

# Distinguished principal series representations for GLn over a p-adic field

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#### 1 Introduction

Let F be a non archimedean local field of characteristic zero and K a quadratic extension of F. If  $\pi$  is a smooth irreducible representation of GL(n, K), the dimension of GL(n, F)-invariant linear forms on its space is known to be at most one (proposition 11, [F]).

One says that  $\pi$  is distinguished if this dimension is one.

In this article, we give a description of distinguished principal series representations of GL(n, K). For the quadratic extension  $\mathbb{C}/\mathbb{R}$ , it is known (cf.[P]) that the irreducible distinguished tempered representations of  $GL(n, \mathbb{C})$  are (up to isomorphism) those unitarily induced from a unitary character  $\chi = (\chi_1, ..., \chi_n)$  of the maximal torus of diagonal matrices, such that there exists  $r \leq n/2$ , for which  $\chi_{i+1}^{\sigma} = \chi_i^{-1}$  for i = 1, 3, ..., 2r - 1, and  $\chi_i|_{F^*} = 1$  for i > 2r. In our case, theorem 3.2 gives an analogous result.

This gives a counter-example to a conjecture of Jacquet (Conjecture 1 in [A]).

We use this occasion to give a different proof of the result for GL(2, K) than the one in [F-H]. To do this, in theorems 4.1 and 4.2, we extend a criterion of Hakim (th.4.1, [H]) characterising smooth unitary irreducible distinguished representations of GL(2, K) in terms of  $\gamma$  factors at 1/2, to all smooth irreducible distinguished representations of GL(2, K).

#### 2 Preliminaries

Let  $\phi$  be a group automorphism, and x an element of the group, we sometimes note  $x^{\phi}$  instead of  $\phi(x)$ , and  $x^{-\phi}$  the inverse of  $x^{\phi}$ .

If  $\phi = x \mapsto h^{-1}xh$  for h in the group, then  $x^h$  designs  $x^{\phi}$ .

Let G be a locally compact totally disconnected group, H a closed subgroup of G. We note  $\Delta_G$  the module of G.

Let X be a locally closed subspace of G, with  $H.X \subset X$ .

If V is a complex vector space, we note D(X, V) the space of smooth V-valued functions on X with compact support (if  $V = \mathbb{C}$ , we simply note it D(X)).

Let  $\rho$  be a smooth representation of H in a complex vector space  $V_{\rho}$ , we note  $D(H \setminus X, \rho, V_{\rho})$ the space of smooth  $V_{\rho}$ -valued functions f on X, with compact support modulo H, which verify  $f(hx) = \rho(h)f(x)$  for  $h \in H$  and  $x \in X$  (if  $\rho$  is a character, we note it  $D(H \setminus X, \rho)$ ). We note  $c - ind_H^G(\rho)$  the representation by right translation of G in  $D(H \setminus G, (\Delta_G / \Delta_H)^{1/2} \rho, V_{\rho})$ .

Let F be a non archimedean local field of characteristic zero, and K a quadratic extension of F. We have  $K = F(\delta)$  with  $\delta^2 = \Delta \in F$ .

We note  $| |_K$  and  $| |_F$  the modules of K and F respectively.

We note  $\sigma$  the non trivial element of the Galois group G(K/F) of K over F, and we use the same letter to design its extension to  $M_n(K)$ .

We note  $N_{K/F}$  the norm of the extension K/F and we note  $\eta_{K/F}$  the nontrivial character of  $F^*$  which is trivial on  $N_{K/F}(K^*)$ .

Whenever G is an algebraic group defined over F, we note G(K) its K-points and G(F) its F-points.

The group GL(n) will be noted  $G_n$ , its standard Borel subgroup will be noted  $B_n$ , the unipotent radical of  $B_n$  will be noted  $U_n$ , and the standard maximal split torus of diagonal matrices  $T_n$ .

We note S the space of matrices M satisfying  $MM^{\sigma} = 1$ .

Everything in this paragraph, and lemma 3.1 of the next paragraph, is contained in [F1], we give detailed proofs here for convenience of the reader.

**Proposition 2.1.** (*|S|*, *ch.10*, *prop.3*)

We have a homeomorphism between  $G_n(K)/G_n(F)$  and S given by the map  $S_n: g \mapsto g^{\sigma}g^{-1}$ .

**Proposition 2.2.** For its natural action on S, each orbit of  $B_n(K)$  contains one and only one element of  $\mathfrak{S}_n$  of order 2 or 1.

*Proof.* We begin with the following:

**Lemma 2.1.** Let w be an element of  $\mathfrak{S}_n \subset G_n(K)$  of order 2.

Let  $\theta'$  be the involution of  $T_n(K)$  given by  $t \mapsto w^{-1}t^{\sigma}w$ , then any  $t \in T_n(K)$  with  $t\theta'(t) = 1$  is of the form  $a/\theta'(a)$  for some  $a \in T_n(K)$ .

Proof of lemma 2.1: There exists  $r \leq n/2$  such that up to conjugacy, w is (1,2)(3,4)...(2r-1,r).



and  $z_j \sigma(z_j) = 1$  for  $j \ge 2r + 1$ .

Hilbert's theorem 90 asserts that each  $z_j, j \ge 2r+1$  is of the form  $u_{j-2r}/\sigma(u_{j-2r})$ , for some



**Lemma 2.2.** Let N be an algebraic connected nilpotent group. Let  $\theta$  be an involutive automorphism of N(K). If  $x \in N(K)$ , verifies  $x\theta(x) = 1_N$  then, there is  $a \in N$  such that  $x = \theta(a^{-1})a$ .

Proof of lemma 2.2: The group N has a composition series  $1_N = N_0 \subset N_1 \subset ... N_{n-1} \subset N_n = N$ , such that each quotient  $N_{i+1}/N_i$  is isomorphic to K, and each commutator subgroup  $[N, N_{i+1}]$  is a subgroup of  $N_i$ .

Now we prove the lemma by induction on n:

If n = 1, then N is isomorphic to (K, +), one concludes taking a = x/2.

 $n \mapsto n+1$ :

suppose the lemma is true for every N of length n.

Let N be of length n + 1, we note  $\bar{x}$  the class of x in  $N/N_1$ .

By induction hypothesis, one gets that there exists an element in  $h \in N_1$ , and an element u in N such that  $x = \theta(u^{-1})uh$ .

Here h lies in the center of N, because  $[N, N_1] = 1_N$ .

As  $x\theta(x) = 1$ , we get  $h\theta(h) = 1$ . By induction hypothesis again, we get  $h = \theta(b^{-1})b$  for  $b \in N_1$ . We then take a = ub. 

We get back to the proof of the proposition 2.2.

For w in  $\mathfrak{S}_n$ , one notes  $U_w$  the subgroup of  $U_n$  generated by the elementary subgroups  $U_\alpha$ , with  $\alpha$  positive, and  $w\alpha$  negative, and  $U'_w$  the subgroup of  $U_n$  generated by the elementary subgroups  $U_{\alpha}$ , with  $\alpha$  positive, and  $w\alpha$  positive. Then  $U_n = U_w' U_w$ .

Let s be in S. According to Bruhat's decomposition, there is w in  $\mathfrak{S}_n$ , and a in  $T_n(K)$ ,  $n_1$  in  $U_n(K)$ and  $n_2^+$  in  $U_w$ , such that  $s = n_1 a w n_2^+$ , with unicity of the decomposition. Then  $s = s^{-\sigma} = n_2^{+-\sigma} w^{-1} a^{-\sigma} n_1^{-\sigma}$ .

Thus we have  $aw = (aw)^{-\sigma}$ , i.e.  $w^2 = 1$  and  $a^w = a^{-\sigma}$ .

Now we write  $n_1^{-\sigma} = u^- u^+$  with  $u^- \in U_w'$  and  $u^+ \in U_w$ , comparing s and  $s^{-\sigma}$ ,  $u^+$  must be equal to  $n_2^+$ .

Hence  $s = n_1 a w u^{-1} n_1^{-\sigma}$ , thus we suppose s = a w n, with n in  $U'_w$ .

From  $s = s^{-\sigma}$ , one has the relation  $awn(aw)^{-1} = n^{-\sigma}$ , applying  $\sigma$  on each side, this becomes  $(aw)^{-1}n^{\sigma}aw = n^{-1}.$ 

But  $\theta: u \mapsto (aw)^{-1}u^{\sigma}aw$  is an involutive automorphism of  $U'_w$ , hence from lemma 2.2, there is u'in  $U'_w$  such that  $n = \theta(u^{-1})u$ .

This gives  $s = u^{-\sigma} a w u$ , so that we suppose s = a w. Again  $w a^{\sigma} w = a^{-1}$ , and applying lemma 2.1 to  $\theta': x \mapsto wx^{\sigma}w$ , we deduce that a is of the form  $y\theta'(y^{-1})$ , and  $s = y^{-\sigma}wy$ .  Let u be the following element of  $M_2(K)$ :  $\begin{pmatrix} 1 & -\delta \\ 1 & \delta \end{pmatrix}$ , one has  $S_2(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

**Remark:**  
If 
$$\tilde{T} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{\sigma} \end{pmatrix} \in G_2(K) | z \in K^* \right\} \simeq K^*$$
, then  $u^{-1}\tilde{T}u = T = \left\{ \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix} \in G_2(F) | x, y \in F \right\}$ .

For  $r \leq n/2$ , one notes  $U_r$  the  $n \times n$  matrix given by the following block decomposition:  $\begin{pmatrix} u & & \\ & \ddots & \\ & & u \\ & & & I_{n-2r} \end{pmatrix}$ 

If w is an element of  $\mathfrak{S}_n$  naturally injected in  $G_n(K)$ , one notes  $U_r^w = w^{-1}U_r w$ .

**Corollary 2.1.** The elements  $U_r^w$  for  $0 \le r \le n/2$ , and  $w \in S_n$  give a complete set of representatives of classes of  $B_n(K) \setminus G_n(K)/G_n(F)$ .

Let  $G_n = \coprod_{w \in \mathfrak{S}_n} B_n w B_n$  be the Bruhat decomposition of  $G_n$ . We call a double-class BwB a Bruhat cell.

**Lemma 2.3.** One can order the Bruhat cells  $C_1$ ,  $C_2$ ,...,  $C_{n!}$  so that for every  $1 \le i \le n!$ , the cell  $C_i$  is closed in  $G_n - \prod_{k=1}^{i-1} C_i$ .

*Proof.* Choose  $C_1 = B_n$ . It is closed in  $G_n$ . Now let  $w_2$  be an element of  $S_n - Id$ , with minimal length. Then from 8.5.5. of [Sp], one has that the Bruhat cell  $Bw_2B$  is closed in  $G_n - B_n$  with respect to the Zariski topology, hence for the p-adic topology, we call it  $C_2$ . We conclude by repeating this process.

**Corollary 2.2.** One can order the classes  $A_1, ..., A_t$  of  $B_n(K) \setminus G_n(K) / G_n(F)$ , so that  $A_i$  is closed in  $G_n(K) - \coprod_{k=1}^{i-1} A_i$ .

*Proof.* From the proof of proposition 2.2, we know that if C is a Bruhat cell of  $G_n$ , then  $S_n \cap C$  is either empty, or it corresponds through the homeorphism  $S_n$  to a class A of  $B_n(K) \setminus G_n(K) / G_n(F)$ . The conclusion follows the preceeding lemma.

**Corollary 2.3.** Each  $A_i$  is locally closed in  $G_n(K)$  for the Zariski topology.

We will also need the following lemma:

**Lemma 2.4.** Let G, H,X, and  $(\rho, V_{\rho})$  be as in the beginning of the section, the map  $\Phi$  from  $D(X) \otimes V_{\rho}$  to  $D(H \setminus X, \rho, V_{\rho})$  defined by  $\Phi : f \otimes v \mapsto (x \mapsto \int_{H} f(hx)\rho(h^{-1})vdh)$  is surjective.

*Proof.* Let  $v \in V_{\rho}$ , U an open subset of G that intersects X, small enough for  $h \mapsto \rho(h)v$  to be trivial on  $H \cap UU^{-1}$ .

Let f' be the function with support in  $H(X \cap U)$  defined by  $hx \mapsto \rho(h)v$ .

Such functions generate  $D(H \setminus X, \rho, V_{\rho})$  as a vector space.

Now let f be the function of  $D(X, V_{\rho})$  defined by  $x \mapsto 1_{U \cap X}(x)v$ , then  $\Phi(f)$  is a multiple of f'. But for x in  $U \cap X$ ,  $\Phi(f)(x) = \int_{H} \rho(h^{-1})f(hx)dh$  because  $h \mapsto \rho(h)v$  is trivial on  $H \cap UU^{-1}$ , plus  $h \mapsto f(hx)$  is a positive function that multiplies v, and f(x) = V, so F(f)(x) is v multiplied by a strictly positive scalar. **Corollary 2.4.** Let Y be a closed subset of X, H-stable, then the restriction map from  $D(H \setminus X, \rho, V_{\rho})$  to  $D(H \setminus Y, \rho, V_{\rho})$  is surjective.

*Proof.* This is a consequence of the known surjectivity of the restriction map from D(X) to D(Y), which implies the surjectivity of the restriction from  $D(X, V_{\rho})$  to  $D(Y, V_{\rho})$  and the commutativity of the diagramm:

$$\begin{array}{cccc} D(X) & \to & D(Y) \\ \downarrow \Phi & & \downarrow \Phi \\ D(H \backslash X, \rho) & \to & D(H \backslash Y, \rho) \end{array} \qquad \Box$$

#### 3 Distinguished principal series

If  $\pi$  is a smooth representation of  $G_n(K)$  of space  $V_{\pi}$ , and  $\chi$  is a character of  $F^*$ , we say that  $\pi$  is  $\chi$ -distinguished if there exists on  $V_{\pi}$  a nonzero linear form L such that  $L(\pi(g)v) = \chi(det(g))L(v)$  whenever g is in  $G_n(F)$  and v belongs to  $V_{\pi}$ . If  $\chi$  is trivial, we simply say that  $\pi$  is distinguished.

We first recall the following:

#### **Theorem 3.1.** (|F|, proposition 12)

Let  $\pi$  be a smooth irreducible representation of  $G_n(K)$  which admits a nonzero  $G_n(F)$ -invariant linear form on its space, then  $\pi^{\sigma} \simeq \check{\pi}$ .

Let  $\chi_1, ..., \chi_n$  be *n* characters of  $K^*$ , with none of their quotients equal to  $| \cdot |_K$ . We note  $\chi$  the character of  $B_n(K)$  defined by  $\chi \begin{pmatrix} b_1 & \star & \star \\ & \ddots & \star \\ & & b_n \end{pmatrix} = \chi_1(b_1)...\chi_n(b_n).$ 

We note  $\pi(\chi_1, ..., \chi_n)$  the representation of  $G_n(K)$  by right translation on the space of functions  $D(B_n(K) \setminus G_n(K), \Delta_{B_n})^{-1/2} \chi$ .

This representation is smooth, irreducible and called the principal series attached to  $\chi$ .

We will need the following lemma (see prop.10.2 of [Ok] for a different proof).

**Lemma 3.1.** Let  $\bar{m} = (m_1, \ldots, m_l)$  be a partition of a positive integer m, let  $P_{\bar{m}}$  be the corresponding standard parabolic subgroup, and  $\pi_i$  be smooth distinguished representations of  $G_{m_i}(K)$ , then  $\pi_1 \times \cdots \times \pi_l = Ind_{P_{\bar{m}}(K)}^{G_m(K)}(\Delta_{P_{\bar{m}}(K)}^{-1/2}(\pi_1 \otimes \cdots \otimes \pi_l))$  is distinguished.

*Proof.* We note  $\pi$  for  $\pi_1 \otimes \cdots \otimes \pi_l$ .

We want to show that  $Hom_{G_m(F)}(c - Ind_{P_{\bar{m}}(K)}^{G_m(K)}(\pi), 1)$  is of dimension at least one. But from the relation  $\Delta_{P_{\bar{m}}(K)} = \Delta_{P_{\bar{m}}(F)}^2$ , we deduce that the restriction map of functions in  $D(P_{\bar{m}}(K) \setminus G_m(K), \Delta_{P_{\bar{m}}(K)}^{-1/2} \pi, V_{\pi})$  to  $G_m(F)$  has value in  $D(P_{\bar{m}}(F) \setminus G_m(F), \Delta_{P_{\bar{m}}(F)}^{-1} \pi, V_{\pi})$ . Let L be a nonzero  $P_{\bar{m}}(F)$ -invariant linear form on  $V_{\pi}$ . If  $\phi$  belongs to  $D(P_{\bar{m}}(K) \setminus G_m(K), \Delta_{P_{\bar{m}}(K)}^{-1/2} \pi)$ , then the function defined on  $G_n(F)$  by  $g \mapsto L(\phi(g))$  belongs to  $D(P_{\bar{m}}(F) \setminus G_m(F), \Delta_{P_{\bar{m}}(F)}^{-1})$ . From [B-H], ch.1, prop.3.4, there is a  $G_m(F)$ -invariant linear form  $\Gamma$  on

$$D(P_{\bar{m}}(F)\backslash G_m(F), \Delta_{P_{\bar{m}}(F)}^{-1})$$

which is positive on positive functions.

Hence  $\phi \mapsto \Gamma(g \mapsto L(\phi(g)))$  gives a  $G_m(F)$ -invariant linear form on

$$D(P_{\bar{m}}(K)\backslash G_m(K), \Delta_{P_{\bar{m}}(K)}^{-1/2}\pi)$$

Now let v be a vector of  $V_{\pi}$  such that L(v) > 0, and let U be a compact open subgroup of  $G_m(K)$ such that  $U \cap P_{\overline{m}}(K)$  is included in the fixator of v.

The function  $\phi_0: pu \mapsto \Delta_{P_{\bar{m}}(K)}^{-1/2}(p)\pi(p)v$  with  $p \in P_{\bar{m}}(K), u \in U$  is well defined and belongs to  $D(P_{\bar{m}}(K) \setminus G_m(K), \Delta_{P_{\bar{m}}(K)}^{-1/2}\pi).$ 

It also verifies that  $g \mapsto L(\phi_0(g))$  is positive and non zero, hence  $\Gamma(g \mapsto L(\phi_0(g)))$  is positive. Thus  $\phi \mapsto \Gamma(g \mapsto L(\phi(g)))$  is nonzero and  $\pi_1 \times \cdots \times \pi_l$  is distinguished. 

**Proposition 3.1.** Suppose that  $\pi(\chi)$  is distinguished, there exists a re-ordering of the  $\chi_i$ 's, and  $r \leq n/2$ , such that  $\chi_{i+1}^{\sigma} = \chi_i^{-1}$  for i = 1, 3, ..., 2r - 1, and that  $\chi_{i|F^*} = 1$  for i > 2r.

*Proof.* We have from corollary 2.4 and lemma 2.2 the following exact sequence of smooth  $G_n(F)$ modules:

$$D(B \setminus G_n - A_1, \Delta_{B_n}^{-1/2} \chi) \hookrightarrow D(B_n(K) \setminus G_n(K), \Delta_{B_n}^{-1/2} \chi) \twoheadrightarrow D(B \setminus A_1, \Delta_{B_n}^{-1/2} \chi).$$

Hence there is a non zero distinguished linear form either on  $D(B \setminus A_1, \Delta_{B_n})^{-1/2} \chi$ , or on  $D(B \setminus G_n - A_1, \Delta_{B_n})^{-1/2} \chi).$ 

In the second case we have the exact sequence

$$D(B \backslash G_n - A_1 \sqcup A_2, \Delta_{B_n}^{-1/2} \chi) \hookrightarrow D(B \backslash G_n - A_1, \Delta_{B_n}^{-1/2} \chi) \twoheadrightarrow D(B \backslash A_2, \Delta_{B_n}^{-1/2} \chi).$$

Repeating the process, we deduce the existence of a non zero distinguished linar form on one of the spaces  $D(B \setminus A_i, \Delta_{B_n}^{-1/2} \chi)$ .

Now there exists w in  $S_n$  and  $r \leq n/2$  such that  $A_i = B_n(K)U_r^w G_n(F)$ . But the application  $f \mapsto [x \mapsto f(U_r^w x)]$  gives an isomorphism of  $G_n(F)$ -modules between  $D(B \setminus A_i, \Delta_{B_n}^{-1/2} \chi)$ and  $D(U_r^{-w}BU_r^{w}\cap G_n(F)\setminus G_n(F), \Delta'\chi')$  where  $\Delta'(x) = \Delta_{B_n}^{-1/2}(U_r^{w}xU_r^{-w})$  and  $\chi'(x) = \chi(U_r^{w}xU_r^{-w})$ . Now there exists a nonzero  $G_n(F)$ -invariant linear form on  $D(U_r^{-w}BU_r^{w}\cap G_n(F)\setminus G_n(F), \Delta'\chi')$  if and only if  $\Delta'\chi'$  is equal to the inverse of the module of  $U_r^{-w}BU_r^{w}\cap G_n(F)$  (cf.[B-H], ch.1, prop.3.4).

From this we deduce that  $\chi'$  is positive on  $U_r^{-w}BU_r^w \cap G_n(F)$  or equivalently  $\chi$  is positive on  $B \cap U_r^{w} G_n(F) U_r^{-w}.$ 



 $2r + t = n, z_i \in K, x_i \in F^*$ , then one has  $\overline{T_r}^w \subset B \cap U_r^w G_n(F) U_r^{-w}$ , so that  $\chi$  must be positive on  $\overline{T_r}^w$ .

We remark that if  $\chi$  is unitary, then  $\chi$  is trivial on  $\overline{T}_r^w$ , and  $\pi(\chi)$  is of the desired form.

For the general case, we deduce from theorem 3.1, that there exists three integers  $2p \ge 0, q \ge 0, s \ge 0$ such that up to reordering, we have  $\chi_{i+1} = \chi_i^{-\sigma}$  for  $1 \le i \le p$ , we have  $\chi_{2p+k|F^*} = 1$  for  $1 \le k \le q$ and these  $\chi_{2p+k}$ 's are different (so that  $\chi_{2p+k} \ne \chi_{2p+k'}^{-\sigma}$  for  $k \ne k'$ ), and  $\chi_{2p+q+j|F^*} = \eta_{K/F}$  for  $1 \le j \le s$ , these  $\chi_{2p+q+j}$ 's being different.

We note  $\mu_k = \chi_{2p+k}$  for  $q \ge k \ge 1$ , and  $\nu'_k = \chi_{2p+q+k'}$  for  $s \ge k' \ge 1$ .

We show that if such a character  $\chi$  is positive on a conjugate of  $\overline{T}_r$  by an element of  $S_n$ , then s = 0. Supose  $\nu_1$  appears, then either  $\nu_1$  is positive on  $F^*$ , but that is not possible, or it is coupled with another  $\chi_i$ , and  $(\nu_1, \chi_i)$  is positive on elements  $(z, z^{\sigma})$ , for z in  $K^*$ .

Suppose  $\chi_i = \nu_j$  for some  $j \neq 1$ , then  $(\nu_1, \chi_i)$  is unitary, so it must be trivial on couples  $(z, z^{\sigma})$ , which implies  $\nu_1 = \nu_j^{-\sigma} = \nu_j$ , which is absurd.

The character  $\chi_i$  cannot be of the form  $\mu_i$ , because it would imply  $\nu_{1|F^*} = 1$ .

The last case is  $i \leq 2p$ , then  $\nu_1^{-\sigma} = \nu_1$  must be the unitary part of  $\chi_i$  because of the positivity of  $(\nu_1, \chi_i)$  on the couples  $(z, z^{\sigma})$ .

But  $\chi_i^{-\sigma}$  also appears, and must be coupled with another character  $\chi_j$  with  $j \leq 2p$  and  $j \neq i$  (for the same reasons as before), such that  $(\chi_i^{-\sigma}, \chi_j)$  is positive on the elements  $(z, z^{\sigma})$ , for z in  $K^*$ , which implies that  $\chi_j$  has unitary part  $\nu_1^{-\sigma} = \nu_1$ .

The character  $\chi_j$  cannot be a  $\mu_k$  because of its unitary part.

If it is a  $\chi_k$  with  $k \leq 2p$ , we consider again  ${\chi_k}^{-\sigma}$ .

But repeating the process lengthily enough, we can suppose that  $\chi_j$  is of the form  $\nu_k$ , for  $k \neq 1$ . Taking unitary parts, we see that  $\nu_k = \nu_1^{-\sigma} = \nu_1$ , which is in contradiction with the fact that all  $\nu_i$ 's are different. We conclude that s = 0.

**Theorem 3.2.** The representation  $\pi(\chi)$  is distinguished if and only if there exists  $r \leq n/2$ , such that  $\chi_{i+1}^{\sigma} = \chi_i^{-1}$  for i = 1, 3, ..., 2r - 1, and that  $\chi_{i|F^*} = 1$  for i > 2r.

*Proof.* There is one implication left.

Suppose  $\chi$  is of the desired form, then  $\pi(\chi)$  is parabolically (unitarily) induced from representations of the type  $\pi(\chi_i, \chi_i^{-\sigma})$  of  $G_2(K)$ , and distinguished characters of  $K^*$ .

Hence, because of lemma 3.1 the theorem will be proved if we know that the representations  $\pi(\chi_i, \chi_i^{-\sigma})$  are distinguished, but this is corollary 4.1 of the next paragraph.

This gives a counter-example to a conjecture of Jacquet (conjecture 1 in [A]), asserting that if an irreducible admissible representation  $\pi$  of  $G_n(K)$  verifies that  $\check{\pi}$  is isomorphic to  $\pi^{\sigma}$ , then it is distinguished if n is odd, and it is distinguished or  $\eta_{K/F}$ -distinguished if n is even.

**Corollary 3.1.** For  $n \geq 3$ , there exist smooth irreducible representations  $\pi$  of  $G_n(K)$ , with central character trivial on  $F^*$ , that are neither distinguished, nor  $\eta_{K/F}$ -distinguished, but verify that  $\check{\pi}$  is isomorphic to  $\pi^{\sigma}$ .

*Proof.* Take  $\chi_1, \ldots, \chi_n$ , all different, such that  $\chi_{1|F^*} = \chi_{2|F^*} = \eta_{K/F}$ , and  $\chi_{j|F^*} = 1$  for  $3 \le j \le n$ . Because each  $\chi_i$  has trivial restriction to  $N_{K/F}(K^*)$ , it is equal to  $\chi_i^{-\sigma}$ , hence  $\check{\pi}$  is isomorphic to  $\pi^{\sigma}$ . Another consequence is that if k and l are two different integers between 1 and n, then  $\chi_k \neq \chi_l^{-\sigma}$ , because we supposed the  $\chi_i$ 's all different.

Then it follows theorem 3.2 that  $\pi = \pi(\chi_1, \ldots, \chi_n)$  is neither distinguished, nor  $\eta_{K/F}$ -distinguished, but clearly, the central character of  $\pi$  is trivial on  $F^*$  and  $\check{\pi}$  is isomrphic to  $\pi^{\sigma}$ .

#### 4 Distinction and gamma factors for GL(2)

As said in the introduction, in this section we generalize to smooth infinite dimensional irreducible representations of  $G_2(K)$  a criterion of Hakim (cf. [H], theorem 4.1) characterising smooth unitary irreducible representations of  $G_2(K)$ .

Let  $\pi$  be a smooth infinite dimensional irreducible represention of  $G_2(K)$ , it is known that it is generic (cf.[Z]) for example).

We note M(K) the mirabolic subgroup of  $G_2(K)$  of matrices of the form  $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$  with a in  $K^*$ and x in K, and M(F) its intersection with  $G_2(F)$ .

We note w the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Let  $K(\pi, \psi)$  be its Kirillov model corresponding to  $\psi$  ([J-L], th. 2.13), it contains the subspace  $D(K^*)$  of functions with compact support on the group  $K^*$ .

If  $\phi$  belongs to  $K(\pi, \psi)$ , and x belongs to K, then  $\phi - \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \phi$  belongs to  $D(K^*)$  ([J-L], prop.2.9, ch.1), from this follows that  $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$ .

We now recall a consequence of the functional equation at 1/2 for Kirillov representations (cf. [B]).

For all  $\phi$  in  $K(\pi, \psi)$  and  $\chi$  character of  $K^*$ , we have whenever both sides converge absolutely:

$$\int_{K^*} \pi(w)\phi(x)(c_\pi\chi)^{-1}(x)d^*x = \gamma(\pi\otimes\chi,\psi)\int_{K^*}\phi(x)\chi(x)d^*x \tag{1}$$

where  $d^*x$  is a Haar measure on  $K^*$ , and  $c_{\pi}$  is the central character of  $\pi$ .

**Theorem 4.1.** Let  $\pi$  be a smooth irreducible representation of infinite dimension with central character trivial on  $F^*$  of  $G_2(K)$ , and  $\psi$  a nontrivial character of K trivial on F. If  $\gamma(\pi \otimes \chi, \psi) = 1$ for every character  $\chi$  of  $K^*$  trivial on  $F^*$ , then  $\pi$  is distinguished.

Proof. In fact, using a Fourier inversion in functional equation (1) and the change of variable  $x \mapsto x^{-1}$ , we deduce that for all  $\phi$  in  $D(K^*) \cap \pi(w)D(K^*)$ , we have

$$c_{\pi}(x) \int_{F^*} \pi(w) \phi(tx^{-1}) d^*t = \int_{F^*} \phi(tx) d^*t$$

 $(d^*t \text{ is a Haar measure on } F^*)$  which for x = 1 gives

$$\int_{F^*} \pi(w)\phi(t)d^*t = \int_{F^*} \phi(t)d^*t.$$

Now we define on  $K(\pi, \psi)$  a linear form  $\lambda$  by:

$$\lambda(\phi_1 + \pi(w)\phi_2) = \int_{F^*} \phi_1(t)d^*t + \int_{F^*} \phi_2(t)d^*t$$

for  $\phi_1$  and  $\phi_2$  in  $D(K^*)$ , which is well defined because of the previous equality and the fact that  $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$ .

It is clear that  $\lambda$  is *w*-invariant.

As the central character of  $\pi$  is trivial on  $F^*$ , it is also clear that  $\lambda$  is  $F^*$ -invariant. Because  $GL_2(F)$  is generated by M(F), its center, and w, it remains to show that  $\lambda$  is M(F)-invariant.

It is clear that if  $\phi \in D(K^*)$ , and  $m \in M(F)$ ,  $\lambda(\pi(m)\phi) = \lambda(\phi)$  (since  $\psi$  is trivial on F). Now if  $\phi = \pi(w)\phi_2 \in \pi(w)D(K^*)$ .

If a belongs to  $F^*$ , then  $\pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \pi(w)\phi_2 = \pi(w)\pi \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \phi_2 = \pi(w)\pi \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \phi_2$  because the central character of  $\pi$  is trivial on  $F^*$ , and  $\lambda(\pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \phi) = \lambda(\phi)$ .

If 
$$x \in F$$
, then  $\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \phi - \phi$  is a function of  $D(F^*)$ , which vanishes on  $F^*$ , hence  $\lambda \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \phi - \phi = 0$ .

Eventually  $\lambda$  is M(F)-invariant, hence  $G_2(F)$ -invariant, it is clear that its restriction to  $D(K^*)$  is non zero.

**Corollary 4.1.** Let  $\mu$  be a character of  $K^*$ , then  $\pi(\mu, \mu^{-\sigma})$  is distinguished.

Proof. Indeed, first we notice that the central character  $\mu\mu^{-\sigma}$  of  $\pi(\mu,\mu^{-\sigma})$  is trivial on  $F^*$ . Now let  $\chi$  be a character of  $K^*/F^*$ , then  $\gamma(\pi(\mu,\mu^{-\sigma})\otimes\chi,\psi) = \gamma(\mu\chi,\psi)\gamma(\mu^{-\sigma}\chi,\psi) = \gamma(\mu\chi,\psi)\gamma(\mu^{-1}\chi^{\sigma},\psi^{\sigma})$ , and as  $\psi|F| = 1$  and  $\chi_{|F^*} = 1$ , one has  $\psi^{\sigma} = \psi^{-1}$  and  $\chi^{\sigma} = \chi^{-1}$ , so that  $\gamma(\pi(\chi,\chi^{-\sigma}),\psi) = \gamma(\mu\chi,\psi)\gamma(\mu^{-1}\chi^{-1},\psi^{-1}) = 1$ . The conclusion falls from proposition 4.1.

The converse of theorem 4.1 is also true:

**Theorem 4.2.** Let  $\pi$  be a smooth irreducible distinguished representation of infinite dimension of  $G_2(K)$  and  $\psi$  a non trivial character of K/F, then  $\gamma(\pi, \psi) = 1$ .

*Proof.* Suppose  $\lambda$  is a non zero  $G_2(F)$ -invariant linear form on  $K(\pi, \psi)$ , it is shown in the proof of the corollary of proposition 3.3 in [H], that its restriction to  $D(K^*)$  must be a multiple of the Haar measure on  $F^*$ .

Hence for any function in  $D(K^*) \cap \pi(w)D(K^*)$ , we must have  $\int_{F^*} \phi(t)d^*t = \int_{F^*} \pi(w)\phi(t)d^*t$ . From this one deduces that for any function in  $D(K^*) \cap \pi(w)D(K^*)$ :  $\int_{K^*} \pi(w)\phi(x)c_{\pi}^{-1}(x)d^*x = \int_{K^*/F^*} c_{\pi}^{-1}(a)\int_{F^*} \pi(w)\phi(ta)d^*tda$ 

$$= \int_{K^*/F^*} c_{\pi}^{-1}(a) \int_{F^*} \pi(\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}) \pi(w) \phi(t) d^* t da$$
$$= \int_{K^*/F^*} c_{\pi}^{-1}(a) \int_{F^*} \pi(w) c_{\pi}(a) \pi(\begin{array}{cc} a^{-1} & 0 \\ 0 & 1 \end{array}) \phi(t) d^* t da$$

This last quantity is equal to  $\int_{K^*/F^*} \int_{F^*} \pi(\begin{array}{c} a^{-1} & 0\\ 0 & 1 \end{array}) \phi(t) d^*t da$ 

 $=\int_{K^*/F^*}\int_{F^*}\phi(ta^{-1})d^*tda = \int_{K^*/F^*}\int_{F^*}\phi(ta)d^*tda = \int_{K^*}\phi(x)d^*x.$ From this we deduce that either  $\gamma(\pi,\psi)$  is equal to one, either  $\int_{K^*}\phi(x)d^*x$  is equal to zero on  $D(K^*) \cap \pi(w)D(K^*).$ 

In the second case, we could define two independant  $K^*$ -invariant linear forms on  $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$ , given by  $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*} \phi_1(x)d^*x$ , and  $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*} \phi_2(x)d^*x$ . This would contradict theorem 1.2 of [A-G].

#### References

- [A], U.K. Anandavardhanan, Distinguished non-Archimedean representations, Proc. Hyderabad Conference on Algebra and Number Theory, 2005, 183-192.
- [A-G] A. Aizenbud and D. Gourevitch, A proof of the multiplicity one conjecture for GL(n) in GL(n+1), preprint, Arxiv.
- [A-K-T] U.K. Anandavardhanan, A.C. Kable and R. Tandon, Distinguished representations and poles of twisted tensor L-functions, Proc. Amer. Math. Soc., 132 (2004), 2875-2883.
- [B] D. Bump, Automorphic forms and representations, Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.
- [B-H] C. Bushnell and G.Henniart, The Local Langlands Correspondence for GL(2), Springer, 2006.
- [B-Z 1] I.N. Bernstein and A.V.Zelevinsky, Representations of the group GL(n,F), where F is a local non-Archimedean field, Russian Math. Surveys, 31 (1976), no 3, p.1-68
- [B-Z 2] I.N. Bernstein and A.V.Zelevinsky, Induced representations of reductive p-adic groups, Ann.Sc.E.N.S., 1977, p.441-472.
- [B] D. Bump, Automorphic forms and representations, Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.
- [F] Y. Flicker, On distinguished representations, Journal fr die reine und angewandte mathematik, 418 (1991), 139-172.
- [F1] Y. Flicker, Distinguished representations and a Fourier summation formula, Bulletin de la S.M.F., 120, (1992), 413-465.
- [F-H] Y. Flicker and J.Hakim, Quaternionic distinguished reppresentations, American Journal of Mathematics, 116 (1994), 683-736.
- [H] J. Hakim, Distinguished p-adic Representations, Duke Math. J., 62 (1991), 1-22.
- [J-L] H. Jacquet and R. Langlands, Automotphic forms on GL(2), Lect. Notes in Math., 114, Springer, 1970.
- [Ok] Y. Ok, Distinction and Gamma factors at 1/2: Supercuspidal Case, Thesis, Columbia University (1997)
- [P] M-N. Panichi, Caractrisations du spectre tempr de  $GL_n(\mathbb{C})/GL_n(\mathbb{R})$ , Thse de Doctorat, Universit de Paris 7, 2001.
- [S] J-P. Serre, Corps locaux, Hermann, 1997.

- [Sp] T.A. Springer, Linear algebraic groups, Birkaser, 1998
- [Z] A.V. Zelevinsky, Induced representations of reductive p-adic groups II, Ann.Sc.E.N.S., 1980.
- [Ok] Y. Ok, Distinction and Gamma factors at 1/2: Supercuspidal Case, Thesis, Columbia University (1997)
- [J-P-S] H. Jacquet, I.I. Piatetskii-Shapiro and J.A. Shalika, Rankin-Selberg Convolutions, Am.J.Math 105(1983), 367-464.