset Z$ as well.

Suppose D meets some other $\mathfrak{C}'_l$. As before, $\mathfrak{C}'_l$ has a common extremity with $\mathfrak{C}'_i$ in $B(y, 2\eta_0|y|/5)$, which we call y_i. Note that $y_i \neq x$ because the three ends at x are already taken. Similarly, $\mathfrak{C}'_l$ has a common endpoint y_j with $\mathfrak{C}'_j$ and a common endpoint y_k with $\mathfrak{C}'_k$, both in $B(y, 2\eta_0|y|/5)$ and different from x. Since $\mathfrak{C}'_l$ has only two ends, two of these points are equal, for instance y_j and y_k. But this is impossible, as the only possible intersections of $\mathfrak{C}'_j$ and $\mathfrak{C}'_k$ are x and $-x$ (they leave from x with 120° angles), and we said that $y_j \in B(y, 2\eta_0|y|/5)$. So D meets no other $\mathfrak{C}'_l$.

Another weird possibility would be that $\mathfrak{C}'_i$, say, is very long and returns to D after leaving it. Then it would have another end in D (because (2.3) says that it is not a full circle), and at this end it would meet two other arcs. We just proved that these other arcs cannot be different from $\mathfrak{C}'_j$ and $\mathfrak{C}'_k$, but also that $\mathfrak{C}'_i$ cannot share two ends in $B(y, 2\eta_0|y|/5)$ with $\mathfrak{C}'_j$ or $\mathfrak{C}'_k$. So $D \cap K'$ is reduced to the three arcs that leave from x, and hence $K' \cap D = Z \cap D$. As before, this implies that $Y \cap B(y, \eta_0|y|/20) = Z \cap B(y, \eta_0|y|/20)$ because Y and Z are cones, and then (12.40) holds in this second case as well.

Because of (12.40), (12.39) implies that

$$d_{y,190\delta}(E, Z) = d_{y,190\delta}(E, Y) \leq \frac{2|y|}{190\delta}\, d_{0,2|y|}(E, Y) \leq C\varepsilon'_0 \tag{12.41}$$

because $\delta = 10^{-4}\eta_0|y|$ (and so $B(0, 190\delta) \subset B(0, 2|y|)$). It does not matter that C depends on η_0.

Let us apply Lemma 16.43 in [D3] to $B(0, 190\delta)$, with $F = Z$ and $10^{-2}\varepsilon'_1$ playing the role of δ there. Recall that ε'_1 was already chosen, a little before (12.37). The assumptions are satisfied if ε_1 is small enough (so that (12.26) controls $h(300\delta)$) and ε'_0 is small enough (so that (12.41) controls the distance to Z). we get that

$$\begin{aligned} H^2(E \cap B(y,&110\delta)) \leq H^2(Z \cap B(y, (1 + 10^{-2}\varepsilon'_1)110\delta)) + 10^{-2}\varepsilon'_1(190\delta)^2 \\ &\leq \frac{3\pi}{2}[(1 + 10^{-2}\varepsilon'_1)110\delta)]^2 + 10^{-2}\varepsilon'_1(190\delta)^2 \leq (110\delta)^2\Big\{\frac{3\pi}{2} + \frac{\varepsilon'_1}{2}\Big\} \end{aligned} \tag{12.42}$$

because Z is of type $\mathbb{Y}$ or $\mathbb{P}$, so its density in any ball is at most $3\pi/2$. We divide by $(110\delta)^2$, add $C_0\delta^b \leq C_0 r_0^b \leq \varepsilon_1 \leq \varepsilon'_1/2$ by (12.26) and if ε_1 is small enough, and get (12.38).

Unfortunately, we cannot use Z directly for (12.37), because we do not know that it is centered at y. It would not matter much if Z were a plane, but if we want to apply Lemma 12.30 on $B(y, \delta)$, it is important to use a cone centered at y. So let us check that

$$Z \text{ is of type } \mathbb{Y}, \text{ and its spine meets } B(y, 10\varepsilon'_1\delta). \tag{12.43}$$

Suppose not. Then Z coincides with a plane in $B(y, 2\varepsilon_1'\delta)$. Let us apply Lemma 16.43 in [D3] again, this time in $B(y, 3\varepsilon_1'\delta)$, again with $F = Z$, and with $\delta = 10^{-2}$. We can still use (12.26) to control $h(3\varepsilon_1'\delta)$, and we observe that

$$d_{0,4\varepsilon_1'\delta}(E, Z) \le \frac{190}{4\varepsilon_1'}\, d_{0,190\delta}(E, Z) \le C\varepsilon_0'/\varepsilon_1' \tag{12.44}$$

by (12.41). Again we take ε_0' small enough, depending on ε_1', apply the lemma, and get that

$$\begin{aligned} H^2(E \cap B(y, \varepsilon_1'\delta)) &\le H^2(Z \cap B(y, (1 + 10^{-2})\varepsilon_1'\delta) + 10^{-2}(3\varepsilon_1'\delta)^2 \\ &\le \pi\,[(1 + 10^{-2})\varepsilon_1'\delta]^2 + 10^{-2}(3\varepsilon_1'\delta)^2 \le \frac{4\pi}{3}\,(\varepsilon_1'\delta)^2. \end{aligned} \tag{12.45}$$

On the other hand, $y \in E_Y$, so

$$3\pi/2 = d(y) \le (\varepsilon_1'\delta)^{-2} H^2(E \cap B(y, \varepsilon_1'\delta)) + Ch_1(\varepsilon_1'\delta) \le \frac{4\pi}{3} + Ch_1(\varepsilon_1'\delta) \tag{12.46}$$

by (12.34), (3.8), and (12.45). This contradiction proves (12.43).

By (12.43), we can translate Z by less than $10\varepsilon_1'\delta$ to get a set Z' of type $\mathbb{Y}$ centered at y. Then

$$d_{y,100\delta}(E, Z') \le \frac{190}{100}\, d_{0,190\delta}(E, Z) + \frac{10\varepsilon_1'\delta}{100\delta} \le C\varepsilon_0' + \frac{\varepsilon_1'}{10} < \varepsilon_1' \tag{12.47}$$

by (12.41), so we can use Z' in (12.37).

We finally checked (12.37) and (12.38), which allows us to apply Lemma 12.30 to $E - y$ in $B(0, \delta)$, and with respect to the cone Z'. Recall that we even arranged to get a constant $[10^{-4}\eta_0]^{b_1/3}\varepsilon_0$ instead of ε_0, so we get a minimal cone $Y(y)$ centered at y, and such that

$$d_{y,r}(E, Y(y)) \le (r/\delta)^{b_1/3}\,[10^{-4}\eta_0]^{b_1/3}\varepsilon_0 \tag{12.48}$$

for $0 \le r \le 100\delta$. This is exactly the same thing as (12.36), because $\delta = 10^{-4}\eta_0|y|$.

To complete the proof of Lemma 12.35, we still need to check that $Y(y)$ is of type $\mathbb{Y}$. But we observed in the proof of Lemma 12.30 (after (12.33)) that we can take $Y(y)$ in $\mathcal{Z}_0(Z', \tau)$ for some small τ, as in (12.10), and then it is clear that $Y(y)$ is of type $\mathbb{Y}$, just like Z'. Lemma 12.35 follows. □

We shall also need to control E in small balls centered on $E \setminus [E_Y \cup \{0\}]$, so we state a third lemma. Set

$$\rho(x) = \operatorname{dist}(x, E_Y \cup \{0\}) \le |x| \le 10r_0 \tag{12.49}$$

for $x \in E \cap B(x, 10r_0)$.

Lemma 12.50. *For $x \in E \cap B(x, 10r_0) \setminus [E_Y \cup \{0\}]$, there is a plane $P(x)$ through x such that*

$$d_{x,r}(E, P(x)) \le (r/\rho(x))^{b_1/3}\varepsilon_0 \quad \text{for } 0 \le r \le 10^{-5}\eta_0\rho(x). \tag{12.51}$$

We shall proceed as in Lemma 12.35, and apply Lemma 12.35 or Lemma 12.30 to a small ball centered at x, where the choice of the lemma will depend on whether x is significantly closer to E_Y than to the origin. The manipulation of quantifiers will be similar.

We start with the case when

$$\rho(x) \geq 10^{-4}\eta_0|x|. \tag{12.52}$$

Then we set $\delta = 10^{-7}\eta_0|x|$, and we want to apply Lemma 12.30 to $E - y$ in $B(0,\delta)$, and with ε_0 replaced with $[10^{-7}\eta_0]^{b_1/3}\varepsilon_0$. We denote by ε_1' the constant in (12.26) and (12.27) that will be enough to do this. This time we want to find a plane P through x such that

$$d_{x,100\delta}(E,P) \leq \varepsilon_1' \tag{12.53}$$

(as before, in (12.37)), and also prove that

$$(110\delta)^{-2}H^2(E \cap B(y,110\delta)) - d(x) + C_0\delta^b \leq \varepsilon_1'. \tag{12.54}$$

(as for (12.38)). Note that (12.1) holds and $B(x,110\delta) \subset U$ as before, and P' automatically satisfies (12.4) (because π is the smallest possible density), and (12.5) (see Section 14). So the other assumptions for Lemma 12.30 are satisfied.

Let us again apply Lemma 12.30 to $B(0,r_0)$, but with ε_0 replaced with a smaller ε_0' to be chosen soon. We take $r = 2|x|$, and (12.31) says that

$$d_{0,2|x|}(E,Y) \leq (2|x|/r_0)^{b_1/3}\varepsilon_0' \leq \varepsilon_0'. \tag{12.55}$$

The same geometric argument as for (12.40) gives a minimal cone Z of type $\mathbb{Y}$ or $\mathbb{P}$, such that

$$Y \cap B(x,200\delta) = Z \cap B(x,200\delta), \tag{12.56}$$

and so

$$d_{x,190\delta}(E,Z) = d_{y,190\delta}(E,Y) \leq \frac{2|x|}{190\delta}\, d_{0,2|x|}(E,Y) \leq C\varepsilon_0' \tag{12.57}$$

as for (12.41). Let us check that

$$Z \text{ is a plane, or its spine does not meet } B(x,170\delta). \tag{12.58}$$

Otherwise, Z is of type $\mathbb{Y}$, and its spine meets $B(x,170\delta)$ at some point ξ. We want to apply Proposition 16.24 in [D3] to the ball $B(\xi,10\delta)$ and get a point of E_Y near ξ. The usual assumptions that $B(\xi,20\delta) \subset U$ and $h_1(20\delta)$ is small enough follow because $\delta = 10^{-7}\eta_0|x| \leq r_0$ and $x \in B(0,10r_0)$, and by (12.26), and the main assumption that $d_{x,10\delta}(E,Z)$ for some cone Z of type $\mathbb{Y}$ centered at ξ is small enough comes directly from (12.57) if ε_2' is small enough. So Proposition 16.24 in [D3] applies, and gives a point of

type $\mathbb{Y}$ in $B(\xi,\delta)$. But then $\operatorname{dist}(x,E_Y)\le|x-\xi|+\delta\le 171\delta=171\cdot 10^{-7}\eta_0|x|<10^{-4}\eta_0|x|$. This contradiction with (12.52) or (12.49) proves (12.58).

By (12.57) and (12.58), there is a plane P through x such that

$$d_{x,160\delta}(E,P)\le C\varepsilon_0' \tag{12.59}$$

(take a plane P' which coincides with Z in $B(x,170\delta)$, and then translate it by $\operatorname{dist}(x,P')\le 190\delta d_{x,190\delta}(E,Z)\le C\varepsilon_0'\delta$ to make it go through x). In particular, (12.53) holds if ε_0' is small enough.

Next we apply Lemma 16.43 in [D3] to $B(0,140\delta)$, with $F=P$ and $\delta=10^{-2}\varepsilon_1'$, where ε_1' was chosen near (12.52). As before, the assumptions are satisfied if ε_1 is small enough (so that (12.26) controls $h(200\delta)$) and ε_0' is small enough (so that (12.59) controls the distance to P). We get that

$$\begin{aligned} H^2(E\cap B(y,&110\delta))\le H^2(P\cap B(y,(1+10^{-2}\varepsilon_1')110\delta))+10^{-2}\varepsilon_1'(140\delta)^2\\ &\le\pi\,[(1+10^{-2}\varepsilon_1')110\delta)]^2+10^{-2}\varepsilon_1'(140\delta)^2\le(110\delta)^2\Big\{\pi+\frac{\varepsilon_1'}{2}\Big\}\end{aligned} \tag{12.60}$$

as for (12.42), and because P is a plane. Thus

$$(110\delta)^{-2}H^2(E\cap B(y,110\delta))+C_0\delta^b\le\pi+\frac{\varepsilon_1'}{2}+\varepsilon_1\le\pi+\varepsilon_1'\le d(x)+\varepsilon_1' \tag{12.61}$$

by (12.26), if ε_1 is small enough, and because $d(x)\ge\pi$ on E. Then (12.54) holds.

So we completed the verification of (12.53) and (12.54), Lemma 12.30 applies to $E-x$ in $B(0,\delta)$, and with $X=P$. We get a minimal cone $P(x)$ centered at x such that

$$\begin{aligned} d_{x,r}(E,P(x))&\le(r/\delta)^{b_1/3}\,[10^{-7}\eta_0]^{b_1/3}\varepsilon_0\\ &=(r/\rho(x))^{b_1/3}\,(\rho(x)/\delta)^{b_1/3}\,[10^{-7}\eta_0]^{b_1/3}\varepsilon_0\\ &\le(r/\rho(x))^{b_1/3}\,(|x|/\delta)^{b_1/3}\,[10^{-7}\eta_0]^{b_1/3}\varepsilon_0\le(r/\rho(x))^{b_1/3}\varepsilon_0\end{aligned} \tag{12.62}$$

for $0\le r\le 100\delta$, because we applied Lemma 12.30 with $[10^{-7}\eta_0]^{b_1/3}\varepsilon_0$ instead of ε_0, because $\rho(x)\le|x|$ by (12.49), and since $\delta=10^{-7}\eta_0|x|$. In addition, $P(x)$ is a plane, for instance because it lies in $\mathcal{Z}_0(P,\tau)$ for some small τ. So we proved (12.51), and the range is large enough, because $100\delta=10^{-5}\eta_0|x|\ge 10^{-5}\eta_0\rho(x)$.

We are left with the case when (12.52) fails, and so

$$\rho(x)<10^{-4}\eta_0|x|. \tag{12.63}$$

Obviously $\rho(x)=\operatorname{dist}(x,E_Y)$, because the origin is much further. Choose $y\in E_Y$ such that

$$|x-y|\le 2\rho(x)\le 2\cdot 10^{-4}\eta_0|x|; \tag{12.64}$$

obviously and $0 < |y| \leq 2|x| \leq 20r_0$, so we can apply Lemma 12.35 to y. We do this with ε_0 replaced with a smaller ε_0', to be chosen soon; we get a minimal cone $Y(y)$ of type $\mathbb{Y}$, centered at y, such that (12.36) holds. We can take $r = 3\rho(x)$ in (12.36), because $\rho(x) \leq 10^{-4}\eta_0|x| < 2 \cdot 10^{-4}\eta_0|y|$ by (12.63) and (12.64), and we get that

$$d_{y,3\rho(x)}(E, Y(y)) \leq \varepsilon_0' \,. \tag{12.65}$$

We claim that

$$\text{the spine of } Y(y) \text{ does not meet } B(x, \rho(x)/2). \tag{12.66}$$

Indeed, otherwise we can find $\xi \in B(x, \rho(x)/2)$ in the spine, and we can apply Proposition 16.24 in [D3] to the ball $B(\xi, \rho)$, as we did near (12.58), to find a point z of type $\mathbb{Y}$ in $B(\xi, \rho(x)/10)$. Then $\rho(x) \leq |x - z| \leq |x - \xi| + \rho(x)/10) \leq 6\rho(x)/10$, a contradiction which proves (12.66).

By (12.65) and (12.66), there is a plane P through x such that

$$d_{y,\rho(x)/3}(E, P) \leq 18\varepsilon_0' \,. \tag{12.67}$$

(take a plane P' that coincides with $Y(y)$ in $B(x, \rho(x)/2)$, and translate it by $\text{dist}(P', x) \leq 3\rho(x) d_{y,3\rho(x)}(E, Y(y)) \leq 3\varepsilon_0'\rho(x)$ to make it go through x).

This is a good analogue of (12.59). Set $\delta = 10^{-3}\rho(x)$. If ε_0' is small enough, we can apply Lemma 12.30 to $E - x$, in the ball $B(0, \delta)$, with the cone $X = P$, and where we replace ε_2 with $10^{-b_1}\varepsilon_2$. The analogue of (12.27) comes directly from (12.67), and the analogue of (12.28) is deduced from (12.67) and Lemma 16.43 in [D3], as we did for (12.54) in (12.60)-(12.61).

We get a plane $P(x)$ through x such that

$$\begin{aligned} d_{x,r}(E, P(x)) &\leq (r/\delta)^{b_1/3}\, 10^{-b_1}\varepsilon_0 = (r/\rho(x))^{b_1/3}\, (\rho(x)/\delta)^{b_1/3}\, 10^{-b_1}\varepsilon_0 \\ &= (r/\rho(x))^{b_1/3}\varepsilon_0 \end{aligned} \tag{12.68}$$

for $0 \leq r \leq 100\delta = 10^{-1}\rho(x)$, as for (12.62). This is better than (12.51); Lemma 12.50 follows. □

Proof of Proposition 12.28.

The proof that follows is not very efficient, but uses a minimal amount of information. See Remark 12.81, and 13.1 concerning possible improvements. Let us first say how we can compute an exponent $\beta > 0$ that works. Set $t = b_1/3$, and define a function F on $V = \big\{(u, v, w) \in [0, 1]^3 \,;\, u + v + w = 1\big\}$ by

$$F(u, v, w) = \text{Max}\big\{tu, tv - u, tw - u - v\big\}. \tag{12.69}$$

we take

$$\beta = \inf\big\{F(u, v, w) \,;\, (u, v, w) \in V\big\}. \tag{12.70}$$

It is easy to check that $\beta > 0$ (for instance, by compactness), but the result of estimates in terms of the already small $t > 0$ is so disappointingly small that we shall not bother.

Let $x \in E \cap B(0, 10r_0)$ and $0 < r_0 \le 10r_0$ be given. We may try to find the desired cone $Z(x, r)$ in for different ways, and we shall decide about which one depending on the relative values of r, $r + \rho(x)$, and $r + \rho(x) + |x|$. Set $\lambda = r/(30r_0) < 1$, and define $(u, v, w) \in V$ by

$$\frac{r}{r + \rho(x)} = \lambda^u \, , \; \frac{r + \rho(x)}{r + \rho(x) + |x|} = \lambda^v \text{ , and } \frac{r + \rho(x) + |x|}{30r_0} = \lambda^w. \tag{12.71}$$

There is a trivial attempt, which will allow us to take care of the case when r is not too small: we can try $Z(x, r) = X$, where X is as in (12.27), This yields

$$\begin{aligned} d_{x,r}(E, Z(x, r)) = d_{x,r}(E, X) &\le (100r_0/r)\, d_{x,100r_0}(E, X) \\ &\le 100\varepsilon_1 r_0/r = \frac{100\varepsilon_1}{\varepsilon_0}\, (r_0/r)^{1+\beta}\, \varepsilon_0 (r/r_0)^\beta, \end{aligned} \tag{12.72}$$

which is enough for (12.29) if

$$(r/r_0)^{1+\beta} \ge 100\varepsilon_1/\varepsilon_0. \tag{12.73}$$

We start with a first case when $F(u, v, w) = tu$ (and hence $\beta \le ut$, by (12.70)). We want to apply Lemma 12.50, with $\varepsilon_0' = \varepsilon_0/60$. We can only do this if $r \le 10^{-5}\eta_0\rho(x)$ (and ε_1 is small enough), but if this is the case, we get a plane $P(x)$ such that

$$\begin{aligned} d_{x,r}(E, P(x)) &\le (r/\rho)^{b_1/3}\varepsilon_0' \le 2(r/(r + \rho(x)))^{b_1/3}\varepsilon_0' = 2\lambda^{tu}\varepsilon_0' \le 2\lambda^\beta\varepsilon_0' \\ &= 2(r/30r_0)^\beta\varepsilon_0' \le 60(r/r_0)^\beta\varepsilon_0' = (r/r_0)^\beta\varepsilon_0 \end{aligned} \tag{12.74}$$

because $r \le \rho(x)$ and $b_1/3 = t$, and by (12.71). This is enough for (12.29).

If instead $F(u, v, w) = tu$ but

$$r > 10^{-5}\eta_0\rho(x), \tag{12.75}$$

we want to use (12.72), so we shall check that r is not too small because r/ρ represents a good part of λ.

Since $F(u, v, w) = tu$, we get that $tu \ge tv - u$ and so $v \le t^{-1}(1 + t)u$. Similarly, $tu \ge tw - u - v$, so $w \le t^{-1}(1 + t)u + t^{-1}v$, and then $1 = u + v + w \le A(t)u$, where we don't need to compute $A(t)$ explicitly. Hence

$$\begin{aligned} r/r_0 = 30\lambda \ge 30\,\lambda^{A(t)u} &= 30\left(\frac{r}{r + \rho(x)}\right)^{A(t)} \\ &= 30\left(\frac{1}{1 + \rho(x)/r}\right)^{A(t)} \ge 30\left(\frac{1}{1 + 10^5\eta_0^{-1}}\right)^{A(t)} \end{aligned} \tag{12.76}$$

by (12.75). Thus (12.73) holds (if ε_1 is small enough), and (12.72) allows us to conclude.

Our next case is when $F(u,v,w) = tv - u$. This time, we want to apply Lemma 12.35 to $B(y, 2\rho(x)+r)$, with the constant $\varepsilon_0' = \varepsilon_0/5$, and where $y \in E_Y \cup \{0\}$ is such that $|y-x| < 2\rho(x)$. For this we shall need to assume that

$$2\rho(x) + r \le 10^{-2}\eta_0 |y| \tag{12.77}$$

Observe that if (12.77) holds, $\rho(x)$ is much smaller than $|y|$ and $|x|$, so $y \in E_Y \cap B(0, 50r_0)$ and we can indeed apply Lemma 12.35 and get a minimal cone $Y(y)$ such that (12.36) holds, and so

$$\begin{aligned} d_{x,r}(E, Y(y)) &\le \frac{2\rho(x)+r}{r}\, d_{y,2\rho(x)+r}(E, Y(y)) \le \frac{2\rho(x)+r}{r}\Big(\frac{2\rho(x)+r}{|y|}\Big)^{b_1/3}\varepsilon_0' \\ &\le 5\,\frac{\rho(x)+r}{r}\Big(\frac{\rho(x)+r}{|x|+\rho(x)+r}\Big)^{b_1/3}\varepsilon_0' = 5\lambda^{-u}\lambda^{b_1 v/3}\varepsilon_0' = 5\lambda^{tv-u}\varepsilon_0' \\ &= 5\lambda^{F(u,v,w)}\varepsilon_0' \le 5\lambda^{\beta}\varepsilon_0' \le 5(r/r_0)^{\beta}\varepsilon_0' = (r/r_0)^{\beta}\varepsilon_0 \end{aligned} \tag{12.78}$$

because $\big||y| - (|x| + \rho(x) + r)\big| \le |y-x| + \rho(x) + r$ is much smaller than $|y|$ (by (12.77)), and by (12.71) and various definitions. So (12.29) holds in this case.

Next assume that $F(u,v,w) = tv - u$, but (12.77) fails. Observe that $tu \le tv - u$ and $tw - u - v \le tv - u$ because $F(u,v,w) = tv - u$, so $u \le t(1+t)^{-1}v$ and $tw \le (1+t)v$, and finally $1 = u + v + w \le B(t)v$ for some positive $B(t)$. Thus

$$r/r_0 = 30\lambda \ge 30\lambda^{B(t)v} = 30\Big(\frac{r+\rho(x)}{r+\rho(x)+|x|}\Big)^{B(t)}. \tag{12.79}$$

In addition, $|x| \le |y| + |x-y| \le |y| + 2\rho(x) \le C(x + \rho(x))$ because (12.77) fails, so $r/r_0 \ge C^{-1}$, (12.73) holds, and (12.72) yields (12.29).

Our last case is when $F(u,v,w) = tw - u - v$. Then we apply Lemma 12.30, with $\varepsilon_0' = \varepsilon_0/30$ and to $B(0, r+|x|)$ and get that

$$\begin{aligned} d_{x,r}(E, Y) &\le \frac{r+|x|}{r}\, d_{x,r+|x|}(E, Y) \le \frac{r+|x|}{r}\Big(\frac{r+|x|}{r_0}\Big)^{b_1/3}\varepsilon_0' \\ &\le 30\,\frac{r+\rho(x)+|x|}{r}\Big(\frac{r+\rho(x)+|x|}{30r_0}\Big)^{b_1/3}\varepsilon_0' = 30\lambda^{-u-v}\lambda^{tw}\varepsilon_0' \\ &= 30\lambda^{F(u,v,w)}\varepsilon_0' \le 30\lambda^{\beta}\varepsilon_0' \le 30(r/r_0)^{\beta}\varepsilon_0' = (r/r_0)^{\beta}\varepsilon_0\,. \end{aligned} \tag{12.80}$$

(12.71), (12.70), and the usual computation. This proves (12.29) in our last case, and establishes Proposition 12.28. □

Remark 12.81. It is clear that the power β that we get here is far from optimal. A typical place where we lose a lot of information is in the proof of Lemma 12.35, when we apply Lemma 12.30, obtain (12.39), and drop the possibly very small $(2|y|/r_0)^{b_1/3}$ from

our estimate. The effect is that we have to start decay estimates for smaller balls from scratch. We also do such cultural revolutions in (12.55) and (12.65).

The author decided not to try to write a more efficient proof in detail; see 13.1 in the next section for suggestions about how to do this. Note that even with this additional work, our final estimates will not look too good, because of α in (4.8).

Proof of Corollary 12.25. Corollary 12.25 follows from Proposition 12.28 and the generalization in [DDT] of Reifenberg's topological disk. More precisely, Section 10 in [DDT] gives sufficient conditions weaker than the conclusions of Proposition 12.28 for the existence of Φ as in the conclusion of Corollary 12.25, but only of class C^1. See in particular the discussion that starts after (10.22) there, up to the end of the paper. The proof applies and gives $C^{1,\beta}$ estimates (apply the Whitney extension theorem to Hölder-continuous functions).

The reader should not pay too much attention to the existence of the mapping Φ; we stated it this way to avoid long descriptions, but what seems important to the author is the decomposition of $E \cap B(0, 2r)$ into $C^{1+\beta}$ faces F_j, which make 120° angles with each other. Here the decomposition into faces comes from the Hölder description in [D3], and both the fact that they are $C^{1+\beta}$ and that they make the right angles along the edges comes from Proposition 12.28. The point is that we have a control on the variations of the tangent plane, because (12.29) also gives the existence of a tangent cone Z to E at x, with $d_{0,1}(Z, Z(x, r)) \leq C(r/r_0)^\beta \varepsilon_0$ (compare the $Z(x, 2^{-k}r)$ and sum a geometric series). □

Proof of Theorem 1.15. Let E be as in the theorem, and let $x \in E$ be given. Without loss of generality, we may assume that $x = 0$. Let X be a blow-up limit of E at x, with the full length property. [See Definition 3.11 for blow-up limits.] When $n = 3$, we simply need to pick any blow-up limit (and such a limit exists, see (3.13)), because every minimal cone has the full length property (see Section 14). When $n > 3$, we don't know that much, so we put the existence of X in the assumptions.

Then there are arbitrarily small radii r_0 that satisfy the assumptions of Theorem 12.8 (notice that $f(r)$ tends to 0 by its definition (3.5), and that (12.5) holds because $H^2(X \cap B(0, 1)) = d(x)$ by Proposition 3.14). Theorem 12.8 says that there is a unique minimal cone Y (which is then equal to X), and (12.9) even says how fast $d_{0,r}(E, X) = d_{0,r}(E, Y)$ tends to 0. Then we can apply Corollary 12.25 to r_0 for every r_0 small enough, and Theorem 1.15 follows. □

13. Small generalizations and improvements

In this section we are concerned about various ways to take the same proofs as above and get slightly better results. The comments in 13.32 and 13.41 are independent from the previous ones, but 13.22 depends on 13.1.

13.1 Better estimates for Proposition 12.28.

We observed in Remark 12.81 that in the proof of Proposition 12.28, we may lose a lot of information when we apply Lemma 12.30 or 12.35 to estimate the analogue of $f(r)$ for a small ball B centered away from the origin, and where we have a good approximation of E by a minimal cone near B. For instance, in (12.39) we had a point $y \in E_Y$ and we

obtained that

$$d_{0,2|y|}(E,Y) \le (2|y|/r_0)^{b_1/3}\varepsilon_0' \le \varepsilon_0', \tag{13.2}$$

and we decided to use the last inequality alone. If instead we keep the whole estimate, the proof of (12.41) gives a minimal cone Z of type $\mathbb{Y}$ such that

$$d_{y,190\delta}(E,Z) \le \frac{2|y|}{190\delta}\, d_{0,2|y|}(E,Y) \le C(2|y|/r_0)^{b_1/3}\varepsilon_0'\,, \tag{13.3}$$

because $|y|$ and $\delta = 10^{-4}\eta_0|y|$ are comparable. Then we decided to use Lemma 16.43 in [D3] to get a mild control on

$$f(110\delta) = (110\delta)^{-2}H^2(E\cap B(y,110\delta)) - \frac{3\pi}{2} \tag{13.4}$$

(see (12.42)). When $|y|/r_0$ is very small, we should do something more clever instead.

First observe that the spine of Z should pass within $C\delta d_{y,190\delta}(E,Z)$ from y, by the same argument as for (12.43); this allows us to replace Z with a cone Y of type $\mathbb{Y}$, parallel to Z, whose spine contains y, and such that

$$d_{y,180\delta}(E,Y) \le C(|y|/r_0)^{b_1/3}\varepsilon_0'\,. \tag{13.5}$$

We claim that in such circumstances,

$$f(110\delta) \le C(|y|/r_0)^{b_1/3}\varepsilon_0'\,. \tag{13.6}$$

Let us even formulate a slightly more general statement to this effect.

Let E be a reduced almost minimal set in U, with gauge function h, suppose that $0\in E$ and $B(0,2r_0)\subset U$, and that

$$f(3r_0/2) \text{ and } h_1(2r_0) \text{ are small enough,} \tag{13.7}$$

to set the stage. Let Y is a minimal cone centered at the origin, and such that

$$H^2(Y\cap B(0,1)) = d(0). \tag{13.8}$$

Then

$$f(r_0) \le Cd_{0,2r_0}(E,Y) + 9h(3r_0). \tag{13.9}$$

When $n=3$, this is rather easy. We want to use Y to construct a competitor, but we shall need to add a small rim around $Y\cap\partial B(y,r_0)$. Set $B = B(y,r_0)$ and

$$Y_\eta = \big\{z\in B\,;\, \mathrm{dist}(z,Y)\le\eta\big\}, \tag{13.10}$$

where we take $\eta = 2r_0d_{y,2r_0}(E,Y)$. Thus $E\cap B\subset Y_\eta$ by definition of $d_{0,2r_0}(E,Y)$.

Set $Y' = [Y \cap B] \cup [Y_\eta \cap \partial B]$. By "elementary geometry", there is a Lipschitz mapping h such that $h(z) = z$ out of B, $h(z) \in \overline{B}$ for $z \in B$, and which maps $Y_\eta \cap B$ to a subset of Y'. Then set $F = h(E)$; it is easy to see that F is a competitor for E in B, an so

$$\begin{aligned} H^2(E \cap B) &\le H^2(F \cap \overline{B}) + 4r_0^2 h(2r_0) \le H^2(Y') + 4r_0^2 h(2r_0) \\ &\le H^2(Y \cap B(0, r_0)) + H^2(Y_\eta \cap \partial B) + 4r_0^2 h(2r_0) \\ &\le r_0^2 H^2(Y \cap B(0,1)) + C r_0^2 d_{0,2r_0}(E, Y) + 4r_0^2 h(2r_0), \end{aligned} \tag{13.11}$$

as needed for (13.9).

When $n > 3$, we need to construct another competitor, because $H^2(Y_\eta \cap \partial B) = +\infty$. What seems the easiest at this point is to use the construction of Sections 6-9. Let us check that we can choose $r \in (r_0, 3r_0/2)$ such that (4.3) and (4.4) hold, and also

$$H^1(E \cap \partial B(0, r)) \le 2(1 + \tau)\, d(0)\, r \tag{13.12}$$

(the analogue of (6.2) here). We don't nee to worry about (4.3) and (4.4), because Lemma 4.12 says that they hold almost everywhere, so let us assume that (13.12) fails for almost every $r \in (r_0, 3r_0/2)$. Set $\theta(r) = r^{-2} H^2(E \cap B(0, r))$ as usual. By (5.7), (5.6), and (5.8),

$$\begin{aligned} f(3r_0/2) - f(r_0) &\ge \int_{r_0}^{3r_0/2} \theta'(r) dr \ge \int_{r_0}^{3r_0/2} [r^{-2} H^1(E \cap \partial B(0, r)) - 2r^{-1}\theta(r)] dr \\ &\ge \int_{r_0}^{3r_0/2} r^{-1} [2(1 + \tau)\, d(0) - 2\theta(r)] dr \end{aligned} \tag{13.13}$$

because (13.12) fails a.e. Recall from (3.5) that $\theta(r) = d(0) + f(r)$; thus

$$f(3r_0/2) - f(r_0) \ge \int_{r_0}^{3r_0/2} 2r^{-1} [\tau\, d(0) - 2f(r)] dr \tag{13.14}$$

This is impossible if $f(3r_0/2)$ and $h_1(2r_0)$ are small enough, because (3.6) says that then $f(r_0)$ and $f(r)$ are very small too.

So we can find r such that (13.12) holds too, and this allows us to follow the construct of Sections 6-9, with $X = Y$. Most of our estimates are probably useless here, but anyway the first line of (9.69), suitably normalized because here $r \neq 1$, says that

$$H^2(E \cap B) \le (3r_0)^2 h(3r_0) + H^2(\Sigma \cap B(0, r)) + 2550 \int_{E \cap \partial B(0,r)} \operatorname{dist}(z, \Gamma)\, dH^1(z) \tag{13.15}$$

for some nice union Σ of Lipschitz graphs, and some union Γ of small Lipschitz graphs in $\partial B(0, r)$. Notice that $\operatorname{dist}(z, \Gamma) \le 4r_0 d_{0,2r_0}(E, Y)$ on $E \cap \partial B(0, r)$, by definition of $d_{0,2r_0}(E, Y)$ and construction of Γ; The construction of Σ, where we tend to minimize areas of surfaces over triangular regions, already gives $H^2(\Sigma \cap B(0, r)) \le H^2(Y \cap B(0, r)) + C r_0^2 d_{0,2r_0}(E, Y)$, but if this were not the case, we could improve on it, or even work directly

on the cone over Γ, to get a deformation Σ' (something that looks like Y' above, but with tilted walls), that satisfies this estimate. The point is that it is much easier to deform a small Lipschitz graph, rather than the set E, when we do not know the topology of E. So (13.15) yields

$$H^2(E \cap B) \le H^2(Y \cap B(0,r)) + Cr_0^2 d_{0,2r_0}(E,Y) + (3r_0)^2 h(3r_0), \tag{13.16}$$

as needed for (13.9).

Return to Proposition 12.28. Let us also modify our estimate near (12.55) and (12.65), as suggested above, and otherwise follow the proof of Proposition 12.28. The worse estimate is when $B(x,r)$ is fairly far from E_Y, and even further from the origin. In this case we choose $y \in E_Y$ such that $|y-x| \le 2\rho(x)$, and first apply Theorem 12.8 (as in Lemma 12.30) to get Y with the property (12.9). Thus

$$d_{0,2|y|}(E,Y) \le C(|y|/r_0)^{b_1/3}\varepsilon_0'. \tag{13.17}$$

We apply (13.9) and get that

$$f(C^{-1}|y|) \le C(|y|/r_0)^{b_1/3}\varepsilon_0', \tag{13.18}$$

where f is computed with respect to y. By Theorem 12.8, we get a cone $Y(y)$ of type $\mathbb{Y}$, with

$$d_{y,\rho}(E,Y(y)) \le C(\rho/|y|)^{b_1/3}\big[(|y|/r_0)^{b_1/3}\varepsilon_0'\big]^{b_1/3}. \tag{13.19}$$

We take ρ a little larger than $|x-y|$, and use $Y(y)$ to get a plane P through x such that

$$d_{x,c\rho}(E,P) \le Cd_{y,\rho}(E,Y(y)) \le C(\rho/|y|)^{b_1/3}\big[(|y|/r_0)^{b_1/3}\varepsilon_0'\big]^{b_1/3} \tag{13.20}$$

for some small $c > 0$. By (13.9), we get a similar estimate for $f(c\rho/2)$, where f is now computed with respect to the center x. We apply Theorem 12.8 one last time, to get a plane $P(x)$ through x such that

$$d_{x,r}(E,P(x)) \le C(r/\rho)^{b_1/3}\,(\rho/|y|)^{b_1^2/9}\,(|y|/r_0)^{b_1^3/27}(\varepsilon_0')^{b_1^3/27} \le \varepsilon_0''(r/r_0)^{b_1^3/27}, \tag{13.21}$$

where $C_0'' = C(\varepsilon_0')^{b_1^3/27}$ is still as small as we want.

Thus, with a little more work (not entirely written down here) we could take $\beta = b_1^3/27$ in Proposition 12.28, and then in Corollary 12.25.

13.22 Larger gauge functions h.

When the gauge function h is larger than a power of r, but not too small, some part of Theorem 1.15 and Corollary 12.25 stays true. Obviously, when h becomes larger, we expect less regularity from the corresponding almost minimal sets E. Trivial counterexamples consist in taking your favorite minimal set E (for instance a line or a plane), distorting it so that some spiraling occurs, and computing a gauge function for the distorted set.

Here we only want to say that if h is not too large, we can keep the same proof as above, and get C^1 estimates (instead of $C^{1+\beta}$), with some control on the modulus of continuity. Let us only take an example, and assume that

$$h(r) \leq C[\log(A/r)]^{-b} \tag{13.23}$$

for some constants $A, b > 0$ and r small enough. Our proof appears to give that E is locally C^1-equivalent to a minimal cone (under the same other assumptions as in Theorem 1.15 and Corollary 12.25) when $b > 30$; however, we shall not check every detail here, and the estimate is unlikely to be optimal.

Notice that the results from Sections 4-11 (and in particular Theorem 4.5) only require the Dini condition (1.16), so we just need to worry about Sections 12 and 13.

We start with the decay rate for f that follows from Theorem 4.5, which was computed in Example 5.21. If f satisfies the differential inequality that comes from Theorem 4.5 and h satisfies (13.23), we get that f decays at least like $C[\log(A/r)]^{-b}$ again. So far, we require $b > 1$, and only because we always assume the Dini condition (1.16). Thus, instead of (12.19), we get that

$$f(x) \leq (x/r_0)^a f(r_0) + C(x/A)^{a/2} + C\big[\log\big(\frac{A}{2x}\big)\big]^{-b} \leq C_{A,r_0}\big[\log\big(\frac{A}{2x}\big)\big]^{-b} \tag{13.24}$$

for $x \in (2^{-l}r_0, r_0)$. Then we apply Corollary 11.85 to get that

$$\begin{aligned} d_{0,2^{-l}r_0}(E, Y_l) + \alpha_+(Y_l) &\leq C2^{-al/3} f(r_0)^{1/3} + C(2^{-l}r_0/A)^{a/6} + C\big[\log\big(\frac{A}{2^{-l+1}r_0}\big)\big]^{-b/3} \\ &\leq C_{A,r_0}\big[\log\big(\frac{A}{2^{-l+1}r_0}\big)\big]^{-b/3} \end{aligned} \tag{13.25}$$

as in (12.21). We get a similar estimate for the relative distances between the successive cones Y_l, and for the existence of a limit Y and (12.22), we simply demand that $b > 3$ to get a converging series, and then sum over l such that $2^{-l}r_0 \leq 2r$ to get that

$$\begin{aligned} d_{0,r}(E, Y) + \alpha_+(Y) &\leq C(r/r_0)^{a/3} f(r_0)^{1/3} + C(r/A)^{a/6} + C\big[\log\big(\frac{A}{2r}\big)\big]^{1-\frac{b}{3}} \\ &\leq C_{A,r_0}\big[\log\big(\frac{A}{2r}\big)\big]^{1-\frac{b}{3}} \end{aligned} \tag{13.26}$$

for $0 < r < r_0/2$, which is supposed to replace (12.9).

For the analogue of Theorem 1.15 and Corollary 12.25, we continue the argument as suggested in 13.1. The analogue of the estimates (13.17)-(13.21) (the worst case scenario) seems (modulo computation mistakes) to be

$$d_{0,2|y|}(E, Y) \leq C(|y|/r_0)^{a/3}(\varepsilon_0')^{1/3} + C(|y|/A)^{a/6} + C\big[\log\big(\frac{A}{4|y|}\big)\big]^{1-\frac{b}{3}}, \tag{13.27}$$

$$f(C^{-1}|y|) \leq C(|y|/r_0)^{a/3}(\varepsilon_0')^{1/3} + C(|y|/A)^{a/6} + C\big[\log\big(\frac{A}{4|y|}\big)\big]^{1-\frac{b}{3}}, \tag{13.28}$$

$$
\begin{aligned}
d_{y,\rho}(E,Y(y)) &\le C(\rho/|y|)^{a/3} f(C^{-1}|y|)^{1/3} + C(\rho/A)^{a/6} + C\big[\log\big(\frac{A}{2\rho}\big)\big]^{1-\frac{b}{3}} \\
&\le C(\rho/|y|)^{a/3}(|y|/r_0)^{a/9}(\varepsilon_0')^{1/9} + C(\rho/|y|)^{a/3}(|y|/A)^{a/18} \\
&\quad + C(\rho/|y|)^{a/3}\big[\log\big(\frac{A}{4|y|}\big)\big]^{\frac{1}{3}-\frac{b}{9}} + C(|y|/A)^{a/6} + C\big[\log\big(\frac{A}{2\rho}\big)\big]^{1-\frac{b}{3}} \\
&\le C(\rho/r_0)^{a/9}(\varepsilon_0')^{1/9} + C(\rho/A)^{a/18} + C(|y|/A)^{a/6} \\
&\quad + C(\rho/|y|)^{a/3}\big[\log\big(\frac{A}{4|y|}\big)\big]^{\frac{1}{3}-\frac{b}{9}} + C\big[\log\big(\frac{A}{2\rho}\big)\big]^{1-\frac{b}{3}},
\end{aligned}
\tag{13.29}
$$

$$
\begin{aligned}
d_{x,c\rho}(E,P) \le C d_{y,\rho}(E,Y(y)) &\le C(\rho/r_0)^{a/9}(\varepsilon_0')^{1/9} + C(\rho/A)^{a/18} + C(|y|/A)^{a/6} \\
&\quad + C(\rho/|y|)^{a/3}\big[\log\big(\frac{A}{4|y|}\big)\big]^{\frac{1}{3}-\frac{b}{9}} + C\big[\log\big(\frac{A}{2\rho}\big)\big]^{1-\frac{b}{3}},
\end{aligned}
\tag{13.30}
$$

a similar estimate for $f(c\rho/2)$, and finally

$$
\begin{aligned}
d_{x,r}(E,P(x)) &\le C(r/\rho)^{a/3} f(c\rho/2)^{1/3} + C(r/A)^{a/6} + C\big[\log\big(\frac{A}{2r}\big)\big]^{1-\frac{b}{3}} \\
&\le C(r/r_0)^{a/27}(\varepsilon_0')^{1/27} + C(r/A)^{a/54} + C(r/\rho)^{a/3}(|y|/A)^{a/18} \\
&\quad + C(r/A)^{a/6} + C(r/\rho)^{a/3}(\rho/|y|)^{a/9}\big[\log\big(\frac{A}{4|y|}\big)\big]^{\frac{1}{9}-\frac{b}{27}} \\
&\quad + C(r/\rho)^{a/3}\big[\log\big(\frac{A}{2\rho}\big)\big]^{1-\frac{b}{3}} + C\big[\log\big(\frac{A}{2r}\big)\big]^{\frac{1}{3}-\frac{b}{9}} \\
&\le C_{A,r_0}\big[\log\big(\frac{A}{2r}\big)\big]^{\frac{1}{9}-\frac{b}{27}}.
\end{aligned}
\tag{13.31}
$$

Recall that we want to apply Section 10 in [DDT] to get C^1 estimates on E and, at least if we do this brutally, we want upper bounds ε_k for $d_{x,r}(E, Z(x,r))$ when $r \sim 2^{-k} r_0$ such that $\sum_k \varepsilon_k < +\infty$. So we want the power in (13.31) to be less than -1, i.e., that $\frac{b}{27} - \frac{1}{9} > 1$, or equivalently $b > 30$. Again, all this is subject to verification and improvement.

13.32 Weaker full length conditions.

The full length condition in Definition 2.10 can be seen as a strange non-degeneracy condition concerning the the length function $H^1(\varphi_*(K))$ on deformations of the cone X through mappings $\varphi \in \Phi(\eta_1)$. If we have less precise estimates, some part of Theorem 4.5 will still hold, but unfortunately not enough to prove the local C^1-equivalence of E to a minimal cone.

Let us say that the (reduced) minimal cone X satisfies a weak full length condition of order $N > 1$ when there is a standard decomposition of $K = X \cap \partial B(0,1)$ as in Section 2, an $\eta_1 < \eta_0/10$, and a constant $C_1 \ge 1$, such that if $\varphi \in \Phi(\eta_1)$ is such that

$$
H^1(\varphi_*(K)) > H^1(K),
\tag{13.33}
$$

then there is a deformation $\widetilde{X}$ of $\varphi_*(X)$ in $B = B(0,1)$ such that

$$
H^2(\widetilde{X} \cap B) \le H^2(\varphi_*(X) \cap B) - C_1^{-1}[H^1(\varphi_*(K)) - H^1(K)]^N.
\tag{13.34}
$$

[The notations are the same as in Definition 2.10, and we only added the exponent N.]

In Theorem 4.5, when we replace the full length condition above (4.8) with a weak full length condition of order N, we get the following weaker form of (4.8):

$$H^2(E\cap B(x,r)) \le \frac{r}{2}\,H^1(E\cap\partial B(x,r)) - \alpha r^2\,[r^{-1}H^1(E\cap\partial B(x,r)) - 2d(x)]_+^N + 4r^2h(2r), \tag{13.35}$$

where the A_+ denotes the nonnegative part of A.

When $H^1(E\cap\partial B(x,r)) \le 2d(x)r$, we claim no gain, and (13.35) follows from (4.7) because (4.6) says that the part inside the brackets is nonnegative. In the other case, notice that (13.35) is weaker than (4.8) (which corresponds to $N=1$).

We proceed as before; there is no need to change anything before the first time we used the full length condition, which happens near (10.16). Here we need to replace (10.17) with

$$H^2(\widetilde{X}\cap B) \le H^2(\varphi_*(X)\cap B) - C_1^{-1}[H^1(\varphi_*(K)) - H^1(K)]_+^N. \tag{13.36}$$

Then we apply Lemma 10.2, as before, and get (10.5) with

$$A = C_1^{-1}[H^1(\varphi_*(K)) - H^1(K)]_+^N = C_1^{-1}[H^1(\rho) - H^1(K)]_+^N \tag{13.37}$$

That is,

$$\begin{aligned} H^2(E\cap\overline{B}) &\le \frac{1}{2}H^1(E\cap\partial B) - 10^{-5}[H^1(E\cap\partial B) - H^1(\rho)] \\ &\qquad - C_1^{-1}\kappa^2[H^1(\rho) - H^1(K)]_+^N + 4h(2) \\ &\le \frac{1}{2}H^1(E\cap\partial B) - C_1^{-1}\kappa^2[H^1(E\cap\partial B) - H^1(\rho)]_+^N \\ &\qquad - C_1^{-1}\kappa^2[H^1(\rho) - H^1(K)]_+^N + 4h(2) \\ &\le \frac{1}{2}H^1(E\cap\partial B) - \alpha[H^1(E\cap\partial B) - H^1(K)]_+^N + 4h(2) \\ &\le \frac{1}{2}H^1(E\cap\partial B) - \alpha[H^1(E\cap\partial B) - 2d(0)]_+^N + 4h(2) \end{aligned} \tag{13.38}$$

because $H^1(E\cap\partial B) - H^1(\rho)$ is small and nonnegative (recall from (9.68) that $H^1(E\cap\partial B) - H^1(\rho) = \delta_1+\delta_2+\delta_3 \ge 0$), with $\alpha = C_1^{-1}\kappa^2$, and then by (10.16). This gives (13.35) after re-scaling (recall that for (13.38) we assumed that $x=0$ and $r=1$).

The new estimate (13.35) yields some decay for the function f, but not as much as before. Notice that (13.35) is the same as (5.27), except that we used $\alpha' = 2^N\alpha$ there. Lemma 5.28 and Remark 5.32 then say that if we use a gauge function h such that

$$h(r) \le C[\mathrm{Log}(\frac{1}{r})]^{-\frac{N}{N-1}} \quad \text{for } r \text{ small} \tag{13.39}$$

(we are allowed larger functions h, but the final estimate is then worse), and as long as we can use (12.35), we get that

$$f(x) \le C_1 \Big[\mathrm{Log}\Big(\frac{C_2}{x}\Big) \Big]^{-\frac{1}{N-1}} \quad \text{for } x \text{ small}, \tag{13.40}$$

as in (5.33) and (5.34). However, two related things happen. First, (13.40) is not enough to get C^1 estimates, or the uniqueness of the tangent cone to E at 0, as in Section 12. But also, we cannot sum up our estimates for smaller balls as we did for Theorem 12.8, and as a consequence we cannot use the same cone X to approximate E in $B(0,r)$ for r small. Thus, if we want to obtain (13.40), we have to find other cones with the weaker full length property, so that we can apply the analogue of Theorem 4.5 and get (13.35) at smaller scales. The simplest way to do this is to assume that every blow up limit of E at 0 is a minimal cone with the weaker full length property, with uniform constants N, C_1, and η_1. Then we get (13.40) for r small, and some estimates on the $\beta_Z(0,r)$ that we can derive from Theorem 11.4.

13.41. Minimal-looking cones.

Recall from Remark 2.14 that a minimal-looking cone is a (reduced) cone E such that $K = E \cap \partial B(0,1)$ is as in the description of minimal cones given at the beginning of Section 2 (up to (2.3)). This comes with a small constant $\eta_0 > 0$ attached (the constant in (2.2) and (2.3)).

Our main results do not use the fact that we deal with true minimal cones. That is, the minimality is only used through the description of Section 2. Thus, for instance, Theorem 4.5 is still true if X in the assumptions is a (full length for (4.8)) minimal-looking cone. Then we can use the differential inequality as before (if we can find an X for almost every radius). Of course, all the standard ways which give us a cone X as in the statement will give a minimal cone.

Similarly, the results of Section 12 go through. For instance, if in Theorem 12.8, we only know that the cone X in (12.3) is a full length minimal-looking cone, we still get the same conclusion. At the beginning of the argument, we only know that the cone Y in the conclusion is minimal-looking, but then it turns out that (12.9) holds, so Y is a blow-up limit of E at x, which forces Y to be minimal. [Again, this also makes it unlikely that we will find X in (12.3) without knowing that it is minimal.] The same remarks apply to Corollary 12.25, Proposition 12.28, and Lemmas 12.30, 12.35, and 12.50.

14. Examples of full length minimal cones

The main purpose of this section is to verify that the standard minimal cones of dimension 2 have the full length property of Definition 2.10. We shall systematically use Lemma 10.23 to do this, so let us review the notation.

We consider a minimal cone X, take a standard decomposition of $K = X \cap \partial B$ (where B is the unit ball), and consider the deformation of $\varphi_*(K)$ and $\varphi_*(X)$ of K and X with a mapping $\varphi \in \Phi(\eta_1)$ (see Section 2). We are allowed to choose η_1, and we shall do it so that the main term in some expansion is larger than the errors.

Lemma 10.23 says that there is a competitor $\widetilde{X}$ for $\varphi_*(X)$ in B such that

$$H^2(\widetilde{X} \cap B) \le H^2(\varphi_*(X) \cap B) - C^{-1}\alpha_+(\varphi)^2, \tag{14.1}$$

where $\alpha_+(\varphi)$ is defined in (10.20)-(10.22), and is the maximum deviation (from the standard position that would occur in a minimal cone) of the position of two or three tangent vectors to $\varphi_*(K)$ at a vertex. So (2.12) and the full length property will follow as soon as we check that

$$H^1(\varphi_*(K)) - H^1(K) \le C\alpha_+(\varphi)^2 \tag{14.2}$$

when

$$H^1(\varphi_*(K)) > H^1(K). \tag{14.3}$$

Lemma 14.4. *The planes in $\mathbb{R}^n$ have the full length property.*

Let P be a plane, choose three vertices z_j, $0 \le j \le 2$, in $K = P \cap \partial B$, for instance at equal distances from each other, to define the standard decomposition of K. Then let $\varphi \in \Phi(\eta_1)$ be given and set $w_j = \varphi(z_j)$ for $0 \le j \le 2$.

We may assume that w_1 and w_2 lie in the horizontal plane $P_0 = \{x \in \mathbb{R}^n\,;\, x_3 = \cdots = x_n = 0\}$, and even, if we identify P_0 with $\mathbb{C}$ for convenience, that for $j = 1, 2$, $w_i = e^{i\theta_j}$ for some θ_j which is close to $2\pi j/3$. Denote by ξ the point of $P_0 \cap \partial B$ that lies closest to w_0, and set $\delta = |\xi - w_0| = \operatorname{dist}(w_0, P_0 \cap \partial B)$. Also denote by d the geodesic distance on ∂B. Then

$$\begin{aligned} H^1(\varphi_*(K)) - H^1(K) &= H^1(\varphi_*(K)) - 2\pi = d(w_0, w_1) + d(w_0, w_2) + d(w_1, w_2) - 2\pi \\ &= d(w_0, w_1) + d(w_0, w_2) + d(w_1, w_2) - [d(\xi, w_1) + d(\xi, w_2) + d(w_1, w_2)] \\ &= [d(w_0, w_1) - d(\xi, w_1)] + [d(w_0, w_2) - d(\xi, w_2)] \le C\delta^2 \le C\alpha_+(\varphi)^2. \end{aligned} \tag{14.5}$$

Thus (14.2) holds, and Lemma 14.4 follows. □

Lemma 14.6. *The cones of type $\mathbb{Y}$ in $\mathbb{R}^n$ have the full length property.*

The proof will be a little unnerving (at least for the author), because we cannot really trust the pictures in $\mathbb{R}^3$. Let Y be a cone of type $\mathbb{Y}$ and $\varphi \in \Phi(\eta_1)$ be given, denote by z^+ and z^- the two vertices of K, and chose additional vertices z_1, z_2, and z_3 near the middle of each arc of K, to define a standard decomposition. Then set $w^\pm = \varphi(z^\pm)$ and $w_j = \varphi(z_j)$; these are the vertices of $\varphi_*(K)$.

Choose coordinates of $\mathbb{R}^n$ so that w^+ and w^- lie in a vertical plane in $\mathbb{R}^3$, and even

$$w^\pm = (\sin\rho, 0, \pm\cos\rho, 0) \tag{14.7}$$

for some $\rho \in [0, 10^{-2}]$. (We just rotate the coordinates in $\mathbb{R}^3$ so that the midpoint of the $w^\pm$ lies on the nonnegative first axis; then ρ is small because the $w^\pm$ are almost antipodal.)

Denote by $\Gamma_j^\pm$ the arc of geodesic between $w^\pm$ and w_j, and set $\Gamma_j = \Gamma_j^+ \cup \Gamma_j^-$. Thus $\varphi_*(K)$ is the union of the Γ_j. Denote by v_j the point of Γ_j whose third coordinate vanishes. Let us check that

$$H^1(\Gamma_j) \le d(v_j, w^+) + d(v_j, w^-) + C\alpha_+(\varphi)^2, \tag{14.8}$$

where we still denote by d the geodesic distance on ∂B. Suppose, for the sake of definiteness, that the third coordinate of w_j is nonnegative, so that v_j lies on Γ_j^- and the the geodesic g^- from v_j to w^- is contained in Γ_j^-. Set $G = \Gamma_j \setminus g^-$; thus G is composed of a small arc g of Γ_j^- that goes from v_j to w_j, followed by Γ_j^+ that goes from w_j to w^+. The angle between the two is different from π by at most $\alpha_+(\varphi)$, by (10.21), so $H^1(G) \le d(v_j, w^+) + C\alpha_+(\varphi)^2$. We add $H^1(g^-) = d(v_j, w^-)$ to both sides and get (14.8).

Next we evaluate $d(v_j, w^\pm)$. Write $v_j = (a_j, b_j, 0, \xi_j)$ with $\xi_j \in \mathbb{R}^{n-3}$. Set $|\xi_j| = \sin\alpha_j$, so that $a_j^2 + b_j^2 = \cos^2\alpha_j$, and choose θ_j so that $a_j = \cos\theta_j \cos\alpha_j$ and $b_j = \sin\theta_j \cos\alpha_j$. Thus

$$v_j = (\cos\theta_j \cos\alpha_j, \sin\theta_j \cos\alpha_j, 0, \xi_j). \tag{14.9}$$

Since $d(v_j, w^\pm)$ is the length of an arc of great circle that goes from v_j to $w^\pm$, $|w^\pm - v_j| = 2\sin(d(v_j, w^\pm)/2)$. Then

$$\cos(d(v_j, w^\pm)) = 1 - 2\sin^2(d(v_j, w^\pm)/2) = 1 - \frac{1}{2}\,|w^\pm - v_j|^2. \tag{14.10}$$

Next (14.7) and (14.9) yield

$$\begin{aligned} |w^\pm - v_j|^2 &= [\cos\theta_j \cos\alpha_j - \sin\rho]^2 + \sin^2\theta_j \cos^2\alpha_j + \cos^2\rho + \sin^2\alpha_j \\ &= 1 + \cos^2\theta_j \cos^2\alpha_j - 2\cos\theta_j \cos\alpha_j \sin\rho + \sin^2\theta_j \cos^2\alpha_j + \sin^2\alpha_j \\ &= 1 + \cos^2\alpha_j - 2\cos\theta_j \cos\alpha_j \sin\rho + \sin^2\alpha_j = 2 - 2\cos\theta_j \cos\alpha_j \sin\rho \end{aligned} \tag{14.11}$$

and (14.10) says that $\cos(d(v_j, w^\pm)) = \cos\theta_j \cos\alpha_j \sin\rho$. Set $t_j^\pm = \pi/2 - d(v_j, w^\pm)$; these will be easier to manipulate because they are small. Record that

$$\sin t_j^\pm = \cos(d(v_j, w^\pm)) = \cos\theta_j \cos\alpha_j \sin\rho, \tag{14.12}$$

so $|\sin t_j^\pm| \le \sin\rho \le 10^{-2}$, hence $|t_j| \le \mathrm{Arcsin}(10^{-2})$ and then

$$|t_j^\pm - \sin t_j^\pm| \le \frac{1}{6}\,|t_j^\pm|^3 \le \frac{1}{3}\,|\sin^3 t_j^\pm| \le \frac{1}{3}\,\sin^3\rho. \tag{14.13}$$

Also,

$$3\pi - \sum_{j\,;\,\pm} d(v_j, w^\pm) = \sum_{j\,;\,\pm} \big[\frac{\pi}{2} - d(v_j, w^\pm)\big] = \sum_{j\,;\,\pm} t_j^\pm \tag{14.14}$$

so

$$\left|3\pi - \sum_{j\,;} d(v_j, w^{\pm}) - 2\sin\rho \sum_j \cos\theta_j \cos\alpha_j\right| \le 2\sin^3\rho, \tag{14.15}$$

by (14.12) and (14.13).

Next we want to evaluate angles. Let us first prove that

$$\rho \le 2\alpha_+(\varphi). \tag{14.16}$$

Set $s_j^{\pm} = w^{\pm} - \langle w_j, w^{\pm}\rangle w_j$; observe that $s_j^{\pm}$ lies in the plane through $w^{\pm}$ and w_j, and is orthogonal to w_j, so it is parallel to the tangent of $\Gamma_j^{\pm}$ at w_j. Also, $|s_j^{\pm}| \ge 9/10$ because $\langle w_j, w^{\pm}\rangle$ is small (recall that φ does not move the points very far, and that z_j is orthogonal to $z^{\pm}$). Let $\pi - \beta_j$ denote the angle of Γ_j^+ and Γ_j^- at w_j, or equivalently of s_j^+ and s_j^-. Then

$$|\beta_j| \le \alpha_+(\varphi) \tag{14.17}$$

by definition of $\alpha_+(\varphi)$. Denote by p the orthogonal projection of s_j^+ on the direction of s_j^-, and set $p^\perp = s_j^+ - p$. Then $|p^\perp| = |s_j^+||\sin\beta_j|$.

We want to use a determinant to evaluate this, so we consider the vector space V spanned by w_j, w^-, and w^+, with the Euclidean structure inherited from $\mathbb{R}^n$ and any choice of orientation. [If the three vectors are not independent, pick any V of dimension 3 that contains them.] Then compute $D_j = \det(w_j, w^-, w+)$ in V. Since we may remove a linear combination of two vectors to the third one, $D_j = \det(w_j, s_j^-, s_j^+) = \det(w_j, s_j^-, p^\perp)$, and so $|D_j| = |s_j^-||p^\perp| = |s_j^-||s_j^+||\sin\beta_j|$ because the three vectors are orthogonal and by the computation above. So

$$|D_j| \le |\sin\beta_j| \le \sin(\alpha_+(\varphi)), \tag{14.18}$$

by (14.17). We can also compute D_j brutally. Denote by e_1 and e_3 the first and third elements of the canonical basis, complete the basis of V, and denote by x, z, and y the coordinates of w_j. Thus $|y| = \operatorname{dist}(w_j, P_v)$, where P_v is the vertical plane spanned by e_1 and e_3. A simple computation using (14.7) says that

$$|D_j| = 2|y|\sin\rho\cos\rho = \sin(2\rho)\operatorname{dist}(w_j, P_v). \tag{14.19}$$

Now we want to select j so that $\operatorname{dist}(w_j, P_v)$ is not too small. Recall that $w_j = \varphi(z_j)$, and that the three z_j lie in a 2-plane orthogonal to the line through z^+, where they make 120° degree angles with each other. At least one z_j lies at distance $\ge 1/2$ from the vector space W through e_1 and z^+. Recall that z^+ is quite close to w^+, which is close to e_3 because $\rho \le 10^{-2}$, and since these vectors are almost orthogonal to e_1, W makes a small angle with P_v. Then $\operatorname{dist}(z_j, P_v) \ge 4/10$, and $\operatorname{dist}(w_j, P_v) \ge 1/3$ for the corresponding w_j. So

$$\sin(2\rho) \le 3\sin(2\rho)\operatorname{dist}(w_j, P_v) \le 3|D_j| \le 3\sin(\alpha_+(\varphi)) \tag{14.20}$$

by (14.19) and (14.18), which proves (14.16).

Next we want to control some other term of (14.15), namely $S = \sum_j \cos\theta_j \cos\alpha_j$. Recall from (14.9) and the definition above (14.8) that $\cos\theta_j \cos\alpha_j = \langle v_j, e_1\rangle$, where v_j is the point of Γ_j such that $\langle v_j, e_3\rangle = 0$.

Denote by $\widehat{\Gamma}_j^+$ the geodesic that leaves from w^+ in the same direction as Γ_j^+, but which we continue a little further, so that its length is $2\pi/3$, say. Let $h_j : [0, 2\pi/3] \to \partial B$ denote the parameterization of $\widehat{\Gamma}_j^+$ by arc-length. We want to check that

$$|v_j - h_j(\pi/2)| \le 3\alpha_+(\varphi). \tag{14.21}$$

First notice that $\langle h_j(t), w^+\rangle = \cos t$, so

$$\cos t - \rho \le \langle h_j(t), e_3\rangle \le \cos t + \rho \tag{14.22}$$

because $|w^+ - e_3| \le \rho$ by (14.7). Let $t_j \in [0, 2\pi/3]$ be such that $\langle h_j(t_j), e_3\rangle = 0$. Then $|\cos t_j| \le \rho$, or equivalently, $|t_j - \pi/2| \le \mathrm{Arcsin}(\rho)$.

If v_j lies in Γ_j^+, then $v_j = h(t_j)$ and $|v_j - h_j(\pi/2)| \le |t_j - \pi/2| \le \mathrm{Arcsin}(\rho) \le 3\alpha_+(\varphi)$, by (14.16) and as needed for (14.21).

Otherwise, if $v_j \in \Gamma_j^-$, we go from w^+ to v_j as follows. First we follow Γ_j^+ up to w_j, and then we turn by at most $\alpha_+(\varphi)$ (by definition of $\alpha_+(\varphi)$) to follow Γ_j^+. Let T be such that $h_j(T) = w_j$; the continuation of the trip can be parameterized by a new function h_j', with $h_j'(T) = h_j(T) = w_j$, and then $|h_j'(t) - h_j(t)| \le 2(t-T)\,\alpha_+(\varphi)$ for $T \le t \le T + 10^{-1}$. We cross the hyperplane plane $x_3 = 0$ much before that, because for $t = T + 10^{-1}$, $h_j(t)$ lies far under the plane, by (14.22).

Now $v_j = h_j'(t)$, where $t \in [T, T+10^{-1}]$ is such that $\langle h_j'(t), e_3\rangle = 0$. Then $|\langle h_j(t), e_3\rangle| \le |h_j'(t) - h_j(t)| \le 2(t-T)\,\alpha_+(\varphi) \le \alpha_+(\varphi)/5$, so $|\cos t| \le \rho + \alpha_+(\varphi)/5$ by (14.22), and $|t - \pi/2| \le 11\alpha_+(\varphi)/5$ by (14.16). Finally

$$\begin{aligned} |v_j - h_j(\pi/2)| = |h_j'(t) - h_j(\pi/2)| &\le |h_j(t) - h_j(\pi/2)| + \alpha_+(\varphi)/5 \\ &\le |t - \pi/2| + \alpha_+(\varphi)/5 \le 3\alpha_+(\varphi), \end{aligned} \tag{14.23}$$

which proves (14.21) in our second case.

We prefer the $h_j(\pi/2)$ because they are easy to localize. In fact, $h_j(\pi/2)$ is precisely the unit direction of Γ_j at w^+, so $\big|\sum_j h_j(\pi/2)\big| \le \alpha_+(\varphi)$ by the definition (10.20) and (10.22). Then $\big|\sum_j v_j\big| \le 10\alpha_+(\varphi)$ by (14.23), and

$$\Big|\sum_j \cos\theta_j \cos\alpha_j\Big| = \Big|\sum_j \langle v_j, e_1\rangle\Big| = \Big|\langle \sum_j v_j, e_1\rangle\Big| \le \Big|\sum_j v_j\Big| \le 10\alpha_+(\varphi) \tag{14.24}$$

by (14.9). Thus

$$\begin{aligned} \big|3\pi - \sum_{j\,;\,\pm} d(v_j, w^\pm)\big| &\le 2\sin\rho \Big|\sum_j \cos\theta_j \cos\alpha_j\Big| + 2\sin^3\rho \\ &\le 20\sin\rho\ \alpha_+(\varphi) + 2\sin^3\rho \le 40\alpha_+(\varphi)^2 + 8\alpha_+(\varphi)^3 \le 41\alpha_+(\varphi)^2 \end{aligned} \tag{14.25}$$

by (14.15), (14.24), and (14.16). We now compare to (14.8) and get that

$$H^1(\varphi_*(K)) = \sum_j H^1(\Gamma_j) \le \sum_j [d(v_j, w^+) + d(v_j, w^-) + C\alpha_+(\varphi)^2] \le 3\pi + C\alpha_+(\varphi)^2 = H^1(K) + C\alpha_+(\varphi)^2 \tag{14.26}$$

because K is a cone of type $\mathbb{Y}$. Thus (14.2) holds, and Lemma 14.6 follows. □

We turn to the cone T that was described in Section 1, near Figure 1.2. That is, $T \subset \mathbb{R}^3$ is the cone over the union of the edges of a regular tetrahedron centered at the origin.

Lemma 14.27. *The isometric images in $\mathbb{R}^n$ of the cone T have the full length property.*

We shall denote by A'_j, $1 \le j \le 4$, the vertices of the regular tetrahedron used to define T; in the present situation, the regular decomposition of $K = T \cap \partial B$ constructed in Section 2 is just its decomposition into six arcs of great circles, i.e., we don't need to add new vertices.

Let $\varphi \in \Phi(\eta_1)$ be given, with η_1 small; then $\varphi_*(K)$ is a tetrahedron in $\mathbb{R}^n$, with the vertices $A_j = \varphi(A'_j) \in \partial B$. We want to show first that there is a regular tetrahedron centered at the origin, with vertices $a_j \in \partial B$, $1 \le j \le 4$, and such that

$$|a_j - A_j| \le C\alpha_+(\varphi) \text{ for } 1 \le j \le 4. \tag{14.28}$$

We shall first reduce to the simpler situation when $\varphi_*(K)$ lies in $\mathbb{R}^3$. We may assume that A_1, A_2, and A_3 lie in $\mathbb{R}^3$. Denote by $w_{i,j}$ the unit direction of the geodesic from A_i to A_j, at the point A_i (and pointing away from A_i). We know from the definitions (10.20) and (10.22) that $|w_{1,2} + w_{1,3} + w_{1,4}| \le \alpha_+(\varphi)$, so $\operatorname{dist}(w_{1,4}, \mathbb{R}^3) \le \alpha_+(\varphi)$ because the other vectors lie in $\mathbb{R}^3$. We follow the geodesic for less than two units of length to go from A_1 to A_4, so $\operatorname{dist}(A_4, \mathbb{R}^3) \le 3\alpha_+(\varphi)$ (compare with the end of a nearby geodesic with the same length and contained in $\mathbb{R}^3$). Let $A' \in \mathbb{R}^3$ be such that $|A' - A_4| \le 3\alpha_+(\varphi)$, and then set $A = A'/|A'|$. Thus $A \in \mathbb{R}^3 \cap \partial B$ and $|A - A_4| \le 6\alpha_+(\varphi)$.

When we replace A_4 with A, we modify the $w_{i,j}$ by less than $C\alpha_+(\varphi)$, so if we modify φ by setting $\widetilde{\varphi}(a_4) = A$, we still have that $\alpha_+(\widetilde{\varphi}) \le C\alpha_+(\varphi)$. If we can prove (14.28) for tetrahedra in $\mathbb{R}^3$, we apply it to $\widetilde{\varphi}_*(K)$ and get a regular tetrahedron which also works for $\varphi_*(K)$ (because $|A - A_4| \le 6\alpha_+(\varphi)$).

So we may assume that the A_j lie in $\mathbb{R}^3$. This will be convenient, because this will allow us to use some identities on spherical triangles.

Consider the spherical triangle (A_1, A_2, A_3) for a moment, and denote by α_j its angle at A_j, and l_j the length of the edge opposite to A_j. We shall use the following two formulae, which we take from [Be]:

$$\frac{\sin l_1}{\sin \alpha_1} = \frac{\sin l_2}{\sin \alpha_2} = \frac{\sin l_3}{\sin \alpha_3} \tag{14.29}$$

and

$$\cos\alpha_1 = \frac{\cos l_1 - \cos l_2\,\cos l_3}{\sin l_2 \sin l_3}, \tag{14.30}$$

which are respectively 18.6.13.4 and 18.6.13.7 in [Be]. Note that (14.30) looks a little strange at first sight, because the denominator on the right-hand side could be small, but if this happens, the cosines of the numerator are all close to 1, and the numerator is small too. Let us check that

$$\big|\alpha_j - \frac{2\pi}{3}\big| \le 5\alpha_+(\varphi). \tag{14.31}$$

Denote by w^1, w^2, and w^3 the unit directions of the three geodesics of $\varphi_*(K)$ that leave from A_j. Observe that

$$|w^1 + w^2 + w^3| \le \alpha_+(\varphi), \tag{14.32}$$

by the definitions (10.20) and (10.22). Denote by α^1, α^2, α^3 the angles of the w^k, with α^k opposite to w^k, and choose the names so that $\alpha_j = \alpha^1$.

Since we now work in $\mathbb{R}^3$, the three vectors w^k lie in a same plane (the plane orthogonal to the direction of A_j), so $\alpha^1 + \alpha^2 + \alpha^3 = 2\pi$. Let us assume that α^2 lies on the other side of $\frac{2\pi}{3}$ than $\alpha_j = \alpha^1$ (if not, α^3 has this property); then $|\alpha^1 - \alpha^2| = |\alpha_j - \alpha^2| \ge \big|\alpha_j - \frac{2\pi}{3}\big|$. Let h denote the coordinate of $w^1 + w^2 + w^3$ on the direction orthogonal to w^3; then

$$\begin{aligned} |h| =& \big|\sin\alpha^1 - \sin\alpha^2\big| = \Big|2\sin\big((\alpha^1 - \alpha^2)/2\big)\cos\big((\alpha^1 + \alpha^2)/2\big)\Big| \\ &\ge \frac{1}{2}\Big|\sin\big((\alpha^1 - \alpha^2)/2\big)\Big| \ge \frac{1}{5}\,|\alpha^1 - \alpha^2| \ge \frac{1}{5}\,\big|\alpha_j - \frac{2\pi}{3}\big| \end{aligned} \tag{14.33}$$

because only w_1 and w_2 contribute, and all the α^j are fairly close to $2\pi/3$. Now $|h| \le |w^1 + w^2 + w^3|$, and (14.31) follows from (14.33) and (14.32).

Return to (14.29) and (14.30). The $\sin\alpha_j$ in the formula are all $5\alpha_+(\varphi)$-close to $\sqrt{3}/2$, so (14.29) says that the three $\sin l_j$ are all close to a same value. That is, set $y = \sin l_1$; then, for $j \ne 1$,

$$|\sin l_j - y| = \Big|\frac{y\sin\alpha_j}{\sin\alpha_1} - y\Big| = \frac{y|\sin\alpha_j - \sin\alpha_1|}{\sin\alpha_1} \le 2|\alpha_j - \alpha_1| \le 20\alpha_+(\varphi) \tag{14.34}$$

by (14.29) and (14.31).

Denote by L the the length of each arc of K. We can compute L, because it is the side-length of an equilateral spherical triangle with $2\pi/3$ angles, and (14.30) says that $X = \cos L$ is a solution of $\frac{X - X^2}{1 - X^2} = \cos(2\pi/3) = -1/2$, so $\frac{X}{1+X} = -1/2$ and $X = -1/3$. Now

$$|l_j - L| \le 10^{-1} \quad \text{for } 1 \le j \le 3 \tag{14.35}$$

because $\varphi \in \Phi(\eta_1)$ for some small η_1 that we can choose. In the range where (14.35) holds, the right-hand side of (14.30) is a Lipschitz function of the three $\sin l_j$. When the three arguments are equal to $y = \sin l_1$, we get the value $\frac{x-x^2}{1-x^2} = \frac{x}{1+x}$, where we set $x = \cos l_1$. Thus

$$\begin{aligned} \Big|\frac{x}{1+x} + \frac{1}{2}\Big| &\le \Big|\frac{x}{1+x} - \cos\alpha_1\Big| + \Big|\cos\alpha_1 + \frac{1}{2}\Big| \\ &\le C\sum_{j=2}^{3} |\sin l_j - y| + \Big|\alpha_1 - \frac{3\pi}{2}\Big| \le C\alpha_+(\varphi) \end{aligned} \tag{14.36}$$

by (14.30), (14.34), and (14.31). We solve for x and get that $|x + 1/2| \le C\alpha_+(\varphi)$, which in turn shows that $|l_1 - L| \le C\alpha_+(\varphi)$ (because $\cos l_1 = x$ and $\cos L = -1/2$). This implies that

$$|l_j - L| \le C\alpha_+(\varphi) \quad \text{for } 1 \le j \le 3 \tag{14.37}$$

because (14.34) gives some control on $|l_j - l_1|$.

Of course what we did for the spherical triangle with vertices A_1, A_2 and A_3 can also be done for the other triangles, and we get that each angle in $\varphi_*(K)$ is $5\alpha_+(\varphi)$-close to $2\pi/3$, and the length of each arc of $\varphi_*(K)$ is $C\alpha_+(\varphi)$-close to L.

The regular tetrahedron promised in (14.28) can now be constructed by hand: we take $a_1 = A_1$, then $a_2 \in \partial B$ in the plane that contains A_1 and A_2, close to A_2 and so that $d(a_1, a_2) = L$. It is automatically $C\alpha_+(\varphi)$-close to A_2. The two last points a_3 and a_4 are the only two points of ∂B that lie at geodesic distance L from a_1 and a_2, and it is easy to see that they also lie within $C\alpha_+(\varphi)$ of A_3 and A_4.

So we get (14.28), and now we need to prove (14.2). Define a function F on $(\partial B)^4$ by $F(a_1, a_2, a_3, a_4) = \sum_{1\le j<k\le 4} d_{\partial B}(a_j, a_k)$ (the sum of the length of the geodesic arcs that connect the vertices). For the tetrahedron of (14.28), we just get $F(a_1, a_2, a_3, a_4) = 6L$.

Now F is smooth near (a_1, a_2, a_3, a_4) , with a bounded second derivative, and (a_1, a_2, a_3, a_4) is a critical point of F; this is easy to check directly, because the geodesic arcs that connect the vertices make angles of 120° when they meet. Then

$$\begin{aligned} |H^1(\varphi_*(K)) - H^1(K)| &= |F(A_1, A_2, A_3, A_4) - F(a_1, a_2, a_3, a_4)| \\ &\le C\sum_{1\le i\le 4} |A_i - a_i|^2 \le C\alpha_+(\varphi)^2 \end{aligned} \tag{14.38}$$

by Taylor's formula and (14.28). This proves (14.2), and Lemma 14.27 follows. □

Remark 14.39. The full length constants in Lemmas 14.4, 14.6, and 14.27 do not depend on n, for instance because the $\varphi_*(K)$ have at most five points, so that we can always assume that they lie in $\mathbb{R}^5$.

Remark 14.40. The sufficient condition (14.2) that implies the full length condition can be checked independently on the connected components of $K = E \cap \partial B$. That is, if E is a

minimal (or minimal-looking) cone, then K satisfies (14.2) if and only if every connected component of K satisfies it.

This is easy to check; the only point is that by the description of Section 2, K is the finite union of its connected components K_j, which lie at distances at least η_0 from each other. Then $\varphi_*(K)$ is the disjoint union of the $\varphi_*(K_j)$, and (14.2) can be verified independently for each K_j.

Thus, when E is a finite union of planes, or sets of type $\mathbb{Y}$ or $\mathbb{T}$ that only meet at the origin, E automatically has the full length property. But even in this case the conditions under which E is a minimal cone are not known. In fact, the only known example seems to be the union of minimal cones that lie in orthogonal spaces.

It is a little less obvious that the full length condition itself can be checked independently on the connected components. Let us merely suggest a strategy that we could use to prove that if every connected component of K satisfies the full length condition, then K itself satisfies it.

Let $\varphi \in \Phi(\eta_1)$ be such that $H^1(\varphi_*(K)) > H^1(K)$; we can find a component K_1 of K and a deformation $\widetilde{X}_1 = f(X_1)$ of the cone X_1 over $\varphi_*(K_1)$ that allows to save some amount A of area, as in (2.12).

Set $K_2 = K \setminus K_1$, and denote by X_2 the cone over $\varphi_*(K_2)$. We would like to take $f(x) = x$ on X_2, and then say that $H^2(f(X_1 \cup X_2) \cap B(0,1)) \leq H^2(f(X_1) \cap B(0,1)) + H^2(f(X_2) \cap B(0,1)) = H^2(\widetilde{X}_1 \cap B(0,1)) + H^2(X_2 \cap B(0,1)) \leq H^2(\varphi_*(K)) - A$ and conclude.

However, X_1 and X_2 meet at the origin, so we cannot take two different definitions of f there, and some surgery near the origin is needed. Notice that the question only arises when $n \geq 4$, because we know the minimal cones when $n = 3$. Then $\varphi_*(K_1)$ and $\varphi_*(K_2)$ are not only disjoint, but (by general position) they are not linked in ∂B. That is, we can deform $\varphi_*(K_1)$ to a point z_1 inside $\partial B \setminus \varphi_*(K_2)$, and then deform $\varphi_*(K_2)$ to a point in $\partial B \setminus \{z_1\}$. This should allow us to modify $\varphi_*(E) = X_1 \cup X_2$ in a small ball near the origin, so that the images of X_1 and X_2 are mostly far apart, except for a small line segment that connects them. [Follow the deformation above, where the distance to the origin plays the role of the time parameter, on a small annulus.] Then the initial plan of taking different definitions of the deformation on the two pieces can be realized: what happens on the line segment does not matter, because its image will have finite H^1-measure.

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