Distinguished principal series representations for $GL_n$ over a $p$–adic field

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1 Introduction

For $K/F$ a quadratic extension of $p$-adic fields, let $\sigma$ be the conjugation relative to this extension, and $\eta_{K/F}$ be the character of $F^*$ with kernel norms of $K^*$.

If $\pi$ is a smooth irreducible representation of $GL(n, K)$, and $\chi$ a character of $F^*$, the dimension of the space of linear forms on its space, which transform by $\chi$ under $GL(n, F)$ (with respect to the action $[(L, g) \mapsto L \circ \pi(g)]$), is known to be at most one (Proposition 11, [F]). One says that $\pi$ is $\chi$-distinguished if this dimension is one, and says that $\pi$ is distinguished if it is 1-distinguished.

In this article, we give a description of distinguished principal series representations of $GL(n, K)$.

The result (Theorem 3.2) is that the irreducible distinguished representations of the principal series of $GL(n, K)$ are (up to isomorphism) those unitarily induced from a character $\chi = (\chi_1, ..., \chi_n)$ of the maximal torus of diagonal matrices, such that there exists $r \leq n/2$, for which $\chi_{i+1}^2 = \chi_i^{-1}$ for $i = 1, 3, ..., 2r - 1$, and $\chi_{i \geq 2r} = 1$ for $i > 2r$. For the quadratic extension $\mathbb{C}/\mathbb{R}$, it is known (cf. [F]) that the analogous result is true for tempered representations.

For $n \geq 3$, this gives a counter-example (Corollary [11] to a conjecture of Jacquet (Conjecture 1 in [A]). This conjecture states that an irreducible representation $\pi$ of $GL(n, K)$ with central character trivial on $F^*$ is isomorphic to $\tilde{\pi}^\sigma$ if and only if it is distinguished or $\eta_{K/F}$-distinguished (where $\eta_{K/F}$ is the character of order 2 of $F^*$, attached by local class field theory to the extension $K/F$). For discrete series representations, the conjecture is verified, it was proved in [K].

Unitary irreducible distinguished principal series representations of $GL(2, K)$ were described in [H], and the general case of distinguished irreducible principal series representations of $GL(2, K)$ was treated in [F-H]. We use this occasion to give a different proof of the result for $GL(2, K)$ than the one in [F-H]. To do this, in Theorems 4.1 and 4.2, we extend a criterion of Hakim (th.4.1, [H]) characterising smooth unitary irreducible distinguished representations of $GL(2, K)$ in terms of $\gamma$ factors at $1/2$, to all smooth irreducible distinguished representations of $GL(2, K)$.

2 Preliminaries

Let $\phi$ be a group automorphism, and $x$ an element of the group, we sometimes note $x^\phi$ instead of $\phi(x)$, and $x^{-\phi}$ the inverse of $x^\phi$. If $\phi = x \mapsto h^{-1}xh$ for $h$ in the group, then $x^h$ designs $x^\phi$. 
Let $G$ be a locally compact totally disconnected group, $H$ a closed subgroup of $G$.

We note $\Delta_G$ the module of $G$, given by the relation $d_G(gx) = \Delta_G(g)d_G(x)$, for a right Haar measure $d_G$ on $G$.

Let $X$ be a locally closed subspace of $G$, with $H.X \subset X$. If $V$ is a complex vector space, we note $D(X,V)$ the space of smooth $V$-valued functions on $X$ with compact support (if $V = \mathbb{C}$, we simply note it $D(X)$).

Let $\rho$ be a smooth representation of $H$ in a complex vector space $V$, we note $D(H \setminus G, \Delta_G/\Delta_H \rho, V \rho)$ the space of smooth $V\rho$-valued functions $f$ on $X$, with compact support modulo $H$, which verify $f(hx) = \rho(h)f(x)$ for $h \in H$ and $x \in X$ (if $\rho$ is a character, we note it $D(H \setminus G, \rho)$).

We note $\text{ind}_{G}^{H}(\rho)$ the representation by right translations of $G$ in $D(H \setminus G, (\Delta_G/\Delta_H)^{1/2} \rho, V \rho)$.

Let $F$ be a non archimedean local field of characteristic zero, and $K$ a quadratic extension of $F$. We have $K = F(\delta)$ with $\delta^2 \in F^*$.

We note $| |_K$ and $| |_F$ the modules of $K$ and $F$ respectively.

We note $\sigma$ the non trivial element of the Galois group $G(K/F)$ of $K$ over $F$, and we use the same letter to design for its action on $M_n(K)$.

We note $N_{K/F}$ the norm of the extension $K/F$ and we note $\eta_{K/F}$ the nontrivial character of $F^*$ which is trivial on $N_{K/F}(K^*)$.

Whenever $G$ is an algebraic group defined over $F$, we note $G(K)$ its $K$-points and $G(F)$ its $F$-points.

The group $GL(n)$ will be noted $G_n$, its standard Borel subgroup will be noted $B_n$, its unipotent radical $U_n$, and the standard maximal split torus of diagonal matrices $T_n$.

We note $S$ the space of matrices $M$ in $G_n(K)$ satisfying $MM^\sigma = 1$.

Everything in this paragraph is more or less contained in [F1], we give detailed proofs here for convenience of the reader.

**Proposition 2.1.** ([S], ch.10, prop.3) We have a homeomorphism between $G_n(K)/G_n(F)$ and $S$ given by the map $S_n : g \mapsto g\sigma g^{-1}$.

**Proposition 2.2.** For its natural action on $S$, each orbit of $B_n(K)$ contains one and only one element of $S_n$ of order 2 or 1.

**Proof.** We begin with the following:

**Lemma 2.1.** Let $w$ be an element of $S_n \subset G_n(K)$ of order at most 2.

Let $\theta'$ be the involution of $T_n(K)$ given by $t \mapsto t^{-1}t^\sigma w$, then any $t \in T_n(K)$ with $t\theta'(t) = 1$ is of the form $a/\theta'(a)$ for some $a \in T_n(K)$.

**Proof of Lemma 2.1.** There exists $r \leq n/2$ such that up to conjugacy, $w$ is $(1,2)(3,4)...(2r-1,2r)$. 

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Let $s$ be in $S$. According to Bruhat’s decomposition, there is $w$ in $\mathfrak{S}_n$, and $a$ in $T_n(K)$, $n_1$ in $U_n(K)$ and $n_2^+$ in $U_w$, such that $s = n_1 w n_2^+$, with unicity of the decomposition.
Corollary 2.1. The elements \( t \) in \( \text{classes of } B \).

The conclusion follows the preceding Lemma.

From the proof of Proposition 2.2, we know that if \( (aw)^{-1}n^\sigma aw = n^{-1} \), applying \( \sigma \) on each side, this becomes \( (aw)^{-1}n^\sigma aw = n^{-1} \).

But \( \theta : u \mapsto (aw)^{-1}u^\sigma aw \) is an involutive automorphism of \( U_w' \), hence from Lemma 2.2 there is \( u' \) in \( U_w' \) such that \( n = \theta(u^{-1})u \).

This gives \( s = u^{-\sigma}awu \), so that we suppose \( s = aw \). Again \( wa^\sigma w = a^{-1} \), and applying Lemma 2.1 to \( \theta' : x \mapsto wx^\sigma w \), we deduce that \( a \) is of the form \( y\theta'(y^{-1}) \), and \( s = ywy^{-\sigma} \).

Let \( u \) be the element \( \begin{pmatrix} 1 & -\delta \\ 1 & \delta \end{pmatrix} \) of \( M_2(K) \); one has \( S_2(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) (cf. Proposition 2.1).

We notice for further use (cf. proof of Proposition 3.1), that if we note \( \tilde{T} \) the subgroup \( \left\{ \begin{pmatrix} z & 0 \\ 0 & z^\sigma \end{pmatrix} \in G_2(K) \mid z \in K^* \right\} \), then \( u^{-1}\tilde{T}u = T = \left\{ \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix} \in G_2(F) \mid x, y \in F \right\} \).

For \( r \leq n/2 \), one notes \( U_r \) the \( n \times n \) matrix given by the following block decomposition:

\[
\begin{pmatrix}
u & \cdots & \cdots \\
\vdots & \ddots & \\
\vdots & & u_{I_n-2r} \\
\end{pmatrix}
\]

If \( w \) is an element of \( \mathfrak{S}_n \) naturally injected in \( G_n(K) \), one notes \( U_r^w = w^{-1}U_r w \).

**Corollary 2.1.** The elements \( U_r^w \) for \( 0 \leq r \leq n/2 \), and \( w \in \mathfrak{S}_n \) give a complete set of representatives of classes of \( B_n(K) \backslash G_n(K)/G_n(F) \).

Let \( G_n = \coprod_{w \in \mathfrak{S}_n} B_n w B_n \) be the Bruhat decomposition of \( G_n \). We call a double-class \( BwB \) a Bruhat cell.

**Lemma 2.3.** One can order the Bruhat cells \( C_1, C_2, \ldots, C_n! \) so that for every \( 1 \leq i \leq n! \), the cell \( C_i \) is closed in \( G_n - \coprod_{k=1}^{i-1} C_k \).

**Proof.** Choose \( C_1 = B_n \). It is closed in \( G_n \). Now let \( w_2 \) be an element of \( \mathfrak{S}_n - Id \), with minimal length. Then from 8.5.5. of [17], one has that the Bruhat cell \( Bw_2B \) is closed in \( G_n - B_n \) with respect to the Zariski topology, hence for the p-adic topology, we call it \( C_2 \). We conclude by repeating this process.

**Corollary 2.2.** One can order the classes \( A_1, A_2, \ldots, A_i \) of \( B_n(K) \backslash G_n(K)/G_n(F) \), so that \( A_i \) is closed in \( G_n(K) - \coprod_{k=1}^{i-1} A_i \).

**Proof.** From the proof of Proposition 2.2 we know that if \( C \) is a Bruhat cell of \( G_n \), then \( S_n \cap C \) is either empty, or it corresponds through the homomorphism \( S_n \) to a class \( A \) of \( B_n(K) \backslash G_n(K)/G_n(F) \).

The conclusion follows the preceeding Lemma.

**Corollary 2.3.** Each \( A_i \) is locally closed in \( G_n(K) \) for the Zariski topology.
We will also need the following Lemma:

**Lemma 2.4.** Let $G$, $H, X$, and $(\rho, V_\rho)$ be as in the beginning of the section, the map $\Phi$ from $D(X) \otimes V_\rho$ to $D(H \setminus X, \rho, V_\rho)$ defined by $\Phi : f \otimes v \mapsto (x \mapsto \int_H f(hx) \rho(h^{-1})vdh)$ is surjective.

**Proof.** Let $v \in V_\rho$, $U$ an open subset of $G$ that intersects $X$, small enough for $h \mapsto \rho(h)v$ to be trivial on $H \cap UU^{-1}$.

Let $f'$ be the function with support in $H(X \cap U)$ defined by $hx \mapsto \rho(h)v$.

Such functions generate $D(H \setminus X, \rho, V_\rho)$ as a vector space.

Now let $f$ be the function of $D(X, V_\rho)$ defined by $x \mapsto 1_{U \cap X}(x)v$, then $\Phi(f)$ is a multiple of $f'$.

But for $x$ in $U \cap X$, $\Phi(f)(x) = \int_H \rho(h^{-1})f(hx)dh$ because $h \mapsto \rho(h)v$ is trivial on $H \cap UU^{-1}$, plus $h \mapsto f(hx)$ is a positive function that multiplies $v$, and $f(x) = V$, so $F(f)(x)$ is $v$ multiplied by a strictly positive scalar. \hfill \Box

**Corollary 2.4.** Let $Y$ be a closed subset of $X$, $H$-stable, then the restriction map from $D(H \setminus X, \rho, V_\rho)$ to $D(H \setminus Y, \rho, V_\rho)$ is surjective.

**Proof.** This is a consequence of the known surjectivity of the restriction map from $D(X)$ to $D(Y)$, which implies the surjectivity of the restriction from $D(X, V_\rho)$ to $D(Y, V_\rho)$ and of the commutativity of the diagram:

$$
\begin{array}{ccc}
D(X) & \rightarrow & D(Y) \\
\downarrow \Phi & & \downarrow \Phi \\
D(H \setminus X, \rho) & \rightarrow & D(H \setminus Y, \rho)
\end{array}
$$

\hfill \Box

3 Distinguished principal series

If $\pi$ is a smooth representation of $G_n(K)$ of space $V_\pi$, and $\chi$ is a character of $F^*$, we say that $\pi$ is $\chi$-distinguished if there exists on $V_\pi$ a nonzero linear form $L$ such that $L(\pi(g)v) = \chi(\text{det}(g))L(v)$ whenever $g$ is in $G_n(F)$ and $v$ belongs to $V_\pi$. If $\chi$ is trivial, we simply say that $\pi$ is distinguished.

We first recall the following:

**Theorem 3.1.** (\cite{I}, Proposition 12)

Let $\pi$ be a smooth irreducible distinguished representation of $G_n(K)$, then $\pi^{\sigma} \simeq \pi$.

Let $\chi_1, \ldots, \chi_n$ be $n$ characters of $K^*$, with none of their quotients equal to $| \cdot |_K$. We note $\chi$ the character of $B_n(K)$ defined by $\chi \left( \begin{array}{ccc}
b_1 & \ast & \ast \\
\ast & \ddots & \ast \\
b_n & & \ast
\end{array} \right) = \chi_1(b_1) \ldots \chi_n(b_n)$.

We note $\pi(\chi)$ the representation of $G_n(K)$ by right translation on the space of functions $D(B_n(K) \setminus G_n(K), \Delta B_n^{-1/2} \chi)$. This representation is smooth, irreducible and called the principal series attached to $\chi$.

If $\pi$ is a smooth representation of $G_n(K)$, we note $\bar{\pi}$ its smooth contragredient.

We will need the following Lemma:

**Lemma 3.1.** (Proposition 26 in \cite{F}) Let $\bar{m} = (m_1, \ldots, m_l)$ be a partition of a positive integer $m$, let $P_{\bar{m}}$ be the corresponding standard parabolic subgroup, and for each $1 \leq i \leq l$, let $\pi_i$ be a smooth...
distinguished representation of $G_{m_1}(K)$, then $\pi_1 \times \cdots \times \pi_t = \text{ind}_{F_{m_1}(K)}^{G_{m_1}(K)}(\Delta_{m_1}^{-1/2}(\pi_1 \otimes \cdots \otimes \pi_t))$ is distinguished.

We now come to the principal results:

**Proposition 3.1.** Let $\chi = (\chi_1, \ldots, \chi_n)$ be a character of $T_n(K)$, suppose that the principal series representation $\pi(\chi)$ is distinguished, there exists a re-ordering of the $\chi_i$'s, and $r \leq n/2$, such that $\chi_{i+1} = \chi_i^{-1}$ for $i = 1, 3, \ldots, 2r - 1$, and that $\chi_i|_{\mathbb{F}_r} = 1$ for $i > 2r$.

*Proof.* We write $B = B_n(K), G = G_n(K)$. We have from Corollary 2.2 and 3.1, the following exact sequence of smooth $G_n(F)$-modules:

$$D(B \setminus G - A_1, \Delta_B^{-1/2}) \hookrightarrow D(B \setminus G, \Delta_B^{-1/2}) \rightarrow D(B \setminus A_1, \Delta_B^{-1/2}).$$

Hence there is a non zero distinguished linear form either on $D(B \setminus A_1, \Delta_B^{-1/2})$, or on $D(B \setminus G - A_1, \Delta_B^{-1/2})$.

In the second case we have the exact sequence

$$D(B \setminus G - A_1 \sqcup A_2, \Delta_B^{-1/2}) \hookrightarrow D(B \setminus G - A_1, \Delta_B^{-1/2}) \rightarrow D(B \setminus A_2, \Delta_B^{-1/2}).$$

Repeating the process, we deduce the existence of a non zero distinguished linear form on one of the spaces $D(B \setminus A_1, \Delta_B^{-1/2})$.

From Corollary 2.2 we choose $\omega$ in $S_n$ and $r \leq n/2$ such that $A_i = BU_r^wG_n(F)$. The application $f \mapsto [x \mapsto f(U_r^w x)]$ gives an isomorphism of $G_n(F)$-modules between $D(B \setminus A_1, \Delta_B^{-1/2})$ and $D(U_r^{-w}B^{-w}U_r^w \cap G_n(F) \setminus G_n(F), \Delta'_\chi')$ where $\Delta'(x) = \Delta_B^{-1/2}(U_r^w x U_r^{-w})$ and $\chi'(x) = \chi(U_r^w x U_r^{-w})$.

Now there exists a nonzero $G_n(F)$-invariant linear form on $D(U_r^{-w}B^{-w}U_r^w \cap G_n(F) \setminus G_n(F), \Delta'_\chi')$ if and only if $\Delta'\chi'$ is equal to the inverse of the module of $U_r^{-w}B^{-w}U_r^w \cap G_n(F)$ (cf. [B-H], ch.1, prop.3.4). From this we deduce that $\chi'$ is positive on $U_r^{-w}B^{-w}U_r^w \cap G_n(F)$ or equivalently $\chi$ is positive on $B \cap U_r^{-w}G_n(F)U_r^{-w}$.

Let $\widehat{T}_r$ be the $F$-torus of matrices of the form

$$\begin{pmatrix}
  z_1 & & & \\
  & \ddots & & \\
  & & z_r & \\
  & & & x_1 \\
  & & & & \ddots \\
  & & & & & x_t
\end{pmatrix},
$$

with $2r + t = n, z_i \in K^*, x_i \in F^*$, then one has $\widehat{T}_r^w \subset B \cap U_r^{-w}G_n(F)U_r^{-w}$, so that $\chi$ must be positive on $\widehat{T}_r^w$.

We remark that if $\chi$ is unitary, then $\chi$ is trivial on $\widehat{T}_r^w$, and $\pi(\chi)$ is of the desired form.

For the general case, we deduce from Theorem 5.1 that there exists three integers $p \geq 0, q \geq 0, s \geq 0$ such that up to reordering, we have $\chi_{2i} = \chi_{2i+1}$ for $1 \leq i \leq p$, we have $\chi_{2p+k}|_{\mathbb{F}_r} = 1$ for $1 \leq k \leq q$.
and these \( \chi_{2p+k} \)'s are different (so that \( \chi_{2p+k} \neq \chi_{2p+k'} \) for \( k \neq k' \)), and \( \chi_{2p+q+j} = \eta_{K/F} \) for \( 1 \leq j \leq s \), these \( \chi_{2p+q+j} \)'s being different.

We note \( \mu_k = \chi_{2p+k} \) for \( q \geq k \geq 1 \), and \( \nu_i' = \chi_{2p+q+i} \) for \( s \geq k' \geq 1 \).

We show that if such a character \( \chi \) is positive on a conjugate of \( T_r \) by an element of \( S_n \), then \( s = 0 \).

Suppose \( \nu_1 \) appears, then either \( \nu_1 \) is positive on \( F^* \), but that is not possible, or it is coupled with another \( \chi_i \), and \( (\nu_1, \chi_i) \) is positive on elements \( (z, z^\sigma) \), for \( z \) in \( K^* \).

Suppose \( \chi_i = \nu_j \) for some \( j \neq 1 \), then \( (\nu_1, \chi_i) \) is unitary, so it must be trivial on couples \( (z, z^\sigma) \), which implies \( \nu_1 = \nu_j = \nu_j', \) which is absurd.

The character \( \chi_i \) cannot be of the form \( \mu_j \), because it would imply \( \nu_{1|F^*} = 1 \).

The last case is \( i \leq 2p \), then \( \nu_1 = \nu_1 \) must be the unitary part of \( \chi_i \) because of the positivity of \( (\nu_1, \chi_i) \) on the couples \( (z, z^\sigma) \).

But \( \chi_i^{-\sigma} \) also appears and is not trivial on \( F^* \), hence must be coupled with another character \( \chi_j \) with \( j \leq 2p \) and \( j \neq i \), such that \( (\chi_i^{-\sigma}, \chi_j) \) is positive on the elements \( (z, z^\sigma) \), for \( z \) in \( K^* \), which implies that \( \chi_j \) has unitary part \( \nu_1^{-\sigma} = \nu_1 \). The character \( \chi_j \) cannot be a \( \mu_k \) because of its unitary part.

If it is a \( \chi_k \) with \( k \leq 2p \), we consider again \( \chi_k^{-\sigma} \).

But repeating the process lengthily enough, we can suppose that \( \chi_j \) is of the form \( \nu_k \), for \( k \neq 1 \).

Taking unitary parts, we see that \( \nu_k = \nu_1^{-\sigma} = \nu_1 \), which is in contradiction with the fact that all \( \nu_i \)'s are different. We conclude that \( s = 0 \).

\[ \text{Theorem 3.2.} \]

Let \( \chi = (\chi_1, \ldots, \chi_n) \) be a character of \( T_n(K) \), the principal series representation \( \pi(\chi) \) is distinguished if and only if there exists \( r \leq n/2 \), such that \( \chi_i^{r+1} = \chi_i^{-1} \) for \( i = 1, 3, \ldots, 2r-1 \), and that \( \chi_{i|F^*} = 1 \) for \( i > 2r \).

\[ \text{Proof.} \] There is one implication left.

Suppose \( \chi \) is of the desired form, then \( \pi(\chi) \) is parabolically (unitarily) induced from representations of the type \( \pi(\chi_i, \chi_i^{-\sigma}) \) of \( G_2(K) \), and distinguished characters of \( K^* \).

Hence, because of Lemma 4.4, the Theorem will be proved if we know that the representations \( \pi(\chi_i, \chi_i^{-\sigma}) \) are distinguished, but this is Corollary 4.1 of the next paragraph.

This gives a counter-example to a conjecture of Jacquet (conjecture 1 in [A]), asserting that if an irreducible admissible representation \( \pi \) of \( G_n(K) \) verifies that \( \tilde{\pi} \) is isomorphic to \( \pi^\sigma \), then it is distinguished if \( n \) is odd, and it is distinguished or \( \eta_{K/F} \)-distinguished if \( n \) is even.

\[ \text{Corollary 3.1.} \]

For \( n \geq 3 \), there exist smooth irreducible representations \( \pi \) of \( G_n(K) \), with central character trivial on \( F^* \), that are neither distinguished, nor \( \eta_{K/F} \)-distinguished, but verify that \( \tilde{\pi} \) is isomorphic to \( \pi^\sigma \).

\[ \text{Proof.} \] Take \( \chi_1, \ldots, \chi_n \), all different, such that \( \chi_1|F^* = \chi_2|F^* = \eta_{K/F} \), and \( \chi_3|F^* = 1 \) for \( 3 \leq j \leq n \).

Because each \( \chi_i \) has trivial restriction to \( N_{K/F}(K^*) \), it is equal to \( \chi_i^{-\sigma} \), hence \( \tilde{\pi} \) is isomorphic to \( \pi^\sigma \).

Another consequence is that if \( k \) and \( l \) are two different integers between 1 and \( n \), then \( \chi_k \neq \chi_l^{-\sigma} \), because we supposed the \( \chi_i \)'s all different.

Then it follows from Theorem 3.2 that \( \pi = \pi(\chi_1, \ldots, \chi_n) \) is neither distinguished, nor \( \eta_{K/F} \)-distinguished, but clearly, the central character of \( \pi \) is trivial on \( F^* \) and \( \tilde{\pi} \) is isomorphic to \( \pi^\sigma \).
4 Distinction and gamma factors for $GL(2)$

As said in the introduction, in this section we generalize to smooth infinite dimensional irreducible representations of $G_2(K)$ a criterion of Hakim (cf. [H], Theorem 4.1) characterising smooth unitary irreducible distinguished representations of $G_2(K)$. In proof of Theorem 4.1 of [H], Hakim deals with unitary representations so that the integrals of Kirillov functions on $F^*$ with respect to a Haar measure of $F^*$ converge. We skip the convergence problems using Proposition 2.9 of chapter 1 of [J-L].

We note $M(K)$ the mirabolic subgroup of $G_2(K)$ of matrices of the form $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$ with $a$ in $K^*$ and $x$ in $K$, and $M(F)$ its intersection with $G_2(F)$. We note $w$ the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let $\pi$ be a smooth infinite dimensional irreducible representation of $G_2(K)$, it is known that it is generic (cf.[Z] for example). Let $K(\pi,\psi)$ be its Kirillov model corresponding to $\psi$ ([J-L], th. 2.13), it contains the subspace $D(K^*)$ of functions with compact support on the group $K^*$. If $\phi$ belongs to $K(\pi,\psi)$, and $x$ belongs to $K$, then $\phi - \pi \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \phi$ belongs to $D(K^*)$ ([J-L], prop.2.9, ch.1), from this follows that $K(\pi,\psi) = D(K^*) + \pi(w)D(K^*)$.

We now recall a consequence of the functional equation at $1/2$ for Kirillov representations (cf. [B], section 4.7).

For all $\phi$ in $K(\pi,\psi)$ and $\chi$ character of $K^*$, we have whenever both sides converge absolutely:

$$\int_{K^*} \pi(w)\phi(x)(c_\pi \chi)^{-1}(x)d^*x = \gamma(\pi \otimes \chi, \psi) \int_{K^*} \phi(x)\chi(x)d^*x$$

(1)

where $d^*x$ is a Haar measure on $K^*$, and $c_\pi$ is the central character of $\pi$.

**Theorem 4.1.** Let $\pi$ be a smooth irreducible representation of $G_2(K)$ of infinite dimension with central character trivial on $F^*$, and $\psi$ a nontrivial character of $K$ trivial on $F$. If $\gamma(\pi \otimes \chi, \psi) = 1$ for every character $\chi$ of $K^*$ trivial on $F^*$, then $\pi$ is distinguished.

**Proof.** In fact, using a Fourier inversion in functional equation [H] and the change of variable $x \mapsto x^{-1}$, we deduce that for all $\phi$ in $D(K^*) \cap \pi(w)D(K^*)$, we have

$$c_\pi(x) \int_{F^*} \pi(w)\phi(tx^{-1})d^*t = \int_{F^*} \phi(t)x d^*t$$

($d^*t$ is a Haar measure on $F^*$) which for $x = 1$ gives

$$\int_{F^*} \pi(w)\phi(t)d^*t = \int_{F^*} \phi(t)d^*t.$$

Now we define on $K(\pi,\psi)$ a linear form $\lambda$ by:

$$\lambda(\phi_1 + \pi(w)\phi_2) = \int_{F^*} \phi_1(t)d^*t + \int_{F^*} \phi_2(t)d^*t$$
for $\phi_1$ and $\phi_2$ in $D(K^*)$, which is well defined because of the previous equality and the fact that $K(\pi, \psi) = D(K^*) + \pi(w)D(K^*)$.

It is clear that $\lambda$ is $w$-invariant. As the central character of $\pi$ is trivial on $F^*$, $\lambda$ is also $F^*$-invariant. Because $GL_2(F)$ is generated by $M(F)$, its center, and $w$, it remains to show that $\lambda$ is $M(F)$-invariant.

Since $\psi$ is trivial on $F$, one has if $\phi \in D(K^*)$ and $m \in M(F)$ the equality $\lambda(\pi(m)\phi) = \lambda(\phi)$.

Now if $\phi = \pi(w)\phi_2 \in \pi(w)D(K^*)$, and if $a$ belongs to $F^*$, then $\pi\left(\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right) \pi(w)\phi_2 = \pi(w)\pi\left(\begin{array}{cc} 1 & 0 \\ 0 & a \end{array}\right) \phi_2 = \pi(w)\pi\left(\begin{array}{cc} a^{-1} & 0 \\ 0 & 1 \end{array}\right) \phi_2$ because the central character of $\pi$ is trivial on $F^*$, and $\lambda(\pi\left(\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right) \phi) = \lambda(\phi)$.

If $x \in F$, then $\pi\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) \phi - \phi$ is a function in $D(K^*)$, which vanishes on $F^*$, hence $\lambda\pi(\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) \phi - \phi) = 0$.

Eventually $\lambda$ is $M(F)$-invariant, hence $G_2(F)$-invariant, it is clear that its restriction to $D(K^*)$ is non zero.

**Corollary 4.1.** Let $\mu$ be a character of $K^*$, then $\pi(\mu, \mu^{-\sigma})$ is distinguished.

**Proof.** Indeed, first we notice that the central character $\mu\mu^{-\sigma}$ of $\pi(\mu, \mu^{-\sigma})$ is trivial on $F^*$.

Now let $\chi$ be a character of $K^*/F^*$, then $\gamma(\pi(\mu, \mu^{-\sigma}) \otimes \chi, \psi) = \gamma(\mu \chi, \psi) \gamma(\mu^{-\sigma} \chi, \psi) = \gamma(\mu \chi, \psi) \gamma(\mu^{-1} \chi^{-\sigma}, \psi^{-1})$, and as $\psi|F = 1$ and $\chi|F^* = 1$, one has $\psi^{-1} = \psi^{-1}$ and $\chi^{-\sigma} = \chi^{-1}$, so that $\gamma(\pi(\chi, \chi^{-\sigma}), \psi) = \gamma(\mu \chi, \psi) \gamma(\mu^{-1} \chi^{-1}, \psi^{-1}) = 1$. The conclusion falls from Proposition 4.1.

Assuming Theorem 1.2 of [A-G], the converse of Theorem 4.1 is also true:

**Theorem 4.2.** Let $\pi$ be a smooth irreducible representation of infinite dimension of $G_2(K)$ with central character trivial on $F^*$ and $\psi$ a non trivial character of $K/F$, it is distinguished if and only if $\gamma(\pi \otimes \chi, \psi) = 1$ for every character $\chi$ of $K^*$ trivial on $F^*$.

**Proof.** It suffices to show that if $\pi$ is a smooth irreducible distinguished representation of infinite dimension of $G_2(K)$, and $\psi$ a non trivial character of $K/F$, then $\gamma(\pi, \psi) = 1$. Suppose $\lambda$ is a non zero $G_2(F)$-invariant linear form on $K(\pi, \psi)$, it is shown in the proof of the corollary of Proposition 3.3 in [H], that its restriction to $D(F^*)$ must be a multiple of the Haar measure on $F^*$. Hence for any function $\phi$ in $D(K^*) \cap \pi(w)D(K^*)$, we must have $\int_{F^*} \phi(t)d^*t = \int_{F^*} \pi(w)\phi(t)d^*t$. 

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From this one deduces that for any function in $D(K^*) \cap \pi(w) D(K^*)$:

$$
\int_{K^*} \pi(w) \phi(x) e^{-1}_\pi(x) d^*x = \int_{K^*/F^*} c^{-1}_\pi(a) \int_{F^*} \pi(w) \phi(ta) d^* tda
$$

$$
= \int_{K^*/F^*} c^{-1}_\pi(a) \int_{F^*} \pi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) \pi(w) \phi(t) d^* tda
$$

$$
= \int_{K^*/F^*} c^{-1}_\pi(a) \int_{F^*} \pi(w) c_\pi(a) \pi(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}) \phi(t) d^* tda
$$

$$
= \int_{K^*/F^*} \int_{F^*} \pi(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}) \phi(t) d^* tda
$$

$$
= \int_{K^*/F^*} \int_{F^*} \phi(ta^{-1}) d^* tda
$$

$$
= \int_{K^*/F^*} \int_{F^*} \phi(ta) d^* tda
$$

$$
= \int_{K^*} \phi(x) d^* x
$$

This implies that either $\gamma(\pi, \psi)$ is equal to one, or $\int_{K^*} \phi(x) d^* x$ is equal to zero on $D(K^*) \cap \pi(w) D(K^*)$. In the second case, we could define two independent $K^*$-invariant linear forms on $K(\pi, \psi) = D(K^*) + \pi(w) D(K^*)$, given by $\phi_1 + \pi(w) \phi_2 \mapsto \int_{K^*} \phi_1(x) d^* x$, and $\phi_1 + \pi(w) \phi_2 \mapsto \int_{K^*} \phi_2(x) d^* x$. This would contradict Theorem 1.2 of [A-G].

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References


