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Distinguished principal series representations for GL_n over a p – adic field

June 9, 2008

1 Introduction

Let F be a non archimedean local field of characteristic zero and K a quadratic extension of F . If π is a smooth irreducible representation of $GL(n, K)$, the dimension of $GL(n, F)$ -invariant linear forms on its space is known to be at most one (proposition 11, [F]).

One says that π is distinguished if this dimension is one.

In this article, we give a description of distinguished principal series representations of $GL(n, K)$. For the quadratic extension \mathbb{C}/\mathbb{R} , it is known (cf.[P]) that the irreducible distinguished tempered representations of $GL(n, \mathbb{C})$ are (up to isomorphism) those unitarily induced from a unitary character $\chi = (\chi_1, \dots, \chi_n)$ of the maximal torus of diagonal matrices, such that there exists $r \leq n/2$, for which $\chi_{i+1}^\sigma = \chi_i^{-1}$ for $i = 1, 3, \dots, 2r - 1$, and $\chi_{i|F^*} = 1$ for $i > 2r$. In our case, theorem 3.2 gives an analogous result.

This gives a counter-example to a conjecture of Jacquet (Conjecture 1 in [A]).

We use this occasion to give a different proof of the result for $GL(2, K)$ than the one in [F-H]. To do this, in theorems 4.1 and 4.2, we extend a criterion of Hakim (th.4.1, [H]) characterising smooth unitary irreducible distinguished representations of $GL(2, K)$ in terms of γ factors at $1/2$, to all smooth irreducible distinguished representations of $GL(2, K)$.

2 Preliminaries

Let ϕ be a group automorphism, and x an element of the group, we sometimes note x^ϕ instead of $\phi(x)$, and $x^{-\phi}$ the inverse of x^ϕ .

If $\phi = x \mapsto h^{-1}xh$ for h in the group, then x^h designs x^ϕ .

Let G be a locally compact totally disconnected group, H a closed subgroup of G .

We note Δ_G the module of G .

Let X be a locally closed subspace of G , with $H.X \subset X$.

If V is a complex vector space, we note $D(X, V)$ the space of smooth V -valued functions on X with compact support (if $V = \mathbb{C}$, we simply note it $D(X)$).

Let ρ be a smooth representation of H in a complex vector space V_ρ , we note $D(H \backslash X, \rho, V_\rho)$ the space of smooth V_ρ -valued functions f on X , with compact support modulo H , which verify $f(hx) = \rho(h)f(x)$ for $h \in H$ and $x \in X$ (if ρ is a character, we note it $D(H \backslash X, \rho)$).

Let u be the following element of $M_2(K)$: $\begin{pmatrix} 1 & -\delta \\ 1 & \delta \end{pmatrix}$, one has $S_2(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Remark:
If $\tilde{T} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^\sigma \end{pmatrix} \in G_2(K) \mid z \in K^* \right\} \simeq K^*$, then $u^{-1}\tilde{T}u = T = \left\{ \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix} \in G_2(F) \mid x, y \in F \right\}$.

For $r \leq n/2$, one notes U_r the $n \times n$ matrix given by the following block decomposition:

$$\begin{pmatrix} u & & & \\ & \ddots & & \\ & & u & \\ & & & I_{n-2r} \end{pmatrix}$$

If w is an element of \mathfrak{S}_n naturally injected in $G_n(K)$, one notes $U_r^w = w^{-1}U_r w$.

Corollary 2.1. *The elements U_r^w for $0 \leq r \leq n/2$, and $w \in S_n$ give a complete set of representatives of classes of $B_n(K) \backslash G_n(K) / G_n(F)$.*

Let $G_n = \coprod_{w \in \mathfrak{S}_n} B_n w B_n$ be the Bruhat decomposition of G_n . We call a double-class BwB a Bruhat cell.

Lemma 2.3. *One can order the Bruhat cells $C_1, C_2, \dots, C_n!$ so that for every $1 \leq i \leq n!$, the cell C_i is closed in $G_n - \coprod_{k=1}^{i-1} C_k$.*

Proof. Choose $C_1 = B_n$. It is closed in G_n . Now let w_2 be an element of $S_n - Id$, with minimal length. Then from 8.5.5. of [Sp], one has that the Bruhat cell Bw_2B is closed in $G_n - B_n$ with respect to the Zariski topology, hence for the p-adic topology, we call it C_2 . We conclude by repeating this process. \square

Corollary 2.2. *One can order the classes A_1, \dots, A_i of $B_n(K) \backslash G_n(K) / G_n(F)$, so that A_i is closed in $G_n(K) - \coprod_{k=1}^{i-1} A_k$.*

Proof. From the proof of proposition 2.2, we know that if C is a Bruhat cell of G_n , then $S_n \cap C$ is either empty, or it corresponds through the homeomorphism S_n to a class A of $B_n(K) \backslash G_n(K) / G_n(F)$. The conclusion follows the preceding lemma. \square

Corollary 2.3. *Each A_i is locally closed in $G_n(K)$ for the Zariski topology.*

We will also need the following lemma:

Lemma 2.4. *Let G, H, X , and (ρ, V_ρ) be as in the beginning of the section, the map Φ from $D(X) \otimes V_\rho$ to $D(H \backslash X, \rho, V_\rho)$ defined by $\Phi : f \otimes v \mapsto (x \mapsto \int_H f(hx) \rho(h^{-1}) v dh)$ is surjective.*

Proof. Let $v \in V_\rho$, U an open subset of G that intersects X , small enough for $h \mapsto \rho(h)v$ to be trivial on $H \cap UU^{-1}$.

Let f' be the function with support in $H(X \cap U)$ defined by $hx \mapsto \rho(h)v$.

Such functions generate $D(H \backslash X, \rho, V_\rho)$ as a vector space.

Now let f be the function of $D(X, V_\rho)$ defined by $x \mapsto 1_{U \cap X}(x)v$, then $\Phi(f)$ is a multiple of f' .

But for x in $U \cap X$, $\Phi(f)(x) = \int_H \rho(h^{-1}) f(hx) dh$ because $h \mapsto \rho(h)v$ is trivial on $H \cap UU^{-1}$, plus $h \mapsto f(hx)$ is a positive function that multiplies v , and $f(x) = V$, so $F(f)(x)$ is v multiplied by a strictly positive scalar. \square

Corollary 2.4. *Let Y be a closed subset of X , H -stable, then the restriction map from $D(H \setminus X, \rho, V_\rho)$ to $D(H \setminus Y, \rho, V_\rho)$ is surjective.*

Proof. This is a consequence of the known surjectivity of the restriction map from $D(X)$ to $D(Y)$, which implies the surjectivity of the restriction from $D(X, V_\rho)$ to $D(Y, V_\rho)$ and the commutativity of the diagramm:

$$\begin{array}{ccc} D(X) & \rightarrow & D(Y) \\ \downarrow \Phi & & \downarrow \Phi \\ D(H \setminus X, \rho) & \rightarrow & D(H \setminus Y, \rho) \end{array} \quad \square$$

3 Distinguished principal series

If π is a smooth representation of $G_n(K)$ of space V_π , and χ is a character of F^* , we say that π is χ -distinguished if there exists on V_π a nonzero linear form L such that $L(\pi(g)v) = \chi(\det(g))L(v)$ whenever g is in $G_n(F)$ and v belongs to V_π . If χ is trivial, we simply say that π is distinguished.

We first recall the following:

Theorem 3.1. (*[F], proposition 12*)

Let π be a smooth irreducible representation of $G_n(K)$ which admits a nonzero $G_n(F)$ -invariant linear form on its space, then $\pi^\sigma \simeq \tilde{\pi}$.

Let χ_1, \dots, χ_n be n characters of K^* , with none of their quotients equal to $|\cdot|_K$. We note χ the character of $B_n(K)$ defined by $\chi \begin{pmatrix} b_1 & \star & \star \\ & \ddots & \star \\ & & b_n \end{pmatrix} = \chi_1(b_1) \dots \chi_n(b_n)$.

We note $\pi(\chi_1, \dots, \chi_n)$ the representation of $G_n(K)$ by right translation on the space of functions $D(B_n(K) \setminus G_n(K), \Delta_{B_n}^{-1/2} \chi)$.

This representation is smooth, irreducible and called the principal series attached to χ .

We will need the following lemma (see prop.10.2 of [Ok] for a different proof).

Lemma 3.1. *Let $\bar{m} = (m_1, \dots, m_l)$ be a partition of a positive integer m , let $P_{\bar{m}}$ be the corresponding standard parabolic subgroup, and π_i be smooth distinguished representations of $G_{m_i}(K)$, then $\pi_1 \times \dots \times \pi_l = \text{Ind}_{P_{\bar{m}}(K)}^{G_m(K)} (\Delta_{P_{\bar{m}}(K)}^{-1/2} (\pi_1 \otimes \dots \otimes \pi_l))$ is distinguished.*

Proof. We note π for $\pi_1 \otimes \dots \otimes \pi_l$.

We want to show that $\text{Hom}_{G_m(F)}(c - \text{Ind}_{P_{\bar{m}}(K)}^{G_m(K)}(\pi), 1)$ is of dimension at least one. But from the relation $\Delta_{P_{\bar{m}}(K)} = \Delta_{P_{\bar{m}}(F)}^2$, we deduce that the restriction map of functions in

$D(P_{\bar{m}}(K) \setminus G_m(K), \Delta_{P_{\bar{m}}(K)}^{-1/2} \pi, V_\pi)$ to $G_m(F)$ has value in $D(P_{\bar{m}}(F) \setminus G_m(F), \Delta_{P_{\bar{m}}(F)}^{-1} \pi, V_\pi)$.

Let L be a nonzero $P_{\bar{m}}(F)$ -invariant linear form on V_π . If ϕ belongs to $D(P_{\bar{m}}(K) \setminus G_m(K), \Delta_{P_{\bar{m}}(K)}^{-1/2} \pi)$, then the function defined on $G_m(F)$ by $g \mapsto L(\phi(g))$ belongs to $D(P_{\bar{m}}(F) \setminus G_m(F), \Delta_{P_{\bar{m}}(F)}^{-1} \pi)$.

From [B-H], ch.1, prop.3.4, there is a $G_m(F)$ -invariant linear form Γ on

$$D(P_{\bar{m}}(F) \setminus G_m(F), \Delta_{P_{\bar{m}}(F)}^{-1} \pi)$$

which is positive on positive functions.

Hence $\phi \mapsto \Gamma(g \mapsto L(\phi(g)))$ gives a $G_m(F)$ -invariant linear form on

$$D(P_{\bar{m}}(K) \backslash G_m(K), \Delta_{P_{\bar{m}}(K)}^{-1/2} \pi).$$

Now let v be a vector of V_π such that $L(v) > 0$, and let U be a compact open subgroup of $G_m(K)$ such that $U \cap P_{\bar{m}}(K)$ is included in the fixator of v .

The function $\phi_0 : pu \mapsto \Delta_{P_{\bar{m}}(K)}^{-1/2}(p)\pi(p)v$ with $p \in P_{\bar{m}}(K), u \in U$ is well defined and belongs to $D(P_{\bar{m}}(K) \backslash G_m(K), \Delta_{P_{\bar{m}}(K)}^{-1/2} \pi)$.

It also verifies that $g \mapsto L(\phi_0(g))$ is positive and non zero, hence $\Gamma(g \mapsto L(\phi_0(g)))$ is positive. Thus $\phi \mapsto \Gamma(g \mapsto L(\phi(g)))$ is nonzero and $\pi_1 \times \cdots \times \pi_l$ is distinguished. \square

Proposition 3.1. *Suppose that $\pi(\chi)$ is distinguished, there exists a re-ordering of the χ_i 's, and $r \leq n/2$, such that $\chi_{i+1}^\sigma = \chi_i^{-1}$ for $i = 1, 3, \dots, 2r - 1$, and that $\chi_{i|F^*} = 1$ for $i > 2r$.*

Proof. We have from corollary 2.4 and lemma 2.2 the following exact sequence of smooth $G_n(F)$ -modules:

$$D(B \backslash G_n - A_1, \Delta_{B_n}^{-1/2} \chi) \hookrightarrow D(B_n(K) \backslash G_n(K), \Delta_{B_n}^{-1/2} \chi) \twoheadrightarrow D(B \backslash A_1, \Delta_{B_n}^{-1/2} \chi).$$

Hence there is a non zero distinguished linear form either on $D(B \backslash A_1, \Delta_{B_n}^{-1/2} \chi)$, or on $D(B \backslash G_n - A_1, \Delta_{B_n}^{-1/2} \chi)$.

In the second case we have the exact sequence

$$D(B \backslash G_n - A_1 \sqcup A_2, \Delta_{B_n}^{-1/2} \chi) \hookrightarrow D(B \backslash G_n - A_1, \Delta_{B_n}^{-1/2} \chi) \twoheadrightarrow D(B \backslash A_2, \Delta_{B_n}^{-1/2} \chi).$$

Repeating the process, we deduce the existence of a non zero distinguished linear form on one of the spaces $D(B \backslash A_i, \Delta_{B_n}^{-1/2} \chi)$.

Now there exists w in S_n and $r \leq n/2$ such that $A_i = B_n(K)U_r^w G_n(F)$.

But the application $f \mapsto [x \mapsto f(U_r^w x)]$ gives an isomorphism of $G_n(F)$ -modules between $D(B \backslash A_i, \Delta_{B_n}^{-1/2} \chi)$ and $D(U_r^{-w} B U_r^w \cap G_n(F) \backslash G_n(F), \Delta' \chi')$ where $\Delta'(x) = \Delta_{B_n}^{-1/2}(U_r^w x U_r^{-w})$ and $\chi'(x) = \chi(U_r^w x U_r^{-w})$.

Now there exists a nonzero $G_n(F)$ -invariant linear form on $D(U_r^{-w} B U_r^w \cap G_n(F) \backslash G_n(F), \Delta' \chi')$ if and only if $\Delta' \chi'$ is equal to the inverse of the module of $U_r^{-w} B U_r^w \cap G_n(F)$ (cf. [B-H], ch.1, prop.3.4).

From this we deduce that χ' is positive on $U_r^{-w} B U_r^w \cap G_n(F)$ or equivalently χ is positive on $B \cap U_r^w G_n(F) U_r^{-w}$.

Let \bar{T}_r be the F -torus of matrices of the form $\left(\begin{array}{ccccccc} z_1 & & & & & & \\ & z_1^\sigma & & & & & \\ & & \ddots & & & & \\ & & & z_r & & & \\ & & & & z_r^\sigma & & \\ & & & & & x_1 & \\ & & & & & & \ddots \\ & & & & & & & x_t \end{array} \right)$ with

$2r + t = n$, $z_i \in K$, $x_i \in F^*$, then one has $\bar{T}_r^w \subset B \cap U_r^w G_n(F) U_r^{-w}$, so that χ must be positive on \bar{T}_r^w .

We remark that if χ is unitary, then χ is trivial on \bar{T}_r^w , and $\pi(\chi)$ is of the desired form.

For the general case, we deduce from theorem 3.1, that there exists three integers $2p \geq 0, q \geq 0, s \geq 0$ such that up to reordering, we have $\chi_{i+1} = \chi_i^{-\sigma}$ for $1 \leq i \leq p$, we have $\chi_{2p+k}|_{F^*} = 1$ for $1 \leq k \leq q$ and these χ_{2p+k} 's are different (so that $\chi_{2p+k} \neq \chi_{2p+k'}^{-\sigma}$ for $k \neq k'$), and $\chi_{2p+q+j}|_{F^*} = \eta_{K/F}$ for $1 \leq j \leq s$, these χ_{2p+q+j} 's being different.

We note $\mu_k = \chi_{2p+k}$ for $q \geq k \geq 1$, and $\nu'_k = \chi_{2p+q+k'}$ for $s \geq k' \geq 1$.

We show that if such a character χ is positive on a conjugate of \bar{T}_r by an element of S_n , then $s = 0$. Suppose ν_1 appears, then either ν_1 is positive on F^* , but that is not possible, or it is coupled with another χ_i , and (ν_1, χ_i) is positive on elements (z, z^σ) , for z in K^* .

Suppose $\chi_i = \nu_j$ for some $j \neq 1$, then (ν_1, χ_i) is unitary, so it must be trivial on couples (z, z^σ) , which implies $\nu_1 = \nu_j^{-\sigma} = \nu_j$, which is absurd.

The character χ_i cannot be of the form μ_j , because it would imply $\nu_1|_{F^*} = 1$.

The last case is $i \leq 2p$, then $\nu_1^{-\sigma} = \nu_1$ must be the unitary part of χ_i because of the positivity of (ν_1, χ_i) on the couples (z, z^σ) .

But $\chi_i^{-\sigma}$ also appears, and must be coupled with another character χ_j with $j \leq 2p$ and $j \neq i$ (for the same reasons as before), such that $(\chi_i^{-\sigma}, \chi_j)$ is positive on the elements (z, z^σ) , for z in K^* , which implies that χ_j has unitary part $\nu_1^{-\sigma} = \nu_1$.

The character χ_j cannot be a μ_k because of its unitary part.

If it is a χ_k with $k \leq 2p$, we consider again $\chi_k^{-\sigma}$.

But repeating the process lengthily enough, we can suppose that χ_j is of the form ν_k , for $k \neq 1$. Taking unitary parts, we see that $\nu_k = \nu_1^{-\sigma} = \nu_1$, which is in contradiction with the fact that all ν_i 's are different. We conclude that $s = 0$. \square

Theorem 3.2. *The representation $\pi(\chi)$ is distinguished if and only if there exists $r \leq n/2$, such that $\chi_{i+1}^\sigma = \chi_i^{-1}$ for $i = 1, 3, \dots, 2r - 1$, and that $\chi_i|_{F^*} = 1$ for $i > 2r$.*

Proof. There is one implication left.

Suppose χ is of the desired form, then $\pi(\chi)$ is parabolically (unitarily) induced from representations of the type $\pi(\chi_i, \chi_i^{-\sigma})$ of $G_2(K)$, and distinguished characters of K^* .

Hence, because of lemma 3.1 the theorem will be proved if we know that the representations $\pi(\chi_i, \chi_i^{-\sigma})$ are distinguished, but this is corollary 4.1 of the next paragraph. \square

This gives a counter-example to a conjecture of Jacquet (conjecture 1 in [A]), asserting that if an irreducible admissible representation π of $G_n(K)$ verifies that $\tilde{\pi}$ is isomorphic to π^σ , then it is distinguished if n is odd, and it is distinguished or $\eta_{K/F}$ -distinguished if n is even.

Corollary 3.1. *For $n \geq 3$, there exist smooth irreducible representations π of $G_n(K)$, with central character trivial on F^* , that are neither distinguished, nor $\eta_{K/F}$ -distinguished, but verify that $\tilde{\pi}$ is isomorphic to π^σ .*

Proof. Take χ_1, \dots, χ_n , all different, such that $\chi_1|_{F^*} = \chi_2|_{F^*} = \eta_{K/F}$, and $\chi_j|_{F^*} = 1$ for $3 \leq j \leq n$. Because each χ_i has trivial restriction to $N_{K/F}(K^*)$, it is equal to $\chi_i^{-\sigma}$, hence $\tilde{\pi}$ is isomorphic to π^σ . Another consequence is that if k and l are two different integers between 1 and n , then $\chi_k \neq \chi_l^{-\sigma}$, because we supposed the χ_i 's all different.

Then it follows theorem 3.2 that $\pi = \pi(\chi_1, \dots, \chi_n)$ is neither distinguished, nor $\eta_{K/F}$ -distinguished, but clearly, the central character of π is trivial on F^* and $\tilde{\pi}$ is isomorphic to π^σ . \square

4 Distinction and gamma factors for $GL(2)$

As said in the introduction, in this section we generalize to smooth infinite dimensional irreducible representations of $G_2(K)$ a criterion of Hakim (cf. [H], theorem 4.1) characterising smooth unitary irreducible representations of $G_2(K)$.

Let π be a smooth infinite dimensional irreducible representation of $G_2(K)$, it is known that it is generic (cf.[Z]) for example).

We note $M(K)$ the mirabolic subgroup of $G_2(K)$ of matrices of the form $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$ with a in K^* and x in K , and $M(F)$ its intersection with $G_2(F)$.

We note w the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let $K(\pi, \psi)$ be its Kirillov model corresponding to ψ ([J-L], th. 2.13), it contains the subspace $D(K^*)$ of functions with compact support on the group K^* .

If ϕ belongs to $K(\pi, \psi)$, and x belongs to K , then $\phi - \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \phi$ belongs to $D(K^*)$ ([J-L], prop.2.9, ch.1), from this follows that $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$.

We now recall a consequence of the functional equation at $1/2$ for Kirillov representations (cf. [B]).

For all ϕ in $K(\pi, \psi)$ and χ character of K^* , we have whenever both sides converge absolutely:

$$\int_{K^*} \pi(w)\phi(x)(c_\pi\chi)^{-1}(x)d^*x = \gamma(\pi \otimes \chi, \psi) \int_{K^*} \phi(x)\chi(x)d^*x \quad (1)$$

where d^*x is a Haar measure on K^* , and c_π is the central character of π .

Theorem 4.1. *Let π be a smooth irreducible representation of infinite dimension with central character trivial on F^* of $G_2(K)$, and ψ a nontrivial character of K trivial on F . If $\gamma(\pi \otimes \chi, \psi) = 1$ for every character χ of K^* trivial on F^* , then π is distinguished.*

Proof. In fact, using a Fourier inversion in functional equation (1) and the change of variable $x \mapsto x^{-1}$, we deduce that for all ϕ in $D(K^*) \cap \pi(w)D(K^*)$, we have

$$c_\pi(x) \int_{F^*} \pi(w)\phi(tx^{-1})d^*t = \int_{F^*} \phi(tx)d^*t$$

(d^*t is a Haar measure on F^*) which for $x = 1$ gives

$$\int_{F^*} \pi(w)\phi(t)d^*t = \int_{F^*} \phi(t)d^*t.$$

Now we define on $K(\pi, \psi)$ a linear form λ by:

$$\lambda(\phi_1 + \pi(w)\phi_2) = \int_{F^*} \phi_1(t)d^*t + \int_{F^*} \phi_2(t)d^*t$$

for ϕ_1 and ϕ_2 in $D(K^*)$, which is well defined because of the previous equality and the fact that $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$.

It is clear that λ is w -invariant.

As the central character of π is trivial on F^* , it is also clear that λ is F^* -invariant.

Because $GL_2(F)$ is generated by $M(F)$, its center, and w , it remains to show that λ is $M(F)$ -invariant.

It is clear that if $\phi \in D(K^*)$, and $m \in M(F)$, $\lambda(\pi(m)\phi) = \lambda(\phi)$ (since ψ is trivial on F).

Now if $\phi = \pi(w)\phi_2 \in \pi(w)D(K^*)$.

If a belongs to F^* , then $\pi\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\pi(w)\phi_2 = \pi(w)\pi\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)\phi_2 = \pi(w)\pi\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\phi_2$ because the central character of π is trivial on F^* , and $\lambda(\pi\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\phi) = \lambda(\phi)$.

If $x \in F$, then $\pi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\phi - \phi$ is a function of $D(F^*)$, which vanishes on F^* , hence $\lambda\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\phi - \phi\right) = 0$.

Eventually λ is $M(F)$ -invariant, hence $G_2(F)$ -invariant, it is clear that its restriction to $D(K^*)$ is non zero. \square

Corollary 4.1. *Let μ be a character of K^* , then $\pi(\mu, \mu^{-\sigma})$ is distinguished.*

Proof. Indeed, first we notice that the central character $\mu\mu^{-\sigma}$ of $\pi(\mu, \mu^{-\sigma})$ is trivial on F^* .

Now let χ be a character of K^*/F^* , then $\gamma(\pi(\mu, \mu^{-\sigma}) \otimes \chi, \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-\sigma}\chi, \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-1}\chi^\sigma, \psi^\sigma)$, and as $\psi|_F = 1$ and $\chi|_{F^*} = 1$, one has $\psi^\sigma = \psi^{-1}$ and $\chi^\sigma = \chi^{-1}$, so that $\gamma(\pi(\mu, \mu^{-\sigma}), \psi) = \gamma(\mu\chi, \psi)\gamma(\mu^{-1}\chi^{-1}, \psi^{-1}) = 1$. The conclusion falls from proposition 4.1. \square

The converse of theorem 4.1 is also true:

Theorem 4.2. *Let π be a smooth irreducible distinguished representation of infinite dimension of $G_2(K)$ and ψ a non trivial character of K/F , then $\gamma(\pi, \psi) = 1$.*

Proof. Suppose λ is a non zero $G_2(F)$ -invariant linear form on $K(\pi, \psi)$, it is shown in the proof of the corollary of proposition 3.3 in [H], that its restriction to $D(K^*)$ must be a multiple of the Haar measure on F^* .

Hence for any function in $D(K^*) \cap \pi(w)D(K^*)$, we must have $\int_{F^*} \phi(t)d^*t = \int_{F^*} \pi(w)\phi(t)d^*t$.

From this one deduces that for any function in $D(K^*) \cap \pi(w)D(K^*)$:

$$\begin{aligned} \int_{K^*} \pi(w)\phi(x)c_\pi^{-1}(x)d^*x &= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi(w)\phi(ta)d^*tda \\ &= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\pi(w)\phi(t)d^*tda \\ &= \int_{K^*/F^*} c_\pi^{-1}(a) \int_{F^*} \pi(w)c_\pi(a)\pi\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\phi(t)d^*tda \end{aligned}$$

This last quantity is equal to $\int_{K^*/F^*} \int_{F^*} \pi\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\phi(t)d^*tda$

$$= \int_{K^*/F^*} \int_{F^*} \phi(ta^{-1})d^*tda = \int_{K^*/F^*} \int_{F^*} \phi(ta)d^*tda = \int_{K^*} \phi(x)d^*x.$$

From this we deduce that either $\gamma(\pi, \psi)$ is equal to one, either $\int_{K^*} \phi(x)d^*x$ is equal to zero on

$D(K^*) \cap \pi(w)D(K^*)$.

In the second case, we could define two independent K^* -invariant linear forms on $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$, given by $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*} \phi_1(x)d^*x$, and $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*} \phi_2(x)d^*x$. This would contradict theorem 1.2 of [A-G]. \square

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