HAL open science

The ground state energy of the mean field spin glass model

Flora Koukiou

▶ To cite this version:

Flora Koukiou. The ground state energy of the mean field spin glass model. 2008. hal-00286116v2

HAL Id: hal-00286116
https://hal.science/hal-00286116v2

Preprint submitted on 4 Jul 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The ground state energy of the mean field spin glass model

F. Koukiou*
Laboratoire de physique théorique et modélisation †
Université de Cergy-Pontoise F-95302 Cergy-Pontoise
(Dated: July 4, 2010)

We establish a functional equation relating the Gibbs measure at a particular low temperature with the one at temperature 1. This equation enables us to calculate the limiting free energy of the Sherrington-Kirkpatrick spin glass model at this particular value of low temperature without making use of the replica method. We get additionally a sharper lower bound for the ground state energy ϵ_0

$$\epsilon_0 \geq -0.7833\cdots,$$

close to the replica symmetry breaking and numerical simulations values.

† UMR 8089, CNRS

INTRODUCTION AND MAIN RESULT

During the last decade, mean field models of spin glasses have motivated increansingly many studies by physicists and mathematicians [1, 4–6, 8, 9, 11]. The rigorous understanding of the infinite volume limit of thermodynamic quantities remained quite insufficient until the recent breakthrough obtained by Guerra and Toninelli [5] on their existence and uniqueness. This major discovery followed by several important results [2, 4] providing a mathematical interpretation of the original formulae proposed by Parisi [8] on the basis of heuristic arguments.

In this note, without making use of the replica approach, we calculate, for a particular value of the (low) temperature, the limiting free energy of the Sherrington-Kirkpatrick model and obtain a lower bound for the density of the ground state energy. Although the limiting free energy is given, for the whole low temperature region, by the rather complicated Parisi formula, we obtain, for a given temperature, a very simple expression. This allows an improvement of all known rigorous bounds for the ground state energy.

We first recall some basic definitions. Suppose that a finite set of n sites is given. Let $\sigma_i \in \{1, -1\}$ be the spin variable on the site i and σ a generic configuration in the configuration space $\Sigma_n = \{-1, 1\}^n$. The finite volume Hamiltonian of the model is given by the following real-valued function on Σ_n

$$H_n(\sigma) = -\frac{1}{\sqrt{n}} \sum_{1 \le i < j \le n} J_{ij}\sigma_i\sigma_j,$$

where the family of couplings J_{ij} are independed centered Gaussian random variables of variance 1. Note however, that the function H_n can equivalently be defined directly on the infinite configuration space $\Sigma_\infty = \{-1, 1\}^{\mathbb{N}}$ and the infinite collection of random variables $J = (J_{ij})_{i,j\in\mathbb{N}}$ just by the trivial modification in the definition of H_n:

$$H_n(\sigma) = -\frac{1}{\sqrt{n}} \sum_{1 \le i < j \le n} J_{ij}\sigma_i\sigma_j \mathbb{1}_{\Sigma_n}(\sigma).$$

This remark will be instrumental in the course of the proof since only changes the sequence of functions $(H_n)_{n\in\mathbb{N}}$ and not the configuration and environment spaces. Henceforth, the Hamiltonian will be defined on these infinite spaces.

For the inverse temperature $\beta = \frac{1}{T} > 0$, the disorder dependent partition function $Z_n(\beta)$, is defined by

$$Z_n(\beta, J) = \sum_\sigma \exp(-\beta H_n(\sigma, J)).$$

Moreover, if E_J denotes the expectation with respect to the randomness J_{ij}, it is very simple to show that $E_J Z_n(\beta, J) = 2^n e^{\frac{\beta^2}{4}(n-1)}$.

We denote by $\mu_{n,\beta}(\sigma|J)$, the corresponding Gibbs probability measure, conditionned on fixed randomness:

$$\mu_{n,\beta}(\sigma|J) = e^{-\beta H_n(\sigma,J)}/Z_n(\beta, J),$$

and, by $S(\mu_{n,\beta}(\sigma|J))$, its entropy, defined by $S(\mu_{n,\beta}(\sigma|J)) = -\sum_\sigma \mu_{n,\beta}(\sigma|J) \log \mu_{n,\beta}(\sigma|J)$.

For fixed randomness, the real functions

$$f_n(\beta) = \frac{1}{n} E_J \log Z_n(\beta, J)$$

and

$$\bar{f}_n(\beta) = \frac{1}{n} \log E_J Z_n(\beta, J),$$

define the quenched average of the free energy per site and the annealed specific free energy respectively.

The ground state energy density $-\epsilon_n(J)$ is defined by

$$-\epsilon_n(J) = \frac{1}{n} \inf_{\sigma\in\Sigma_n} H_n(\sigma, J).$$

For the low temperature region ($\beta > 1$), the J-almost sure existence of the infinite volume limits

$$\lim_{n\to\infty} f_n(\beta) = f_\infty(\beta),$$

and,

$$-\lim_{n\to\infty}\epsilon_n(J)=\lim_{\beta\to\infty}\frac{f_\infty(\beta)}{\beta}=-\epsilon_0$$

was first proved by Guerra and Toninelli [5]. More recently, Aizenman, Sims and Starr [2] gave a clear mathematical interpretation of the limit $f_\infty(\beta)$ in terms of the variational formula proposed by Parisi.

In the following section we prove the

Theorem : *Let $\beta_*=4\log 2=2.77258\cdots$ Almost surely, the infinite volume limit $f_\infty(\beta_*)$ is given by*

$$f_\infty(\beta_*)=\lim_{n\to\infty}\frac{1}{n}E_J\log Z_n(\beta_*,J)=\frac{\beta_*^2}{4}+\frac{1}{4}.$$

A directly related result is the following corollary, which improves all the rigorous lower bounds for the ground state energy.

Corollary : *Almost surely, the ground state energy density of the Sherrington-Kirkpatrick spin glass model is bounded by*

$$\epsilon_0\geq=-0.7833\cdots.$$

PROOF OF THE MAIN RESULT

Notice first that for all $\beta>0$, the limit $f_\infty(\beta)$ exists and it is a convex function of β [5]. Let $\beta=\beta_1\equiv 1$. From the high temperature results [1], we have, almost surely, that

$$f_\infty(\beta_1)=\lim_{n\to\infty}\frac{1}{n}E_J\log Z_n(\beta_1,J)=\bar{f}_\infty(\beta_1)=\lim_{n\to\infty}\frac{1}{n}\log E_J Z_n(\beta_1,J)=\log 2+\frac{\beta_1^2}{4}=\log 2+\frac{1}{4}. \tag{1}$$

The following figure 1 illustrates the definition of the inverse temperature β_*; the annealed free energy $\bar{f}_\infty(\beta)=\log 2+\frac{\beta^2}{4}$ is plotted as a function of β and the straight line is defined by $\frac{\beta}{\beta_1}f_\infty(\beta_1)=\beta f_\infty(\beta_1)$.

One can remark that, for $\beta_*=2,77258\cdots=4\log 2$, the annealed free energy $\bar{f}_\infty(\beta_*)$ is simply related to $f_\infty(\beta_1)$ via the following relation

$$\bar{f}_\infty(\beta_*)=\frac{\beta_*^2}{4}+\log 2=\frac{\beta_*}{\beta_1}(\frac{\beta_*\beta_1}{4}+\frac{\beta_1}{\beta_*}\log 2)=\frac{\beta_*}{\beta_1}(\log 2+\frac{1}{4})=\frac{\beta_*}{\beta_1}f_\infty(\beta_1). \tag{2}$$

By making use of this remark we define the Gibbs probability measure $\mu_{n,\beta_*}(\sigma|J)$ by the functional equation

$$\mu_{n,\beta_*}(\sigma|J):=\frac{\exp(\beta_* H_n(\sigma,J))}{Z_n(\beta_*,J)}=\mu_{n,\beta_1}^{\frac{\beta_*}{\beta_1}}(\sigma|J)\frac{Z_n^{\frac{\beta_*}{\beta_1}}(\beta_1,J)}{Z_n(\beta_*,J)}, \tag{3}$$

induced by the mapping $T:\exp(\beta_1 H_n(\sigma,J))\mapsto(\exp(\beta_1 H_n(\sigma,J)))^{\beta_*/\beta_1}$ among Boltzmann factors. Since μ_{n,β_*} is a probability on the configuration space, summing over σ, we have indeed

$$\lim_{n\to\infty}\frac{1}{n}E_J\log\sum_\sigma\mu_{n,\beta_1}^{\frac{\beta_*}{\beta_1}}(\sigma|J)=-\alpha_\infty(\frac{\beta_*}{\beta_1}), \tag{4}$$

where the limit $\alpha_\infty(\frac{\beta_*}{\beta_1})$ gives the deviation of the free energy $f_\infty(\beta_*)$ from its mean value:

$$\begin{aligned}\alpha_\infty:=\alpha_\infty(\frac{\beta_*}{\beta_1})&=\lim_{n\to\infty}\frac{\beta_*}{\beta_1}\frac{1}{n}E_J\log Z_n(\beta_1,J)-\lim_{n\to\infty}\frac{1}{n}E_J\log Z_n(\beta_*,J)\\&=\frac{\beta_*}{\beta_1}f_\infty(\beta_1)-f_\infty(\beta_*)\\&=\bar{f}_\infty(\beta_*)-f_\infty(\beta_*).\end{aligned}$$

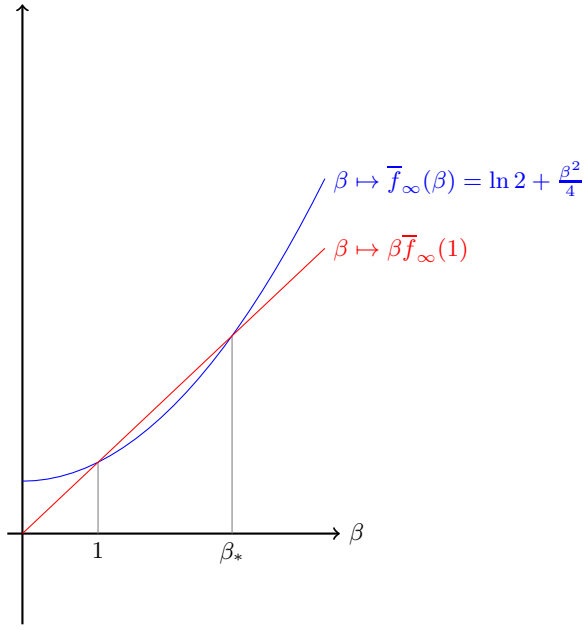

FIG. 1. The value $\beta_* = 2,77258\cdots$, is given by the intersection of the graph of the annealed free energy $\bar{f}_\infty(\beta)$ with the straight line $\beta f_\infty(1)$.

The reader can remark that the value β_1 fixes the temperature scale; the temperature β_* is expressed in units where $\beta_1 \equiv 1$. Therefore, $\frac{\beta_*}{\beta_1}$ is a non dimensional quantity.

We can now calculate the limit $f_\infty(\beta_*)$. The idea is the following. For $\eta > 0$, consider the set S of configurations σ where the Boltzmann factor is close to its mean value: $|\exp(-\beta H_n(\sigma, J)) - \exp(\frac{\beta^2}{4}(n-1)| \leq \eta$. For $\beta = 1$, the set S is of full measure and moreover, the Gibbs measure behaves, at the thermodynamic limit, as $\lim_{n\to\infty} \frac{1}{n} \log \mu_{n,\beta_1}(\sigma|J) = \frac{\beta_1^2}{4} - f_\infty(\beta_1) = -\log 2$.

Taking now the image under T, we have the following behaviour of the limit

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log \mu_{n,\beta_*}(\sigma|J) &= \frac{\beta_*\beta_1}{4} - f_\infty(\beta_*) \\ &= -\frac{\beta_*^2}{4} + \alpha_\infty, \end{aligned}$$

i.e. this image differs from the limit $-f_\infty(\beta_*)$ by $\frac{\beta_*\beta_1}{4} = \log 2$. Now, from the previous equation and the easily verified relation

$$\frac{\beta_*^2}{4} - (\alpha_\infty + \frac{\beta_1^2}{4})) = f_\infty(\beta_*) - f_\infty(\beta_1), \tag{5}$$

we can deduce the behaviour of the limit α_∞ by making use of the following argument. First, the difference of the limit $-\frac{\beta_*^2}{4} + \alpha_\infty$ from $-\log 2 + \alpha_\infty$ is given by $(\beta_* - 1)\log 2$. Moreover, the difference between the limiting values $-\frac{\beta_*^2}{4} + \alpha_\infty + \frac{\beta_1^2}{4}$ and $-\log 2 + \alpha_\infty + \frac{\beta_1^2}{4}$ is also $(\beta_* - 1)\log 2$.

Second, from equation (5), since the limit $-\frac{\beta_*^2}{4} + \alpha_\infty + \frac{\beta_1^2}{4}$ is equal to $-f_\infty(\beta_*) + f_\infty(\beta_1)$, it follows that $-\log 2 + \alpha_\infty + \frac{\beta_1^2}{4} = 0$. We have indeed,

$$\alpha_\infty + \frac{\beta_1^2}{4} = \log 2 = \frac{\beta_1\beta_*}{4}, \tag{6}$$

and

$$f_\infty(\beta_*) - f_\infty(\beta_1) = (\beta_* - 1)\log 2 \tag{7}$$

as illustrated in figure (2).

One can now check that the obtained value of the limit

$$f_\infty(\beta_*) = \beta_* \log 2 + \frac{\beta_1^2}{4} = \frac{\beta_*^2}{4} + \frac{\beta_1^2}{4} = 2.1718\cdots, \tag{8}$$

is lower than the spherical model bound $(2.2058\cdots)$.

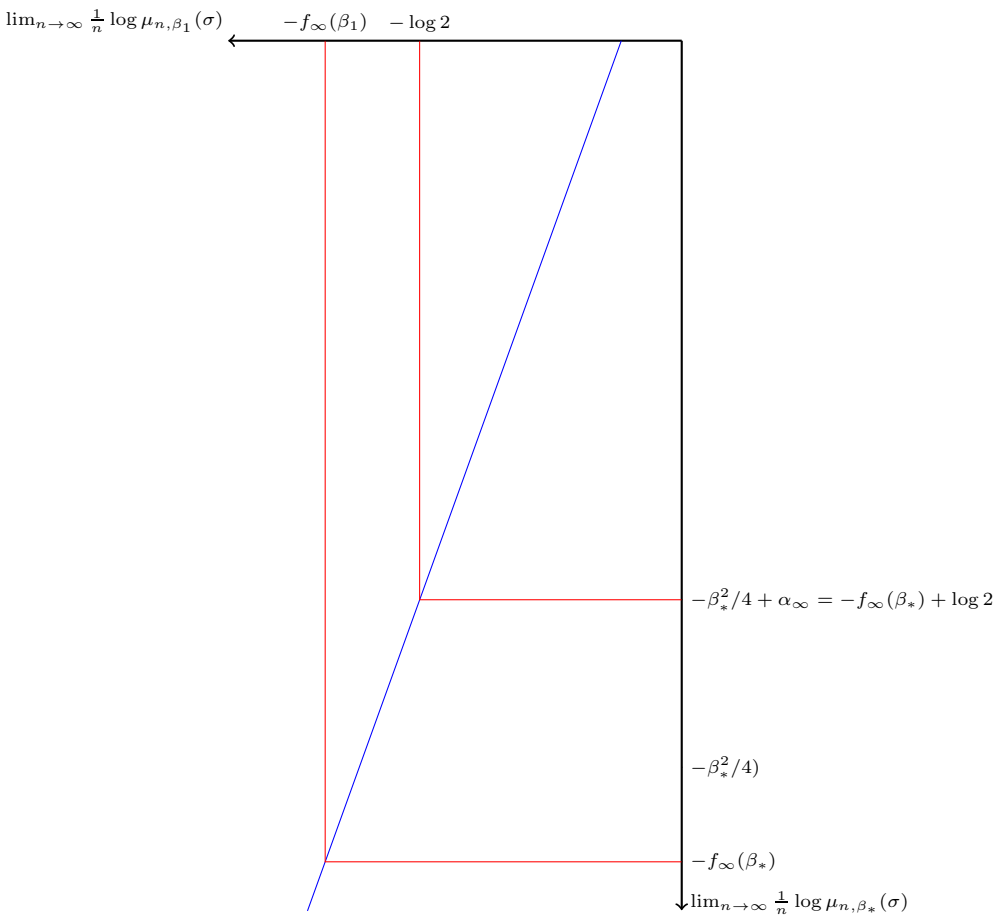

FIG. 2. The straight oblique line represents the effect of the mapping T on measures.

Remark 1: Our result can also be obtained using a large deviations approach, namely by comparing the Gibbs measures at the inverse temperatures 1 and β_* when the corresponding Boltzmann factors $\exp(-1H_n(\sigma, J))$ and $\exp(-\beta_* H_n(\sigma, J))$ behave as their mean values $\exp(\frac{n-1}{4})$ and $\exp(\frac{\beta_*^2}{4}(n-1))$ respectively. In this case, the two limits $\lim_{n\to\infty} \frac{1}{n} \log \mu_{n,\beta_*}(\sigma|J)$ and $\lim_{n\to\infty} \frac{1}{n} \log \mu_{n,\beta_1}(\sigma|J)$ differ by α_∞.

Remark 2: The point where these two limits are equal corresponds to the fixed point of the functional equation (3) and arises when $\lim_{n\to\infty} \frac{1}{n} \log \mu_{n,\beta_*}(\sigma|J) = -\log 2 + \alpha_\infty$.

We can now obtain a lower bound for the ground state energy density $-\epsilon_n(J)$. Notice that

$$f_\infty(\beta_*) = -\lim_{n\to\infty} \frac{\beta_*}{n} \sum_\sigma H_n(\sigma|J)\mu_{n,\beta_*}(\sigma|J) + s(\mu_{\beta_*}) = \frac{\beta_*^2}{4} + \frac{1}{4}, \tag{9}$$

where the limit

$$s(\mu_{\beta_*}) = \lim_{n\to\infty} \frac{1}{n} S(\mu_{n,\beta_*}(\sigma|J)), \tag{10}$$

gives the (mean) entropy of the Gibbs measure.

We have indeed, by the positivity of the entropy $s(\mu_{\beta_*}) \geq 0$, that

$$\epsilon_0 = -\lim_{\beta\to\infty} \frac{f_\infty(\beta)}{\beta} \geq -\frac{\beta_*}{4} - \frac{1}{4\beta_*} = -0.7833\cdots; \tag{11}$$

close to the value $-0.7633\cdots$ obtained by numerical simulations based on the replica approach.

CONCLUDING REMARKS

In this note we obtained, under the assumption of minimal entropy, a rigorous lower bound for the ground state energy density which improves all the previous estimations.

A last observation concerns the value of the temperature β_*: it is obtained from the relation (2) between the free energies $\bar{f}_\infty(\beta_*)$ and $f_\infty(1)$; moreover, one can readily check that β_* is given by $\beta_* = \beta_c^2$, where $\beta_c = 2\sqrt{\log 2}$ is the critical temperature of the Random- Energy Model (REM) [3]. The REM is defined by 2^n energy levels $E_i (i = 1, \cdots, n)$, a family of random, independent, identically distributed random variables; many results are qualitatively the same as those of the SK model. It would be interesting to clarify this relationship in order to obtain some information on the behaviour and properties of the Gibbs measure at low temperatures. Both β_c and β_* are to be compared with the value at $\beta_1 \equiv 1$, *i.e.* the maximum value of β where the free energies of the two models coincide. What we learn by the comparison of the two models is that the Gibbs measure of the SK has seemingly a richer structure than for the REM. As a matter of fact, the entropy of the REM vanishes at β_c while the entropy of the SK model is still strictly positive at this point. We expect moreover that the entropy of the SK model vanishes at β_* but this remains an open problem that is discussed in a forthcoming paper [7].

* flora.koukiou@u-cergy.fr

[1] Aizenman, M., Lebowitz, J. L., Ruelle, D.: Some Rigorous Results on the Sherrington-Kirkpatrick Spin Glass Model. Commun. Math. Phys. **112**, 3–20 (1987).

[2] Aizenman, M., Sims, R., Starr, S.,L.: An extended variational principle for the SK spin-glass model. Phys. Rev. B, **6821**(21): 4403, (2003).

[3] Derrida, B., Random energy model: An exactly solvable model of disordered systems. Phys. Rev. **B4**, 2613–2626 (1981).

[4] Guerra, F.: Broken replica symmetry bounds in the mean field spin glass model. Comm. Math Phys. **233**(1), 1–12 (2003).

[5] Guerra, F., Toninelli, F.: The thermodynamic limit in mean field spin glass models. Commun. Math. Phys. **230**(1), 71–79 (2002).

[6] Koukiou, F.: The low temperature free energy of the Sherrington- Kirkpatrick spin glass model. Eur. Lett. **33**, 95–98 (1996).

[7] Koukiou, F.: The entropy of the Gibbs measure of mean-field spin glasses. To appear.

[8] Parisi, G.: A sequence of approximated solutions to the Sherrington-Kirkpatrick model for spin glasses. J. Phys. **A 13**, L115–L121 (1980). The order parameter for spin glasses: a function on the interval $[0, 1]$. J. Phys. **A 13**, 1101–1112 (1980).

[9] Pastur, L., Shcherbina, M.V.: Absence of self-averaging of the order parameter in the Sherrington-Kirkpatrick model. J. Stat. Phys. **62**, 1–19 (1991).

[10] Sherrington, D., Kirkpatrick, S.: Solvable model of a spin glass. Phys. Rev. Lett. **35**, 1792–1796 (1975). Infinite-ranged models of spin-glasses. Phys.Rev. **17**, 4384–4403 (1978).

[11] Talagrand, M.: *Spin Glasses: A Challenge for Mathematicians. Mean Field Models and Cavity Method* Springer-Verlag (2003).