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Guy Barles, Pierre Cardaliaguet, Olivier Ley, Aurélien Monteillet

EXISTENCE OF WEAK SOLUTIONS FOR GENERAL NONLOCAL AND NONLINEAR SECOND-ORDER PARABOLIC EQUATIONS

GUY BARLES, PIERRE CARDALIAGUET, OLIVIER LEY, AND AURÉLIEN MONTEILLET

Abstract. In this article, we provide existence results for a general class of nonlocal and nonlinear second-order parabolic equations. The main motivation comes from front propagation theory in the cases when the normal velocity depends on the moving front in a nonlocal way. Among applications, we present level-set equations appearing in dislocations' theory and in the study of Fitzhugh-Nagumo systems.

1. Introduction

We are concerned with a class of nonlocal and nonlinear parabolic equations which can be written as

$$(1.1)\qquad \begin{cases} u_t = H[\mathbb{1}_{\{u\geq 0\}}](x,t,u,Du,D^2u) & \text{in } \mathbb{R}^N\times(0,T),\\ u(\cdot,0)=u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

where u_t, Du and D^2u stand respectively for the time derivative, gradient and Hessian matrix with respect to the space variable x of $u:\mathbb{R}^N\times[0,T]\to\mathbb{R}$ and where $\mathbb{1}_A$ denotes the indicator function of a set A. The initial datum u_0 is a bounded and Lipschitz continuous function on $\mathbb{R}^N$.

For any indicator function $\chi:\mathbb{R}^N\times[0,T]\to\mathbb{R}$, or more generally for any $\chi\in L^\infty(\mathbb{R}^N\times[0,T];[0,1])$, $H[\chi]$ denotes a function of $(x,t,r,p,A)\in\mathbb{R}^N\times[0,T]\times\mathbb{R}\times\mathbb{R}^N\setminus\{0\}\times\mathcal{S}_N$, where $\mathcal{S}_N$ is the set of real, $N\times N$ symmetric matrices. For almost any $t\in[0,T]$, $(x,r,p,A)\mapsto H[\chi](x,t,r,p,A)$ is a continuous function on $\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\setminus\{0\}\times\mathcal{S}_N$ with a possible singularity at $p=0$ (when considering geometrical equations, see for instance Giga [11]), while $t\mapsto H[\chi](x,t,r,p,A)$ is a bounded measurable function for all

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$(x, r, p, A) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}_N$. We recall that the equation is said to be degenerate elliptic (or here parabolic) if, for any $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$, for any $(x, r, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$, for almost every $t \in [0,T]$ and for all $A, B \in \mathcal{S}_N$, one has

$$H[\chi](x, t, r, p, A) \le H[\chi](x, t, r, p, B) \quad \text{if } A \le B,$$

where $\le$ stands for the usual partial ordering for symmetric matrices.

Such equations arise typically when one aims at describing, through the "level-set approach", the motion of a family $\{K(t)\}_{t\in[0,T]}$ of closed subsets of $\mathbb{R}^N$ evolving with a nonlocal velocity. Indeed, following the main idea of the level-set approach, it is natural to introduce a function u such that

$$K(t) = \{x \in \mathbb{R}^N;\ u(x,t) \ge 0\} \ ,$$

and (1.1) can be seen as the level-set equation for u. In this framework, the nonlinearity H corresponds to the velocity and, in the applications we have in mind, it depends not only on the time, the position of the front, the normal direction and the curvature tensor but also on nonlocal properties of $K(t)$ which are carried by the dependence in $\mathbb{1}_{\{u\ge 0\}}$. We may face rather different nonlocal dependences and this is why we have chosen this formulation: in any case, the equation appears as a well-posed equation if we would consider the nonlocal dependence (i.e. $\mathbb{1}_{\{u\ge 0\}}$) as being fixed; in other words, the $H[\chi]$-equation enjoys "good" properties.

Finally, we recall that, still in the case of level-set equations, the function u_0 is used to represent the initial front, *i.e.*

$$\{u_0 \ge 0\} = K_0 \quad \text{and} \quad \{u_0 = 0\} = \partial K_0 \tag{1.2}$$

for some fixed compact set $K_0 \subset \mathbb{R}^N$. We refer the reader to [11] and the references therein for precisions.

Now we turn to the main examples we have in mind.

1. Dislocation dynamics equations

$$u_t = (c_0(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t)\ge 0\}}(x) + c_1(x,t))|Du|,$$

$$\text{or} \quad u_t = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right) + c_0(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t)\ge 0\}}(x) + c_1(x,t)\right]|Du|,$$

where

$$c_0(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t)\ge 0\}}(x) = \int_{\mathbb{R}^N} c_0(x-y, t)\mathbb{1}_{\{u(\cdot,t)\ge 0\}}(y)dy$$

and $\operatorname{div}\left(\frac{Du}{|Du|}\right)(x,t)$ is the mean curvature of the set $\{u(\cdot,t) = u(x,t)\}$ at x.

Typically, the reasonnable assumptions in this context (see, for example, [4]) are the following: c_0, c_1 are bounded, continuous functions which are

Lipschitz continuous in x (uniformly with respect to t) and c_0, $D_x c_0 \in L^\infty([0,T]; L^1(\mathbb{R}^N))$. In particular, and this is a key difference with the second example below, c_0 is bounded.

2. Fitzhugh-Nagumo type systems, which, in a simplified form, reduce to the nonlocal equation

$$u_t = \alpha \left(\int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)\, \mathbb{1}_{\{u(y,s)\geq 0\}}\, dyds \right) |Du|,$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and G is the Green function of the heat equation (see (3.10)).

There are two key differences with the first example: the convolution kernel acts in space and time, and G is not bounded. This difference plays a central role when one tries to prove uniqueness (cf. [5]).

3. Equations of the form

$$\begin{aligned} & u_t = (k - \mathcal{L}^N(\{u(\cdot,t) \geq 0\}))|Du| \qquad (k \in \mathbb{R}), \\ \text{or} \quad & u_t = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right) - \mathcal{L}^N(\{u(\cdot,t)\geq 0\}) \right] |Du|, \end{aligned}$$

where $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure and therefore the velocity of the front at time t depends on the volume of $K(t) = \{u(\cdot,t) \geq 0\}$.

In the classical cases, the equations of the level-set approach are solved by using the theory of viscosity solutions. Nevertheless there are two key features which may prevent a direct use of viscosity solutions' theory to treat the above examples: the main problem is that these examples do not satisfy the right monotonicity property. This can be seen either through the fact that $\{u \geq 0\} \subset \{v \geq 0\}$ does not imply that $H[\mathbb{1}_{\{u\geq 0\}}] \leq H[\mathbb{1}_{\{v \geq 0\}}]$, or by remarking that the associated front propagations do not satisfy the "inclusion principle" (geometrical monotonicity). Indeed, in the dislocation dynamics case, the kernel c_0 changes sign, which implies the two above facts. Therefore the classical comparison arguments of viscosity solutions' theory fail, and since existence is also based on these arguments through the Perron's method, the existence of viscosity solutions to these equations becomes an issue too.

The second (and less important) feature which prevents a direct use of the standard level-set approach arguments is the form of the nonlocal dependence in $\mathbb{1}_{\{u\geq 0\}}$: as shown in Slepcev [17] and used in the present framework (but in the monotone case) in [4], a dependence in $\mathbb{1}_{\{u \geq u(x,t)\}}$ is the most adapted to the level-set approach since all the level sets of the solutions are treated similarly instead of having the 0-level set playing a particular role.

As a consequence, we are going to use a notion of weak solutions for (1.1) introduced in [4] (see Definition 2.1), and prove two general existence results. As an simple application of this theorem, we recover existence results for dislocation equations and the Fitzhugh-Nagumo system obtained by Giga, Goto and Ishii [12], Soravia and Souganidis [18] and in [4]. Let us mention that the technique of proof of our results, using Kakutani's fixed point theorem, is the same that is used in [12]. Here we generalize its range of application and combine it with a new stability result of Barles [3]. In [5], we prove the uniqueness of such weak solutions for these two model equations. Note that the issue of uniqueness is a difficult problem and, in general, uniqueness does not hold as shown by the counterexample developed in [4].

Another issue of these nonlocal equations is connected to the behavior of H with respect to the size of the set $\{u \geq 0\}$. Indeed, in the dislocation dynamics case, if $c_0 \in L^\infty([0,T]; L^1(\mathbb{R}^N))$, then $H[\mathbb{1}_{\{u\geq 0\}}]$ is defined without restriction on the size of $\{u \geq 0\}$. The situation is the same for the Fitzhugh-Nagumo system. However, if c_0 is only bounded and not in L^1, or in volume-dependent equations, then the support of $\{u \geq 0\}$ has to be bounded for $H[\mathbb{1}_{\{u\geq 0\}}]$ to be defined. This leads us to distinguish two cases, that we will call respectively the unbounded and the bounded case.

The paper is organized as follows: in Section 2 we give a general definition of a weak solution to (1.1). In Section 3 we prove existence of such solutions in the unbounded case, and apply our result to dislocation equations and the Fitzhugh-Nagumo system. In Section 4 we treat the bounded case and give as an application an existence result for volume-dependent equations.

Notation: In the sequel, $|\cdot|$ denotes the standard euclidean norm in $\mathbb{R}^N$, $B(x,R)$ (resp. $\bar{B}(x,R)$) is the open (resp. closed) ball of radius R centered at $x \in \mathbb{R}^N$. The notation $\mathcal{S}_N$ denotes the space of real $N \times N$ symmetric matrices.

2. Definition of weak solutions

We will use the following definition of weak solutions introduced in [4]. To do so, we use the notion of viscosity solutions for equations with a measurable dependence in time which we call below "L^1-viscosity solution". We refer the reader to [4, Appendix] for the definition of L^1-viscosity solutions and [13, 15, 16, 8, 9] for a complete presentation of the theory.

Definition 2.1. *Let $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ be a continuous function. We say that u is a weak solution of* (1.1) *if there exists $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ such that*

(1) *u is a L^1-viscosity solution of*

$$\begin{cases} u_t(x,t) = H[\chi](x,t,u,Du,D^2u) & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

(2) *For almost all $t \in [0,T]$,*

$$\mathbb{1}_{\{u(\cdot,t)>0\}} \leq \chi(\cdot,t) \leq \mathbb{1}_{\{u(\cdot,t)\geq 0\}} \quad \text{a.e. in } \mathbb{R}^N. \tag{2.2}$$

Moreover, we say that u is a classical solution of (1.1) *if in addition, for almost all $t \in [0,T]$,*

$$\mathbb{1}_{\{u(\cdot,t)>0\}} = \mathbb{1}_{\{u(\cdot,t)\geq 0\}} \quad \text{a.e. in } \mathbb{R}^N.$$

Remark 2.2. If for any fixed $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ the map $H[\chi]$ is geometric, then the map χ defined by (2.1)-(2.2) only depends on the $0-$level-set of the initial condition u_0, as in the classical level-set approach. Indeed, let $u_0^1 : \mathbb{R}^N \to \mathbb{R}$ be another bounded continuous map such that

$$\{u_0 \geq 0\} = \{u_0^1 \geq 0\} \text{ and } \{u_0 > 0\} = \{u_0^1 > 0\}\ ,$$

and u^1 be the solution to (2.1) with the same χ but with initial condition u_0^1 (under the assumptions of Theorem 3.1 or Theorem 4.1 such a solution exists and is unique). Then from the key property of geometric equations (see for instance [11]) we will have, for almost all $t \in [0,T]$,

$$\mathbb{1}_{\{u(\cdot,t)>0\}} = \mathbb{1}_{\{u^1(\cdot,t)>0\}} \leq \chi(\cdot,t) \leq \mathbb{1}_{\{u(\cdot,t)\geq 0\}} = \mathbb{1}_{\{u^1(\cdot,t)\geq 0\}} \quad \text{a.e. in } \mathbb{R}^N$$

This means that the map χ can be interpreted as the weak solution of a nonlocal geometric flow.

3. Existence of weak solutions to (1.1) (unbounded case)

In this section, we are interested in the case where the Hamiltonian $H[\chi]$ is defined without any restriction on the size of the support of χ.

3.1. **The existence theorem.**

We first state some assumptions which we use here but also in the next sections. To avoid repeating them, we are going to formulate assumptions on the nonlinearities $H[\chi]$ which have to be satisfied for any $\chi \in X$ (and uniformly for such χ) where X is a subset of $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$. We use a different X in this section and for the "bounded" case.

(H1-X) (i) For any $\chi \in X$, Equation (2.1) has a bounded uniformly continuous L^1-viscosity solution u. Moreover, there exists a constant $L > 0$ independent of $\chi \in X$ such that $|u|_\infty \le L$.

(ii) For any fixed $\chi \in X$, a comparison principle holds for Equation (2.1): if u is a bounded, upper-semicontinuous L^1-viscosity subsolution of (2.1) in $\mathbb{R}^N \times (0,T)$ and v is a bounded, lower-semicontinuous L^1-viscosity supersolution of (2.1) in $\mathbb{R}^N \times (0,T)$ with $u(\cdot,0) \le v(\cdot,0)$ in $\mathbb{R}^N$, then $u \le v$ in $\mathbb{R}^N \times [0,T]$.

In the same manner, if u is a bounded, upper-semicontinuous L^1-viscosity subsolution of (2.1) in $B(x,R) \times (0,T)$ for some $x \in \mathbb{R}^N$ and $R > 0$, and v is a bounded, lower-semicontinuous L^1-viscosity supersolution of (2.1) in $B(x,R) \times (0,T)$ with $u(y,t) \le v(y,t)$ if $t = 0$ or $|y-x| = R$, then $u \le v$ in $B(x,R) \times [0,T]$.

(H2-X) (i) For any compact subset $K \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}_N$, there exists a (locally bounded) modulus of continuity $m_K : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $m_K(\cdot,\varepsilon) \to 0$ in $L^1(0,T)$ as $\varepsilon \to 0$, and

$$|H[\chi](x_1,t,r_1,p_1,A_1) - H[\chi](x_2,t,r_2,p_2,A_2)| \le m_K(t, |x_1-x_2| + |r_1-r_2| + |p_1-p_2| + |A_1-A_2|)$$

for any $\chi \in X$, for almost all $t \in [0,T]$ and all $(x_1,r_1,p_1,A_1), (x_2,r_2,p_2,A_2) \in K$.

(ii) There exists a bounded function $f(x,t,r)$, which is continuous in x and r for almost every t and mesurable in t, such that: for any neighborhood V of $(0,0)$ in $\mathbb{R}^N \setminus \{0\} \times \mathcal{S}_N$ and any compact subset $K \subset \mathbb{R}^N \times \mathbb{R}$, there exists a modulus of continuity $m_{K,V} : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $m_{K,V}(\cdot,\varepsilon) \to 0$ in $L^1(0,T)$ as $\varepsilon \to 0$, and

$$|H[\chi](x,t,r,p,A) - f(x,t,r)| \le m_{K,V}(t, |p| + |A|)$$

for any $\chi \in X$, for almost all $t \in [0,T]$, all $(x,r) \in K$ and $(p,A) \in V$.

(iii) If $\chi_n \rightharpoonup \chi$ weakly-$*$ in $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ with $\chi_n, \chi \in X$ for all n, then for all $(x,t,r,p,A) \in \mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}_N$,

$$\int_0^t H[\chi_n](x,s,r,p,A)ds \underset{n\to+\infty}{\longrightarrow} \int_0^t H[\chi](x,s,r,p,A)ds$$

locally uniformly for $t \in [0,T]$.

We finally add an assumption which is not optimal on the behavior of $H[\chi]$ with respect to r but to which we can reduce, in most cases, after some change of unknown functions like $u \to u\exp(\gamma t)$:

(H3-X) For any $\chi \in X$, for almost every $t \in [0,T]$, for all $(x,p,A) \in \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}_N$, and for any $r_1 \geq r_2$

$$H[\chi](x,t,r_1,p,A) \geq H[\chi](x,t,r_2,p,A) \ .$$

Of course, gathering **(H2-X)** and **(H3-X)**, it is easy to show that f satisfies the same property.

We have chosen to state Assumption **(H1-X)** in this form which may look artificial: it means that we have existence, uniqueness of a continuous L^1-viscosity solution u associated to any measurable *fixed* function $0 \leq \chi \leq 1$. For conditions on H under which **(H1-X)** is verified, we refer to [8, 16] and Section 3.2. Moreover, **(H1-X)** (i) states that the u's are bounded uniformly with respect to $\chi \in X$: for the geometrical equations of the level-set approach, this uniform bound on u is automatically satisfied with $L = ||u_0||_\infty$ if **(H1-X)** (ii) holds, using that, in this case, constants are L^1-viscosity solutions of (2.1). Assumption **(H2-X)** comes from [3] and will be used to apply a stability result for equations with L^1-dependence in time.

Our general existence theorem is the following:

Theorem 3.1. *Assume that* **(H1-X)***,* **(H2-X)** *and* **(H3-X)** *hold with* $X = L^\infty(\mathbb{R}^N \times [0,T];[0,1])$*. Then there exists at least a weak solution to* (1.1)*.*

Remark 3.2. See also [3] for the stability of weak solutions.

Proof. From **(H1-X)**, the set-valued mapping

$$\begin{array}{rccl} \xi : & X & \rightrightarrows & X \\ & \chi & \mapsto & \big\{\chi' \ : \ \mathbb{1}_{\{u(\cdot,t)>0\}} \leq \chi'(\cdot,t) \leq \mathbb{1}_{\{u(\cdot,t)\geq 0\}} \text{ for almost all } t \in [0,T], \\ & & & \qquad \text{where } u \text{ is the } L^1\text{-viscosity solution of (2.1)}\big\}, \end{array}$$

is well-defined. Clearly, there exists a weak solution to (1.1) if there exists a fixed point χ of ξ, which means that $\chi \in \xi(\chi)$. In this case the corresponding u is a weak solution of (1.1). We therefore aim at using Kakutani's fixed point theorem for set-valued mappings (see [2, Theorem 3 p. 232]).

In the Hausdorff convex space $L^\infty(\mathbb{R}^N \times [0,T];\mathbb{R})$, the subset X is convex and compact for the L^∞-weak-$*$ topology (since it is closed and bounded). In the same way, for any $\chi \in X$, $\xi(\chi)$ is a non-empty convex compact subset of X for the L^∞-weak-$*$ topology.

Let us check that ξ is upper semicontinuous for this topology. It suffices to show that, if

$$\chi_n \in X \underset{L^\infty\text{-weak-}*}{\rightharpoonup} \chi \quad \text{and} \quad \chi'_n \in \xi(\chi_n) \underset{L^\infty\text{-weak-}*}{\rightharpoonup} \chi',$$

then

$$\chi' \in \xi(\chi).$$

Let u_n be the unique L^1-viscosity solution of (2.1) associated to χ_n by **(H1-**X**)**. Using **(H1-**X**)**, we know that the u_n's are uniformly bounded. We can therefore define the half-relaxed limits[1]

$$\overline{u} = \limsup{}^* u_n \quad \text{and } \underline{u} = \liminf{}_* u_n.$$

From **(H2-**X**)** (convergence of the Hamiltonians), we can apply Barles' stability result [3, Theorem 1.1] to obtain that $\overline{u}$ (respectively $\underline{u}$) is a L^1-viscosity subsolution (respectively supersolution) of (2.1) associated to χ.

In order to apply **(H1-**X**)** (ii) (comparison), we first have to show that

$$\overline{u}(x,0) \le u_0(x) \le \underline{u}(x,0) \quad \text{in } \mathbb{R}^N .$$

To do so, we examine **(H2-**X**)** and deduce that $H[\chi](x,t,0,p,A)$ is bounded if p and A are bounded, uniformly with respect to $\chi \in X$. Moreover u_0 is Lipschitz continuous, and therefore for all $0 < \varepsilon \le 1$, we have, for any $x, y \in \mathbb{R}^N$,

$$u_0(y) \le u_0(x) + ||Du_0||_\infty |x-y| \le u_0(x) + \frac{|x-y|^2}{2\varepsilon^2} + \frac{||Du_0||_\infty \varepsilon^2}{2} .$$

We fix x and we argue in the ball $B(x,\varepsilon)$. Using **(H3-**X**)** and the fact that the $H[\chi_n]$'s are locally bounded, the function

$$\psi_\varepsilon(y,t) = u_0(x) + \frac{|x-y|^2}{2\varepsilon^2} + \frac{||Du_0||_\infty \varepsilon^2}{2} + C_\varepsilon t$$

is a supersolution of the $H[\chi_n]$-equation in the ball $B(x,\varepsilon)$ provided that C_ε is large enough. By **(H1-**X**)** (ii) (comparison), we obtain

$$u_n(y,t) \le \psi_\varepsilon(y,t) \quad \text{in } B(x,\varepsilon) \times [0,T] ,$$

and then

$$\overline{u}(y,t) \le \psi_\varepsilon(y,t) \quad \text{in } B(x,\varepsilon) \times [0,T] .$$

Examining the right-hand side at $(y,t) = (x,0)$ and letting $\varepsilon \to 0$ provides the inequality $\overline{u}(x,0) \le u_0(x)$. An analogous argument gives $\underline{u}(x,0) \ge u_0(x)$.

[1] $\limsup{}^* u_n(x,t) := \limsup\limits_{\substack{(y,s)\to(x,t)\\ n\to\infty}} u_n(y,s)$ and $\liminf{}_* u_n(x,t) := \liminf\limits_{\substack{(y,s)\to(x,t)\\ n\to\infty}} u_n(y,s)$.

By **(H1-X)** (ii) (comparison), we therefore have $\overline{u} \le \underline{u}$ in $\mathbb{R}^N$, which implies that $u := \overline{u} = \underline{u}$ is the unique continuous L^1-viscosity solution of (2.1) associated to χ, as well as the local uniform convergence of u_n to u.

Moreover, since $\chi'_n \in \xi(\chi_n)$, we have, for all $\varphi \in L^1(\mathbb{R}^N \times [0,T]; \mathbb{R}_+)$,

$$\int_0^T \int_{\mathbb{R}^N} \varphi\, 1\!\!1_{\{u_n(\cdot,t)>0\}} \le \int_0^T \int_{\mathbb{R}^N} \varphi\, \chi'_n \le \int_0^T \int_{\mathbb{R}^N} \varphi\, 1\!\!1_{\{u_n(\cdot,t)\ge 0\}}.$$

Since $\chi'_n \underset{L^\infty\text{-weak-}*}{\rightharpoonup} \chi'$, applying Fatou's Lemma, we get

$$\int_0^T \int_{\mathbb{R}^N} \varphi \liminf 1\!\!1_{\{u_n(\cdot,t)>0\}} \le \int_0^T \int_{\mathbb{R}^N} \varphi\, \chi' \le \int_0^T \int_{\mathbb{R}^N} \varphi \limsup 1\!\!1_{\{u_n(\cdot,t)\ge 0\}}.$$

But $1\!\!1_{\{u(\cdot,t)>0\}} \le \liminf 1\!\!1_{\{u_n(\cdot,t)>0\}}$ and $\limsup 1\!\!1_{\{u_n(\cdot,t)\ge 0\}} \le 1\!\!1_{\{u(\cdot,t)\ge 0\}}$. It follows that

$$1\!\!1_{\{u(\cdot,t)>0\}} \le \chi' \le 1\!\!1_{\{u(\cdot,t)\ge 0\}} \quad \text{for a.e. } t \in [0,T],$$

and therefore $\chi' \in \xi(\chi)$.

We infer the existence of a weak solution of (1.1) by Kakutani's fixed point theorem [2, Theorem 3 p. 232], as announced. □

3.2. **Applications.**

3.2.1. *Dislocation dynamics equations.* One important example for which Theorem 3.1 provides a weak solution is the dislocation dynamics equation (see [1], [4] and the references therein), which reads

$$\left\{ \begin{array}{ll} u_t = (c_0(\cdot,t) \star 1\!\!1_{\{u(\cdot,t)\ge 0\}}(x) + c_1(x,t))|Du| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{array} \right. \tag{3.1}$$

where

$$c_0(\cdot,t) \star 1\!\!1_{\{u(\cdot,t)\ge 0\}}(x) = \int_{\mathbb{R}^N} c_0(x-y,t) 1\!\!1_{\{u(\cdot,t)\ge 0\}}(y) dy.$$

We assume that c_0 and c_1 satisfy the following assumptions:

(A) (i) $c_0 \in C^0([0,T]; L^1(\mathbb{R}^N))$, $c_1 \in C^0(\mathbb{R}^N \times [0,T]; \mathbb{R})$.

(ii) $Dc_0 \in L^\infty([0,T]; L^1(\mathbb{R}^N))$.

(iii) There exists a constant M such that, for any $x, y \in \mathbb{R}^N$ and $t \in [0,T]$

$$|c_1(x,t)| \le M \quad \text{and} \quad |c_1(x,t) - c_1(y,t)| \le M|x-y|.$$

Theorem 3.3. *Under assumption* **(A)**, *Equation* (3.1) *has at least a weak solution. Moreover, if, for all* $(x,t,\chi) \in \mathbb{R}^N \times [0,T] \times L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$,

$$c_0(\cdot,t) \star \chi(\cdot,t)(x) + c_1(x,t) \ge 0, \tag{3.2}$$

and if there exists $\eta > 0$ with

$$|u_0| + |Du_0| \geq \eta \quad \text{in } \mathbb{R}^N \text{ in the viscosity sense,} \tag{3.3}$$

then any weak solution is classical.

Proof. This theorem is proved by Barles, Cardaliaguet, Ley and Monneau in [4, Theorem 1.2]. Another proof can be done using Theorem 3.1. First we note that Equation (3.1) is a first-order particular case of (1.1) with, for all $(x,t,p,\chi) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \times L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$,

$$H[\chi](x,t,p) = [c_0(\cdot,t) \star \chi(\cdot,t)(x) + c_1(x,t)]|p|.$$

Assumption **(H1-X)** (i) is given by [4, Theorem 5.4] and [14] (for the regularity part), while assumption **(H1-X)** (ii) holds thanks to the results of [16]. Assumption **(H2-X)** is given by [4, proof of Theorem 1.2]. It essentially amounts to noticing that if $\chi_n \rightharpoonup \chi$ in L^∞-weak-$*$, then by the definition of this convergence

$$\begin{aligned}
\int_0^t c_0(\cdot,s) \star \chi_n(\cdot,s)(x)\, ds &= \int_0^t \int_{\mathbb{R}^N} c_0(x-y,s)\chi_n(y,s)\, dyds \\
&\to \int_0^t \int_{\mathbb{R}^N} c_0(x-y,s)\chi(y,s)\, dyds \\
&= \int_0^t c_0(\cdot,s) \star \chi(\cdot,s)(x)\, ds.
\end{aligned}$$

Finally, if (3.2) and (3.3) hold, the solutions are classical by [4, Theorem 1.3]. $\square$

We can also consider the dislocation dynamics equation with an additionnal mean curvature term,

$$u_t = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right) + c_0(\cdot,t) \star \mathbb{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)\right] |Du|, \tag{3.4}$$

which has been studied by Forcadel and Monteillet [10] for instance. Theorem 3.1 also provides a weak solution to (3.4). In [10], however, the authors study the problem with the particular tool of minimizing movements, which enables them to construct a weak solution with χ of the form $\mathbb{1}_E$, with good regularity properties of $t \mapsto E(t)$. This is due to the particular structure of (3.4), namely the presence of the regularizing mean curvature term. Here we can deal with more general nonlocal degenerate parabolic equations.

3.2.2. *A Fitzhugh-Nagumo type system.* We are also interested in the following system,

$$(3.5)\begin{cases} u_t = \alpha(v)|Du| & \text{in } \mathbb{R}^N \times (0,T), \\ v_t - \Delta v = (g^+(v)\mathbb{1}_{\{u\geq 0\}} + g^-(v)(1-\mathbb{1}_{\{u\geq 0\}})) & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0)=u_0,\ v(\cdot,0)=v_0 & \text{in } \mathbb{R}^N, \end{cases}$$

which is obtained as the asymptotics as $\varepsilon \to 0$ of the following Fitzhugh-Nagumo system arising in neural wave propagation or chemical kinetics (see [18]):

$$(3.6)\qquad \begin{cases} u_t^\varepsilon - \varepsilon\Delta u^\varepsilon = \varepsilon^{-1} f(u^\varepsilon, v^\varepsilon), \\ v_t^\varepsilon - \Delta v^\varepsilon = g(u^\varepsilon, v^\varepsilon) \end{cases}$$

in $\mathbb{R}^N \times (0,T)$, where for $(u,v) \in \mathbb{R}^2$,

$$\begin{cases} f(u,v) = u(1-u)(u-a) - v & (0<a<1), \\ g(u,v) = u - \gamma v & (\gamma > 0). \end{cases}$$

The functions α, g^+ and $g^- : \mathbb{R} \to \mathbb{R}$ appearing in (3.5) are associated with f and g. This system has been studied in particular by Giga, Goto and Ishii [12] and Soravia, Souganidis [18]. They proved existence of a weak solution to (3.5). Here we recover their result as an application of Theorem 3.1.

If for $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$, v denotes the solution of

$$(3.7)\qquad \begin{cases} v_t - \Delta v = g^+(v)\chi + g^-(v)(1-\chi) & \text{in } \mathbb{R}^N \times (0,T), \\ v(\cdot,0) = v_0 & \text{in } \mathbb{R}^N, \end{cases}$$

and if $c[\chi](x,t) := \alpha(v(x,t))$, then Problem (3.5) reduces to

$$(3.8)\qquad \begin{cases} u_t(x,t) = c[\mathbb{1}_{\{u\geq 0\}}](x,t)|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

which is a particular case of (1.1). Let us now state the existence theorem of [12] and [18] that we can recover from our general existence theorem. We first gather the assumptions satisfied by α, g^-, g^+ and v_0:

(B) (i) α is Lipschitz continuous on $\mathbb{R}$.

(ii) g^+ and g^- are Lipschitz continuous on $\mathbb{R}^N$, and there exist $\underline{g}$ and $\overline{g}$ in $\mathbb{R}$ such that

$$\underline{g} \leq g^-(r) \leq g^+(r) \leq \overline{g} \quad \text{for all } r \text{ in } \mathbb{R}.$$

(iii) v_0 is bounded and of class C^1 with $\|Dv_0\|_\infty < +\infty$.

Theorem 3.4. *Under assumption* **(B)**, *the problem* (3.8), *or equivalently the system* (3.5), *has at least a weak solution. If in addition* (3.3) *holds and* $\alpha \geq 0$, *then any weak solution is classical.*

Proof. The explicit resolution of the heat equation (3.7) shows that for any $(x,t) \in \mathbb{R}^N \times (0,T)$,

$$
\begin{aligned}
(3.9)\quad v(x,t) = & \int_{\mathbb{R}^N} G(x-y,t)\, v_0(y)\, dy \\
& + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\, [g^+(v)\chi + g^-(v)(1-\chi)](y,s)\, dyds,
\end{aligned}
$$

where G is the Green function defined by

$$
G(y,s) = \frac{1}{(4\pi s)^{N/2}} e^{-\frac{|y|^2}{4s}}. \tag{3.10}
$$

It is then easy to obtain the following lemma:

Lemma 3.5. *Assume that g^-, g^+ and v_0 satisfy* **(B)**. *For $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$, let v be the solution of (3.7). Set $\gamma = \max\{|\underline{g}|, |\overline{g}|\}$. Then there exists a constant k_N depending only on N such that:*

(*i*) *v is uniformly bounded: for all $(x,t) \in \mathbb{R}^N \times [0,T]$,*

$$
|v(x,t)| \le \|v_0\|_\infty + \gamma t.
$$

(*ii*) *v is continuous on $\mathbb{R}^N \times [0,T]$.*

(*iii*) *For any $t \in [0,T]$, $v(\cdot,t)$ is of class C^1 in $\mathbb{R}^N$.*

(*iv*) *For all $t \in [0,T]$, for all $x,y \in \mathbb{R}^N$,*

$$
|v(x,t) - v(y,t)| \le (\, \|Dv_0\|_\infty + \gamma k_N \sqrt{t})\, |x-y|.
$$

(*v*) *For all $0 \le s \le t \le T$, for all $x \in \mathbb{R}^N$,*

$$
|v(x,t) - v(x,s)| \le k_N(\|Dv_0\|_\infty + \gamma k_N \sqrt{s})\sqrt{t-s} + \gamma(t-s).
$$

In particular the velocity $c[\chi]$ in (3.8) is bounded, continuous on $\mathbb{R}^N \times [0,T]$ and Lipschitz continuous in space, uniformly with respect to χ. From general results on existence and comparison for classical viscosity solutions of the eikonal equation with Lipschitz continuous initial datum (see for instance [4, Theorem 2.1]), we obtain that **(H1-X)** is satisfied.

Let us check **(H2-X)** (*iii*) ((*i*) and (*ii*) are straightforward): we claim that, if $\chi_n \rightharpoonup \chi$ in L^∞-weak-$*$, then

$$
\int_0^t c[\chi_n](x,s)\, ds \to \int_0^t c[\chi](x,s)\, ds
$$

locally uniformly in $[0,T]$. Indeed, let v_n (resp. v) be the solution of (3.7) with χ_n (resp. χ) in the right-hand side. The estimates (*iv*) and (*v*) of Lemma 3.5 on the heat equation imply that we can extract by a diagonal argument a subsequence, still denoted (v_n), which converges uniformly to some w in $\bar{B}(0,R)\times[0,T]$ for any $R>0$. We know that for any $(x,t)\in\mathbb{R}^N\times(0,T)$,

$$\begin{aligned} v_n(x,t) = &\int_{\mathbb{R}^N} G(x-y,t)\,v_0(y)\,dy \\ &+\int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)\,[g^+(v_n)\chi_n+g^-(v_n)(1-\chi_n)](y,s)\,dyds \end{aligned}$$

where G is the Green function defined by (3.10). As n goes to infinity, we obtain

$$\begin{aligned} w(x,t) = &\int_{\mathbb{R}^N} G(x-y,t)\,v_0(y)\,dy \\ &+\int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)\,[g^+(w)\chi+g^-(w)(1-\chi)](y,s)\,dyds. \end{aligned}$$

Indeed

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)\,[g^+(v_n)\chi_n+g^-(v_n)(1-\chi_n)](y,s)\,dyds \\ &-\int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)\,[g^+(w)\chi+g^-(w)(1-\chi)](y,s)\,dyds \\ = &\int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)\,[(g^+(w)-g^-(w))(\chi_n-\chi)](y,s)\,dyds \\ &+\int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)[\chi_n\,(g^+(v_n)-g^+(w))+ \\ &\qquad\qquad \chi_n\,(g^-(w)-g^-(v_n))+(g^-(v_n)-g^-(w))](y,s)\,dyds. \end{aligned}$$

The term

$$\int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)\,[(g^+(w)-g^-(w))(\chi_n-\chi)](y,s)\,dyds$$

converges to 0 since $\chi_n \rightharpoonup \chi$ in L^∞-weak-$*$ and

$$|G(x-y,t-s)\,(g^+(w)-g^-(w))(y,s)|\le(\overline{g}-\underline{g})\,G(x-y,t-s),$$

which is an integrable function of (y,s). The rest of the terms converges to 0 by dominated convergence since $v_n\to w$ pointwise in $\mathbb{R}^N\times[0,T]$ and

$$\begin{aligned} &|\chi_n\,(g^+(v_n)-g^+(w))+\chi_n\,(g^-(w)-g^-(v_n))+(g^-(v_n)-g^-(w))| \\ \le\;& 3M|v_n-w| \\ \le\;& 6MC, \end{aligned}$$

where M is a Lipschitz constant for g^+ and g^-, and C is a uniform bound for v_n and w given by Lemma 3.5 (i).

This shows that w is the solution of (3.7), so that $w = v$. In particular (v_n) converges locally uniformly to v. We conclude that

$$\int_0^t c[\chi_n](x,s)\,ds \to \int_0^t c[\chi](x,s)\,ds$$

locally uniformly for $t \in [0,T]$ thanks to the Lipschitz continuity of α. This proves the claim that **(H2-X)** holds, and we obtain existence of weak solutions to (3.8) according to Theorem 3.1.

If $\alpha \geq 0$ and (3.3) holds, then the fattening phenomenon for (2.1) does not happen (see [7, 14]) so that, if u is any weak solution of (2.1), then for almost all $t \in [0,T]$ and almost everywhere in $\mathbb{R}^N$,

$$\mathbb{1}_{\{u(\cdot,t)>0\}} = \mathbb{1}_{\{u(\cdot,t)\geq 0\}},$$

which means that u is a classical solution of (3.8). This completes the proof. $\square$

4. Existence of weak solution to (1.1) (bounded case)

It may happen that our Hamiltonian $H[\mathbb{1}_{\{u\geq 0\}}]$ is only defined when the set $\{u \geq 0\}$ remains bounded: this is typically the case when a volume term is involved. For such cases, the existence of weak solutions may remain true, due to a particular framework.

4.1. **The existence theorem.** We use the following assumption:

(H4) There exists a bounded function $\overline{v} : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ and $R_0 > 0$ such that

(i) $\overline{v}(x,t) < 0$ if $|x| \geq R_0$, for any $t \in [0,T]$,

(ii) $\overline{v}(x,0) \geq u_0(x)$ in $\mathbb{R}^N$,

(iii) $\overline{v}$ is a supersolution of (2.1) for all $\chi \in X$, where

$$X = \{\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1]) \ : \ \chi = 0 \text{ a.e in } \{\overline{v} < 0\}\}\,.$$

Assumption **(H4)** is some kind of compatibility condition between the equation and the initial condition: of course, it implies that $u_0(x) < 0$ if $|x| \geq R_0$ and, more or less, that the equation preserves this property (this is the meaning of $\overline{v}$).

Under this assumption, we obtain the following existence result:

Theorem 4.1. *Assume* **(H4)** *and that* **(H1-X)**, **(H2-X)** *and* **(H3-X)** *hold with X given in* **(H4)**. *Then there exists at least a weak solution to* (1.1).

Proof. The proof follows the arguments of the proof of Theorem 3.1: essentially, the only change is the choice of X.

From **(H4)**, the set-valued mapping

$$\begin{array}{rcl}\xi: & X & \rightrightarrows & X \\ & \chi & \mapsto & \big\{\chi' \;:\; \mathbb{1}_{\{u(\cdot,t)>0\}} \leq \chi'(\cdot,t) \leq \mathbb{1}_{\{u(\cdot,t)\geq 0\}} \text{ for almost all } t \in [0,T], \\ & & & \quad \text{where } u \text{ is the } L^1\text{-viscosity solution of (2.1)}\big\},\end{array}$$

is well-defined : indeed, for any $\chi \in X$, $\overline{v}$ is a supersolution of the $H[\chi]$-equation and we have $u_0(x) \leq \overline{v}(x,0)$ in $\mathbb{R}^N$. Therefore, by **(H1-X)** (comparison),

$$u(x,t) \leq \overline{v}(x,t) \quad \text{in } \mathbb{R}^N \times [0,T] \; .$$

In particular, $u(x,t) < 0$ if $\overline{v}(x,t) < 0$ and clearly any χ' in $\xi(\chi)$ is in X.

We conclude exactly as in the proof of Theorem 3.1. ☐

4.2. **Applications.** The typical cases we have in mind are geometrical equations; for instance,

$$u_t = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right) + \beta\left(\mathcal{L}^N(\{u(\cdot,t) \geq 0\})\right)\right] |Du| \; ,$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbb{R}^N$.

In order to test condition **(H4)**, it is natural to consider radially symmetric supersolutions and typically, we look for supersolutions of the following form:

$$\psi(x,t) := R(t) - |x| \; ,$$

where $R(\cdot)$ is a C^1-function of t. We point out two key arguments to justify this choice: first ψ is concave in x, and checking the viscosity supersolution property is equivalent to checking it at points where ψ is smooth (because of the form of ψ). Next if ψ is a supersolution of the above pde, one can use (if necessary) a change of function $\psi \to \varphi(\psi)$, with $\varphi' > 0$ to ensure that $\varphi(\psi) \geq u_0$ in $\mathbb{R}^N$, and such that φ is bounded, in order to use the comparison principle.

Plugging ψ in the equation, we obtain that ψ is a supersolution if

$$R'(t) \geq -\frac{(N-1)}{|x|} + \beta(\omega_N R^N(t)) \; ,$$

where $\omega_N = \mathcal{L}^N(B(0,1))$. The curvature term $(N-1)/|x|$ is not going to play any major role here since we are concerned with large R's and the equation should hold on the 0-level set of ψ, i.e. for $|x| = R$.

Therefore let us consider the ordinary differential equation (ode)

$$R'(t) = \beta(\omega_N R^N(t)) \quad \text{with } R(0) = R_0 \; .$$

A natural condition for this ode to have solutions which do not blow up in finite time is the sublinearity in R of the right-hand side. This leads to the following conditions on β:

$$\beta(t) \leq L_1 + L_2 t^{1/N} \quad \text{for any } t > 0 \ ,$$

for some constants $L_1, L_2 > 0$. Under this condition, we easily build a function $\overline{v}$ satisfying (**H4**). However if this condition is not satisfied, Theorem 4.1 provides only the small time existence of solutions.

We complete this example by recalling that, for all $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$ with bounded support, we have a comparison result for the equation

$$u_t = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right) + \beta\left(\int_{\mathbb{R}^N} \chi(x,t)dx\right)\right]|Du| \quad \text{in } \mathbb{R}^N \times [0,T]$$

(See Nunziante [15] and Bourgoing [8, 9]).

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(G. Barles, O. Ley) Laboratoire de Mathématiques et Physique Théorique, Fédération Denis Poisson, Université de Tours, Parc de Grandmont, 37200 Tours, France, {barles,ley}@lmpt.univ-tours.fr

(P. Cardaliaguet, A. Monteillet) Laboratoire de Mathématiques, CNRS UMR 6205, Université de Brest, 6 Av. Le Gorgeu BP 809, 29285 Brest, France, {pierre.cardaliaguet, aurelien.monteillet}@univ-brest.fr