



**HAL**  
open science

# Distinguished principal series representations for $GL_n$ over a $p$ -adic field

Nadir Matringe

► **To cite this version:**

Nadir Matringe. Distinguished principal series representations for  $GL_n$  over a  $p$ -adic field. 2008.  
hal-00286597v2

**HAL Id: hal-00286597**

**<https://hal.science/hal-00286597v2>**

Preprint submitted on 10 Jul 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Distinguished principal series representations for $GL_n$ over a $p$ – adic field

Nadir Matringe

July 10, 2008

## 1 Introduction

For  $K/F$  a quadratic extension of  $p$ -adic fields, let  $\sigma$  be the conjugation relative to this extension, and  $\eta_{K/F}$  be the character of  $F^*$  with kernel norms of  $K^*$ .

If  $\pi$  is a smooth irreducible representation of  $GL(n, K)$ , and  $\chi$  a character of  $F^*$ , the dimension of the space of linear forms on its space, which transform by  $\chi$  under  $GL(n, F)$  (with respect to the action  $[(L, g) \mapsto L \circ \pi(g)]$ ), is known to be at most one (Proposition 11, [F]). One says that  $\pi$  is  $\chi$ -distinguished if this dimension is one, and says that  $\pi$  is distinguished if it is 1-distinguished.

In this article, we give a description of distinguished principal series representations of  $GL(n, K)$ . The result (Theorem 3.2) is that the irreducible distinguished representations of the principal series of  $GL(n, K)$  are (up to isomorphism) those unitarily induced from a character  $\chi = (\chi_1, \dots, \chi_n)$  of the maximal torus of diagonal matrices, such that there exists  $r \leq n/2$ , for which  $\chi_{i+1}^\sigma = \chi_i^{-1}$  for  $i = 1, 3, \dots, 2r-1$ , and  $\chi_{i|F^*} = 1$  for  $i > 2r$ . For the quadratic extension  $\mathbb{C}/\mathbb{R}$ , it is known (cf. [P]) that the analogous result is true for tempered representations.

For  $n \geq 3$ , this gives a counter-example (Corollary 3.1) to a conjecture of Jacquet (Conjecture 1 in [A]). This conjecture states that an irreducible representation  $\pi$  of  $GL(n, K)$  with central character trivial on  $F^*$  is isomorphic to  $\tilde{\pi}^\sigma$  if and only if it is distinguished or  $\eta_{K/F}$ -distinguished (where  $\eta_{K/F}$  is the character of order 2 of  $F^*$ , attached by local class field theory to the extension  $K/F$ ). For discrete series representations, the conjecture is verified, it was proved in [K].

Unitary irreducible distinguished principal series representations of  $GL(2, K)$  were described in [H], and the general case of distinguished irreducible principal series representations of  $GL(2, K)$  was treated in [F-H]. We use this occasion to give a different proof of the result for  $GL(2, K)$  than the one in [F-H]. To do this, in Theorems 4.1 and 4.2, we extend a criterion of Hakim (th.4.1, [H]) characterising smooth unitary irreducible distinguished representations of  $GL(2, K)$  in terms of  $\gamma$  factors at  $1/2$ , to all smooth irreducible distinguished representations of  $GL(2, K)$ .

## 2 Preliminaries

Let  $\phi$  be a group automorphism, and  $x$  an element of the group, we sometimes note  $x^\phi$  instead of  $\phi(x)$ , and  $x^{-\phi}$  the inverse of  $x^\phi$ . If  $\phi = x \mapsto h^{-1}xh$  for  $h$  in the group, then  $x^h$  designs  $x^\phi$ .

Let  $G$  be a locally compact totally disconnected group,  $H$  a closed subgroup of  $G$ . We note  $\Delta_G$  the module of  $G$ , given by the relation  $d_G(gx) = \Delta_G(g)d_G(x)$ , for a right Haar measure  $d_G$  on  $G$ .

Let  $X$  be a locally closed subspace of  $G$ , with  $H.X \subset X$ . If  $V$  is a complex vector space, we note  $D(X, V)$  the space of smooth  $V$ -valued functions on  $X$  with compact support (if  $V = \mathbb{C}$ , we simply note it  $D(X)$ ).

Let  $\rho$  be a smooth representation of  $H$  in a complex vector space  $V_\rho$ , we note  $D(H \backslash X, \rho, V_\rho)$  the space of smooth  $V_\rho$ -valued functions  $f$  on  $X$ , with compact support modulo  $H$ , which verify  $f(hx) = \rho(h)f(x)$  for  $h \in H$  and  $x \in X$  (if  $\rho$  is a character, we note it  $D(H \backslash X, \rho)$ ).

We note  $\text{ind}_H^G(\rho)$  the representation by right translations of  $G$  in  $D(H \backslash G, (\Delta_G/\Delta_H)^{1/2}\rho, V_\rho)$ .

Let  $F$  be a non archimedean local field of characteristic zero, and  $K$  a quadratic extension of  $F$ . We have  $K = F(\delta)$  with  $\delta^2$  in  $F^*$ .

We note  $|\cdot|_K$  and  $|\cdot|_F$  the modules of  $K$  and  $F$  respectively.

We note  $\sigma$  the non trivial element of the Galois group  $G(K/F)$  of  $K$  over  $F$ , and we use the same letter to design for its action on  $M_n(K)$ .

We note  $N_{K/F}$  the norm of the extension  $K/F$  and we note  $\eta_{K/F}$  the nontrivial character of  $F^*$  which is trivial on  $N_{K/F}(K^*)$ .

Whenever  $G$  is an algebraic group defined over  $F$ , we note  $G(K)$  its  $K$ -points and  $G(F)$  its  $F$ -points.

The group  $GL(n)$  will be noted  $G_n$ , its standard Borel subgroup will be noted  $B_n$ , its unipotent radical  $U_n$ , and the standard maximal split torus of diagonal matrices  $T_n$ .

We note  $S$  the space of matrices  $M$  in  $G_n(K)$  satisfying  $MM^\sigma = 1$ .

Everything in this paragraph is more or less contained in [F1], we give detailed proofs here for convenience of the reader.

**Proposition 2.1.** ([S], ch.10, prop.3)

We have a homeomorphism between  $G_n(K)/G_n(F)$  and  $S$  given by the map  $S_n : g \mapsto g^\sigma g^{-1}$ .

**Proposition 2.2.** For its natural action on  $S$ , each orbit of  $B_n(K)$  contains one and only one element of  $\mathfrak{S}_n$  of order 2 or 1.

*Proof.* We begin with the following:

**Lemma 2.1.** Let  $w$  be an element of  $\mathfrak{S}_n \subset G_n(K)$  of order at most 2.

Let  $\theta'$  be the involution of  $T_n(K)$  given by  $t \mapsto w^{-1}t^\sigma w$ , then any  $t \in T_n(K)$  with  $t\theta'(t) = 1$  is of the form  $a/\theta'(a)$  for some  $a \in T_n(K)$ .

*Proof of Lemma 2.1.* There exists  $r \leq n/2$  such that up to conjugacy,  $w$  is  $(1, 2)(3, 4)\dots(2r - 1, 2r)$ .



Then  $s = s^{-\sigma} = n_2^{+\sigma} w^{-1} a^{-\sigma} n_1^{-\sigma}$ .

Thus we have  $aw = (aw)^{-\sigma}$ , i.e.  $w^2 = 1$  and  $a^w = a^{-\sigma}$ .

Now we write  $n_1^{-\sigma} = u^- u^+$  with  $u^- \in U_w'$  and  $u^+ \in U_w$ , comparing  $s$  and  $s^{-\sigma}$ ,  $u^+$  must be equal to  $n_2^+$ .

Hence  $s = n_1 a w u^{-1} n_1^{-\sigma}$ , thus we suppose  $s = awn$ , with  $n$  in  $U_w'$ .

From  $s = s^{-\sigma}$ , one has the relation  $awn(aw)^{-1} = n^{-\sigma}$ , applying  $\sigma$  on each side, this becomes  $(aw)^{-1} n^\sigma aw = n^{-1}$ .

But  $\theta : u \mapsto (aw)^{-1} u^\sigma aw$  is an involutive automorphism of  $U_w'$ , hence from Lemma 2.2, there is  $u'$  in  $U_w'$  such that  $n = \theta(u^{-1})u$ .

This gives  $s = u^{-\sigma} awu$ , so that we suppose  $s = aw$ . Again  $wa^\sigma w = a^{-1}$ , and applying Lemma 2.1 to  $\theta' : x \mapsto wx^\sigma w$ , we deduce that  $a$  is of the form  $y\theta'(y^{-1})$ , and  $s = ywy^{-\sigma}$ .  $\square$

Let  $u$  be the element  $\begin{pmatrix} 1 & -\delta \\ 1 & \delta \end{pmatrix}$  of  $M_2(K)$ ; one has  $S_2(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  (cf. Proposition 2.1).

We notice for further use (cf. proof of Proposition 3.1), that if we note  $\tilde{T}$  the subgroup  $\left\{ \begin{pmatrix} z & 0 \\ 0 & z^\sigma \end{pmatrix} \in G_2(K) \mid z \in K^* \right\}$ ,

then  $u^{-1} \tilde{T} u = T = \left\{ \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix} \in G_2(F) \mid x, y \in F \right\}$ .

For  $r \leq n/2$ , one notes  $U_r$  the  $n \times n$  matrix given by the following block decomposition:

$$\begin{pmatrix} u & & & \\ & \ddots & & \\ & & u & \\ & & & I_{n-2r} \end{pmatrix}$$

If  $w$  is an element of  $\mathfrak{S}_n$  naturally injected in  $G_n(K)$ , one notes  $U_r^w = w^{-1} U_r w$ .

**Corollary 2.1.** *The elements  $U_r^w$  for  $0 \leq r \leq n/2$ , and  $w \in \mathfrak{S}_n$  give a complete set of representatives of classes of  $B_n(K) \backslash G_n(K) / G_n(F)$ .*

Let  $G_n = \coprod_{w \in \mathfrak{S}_n} B_n w B_n$  be the Bruhat decomposition of  $G_n$ . We call a double-class  $BwB$  a Bruhat cell.

**Lemma 2.3.** *One can order the Bruhat cells  $C_1, C_2, \dots, C_{n!}$  so that for every  $1 \leq i \leq n!$ , the cell  $C_i$  is closed in  $G_n - \coprod_{k=1}^{i-1} C_k$ .*

*Proof.* Choose  $C_1 = B_n$ . It is closed in  $G_n$ . Now let  $w_2$  be an element of  $\mathfrak{S}_n - Id$ , with minimal length. Then from 8.5.5. of [Sp], one has that the Bruhat cell  $Bw_2B$  is closed in  $G_n - B_n$  with respect to the Zariski topology, hence for the p-adic topology, we call it  $C_2$ . We conclude by repeating this process.  $\square$

**Corollary 2.2.** *One can order the classes  $A_1, \dots, A_t$  of  $B_n(K) \backslash G_n(K) / G_n(F)$ , so that  $A_i$  is closed in  $G_n(K) - \coprod_{k=1}^{i-1} A_k$ .*

*Proof.* From the proof of Proposition 2.2, we know that if  $C$  is a Bruhat cell of  $G_n$ , then  $S_n \cap C$  is either empty, or it corresponds through the homomorphism  $S_n$  to a class  $A$  of  $B_n(K) \backslash G_n(K) / G_n(F)$ . The conclusion follows the preceding Lemma.  $\square$

**Corollary 2.3.** *Each  $A_i$  is locally closed in  $G_n(K)$  for the Zariski topology.*

We will also need the following Lemma:

**Lemma 2.4.** *Let  $G, H, X$ , and  $(\rho, V_\rho)$  be as in the beginning of the section, the map  $\Phi$  from  $D(X) \otimes V_\rho$  to  $D(H \backslash X, \rho, V_\rho)$  defined by  $\Phi : f \otimes v \mapsto (x \mapsto \int_H f(hx) \rho(h^{-1}) v dh)$  is surjective.*

*Proof.* Let  $v \in V_\rho$ ,  $U$  an open subset of  $G$  that intersects  $X$ , small enough for  $h \mapsto \rho(h)v$  to be trivial on  $H \cap UU^{-1}$ .

Let  $f'$  be the function with support in  $H(X \cap U)$  defined by  $hx \mapsto \rho(h)v$ .

Such functions generate  $D(H \backslash X, \rho, V_\rho)$  as a vector space.

Now let  $f$  be the function of  $D(X, V_\rho)$  defined by  $x \mapsto 1_{U \cap X}(x)v$ , then  $\Phi(f)$  is a multiple of  $f'$ .

But for  $x$  in  $U \cap X$ ,  $\Phi(f)(x) = \int_H \rho(h^{-1}) f(hx) dh$  because  $h \mapsto \rho(h)v$  is trivial on  $H \cap UU^{-1}$ , plus  $h \mapsto f(hx)$  is a positive function that multiplies  $v$ , and  $f(x) = V$ , so  $\Phi(f)(x)$  is  $v$  multiplied by a strictly positive scalar.  $\square$

**Corollary 2.4.** *Let  $Y$  be a closed subset of  $X$ ,  $H$ -stable, then the restriction map from  $D(H \backslash X, \rho, V_\rho)$  to  $D(H \backslash Y, \rho, V_\rho)$  is surjective.*

*Proof.* This is a consequence of the known surjectivity of the restriction map from  $D(X)$  to  $D(Y)$ , which implies the surjectivity of the restriction from  $D(X, V_\rho)$  to  $D(Y, V_\rho)$  and of the commutativity of the diagram:

$$\begin{array}{ccc} D(X) & \rightarrow & D(Y) \\ \downarrow \Phi & & \downarrow \Phi \\ D(H \backslash X, \rho) & \rightarrow & D(H \backslash Y, \rho) \end{array} \quad \square$$

### 3 Distinguished principal series

If  $\pi$  is a smooth representation of  $G_n(K)$  of space  $V_\pi$ , and  $\chi$  is a character of  $F^*$ , we say that  $\pi$  is  $\chi$ -distinguished if there exists on  $V_\pi$  a nonzero linear form  $L$  such that  $L(\pi(g)v) = \chi(\det(g))L(v)$  whenever  $g$  is in  $G_n(F)$  and  $v$  belongs to  $V_\pi$ . If  $\chi$  is trivial, we simply say that  $\pi$  is distinguished.

We first recall the following:

**Theorem 3.1.** (*[F], Proposition 12*)

*Let  $\pi$  be a smooth irreducible distinguished representation of  $G_n(K)$ , then  $\pi^\sigma \simeq \tilde{\pi}$ .*

Let  $\chi_1, \dots, \chi_n$  be  $n$  characters of  $K^*$ , with none of their quotients equal to  $|\cdot|_K$ . We note  $\chi$  the

character of  $B_n(K)$  defined by  $\chi \begin{pmatrix} b_1 & \star & \star \\ & \ddots & \star \\ & & b_n \end{pmatrix} = \chi_1(b_1) \dots \chi_n(b_n)$ .

We note  $\pi(\chi)$  the representation of  $G_n(K)$  by right translation on the space of functions  $D(B_n(K) \backslash G_n(K), \Delta_{B_n}^{-1/2} \chi)$ . This representation is smooth, irreducible and called the principal series attached to  $\chi$ .

If  $\pi$  is a smooth representation of  $G_n(K)$ , we note  $\check{\pi}$  its smooth contragredient.

We will need the following Lemma:

**Lemma 3.1.** (*Proposition 26 in [F1]*) *Let  $\bar{m} = (m_1, \dots, m_l)$  be a partition of a positive integer  $m$ , let  $P_{\bar{m}}$  be the corresponding standard parabolic subgroup, and for each  $1 \leq i \leq l$ , let  $\pi_i$  be a smooth*



and these  $\chi_{2p+k}$ 's are different (so that  $\chi_{2p+k} \neq \chi_{2p+k'}^{-\sigma}$  for  $k \neq k'$ ), and  $\chi_{2p+q+j}|_{F^*} = \eta_{K/F}$  for  $1 \leq j \leq s$ , these  $\chi_{2p+q+j}$ 's being different.

We note  $\mu_k = \chi_{2p+k}$  for  $q \geq k \geq 1$ , and  $\nu'_k = \chi_{2p+q+k'}$  for  $s \geq k' \geq 1$ .

We show that if such a character  $\chi$  is positive on a conjugate of  $\bar{T}_r$  by an element of  $S_n$ , then  $s = 0$ . Suppose  $\nu_1$  appears, then either  $\nu_1$  is positive on  $F^*$ , but that is not possible, or it is coupled with another  $\chi_i$ , and  $(\nu_1, \chi_i)$  is positive on elements  $(z, z^\sigma)$ , for  $z$  in  $K^*$ .

Suppose  $\chi_i = \nu_j$  for some  $j \neq 1$ , then  $(\nu_1, \chi_i)$  is unitary, so it must be trivial on couples  $(z, z^\sigma)$ , which implies  $\nu_1 = \nu_j^{-\sigma} = \nu_j$ , which is absurd.

The character  $\chi_i$  cannot be of the form  $\mu_j$ , because it would imply  $\nu_1|_{F^*} = 1$ .

The last case is  $i \leq 2p$ , then  $\nu_1^{-\sigma} = \nu_1$  must be the unitary part of  $\chi_i$  because of the positivity of  $(\nu_1, \chi_i)$  on the couples  $(z, z^\sigma)$ .

But  $\chi_i^{-\sigma}$  also appears and is not trivial on  $F^*$ , hence must be coupled with another character  $\chi_j$  with  $j \leq 2p$  and  $j \neq i$ , such that  $(\chi_i^{-\sigma}, \chi_j)$  is positive on the elements  $(z, z^\sigma)$ , for  $z$  in  $K^*$ , which implies that  $\chi_j$  has unitary part  $\nu_1^{-\sigma} = \nu_1$ . The character  $\chi_j$  cannot be a  $\mu_k$  because of its unitary part.

If it is a  $\chi_k$  with  $k \leq 2p$ , we consider again  $\chi_k^{-\sigma}$ .

But repeating the process lengthily enough, we can suppose that  $\chi_j$  is of the form  $\nu_k$ , for  $k \neq 1$ . Taking unitary parts, we see that  $\nu_k = \nu_1^{-\sigma} = \nu_1$ , which is in contradiction with the fact that all  $\nu_i$ 's are different. We conclude that  $s = 0$ .  $\square$

**Theorem 3.2.** *Let  $\chi = (\chi_1, \dots, \chi_n)$  be a character of  $T_n(K)$ , the principal series representation  $\pi(\chi)$  is distinguished if and only if there exists  $r \leq n/2$ , such that  $\chi_{i+1}^\sigma = \chi_i^{-1}$  for  $i = 1, 3, \dots, 2r-1$ , and that  $\chi_i|_{F^*} = 1$  for  $i > 2r$ .*

*Proof.* There is one implication left.

Suppose  $\chi$  is of the desired form, then  $\pi(\chi)$  is parabolically (unitarily) induced from representations of the type  $\pi(\chi_i, \chi_i^{-\sigma})$  of  $G_2(K)$ , and distinguished characters of  $K^*$ .

Hence, because of Lemma 3.1 the Theorem will be proved if we know that the representations  $\pi(\chi_i, \chi_i^{-\sigma})$  are distinguished, but this is Corollary 4.1 of the next paragraph.  $\square$

This gives a counter-example to a conjecture of Jacquet (conjecture 1 in [A]), asserting that if an irreducible admissible representation  $\pi$  of  $G_n(K)$  verifies that  $\tilde{\pi}$  is isomorphic to  $\pi^\sigma$ , then it is distinguished if  $n$  is odd, and it is distinguished or  $\eta_{K/F}$ -distinguished if  $n$  is even.

**Corollary 3.1.** *For  $n \geq 3$ , there exist smooth irreducible representations  $\pi$  of  $G_n(K)$ , with central character trivial on  $F^*$ , that are neither distinguished, nor  $\eta_{K/F}$ -distinguished, but verify that  $\tilde{\pi}$  is isomorphic to  $\pi^\sigma$ .*

*Proof.* Take  $\chi_1, \dots, \chi_n$ , all different, such that  $\chi_1|_{F^*} = \chi_2|_{F^*} = \eta_{K/F}$ , and  $\chi_j|_{F^*} = 1$  for  $3 \leq j \leq n$ . Because each  $\chi_i$  has trivial restriction to  $N_{K/F}(K^*)$ , it is equal to  $\chi_i^{-\sigma}$ , hence  $\tilde{\pi}$  is isomorphic to  $\pi^\sigma$ . Another consequence is that if  $k$  and  $l$  are two different integers between 1 and  $n$ , then  $\chi_k \neq \chi_l^{-\sigma}$ , because we supposed the  $\chi_i$ 's all different.

Then it follows from Theorem 3.2 that  $\pi = \pi(\chi_1, \dots, \chi_n)$  is neither distinguished, nor  $\eta_{K/F}$ -distinguished, but clearly, the central character of  $\pi$  is trivial on  $F^*$  and  $\tilde{\pi}$  is isomorphic to  $\pi^\sigma$ .  $\square$



## 4 Distinction and gamma factors for $GL(2)$

As said in the introduction, in this section we generalize to smooth infinite dimensional irreducible representations of  $G_2(K)$  a criterion of Hakim (cf. [H], Theorem 4.1) characterising smooth unitary irreducible distinguished representations of  $G_2(K)$ . In proof of Theorem 4.1 of [H], Hakim deals with unitary representations so that the integrals of Kirillov functions on  $F^*$  with respect to a Haar measure of  $F^*$  converge. We skip the convergence problems using Proposition 2.9 of chapter 1 of [J-L].

We note  $M(K)$  the mirabolic subgroup of  $G_2(K)$  of matrices of the form  $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$  with  $a$  in  $K^*$  and  $x$  in  $K$ , and  $M(F)$  its intersection with  $G_2(F)$ . We note  $w$  the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

Let  $\pi$  be a smooth infinite dimensional irreducible representation of  $G_2(K)$ , it is known that it is generic (cf. [Z] for example). Let  $K(\pi, \psi)$  be its Kirillov model corresponding to  $\psi$  ([J-L], th. 2.13), it contains the subspace  $D(K^*)$  of functions with compact support on the group  $K^*$ . If  $\phi$  belongs to  $K(\pi, \psi)$ , and  $x$  belongs to  $K$ , then  $\phi - \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \phi$  belongs to  $D(K^*)$  ([J-L], prop.2.9, ch.1), from this follows that  $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$ .

We now recall a consequence of the functional equation at 1/2 for Kirillov representations (cf. [B], section 4.7).

For all  $\phi$  in  $K(\pi, \psi)$  and  $\chi$  character of  $K^*$ , we have whenever both sides converge absolutely:

$$\int_{K^*} \pi(w)\phi(x)(c_\pi\chi)^{-1}(x)d^*x = \gamma(\pi \otimes \chi, \psi) \int_{K^*} \phi(x)\chi(x)d^*x \quad (1)$$

where  $d^*x$  is a Haar measure on  $K^*$ , and  $c_\pi$  is the central character of  $\pi$ .

**Theorem 4.1.** *Let  $\pi$  be a smooth irreducible representation of  $G_2(K)$  of infinite dimension with central character trivial on  $F^*$ , and  $\psi$  a nontrivial character of  $K$  trivial on  $F$ . If  $\gamma(\pi \otimes \chi, \psi) = 1$  for every character  $\chi$  of  $K^*$  trivial on  $F^*$ , then  $\pi$  is distinguished.*

*Proof.* In fact, using a Fourier inversion in functional equation 1 and the change of variable  $x \mapsto x^{-1}$ , we deduce that for all  $\phi$  in  $D(K^*) \cap \pi(w)D(K^*)$ , we have

$$c_\pi(x) \int_{F^*} \pi(w)\phi(tx^{-1})d^*t = \int_{F^*} \phi(tx)d^*t$$

( $d^*t$  is a Haar measure on  $F^*$ ) which for  $x = 1$  gives

$$\int_{F^*} \pi(w)\phi(t)d^*t = \int_{F^*} \phi(t)d^*t.$$

Now we define on  $K(\pi, \psi)$  a linear form  $\lambda$  by:

$$\lambda(\phi_1 + \pi(w)\phi_2) = \int_{F^*} \phi_1(t)d^*t + \int_{F^*} \phi_2(t)d^*t$$

for  $\phi_1$  and  $\phi_2$  in  $D(K^*)$ , which is well defined because of the previous equality and the fact that  $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$ .

It is clear that  $\lambda$  is  $w$ -invariant. As the central character of  $\pi$  is trivial on  $F^*$ ,  $\lambda$  is also  $F^*$ -invariant. Because  $GL_2(F)$  is generated by  $M(F)$ , its center, and  $w$ , it remains to show that  $\lambda$  is  $M(F)$ -invariant.

Since  $\psi$  is trivial on  $F$ , one has if  $\phi \in D(K^*)$  and  $m \in M(F)$  the equality  $\lambda(\pi(m)\phi) = \lambda(\phi)$ .

Now if  $\phi = \pi(w)\phi_2 \in \pi(w)D(K^*)$ , and if  $a$  belongs to  $F^*$ , then  $\pi\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\pi(w)\phi_2 = \pi(w)\pi\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)\phi_2 = \pi(w)\pi\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\phi_2$  because the central character of  $\pi$  is trivial on  $F^*$ , and  $\lambda(\pi\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\phi) = \lambda(\phi)$ .

If  $x \in F$ , then  $\pi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\phi - \phi$  is a function in  $D(K^*)$ , which vanishes on  $F^*$ , hence  $\lambda\pi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\phi - \phi = 0$ .

Eventually  $\lambda$  is  $M(F)$ -invariant, hence  $G_2(F)$ -invariant, it is clear that its restriction to  $D(K^*)$  is non zero.  $\square$

**Corollary 4.1.** *Let  $\mu$  be a character of  $K^*$ , then  $\pi(\mu, \mu^{-\sigma})$  is distinguished.*

*Proof.* indeed, first we notice that the central character  $\mu\mu^{-\sigma}$  of  $\pi(\mu, \mu^{-\sigma})$  is trivial on  $F^*$ . Now let  $\chi$  be a character of  $K^*/F^*$ , then  $\gamma(\pi(\mu, \mu^{-\sigma}) \otimes \chi, \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-\sigma}\chi, \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-1}\chi^\sigma, \psi^\sigma)$ , and as  $\psi|_F = 1$  and  $\chi|_{F^*} = 1$ , one has  $\psi^\sigma = \psi^{-1}$  and  $\chi^\sigma = \chi^{-1}$ , so that  $\gamma(\pi(\chi, \chi^{-\sigma}), \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-1}\chi^{-1}, \psi^{-1}) = 1$ . The conclusion falls from Proposition 4.1.  $\square$

Assuming Theorem 1.2 of [A-G], the converse of Theorem 4.1 is also true:

**Theorem 4.2.** *Let  $\pi$  be a smooth irreducible representation of infinite dimension of  $G_2(K)$  with central character trivial on  $F^*$  and  $\psi$  a non trivial character of  $K/F$ , it is distinguished if and only if  $\gamma(\pi \otimes \chi, \psi) = 1$  for every character  $\chi$  of  $K^*$  trivial on  $F^*$ .*

*Proof.* It suffices to show that if  $\pi$  is a smooth irreducible distinguished representation of infinite dimension of  $G_2(K)$ , and  $\psi$  a non trivial character of  $K/F$ , then  $\gamma(\pi, \psi) = 1$ . Suppose  $\lambda$  is a non zero  $G_2(F)$ -invariant linear form on  $K(\pi, \psi)$ , it is shown in the proof of the corollary of Proposition 3.3 in [H], that its restriction to  $D(F^*)$  must be a multiple of the Haar measure on  $F^*$ . Hence for any function  $\phi$  in  $D(K^*) \cap \pi(w)D(K^*)$ , we must have  $\int_{F^*} \phi(t)d^*t = \int_{F^*} \pi(w)\phi(t)d^*t$ .

From this one deduces that for any function in  $D(K^*) \cap \pi(w)D(K^*)$ :

$$\begin{aligned}
\int_{K^*} \pi(w)\phi(x)c_\pi^{-1}(x)d^*x &= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi(w)\phi(ta)d^*tda \\
&= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\pi(w)\phi(t)d^*tda \\
&= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi(w)c_\pi(a)\pi\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\phi(t)d^*tda \\
&= \int_{K^*/F^*} \int_{F^*} \pi\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\phi(t)d^*tda \\
&= \int_{K^*/F^*} \int_{F^*} \phi(ta^{-1})d^*tda \\
&= \int_{K^*/F^*} \int_{F^*} \phi(ta)d^*tda \\
&= \int_{K^*} \phi(x)d^*x
\end{aligned}$$

This implies that either  $\gamma(\pi, \psi)$  is equal to one, or  $\int_{K^*} \phi(x)d^*x$  is equal to zero on  $D(K^*) \cap \pi(w)D(K^*)$ . In the second case, we could define two independant  $K^*$ -invariant linear forms on  $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$ , given by  $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*} \phi_1(x)d^*x$ , and  $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*} \phi_2(x)d^*x$ . This would contradict Theorem 1.2 of [A-G].  $\square$

## Acknowledgements

I would like to thank Corinne Blondel and Paul Gérardin, for many helpfull comments and suggestions.

## References

- [A] U.K. Anandavardhanan, *Distinguished non-Archimedean representations*, Proc. Hyderabad Conference on Algebra and Number Theory, 2005, 183-192.
- [A-G] A. Aizenbud and D. Gourevitch, *A proof of the multiplicity one conjecture for  $GL(n)$  in  $GL(n+1)$* , preprint, Arxiv.
- [B] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.
- [B-H] C. Bushnell and G. Henniart, *The Local Langlands Conjecture for  $GL(2)$* , Springer, 2006.
- [F] Y. Flicker, *On distinguished representations*, Journal für die reine und angewandte mathematik, **418** (1991), 139-172.
- [F1] Y. Flicker, *Distinguished representations and a Fourier summation formula*, Bulletin de la S.M.F., **120**, (1992), 413-465.

- [F-H] Y. Flicker and J. Hakim, *Quaternionic distinguished representations*, American Journal of Mathematics, **116** (1994), 683-736.
- [H] J. Hakim, *Distinguished  $p$ -adic Representations*, Duke Math. J., **62** (1991), 1-22.
- [J-L] H. Jacquet and R. Langlands, *Automorphic forms on  $GL(2)$* , Lect. Notes in Math., **114**, Springer, 1970.
- [K] A.C. Kable, *Asai  $L$ -functions and Jacquet's conjecture*, Amer. J. Math., **126**, (2004), 789-820.
- [P] M-N. Panichi, *Caractrisations du spectre tempr de  $GL_n(\mathbb{C})/GL_n(\mathbb{R})$* , Thse de Doctorat, Universit de Paris 7, 2001.
- [S] J-P. Serre, *Corps locaux*, Hermann, 1997.
- [Sp] T.A. Springer, *Linear algebraic groups*, Birkaser, 1998
- [Z] A.V. Zelevinsky, *induced representations of reductive  $p$ -adic groups II*, Ann.Sc.E.N.S., 1980.