Spectral analysis of transport equations with bounce-back boundary conditions

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SPECTRAL ANALYSIS OF TRANSPORT EQUATIONS WITH BOUNCE-BACK BOUNDARY CONDITIONS.

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ABSTRACT. We investigate the spectral properties of the time-dependent linear transport equation with bounce-back boundary conditions. A fine analysis of the spectrum of the streaming operator is given and the explicit expression of the strongly continuous streaming semigroup is derived. Next, making use of a recent result from [1], we prove, via a compactness argument, that the essential spectrum of the transport semigroup and that of the streaming semigroup coincide on all L^p-spaces with $1 < p < \infty$.

KEYWORDS: Transport operator, bounce-back boudary conditions, transport semigroup, essential spectrum, compactness.
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1. INTRODUCTION

The spectral theory of transport equations with *no-reentry boundary conditions* (i.e. with zero incoming flux in the spatial domain) received a lot attention in the last decades (see, for example, the works [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein). The picture is fairly complete by now and almost optimal results have been obtained in [12] for bounded spatial domains and in [13] for unbounded domains.

When dealing with reentry boundary conditions (including periodic boundary conditions, specular reflections, diffuse reflections, generalized or mixed type boundary conditions), many progress have been made in the recent years in the understanding of the spectral features of one-dimensional models [14, 15, 16, 17, 18, 19, 20]. However, to our knowledge, for higher dimensions, only few partial results are available in the literature [21, 22, 23, 24], dealing in particular with very peculiar shapes of the spatial domain. Our paper deals with the following two problems concerning multidimensional transport equations with *bounce-back (reverse) boundary conditions* in convex bounded domains:

(1) The spectral analysis of the streaming operator subjected to bounce-back boundary conditions and the explicit expression of the streaming semigroup.
(2) The compactness of the difference of the (perturbed) transport semigroup and the streaming semigroup.

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To be more precise, we are concerned with the following initial-boundary-value problem in L^p–spaces $(1 \leqslant p < \infty)$

$$\begin{aligned}\frac{\partial \psi}{\partial t}(x,v,t) &= -v\cdot\nabla_x\psi(x,v,t) - \Sigma(x,v)\psi(x,v,t) + \int_{\mathbb{R}^N}\kappa(x,v,v')\psi(x,v',t)dv' \\ &= T\psi(x,v,t) + K\psi(x,v,t), \qquad\qquad (x,v)\in\mathcal{D}\times\mathbb{R}^N,\, t>0;\end{aligned} \tag{1.1a}$$

with bounce-back boundary conditions:

$$\psi_{|\Gamma_-}(x,v,t) = \gamma\psi_{|\Gamma_+}(x,-v,t), \qquad (x,v)\in\Gamma_-, t>0; \tag{1.1b}$$

and the initial condition

$$\psi(x,v,0) = \psi_0(x,v) \in L^p(\mathcal{D}\times\mathbb{R}^N). \tag{1.1c}$$

Here $\mathcal{D}$ is a smooth *convex* open subset of $\mathbb{R}^N$ $(N \geqslant 1)$, γ is a real constant belonging to $(0,1)$ and $\Gamma_\mp$ represent the incoming and outgoing parts of the boundary of the phase space (see Section 2 for details). The collision frequency $\Sigma(\cdot,\cdot) \in L^\infty(\mathcal{D}\times\mathbb{R}^N)$ is a non-negative function. The scattering kernel $\kappa(\cdot,\cdot,\cdot)$ is nonnegative and defines the linear operator K called the collision operator which is assumed to be bounded on $L^p(\mathcal{D}\times\mathbb{R}^N)$ $(1 \leqslant p < +\infty)$. The operator T appearing in (1.1a) is called the *streaming operator* while $T+K$ denotes the (full) transport operator. It is well-known [25] that, since $\gamma < 1$, T generates a C_0-semigroup of contractions $(U(t))_{t\geqslant 0}$ in $L^p(\mathcal{D}\times\mathbb{R}^N)$ (*streaming semigroup*) and, since K is bounded, $T+K$ is also the generator of a C_0-semigroup $(V(t))_{t\geqslant 0}$ in $L^p(\mathcal{D}\times\mathbb{R}^N)$ (*transport semigroup*).

Our work is displayed into two parts, referring to the above points (1) and (2). First, we present a fine description of the spectrum $\sigma(T)$ of the streaming operator in $L^p(\mathcal{D}\times\mathbb{R}^N)$, $1 \leqslant p < \infty$. Moreover, we derive the explicit expression of the streaming semigroup $(U(t))_{t\geqslant 0}$ for the particular case of a space-homogenous collision frequency. Second, we prove that the difference $R_1(t) = V(t) - U(t)$ is compact in $L^p(\mathcal{D}\times\mathbb{R}^N)$ $(1 < p < \infty)$ for any $t \geqslant 0$ under natural assumptions on the scattering operator K. The interest of such a compactness result lies in the fact that it implies that the streaming semigroup and the transport semigroup possess the same essential spectrum (see [26] for a precise definition). In particular, their essential types coincide. This shows that the part of the spectrum of the transport semigroup outside the spectral disc of the streaming semigroup consists of, at most, eigenvalues with finite algebraic mutiplicities. Assuming the existence of such eigenvalues, the transport semigroup can be decomposed into two parts: the first containing the time development of finitely many eigenmodes, the second being of faster decay.

Although the well-posedness of the problem (1.1) is a known fact [27, 25, 22, 28], the description of the spectrum of the streaming operator and the analytic expression of its semigroup seem to be new. Let us also notice that besides the interesting consequences of the compactness of $R_1(t)$ on the behavior for large times of the solution of the problem (1.1), it is an interesting result in itself. *Actually, it is the first time that the compactness of the first order remainder term of the Dyson-Phillips expansion of the transport operator with reentry boundary conditions is discussed in higher dimensions*. For the one-dimensional case we refer to the work [19] while the compactness of $R_1(t)$ in the case of non-reentry boundary condition was established in [12].

As in [19], the mathematical analysis is based upon a recent result owing to M. Sbihi [1] (see also Section 4) valid for Hilbert spaces. Actually, under some natural assumptions on the collision operator, we prove, via approximation arguments, the compactness of $R_1(t)$ on $L^2(\mathcal{D}\times\mathbb{R}^N, dx\otimes dv)$. The result is then extended to $L^p(\mathcal{D}\times\mathbb{R}^N, dx\otimes dv)$ with $p\in(1,2)\cup(2,\infty)$ by an interpolation argument. Unfortunately, the limiting case $p=1$ is not covered by our analysis and requires certainly another approach. Notice that the compactness of $R_1(t)$ for $p=1$ is also an open problem in the one-dimensional case [19] and for no-reentry boundary condition [12].

The outline of this work is as follows. In Section 2 we introduce the functional setting of the problem and fix the different notations and facts needed in the sequel. Section 3 is devoted to the spectral analysis of the streaming operator with bounce-back boundary conditions and to the analytic expression of the streaming semigroup. The compactness of the first order remainder term of the Dyson-Phillips expansion is the topic of Section 4.

2. Preliminary results

For the definitions of the different spectral notions used throughout this paper we refer, for example, to the book [26]. If X is a Banach space, $\mathcal{B}(X)$ will denote the set of all bounded linear operators on X.

Let $\mathcal{D}$ be a smooth bounded open subset of $\mathbb{R}^N$. We define the partial Sobolev space

$$W_p=\{\psi\in X_p\,;\, v\cdot\nabla_x\psi\in X_p\}$$

where $X_p=L^p(\mathcal{D}\times\mathbb{R}^N,\mathrm{d}x\otimes\mathrm{d}v)$ $(1\leqslant p<\infty)$. Let us denote by Γ_- (respectively Γ_+) the incoming (resp. outgoing) part of the boundary of the phase space $\mathcal{D}\times\mathbb{R}^N$

$$\Gamma_\pm=\big\{(x,v)\ \in\partial\mathcal{D}\times\mathbb{R}^N\ ;\ \pm v\cdot n(x)\geqslant 0\big\}$$

where $n(x)$ stands for the outward normal unit at $x\in\partial\mathcal{D}$. Suitable L^p-spaces for the traces on $\Gamma_\pm$ are defined as

$$L^p_\pm=L^p\big(\Gamma_\pm;|v\cdot n(x)|\mathrm{d}\gamma(x)\otimes\mathrm{d}v\big),$$

$\mathrm{d}\gamma(\cdot)$ being the Lebesgue measure on $\partial\mathcal{D}$. For any $\psi\in W_p$, one can define the traces $\psi_{|\Gamma_\pm}$ on $\Gamma_\pm$, however these traces do not belong to $L^p_\pm$ but to a certain weighted space (see [25, 29, 30]). For this reason, one defines

$$\widetilde{W}_p=\left\{\psi\in W_p\,;\,\psi_{|\Gamma_\pm}\in L^p_\pm\right\}.$$

In all the sequel, we shall assume that $\Sigma(\cdot,\cdot)$ is a measurable *non-negative* function on $\mathcal{D}\times\mathbb{R}^N$ that fulfills the following.

Assumption 2.1. *The collision frequency $\Sigma(\cdot,\cdot)$ is an even function of the velocity, i.e. for any $(x,v)\in\mathcal{D}\times\mathbb{R}^N$, $\Sigma(x,v)=\Sigma(x,-v)$.*

Let us define the absorption operator with *bounce-back boundary conditions:*

$$\begin{cases} T\ :\ \mathscr{D}(T) & \subset X_p\longrightarrow X_p\\ & \varphi\longmapsto T\varphi(x,v):=-v\cdot\nabla_x\varphi(x,v)-\Sigma(x,v)\varphi(x,v),\end{cases}$$

with domain

$$\mathscr{D}(T) := \left\{ \psi \in \widetilde{W}_p \ \text{ such that } \psi_{|\Gamma_-}(x,v) = \gamma\, \psi_{|\Gamma_+}(x,-v) \right\}$$

where $0 < \gamma < 1$. We recall (see e.g. [25, 22]) that T is a generator of a non-negative C_0-semigroup of contractions $(U(t))_{t\geqslant 0}$ in X_p $(1 \leqslant p < \infty)$

Remark 2.2. *Notice that our analysis also applies to the more general case $\gamma \geqslant 1$ provided the associated transport operator T generates a C_0-semigroup in X_p. Practical conditions on the geometry of $\mathcal{D}$ ensuring the latter to hold are given in* [28].

Definition 2.3. *For any $(x,v) \in \overline{\mathcal{D}} \times \mathbb{R}^N$, define*

$$t_\pm(x,v) = \sup\{\, t > 0\,; x \pm sv \in \mathcal{D},\ \ \forall\, 0 < s < t\,\} = \inf\{\, s > 0\,;\, x \pm sv \notin \mathcal{D}\}.$$

For the sake of convenience, we will set

$$\tau(x,v) := t_-(x,v) + t_+(x,v) \quad \textit{for any } (x,v) \in \overline{\mathcal{D}} \times \mathbb{R}^N.$$

Let us define

$$\vartheta(x,v) = \int_{-t_+(x,v)}^{t_-(x,v)} \Sigma(x - sv, v)\mathrm{d}s, \qquad\qquad (x,v) \in \overline{\mathcal{D}} \times \mathbb{R}^N.$$

One proves easily the following thanks to Assumption 2.1.

Lemma 2.4. *For any $(x,v) \in \mathcal{D} \times \mathbb{R}^N$,*

$$\vartheta(x,v) = \vartheta(x,-v) = \int_0^{\tau(x,v)} \Sigma(x + t_-(x,v)v - sv, v)\mathrm{d}s.$$

In particular, for any $(x,v) \in \Gamma_-$

$$\vartheta(x,v) = \int_0^{t_+(x,v)} \Sigma(x + sv, v)\mathrm{d}s.$$

Now let us investigate the resolvent of T. For any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > 0$, let us define $M_\lambda \in \mathscr{B}(L^p_-, L^p_+)$ by

$$M_\lambda u(x,v) = u(x - \tau(x,v)v, v) \exp\left\{ - \int_0^{\tau(x,v)} \lambda + \Sigma(x - sv, v)\mathrm{d}s \right\}, \qquad (x,v) \in \Gamma_+,$$

and let $B_\lambda \in \mathscr{B}(L^p_-, X_p)$ be given by

$$B_\lambda u(x,v) = u(x - t_-(x,v)v, v) \exp\left\{ - \int_0^{t_-(x,v)} \lambda + \Sigma(x - sv, v)\mathrm{d}s \right\}, \quad (x,v) \in \mathcal{D}\ .$$

In the same way, let $G_\lambda \in \mathscr{B}(X_p, L^p_+)$ be given as

$$G_\lambda \varphi(x,v) = \int_0^{\tau(x,v)} \varphi(x - sv, v) \exp\left\{ - \int_0^s \lambda + \Sigma(x - tv, v)\mathrm{d}t \right\} \mathrm{d}s, \quad (x,v) \in \Gamma_+\ ;$$

and $C_\lambda \in \mathscr{B}(X_p)$ be defined as

$$C_\lambda \varphi(x,v) = \int_0^{t(x,v)} \varphi(x-tv,v) \exp\left\{ \int_0^t \lambda + \Sigma(x-sv,v)\mathrm{d}s \right\} \mathrm{d}t, \quad (x,v) \in \mathcal{D}\ .$$

It is not difficult to show the following in the spirit of [31].

Proposition 2.5. *Let $0 < \gamma < 1$ be fixed and let $H \in \mathscr{B}(L^p_+, L^p_-)$ be defined by:*

$$H(\phi_+)(x,v) = \gamma \phi_+(x,-v) \qquad \text{for any} \quad (x,v) \in \Gamma_-. \tag{2.1}$$

If $\lambda \in \mathbb{C}$ is such that $1 \in \varrho(M_\lambda H)$, then $\lambda \in \varrho(T)$ with

$$(\lambda - T)^{-1} = B_\lambda H (I - M_\lambda H)^{-1} G_\lambda + C_\lambda. \tag{2.2}$$

In particular, if there is $\lambda_0 \in \mathbb{R}$ such that

$$r_\sigma(M_\lambda H) < 1 \qquad \forall\, \mathrm{Re}\lambda > \lambda_0,$$

then $\{\lambda \in \mathbb{C}\,;\, \mathrm{Re}\lambda > \lambda_0\} \subset \varrho(T)$ and the resolvent of T is given by (2.2).

3. Study of the streaming operator and semigroup

We shall focus in this section on the streaming operator associated with bounce-back boundary conditions and the associated semigroup.

3.1. Description of the spectrum of T.

To discuss the spectrum of T, we provide a more precise description of the inverse operator of $(I - M_\lambda H) \in \mathscr{B}(L^p_+)$. Precisely, let us define the measurable function

$$m_\lambda(x,v) = \gamma\, \exp\left\{ - \int_0^{\tau(x,v)} \lambda + \Sigma(x-sv,v)\mathrm{d}s \right\}, \qquad (x,v) \in \Gamma_+.$$

Before stating our first result we recall that the essential range of the measurable function $m_\lambda(\cdot,\cdot)$, $\mathcal{R}_{\text{ess}}(m_\lambda)$, is the set

$$\left\{ u \in \mathbb{C} \,:\, \Big|\, \{(x,v) \in \Gamma_+;\, |m_\lambda(x,v) - u| < \varepsilon\} \,\Big| \neq 0 \ \ \forall \varepsilon > 0 \right\}$$

where $|A|$ denotes the Lebesgue measure of the set A. Then, one has the following:

Proposition 3.1. *Let $\lambda \in \mathbb{C}$ be such that $1 \notin \mathcal{R}_{\text{ess}}(m_\lambda)$. Then, $(I - M_\lambda H) \in \mathscr{B}(L^p_+)$ is invertible with inverse given by*

$$\left[(I - M_\lambda H)^{-1}\psi\right](x,v) = (1 - m_\lambda^2(x,v))^{-1} \left[(I + M_\lambda H)\psi\right](x,v), \quad \forall\, \psi \in L^p_+, \quad (x,v) \in \Gamma_+.$$

Proof. Let us fix $\lambda \in \mathbb{C}$ and consider the equation:

$$\psi - (M_\lambda H)\psi = g, \tag{3.1}$$

where $g \in L^p(\Gamma_+)$ as well as the unknown function ψ. From (3.1), one sees that

$$\psi(x,v) - m_\lambda(x,v)\psi(x - \tau(x,v)v, -v) = g(x,v), \qquad (x,v) \in \Gamma_+.$$

For any *fixed* $(x,v)\in\Gamma_+$, one has

$$g(x-\tau(x,v)v,-v)=\psi(x-\tau(x,v)v,-v)-m_\lambda(x-\tau(x,v)v,-v)\times \\ \times\,\psi(x-\tau(x,v)v+\tau(x-\tau(x,v)v,-v)v,+v).$$

Now, one sees easily that $m_\lambda(x-\tau(x,v)v,-v)=m_\lambda(x,v)$ while

$$\psi(x-\tau(x,v)v+\tau(x-\tau(x,v)v,-v)v,+v)=\psi(x,v).$$

Therefore, one has

$$g(x-\tau(x,v)v,-v)=\psi(x-\tau(x,v)v,-v)-m_\lambda(x,v)\psi(x,v)$$

so that

$$\begin{aligned}[M_\lambda Hg]\,(x,v)&=m_\lambda(x,v)g(x-\tau(x,v)v,-v)\\&=m_\lambda(x,v)\psi(x-\tau(x,v)v,-v)-m_\lambda^2(x,v)\psi(x,v).\end{aligned}$$

Since $m_\lambda(x,v)\psi(x-\tau(x,v)v,-v)=[M_\lambda H\psi]\,(x,v)$, one obtains from (3.1) that

$$g(x,v)+[M_\lambda Hg]\,(x,v)=\left(1-m_\lambda^2(x,v)\right)\psi(x,v),\qquad (x,v)\in\Gamma_+.$$

This leads to an explicit expression of the solution to (3.1):

$$\psi(x,v)=(1-m_\lambda^2(x,v))^{-1}\left[(I+M_\lambda H)\,g\right](x,v).$$

Defining

$$\mathscr{R}_\lambda g(x,v)=(1-m_\lambda^2(x,v))^{-1}\left[(I+M_\lambda H)\,g\right](x,v)$$

it is not difficult to see that

$$1\notin\mathcal{R}_{\rm ess}(m_\lambda)\implies\mathscr{R}_\lambda\in\mathscr{B}(L^p_+),$$

and the above calculations show that $\mathscr{R}_\lambda=(I-M_\lambda H)^{-1}$. □

The precise picture of the spectrum of T is given by the following, which is in the spirit of [20]

Theorem 3.2. *For any $k\in\mathbb{Z}$, let us define*

$$F_k(x,v)=\frac{\log\gamma-\vartheta(x,v)}{\tau(x,v)}-i\frac{2k\pi}{\tau(x,v)},\qquad\forall(x,v)\in\mathcal{D}\times\mathbb{R}^N.$$

Then,

$$\sigma(T)=\overline{\bigcup_{k\in\mathbb{Z}}\mathcal{R}_{\rm ess}(F_k)}$$

where $\mathcal{R}_{\rm ess}(F_k)$ stands for the essential range of F_k.

Proof. Let us begin with the inclusion $\supset$. Given $\lambda\in\mathcal{R}_{\rm ess}(F_k)$ $(k\in\mathbb{Z})$. Let $\varepsilon>0$ and define

$$\Lambda_\varepsilon:=\{(x,v)\in\overline{\mathcal{D}}\times\mathbb{R}^N\,;\,|\lambda-F_k(x,v)|\leqslant\varepsilon\}.$$

By the definition of $\mathcal{R}_{\rm ess}(F_k)$, $|\Lambda_\varepsilon|\neq 0$ for any $\varepsilon>0$. For any integer $n\in\mathbb{N}$, define $B_n=\{(x,v)\in\mathcal{D}\times\mathbb{R}^N,\tau(x,v)\geqslant 1/n\}$. Since $\tau(x,v)\geqslant 0$ for a. e. $(x,v)\in\mathcal{D}\times\mathbb{R}^N$, one has

$$\Lambda_\varepsilon=\bigcup_n\Big(B_n\cap\Lambda_\varepsilon\Big)$$

so that, for any $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $|B_{n(\varepsilon)} \cap \Lambda_\varepsilon| \neq 0$. Define

$$A_\varepsilon := B_{n(\varepsilon)} \cap \Lambda_\varepsilon, \qquad \varepsilon > 0$$

so that $|A_\varepsilon| \neq 0$ for any $\varepsilon > 0$, while

$$\operatorname{ess-inf} \left\{ \tau(x,v) \ ; \ (x,v) \in A_\varepsilon \right\} \geqslant 1/n(\varepsilon) > 0, \qquad \text{for any } \varepsilon > 0. \tag{3.2}$$

Then, let χ_ε stand for the characteristic function of the measurable set A_ε ($\varepsilon > 0$). One sees that

$$\chi_\varepsilon(x,v) = \chi_\varepsilon(x,-v), \text{ and } \quad \chi_\varepsilon(x+tv,v) = \chi_\varepsilon(x,v)$$

for $t > 0$ small enough. Now, for any $\varepsilon > 0$, one can define

$$\varphi_\varepsilon(x,v) = \chi_\varepsilon(x,v) \exp\left\{ - \int_0^{t_-(x,v)} \Big(F_k(x,v) + \Sigma(x-sv,v) \Big) \mathrm{d}s \right\}, \quad (x,v) \in \mathcal{D} \times \mathbb{R}^N.$$

One sees that $\varphi_\varepsilon \in L^\infty(\mathcal{D} \times \mathbb{R}^N)$ for any $\varepsilon > 0$, since

$$\operatorname*{ess-sup}_{(x,v) \in A_\varepsilon} \left| \exp\left\{ - \int_0^{t_-(x,v)} \Big(F_k(x,v) + \Sigma(x-sv,v) \Big) \mathrm{d}s \right\} \right| < \infty$$

by virtue of (3.2). Moreover, since $t_-(x,v) = 0$ for any $(x,v) \in \Gamma_-$, one sees that

$$\varphi_{\varepsilon|\Gamma_-} = \chi_\varepsilon.$$

Given $(x,v) \in \Gamma_-$ and $\varepsilon > 0$, one has

$$\varphi_\varepsilon(x,-v) = \chi_\varepsilon(x,-v) \exp\left\{ - \int_0^{t_-(x,-v)} \Big(F_k(x,-v) + \Sigma(x+sv,-v) \Big) \mathrm{d}s \right\}.$$

Since $t_-(x,-v) = t_+(x,v) = \tau(x,v)$ and $F_k(x,-v) = F_k(x,v)$ (see Lemma 2.4), one has

$$\varphi_\varepsilon(x,-v) = \chi_\varepsilon(x,v) \exp\left\{ -\tau(x,v) F_k(x,v) - \int_0^{t_+(x,v)} \Sigma(x+sv,-v) \, \mathrm{d}s \right\}.$$

Now, one checks that

$$\begin{aligned} -\tau(x,v) F_k(x,v) &= -\log\gamma + \vartheta(x,v) + 2ik\pi \\ &= -\log\gamma + 2ik\pi + \int_0^{t_+(x,v)} \Sigma(x+sv,v) \mathrm{d}s \qquad \forall (x,v) \in \Gamma_-, \end{aligned}$$

where we used again Lemma 2.4. Thus, one sees that, for any $(x,v) \in \Gamma_-$,

$$\varphi_\varepsilon(x,-v) = \frac{1}{\gamma} \, \varphi_\varepsilon(x,v)$$

which exactly means that φ_ε fulfils the boundary conditions (2.1). Finally, it is easy to see that $\varphi_\varepsilon \in \mathscr{D}(T)$. Define now the net $(\psi_\varepsilon)_\varepsilon$ by

$$\psi_\varepsilon = \varphi_\varepsilon / \|\varphi_\varepsilon\|, \qquad \varepsilon > 0 \tag{3.3}$$

and let $g_\varepsilon = (\lambda - T)\psi_\varepsilon$, i. e.

$$g_\varepsilon(x,v) = (\lambda + \Sigma(x,v)) \, \psi_\varepsilon(x,v) + v \cdot \nabla_x \psi_\varepsilon(x,v).$$

Using the fact, for any $(x,v) \in \mathcal{D} \times \mathbb{R}^N$, and for any $t > 0$ small enough, $\vartheta(x+tv,v) = \vartheta(x,v)$ (note that the same occurs for F_k) one can prove that

$$g_\varepsilon(x,v) = \big(\lambda - F_k(x,v)\big)\psi_\varepsilon(x,v), \qquad (x,v) \in \mathcal{D} \times \mathbb{R}^N, \quad \varepsilon > 0$$

and so

$$\|g_\varepsilon\| \leqslant \operatorname*{ess\,sup}_{(x,v)\in A_\varepsilon} |\lambda - F_k(x,v)| \; \|\psi_\varepsilon\| \leqslant \varepsilon.$$

This, together with (3.3) achieves to show that $(\psi_\varepsilon)_{\varepsilon>0}$ is a *singular net* of T so that $\lambda \in \sigma(T)$. The closedness of the spectrum ensures that

$$\overline{\bigcup_{k\in\mathbb{Z}} \mathcal{R}_{\rm ess}(F_k)} \subset \sigma(T).$$

Let us prove now the converse inclusion. Assume that $\lambda \notin \mathcal{R}_{\rm ess}(F_k)$ for any $k \in \mathbb{Z}$. Then, for any $k \in \mathbb{Z}$, there exists $\beta_k > 0$ such that

$$|\lambda - F_k(x,v)| \geqslant \beta_k \qquad \text{a.e. } (x,v) \in \mathcal{D} \times \mathbb{R}^N,$$

i.e.

$$\left|\frac{\log\gamma - \vartheta(x,v) - 2ik\pi}{\tau(x,v)} - \lambda\right| \geqslant \beta_k \qquad \text{a.e. } (x,v) \in \mathcal{D} \times \mathbb{R}^N.$$

Then,

$$|\log\gamma - \vartheta(x,v) - 2ik\pi - \lambda\,\tau(x,v)| \geqslant \tau(x,v)\,\beta_k \qquad \text{a. e. } (x,v) \in \mathcal{D} \times \mathbb{R}^N.$$

This means that, for any integer $n \geqslant 0$, there exists $c_n > 0$ such that

$$\left|\frac{\log\gamma - \vartheta(x,v) - \lambda\,\tau(x,v)}{2\pi\,n} \pm i\right| \geqslant c_n\tau(x,v) \text{ a. e. } (x,v) \in \mathcal{D} \times \mathbb{R}^N,\ n \geqslant 1,$$

and

$$|\log\gamma - \vartheta(x,v) - \lambda\,\tau(x,v)| \geqslant c_0\,\tau(x,v) \qquad \text{a. e. } (x,v) \in \mathcal{D} \times \mathbb{R}^N.$$

Arguing as in [20], one can choose $M > 0$ such that

$$|\log\gamma - \vartheta(x,v) - \lambda\tau(x,v)| \leqslant M \qquad \text{a. e. } (x,v) \in \mathcal{D} \times \mathbb{R}^N,$$

and

$$|\exp\{\log\gamma - \vartheta(x,v) - \lambda\tau(x,v)\} - 1| \geqslant \frac{C}{2}\prod_{n=1}^{N} c_n^2\tau(x,v)^2 \qquad \text{a. e. } (x,v) \in \mathcal{D} \times \mathbb{R}^N,$$

where

$$C = \operatorname*{ess\,inf}_{(x,v)\in\mathcal{D}\times\mathbb{R}^N} \left|\exp\left\{\frac{1}{2}(\log\gamma - \vartheta(x,v) - \lambda\tau(x,v))\right\}\right|.$$

Moreover, one can easily see that

$$\liminf_{\tau(x,v)\to 0} |\exp\{\log\gamma - \vartheta(x,v) - \lambda\tau(x,v)\} - 1| \geqslant |1-\gamma| > 0.$$

In particular,

$$\operatorname*{ess\,inf}_{(x,v)\in\mathcal{D}\times\mathbb{R}^N} |\exp\{\log\gamma - \vartheta(x,v) - \lambda\tau(x,v)\} - 1| > 0.$$

This proves that $1 \notin \mathcal{R}_{\rm ess}(m_\lambda)$. From Proposition 3.1, one gets that $(I - M_\lambda H)$ is invertible and $\lambda \in \varrho(T)$. □

3.2. **Explicit expression of the semigroup $(U(t))_{t\geqslant 0}$.** We derive in this section the explicit expression of the semigroup $(U(t))_{t\geqslant 0}$ generated by the streaming operator T associated to the bounce-back boundary conditions H given by (2.1). For simplicity, we shall restrict ourselves to the case of a *homogeneous collision frequency*:

$$\Sigma(x,v) = \Sigma(v), \qquad \forall (x,v) \in \mathcal{D}\times\mathbb{R}^N.$$

Theorem 3.3. *The C_0-semigroup $(U(t))_{t\geqslant 0}$ in X_p generated by T is given by*

$$U(t) = \sum_{n=0}^{\infty} U_n(t), \qquad \forall t \geqslant 0,$$

where, for any fixed $t \geqslant 0$,

$$[U_0(t)\varphi](x,v) = \varphi(x-tv,v)\exp(-\Sigma(v)t)\chi_{\{t<t_-(x,v)\}}, \qquad \varphi\in X_p, \quad (x,v)\in\mathcal{D}\times\mathbb{R}^N$$

while, for any $n \geqslant 0$

$$[U_{2n+2}(t)\varphi](x,v) = \gamma^{2n+2}\exp(-\Sigma(v)t)\chi_{\mathcal{I}_{2n+1}(x,v)}(t)\varphi\big(x-tv+(2n+2)\tau(x,v)v;v\big),$$

and

$$[U_{2n+1}(t)\varphi](x,v) = \gamma^{2n+1}\exp(-\Sigma(v)t)\chi_{\mathcal{I}_{2n}(x,v)}(t)\varphi\big(x+tv-2t_-(x,v)v-2n\tau(x,v)v,-v\big)$$

for any $\varphi\in X_p$, and any $(x,v)\in\mathcal{D}\times\mathbb{R}^N$, with

$$\mathcal{I}_k(x,v) = [k\tau(x,v)+t_-(x,v);(k+1)\tau(x,v)+t_-(x,v)], \qquad \text{for any } k\in\mathbb{N}.$$

Proof. The proof is based upon the representation of the resolvent (2.2) and the use of the uniqueness of the Laplace transform. Precisely, let $\lambda > 0$ be fixed. According to Proposition 3.1,

$$\begin{aligned}\left[(I-M_\lambda H)^{-1}\psi\right](x,v) &= (1-m_\lambda^2(x,v))^{-1}\left[(I+M_\lambda H)\psi\right](x,v)\\ &= \sum_{n=0}^{\infty}\gamma^{2n}\exp(-2n(\lambda+\Sigma(v))\tau(x,v))\left[(I+M_\lambda H)\psi\right](x,v),\end{aligned}$$

for any nonnegative $\psi\in L^p_+$, i.e.

$$\begin{aligned}\left[(I-M_\lambda H)^{-1}\psi\right](x,v) &= \sum_{n=0}^{\infty}\gamma^{2n}\exp(-2n(\lambda+\Sigma(v))\tau(x,v))\psi(x,v)+\\ &\sum_{n=0}^{\infty}\gamma^{2n+1}\exp(-(2n+1)(\lambda+\Sigma(v))\tau(x,v))\psi(x-\tau(x,v)v,-v), \qquad (x,v)\in\Gamma_+.\end{aligned}$$

It is then easy to see that, for any fixed $\varphi \in X_p$:

$$B_\lambda H(I - M_\lambda H)^{-1} G_\lambda \varphi = \sum_{n=0}^{\infty} \mathscr{J}_n(\lambda)\varphi \tag{3.4}$$

where, for any $n \geqslant 0$,

$$\left[\mathscr{J}_{2n+1}(\lambda)\varphi\right](x,v) = \gamma^{2n+1} \exp\left\{-2n(\lambda + \Sigma(v))\tau(x,v)\right\} \exp\left\{-(\lambda + \Sigma(v))t_-(x,v)\right\} \times \\ \times \int_0^{\tau(x,v)} \varphi(x - t_-(x,v)v + sv, -v) \exp\left\{-(\lambda + \Sigma(v))s\right\} \mathrm{d}s,$$

and

$$\mathscr{J}_{2n}(\lambda)\varphi(x,v) = \gamma^{2n+2} \exp\left\{-(2n+1)(\lambda + \Sigma(v))\tau(x,v)\right\} \exp\left\{-(\lambda + \Sigma(v))t_-(x,v)\right\} \times \\ \times \int_0^{\tau(x,v)} \varphi(x - t_-(x,v)v + \tau(x,v)v - sv, v) \exp\left\{-(\lambda + \Sigma(v))s\right\} \mathrm{d}s.$$

For fixed $(x,v) \in \mathcal{D} \times \mathbb{R}^N$, performing the change of variable

$$t = 2n\tau(x,v) + t_-(x,v) + s, \quad \mathrm{d}t = \mathrm{d}s, \quad t \in \mathcal{I}_{2n+1}(x,v)$$

in the above expression of $\mathscr{J}_{2n+1}(\lambda)$ leads easily to

$$\left[\mathscr{J}_{2n+1}(\lambda)\varphi\right](x,v) = \int_0^{\infty} \exp(-\lambda t) \left[U_{2n+1}(t)\varphi\right](x,v) \mathrm{d}t.$$

In the same way, the change of variable

$$t = (2n+1)\tau(x,v) + t_-(x,v) + s, \quad \mathrm{d}t = \mathrm{d}s, \quad t \in \mathcal{I}_{2n+2}(x,v)$$

in the above expression of $\mathscr{J}_{2n+2}(\lambda)$ allows to prove that

$$\left[\mathscr{J}_{2n+2}(\lambda)\varphi\right](x,v) = \int_0^{\infty} \exp(-\lambda t) \left[U_{2n+2}(t)\varphi\right](x,v) \, \mathrm{d}t, \qquad \forall n \geqslant 0.$$

Finally, it is easily seen that

$$\left[C_\lambda \varphi\right](x,v) = \int_0^{\infty} \exp(-\lambda t) \left[U_0(t)\varphi\right](x,v) \, \mathrm{d}t.$$

Therefore,

$$(\lambda - T)^{-1}\varphi = \sum_{n=0}^{\infty} \int_0^{\infty} \exp(-\lambda t) U_n(t)\varphi \, \mathrm{d}t$$

for any $\varphi \in X_p$ for which the series converges. Moreover, since T generates a C_0-semigroup $(U(t))_{t \geqslant 0}$ in X_p, one also has

$$(\lambda - T)^{-1}\varphi = \int_0^{\infty} \exp(-\lambda t) U(t)\varphi \, \mathrm{d}t.$$

From the uniqueness of the Laplace transform, this yields

$$U(t)\varphi = \sum_{n=0}^{\infty} U_n(t)\varphi$$

for any nonnegative $\varphi \in X_p$ and, since all the operators involved are clearly nonnegative and the positive cone of X_p is generating, the result holds for arbitrary $\varphi \in X_p$. □

4. Spectral analysis of the perturbed semigroup

We investigate now the spectral properties of the full semigroup governing the problem (1.1). Let us define the collision operator K by

$$K\varphi(x,v) = \int_{\mathbb{R}^N} \kappa(x,v,w)\varphi(x,w)\,\mathrm{d}w$$

where the kernel $\kappa(\cdot,\cdot,\cdot)$ is nonnegative over $\mathcal{D}\times\mathbb{R}^N\times\mathbb{R}^N$. We shall assume here that K is a bounded operator, $K \in \mathscr{B}(X_p)$, $1 \leqslant p < \infty$, so that by the standard bounded perturbation theory, the operator $(T+K, \mathscr{D}(T))$ generates a C_0-semigroup $(V(t))_{t\geqslant 0}$ given by the following Dyson-Phillips expansion series:

$$V(t) = \sum_{j=0}^{\infty} V_j(t)$$

where $V_0(t) = U(t)$,

$$V_j(t) = \int_0^t U(t-s)KV_{j-1}(s)\mathrm{d}s, \qquad (j \geqslant 1).$$

As indicated in the Introduction, our analysis does not cover the case of transport equation in L^1-spaces, so we shall assume in all this section that

$$1 < p < \infty.$$

Throughout the sequel, we shall assume that K is a regular operator in the following sense:

Definition 4.1. *An operator $K \in \mathscr{B}(X_p)$ $(1 < p < \infty)$ is said to be regular if K can be approximated in the operator norm by operators of the form:*

$$\varphi \in X_p \longmapsto \sum_{i\in I} \alpha_i(x)\beta_i(v) \int_{\mathbb{R}^N} \theta_i(w)\varphi(x,w)\mathrm{d}w \in X_p \tag{4.1}$$

where I is finite, $\alpha_i \in L^\infty(\mathcal{D})$, $\beta_i \in L^p(\mathbb{R}^N,\mathrm{d}v)$ and $\theta_i \in L^q(\mathbb{R}^N,\mathrm{d}v)$, $1/p+1/q=1$.

Remark 4.2. *Since $1 < p < \infty$, one notes that the set $\mathcal{C}_c(\mathbb{R}^N)$ of continuous functions with compact support in $\mathbb{R}^N$ is dense in $L^q(\mathbb{R}^N,\mathrm{d}v)$ as well as in $L^p(\mathbb{R}^N,\mathrm{d}v)$ $(1/p+1/q=1)$. Consequently, one may assume in the above definition that $\beta_i(\cdot)$ and $\theta_i(\cdot)$ are continuous functions with compact supports in $\mathbb{R}^N$.*

We prove in this section the following compactness result, generalizing known ones for $1D$-transport problems [19]

Theorem 4.3. *Assume $1 < p < \infty$. If $K \in \mathscr{B}(X_p)$ is a regular operator, then the difference $V(t) - U(t)$ is compact for any $t \geqslant 0$. As a consequence, $\sigma_{\rm ess}(V(t)) = \sigma_{\rm ess}(U(t))$ for any $t \geqslant 0$.*

Remark 4.4. *Notice that the compactness of the difference $V(t) - U(t)$ for any $t \geqslant 0$ implies that of $(\lambda - T - K)^{-1} - (\lambda - T)^{-1}$ for sufficiently large λ. In particular,*

$$\sigma_{\rm ess}(T + K) = \sigma_{\rm ess}(T).$$

This result was already obtained in [32] *and is valid for more general reentry boundary conditions.*

The rest of the paper is devoted to the proof of the above Theorem. We shall adopt the so-called resolvent approach which allows to infer the compactness of

$$R_1(t) = V(t) - U(t), \qquad t \geqslant 0$$

from properties of the resolvent $(\lambda - T)^{-1}$ and K only. The basis of our approach is a fundamental result owing to M. Sbihi [1, Theorem 2.2, Corollary 2.1] which, applied to our case, asserts that, for $p = 2$, if T is dissipative and there exists $\alpha > w(U)$ ($w(U)$ denoting the type of the semigroup $(U(t))_{t \geqslant 0}$) such that

$$\alpha + i\beta - T)^{-1} K (\alpha + i\beta - T)^{-1} \qquad \text{is compact for all } \beta \in \mathbb{R} \tag{4.2}$$

and

$$\lim_{\beta \to \infty} (\|K^*(\alpha + i\beta - T)^{-1} K\| + \|K(\alpha + i\beta - T)^{-1} K^*\|) = 0, \tag{4.3}$$

then $R_1(t) = V(t) - U(t)$ is compact on X_2 for all $t \geqslant 0$.

Notice that here, the streaming operator T is dissipative on X_p, $p \in (1, \infty)$, in particular for $p = 2$. Moreover, the compactness assumption (4.2) follows from Theorem 3.1 in [32] and holds true for more general boundary conditions. Therefore, we have only to check that (4.3) holds true provided K is a regular collision operator.

Though M. Sbihi's result is a purely Hilbertian one, it has already been noticed in [19] that it can be applied successfully to neutron transport problems in L^p-spaces for any $1 < p < \infty$. Actually, since K is regular and $R_1(t)$ depends continuously on $K \in \mathscr{B}(X_p)$, one may assume that K is of the form (4.1) where, according to Remark 4.2, the functions β_i and θ_i are continuous with compact supports in $\mathbb{R}^N$. In this case, K is bounded in any $\mathscr{B}(X_r)$ and, by an interpolation argument already used in [19], we may restrict ourselves to prove the compactness of $R_1(t)$ in X_2. Moreover, using a domination argument as in [19], there is no loss of generality in proving the compactness of $R_1(t)$ in the special case

$$\Sigma(v) = \sigma > 0, \qquad \gamma = 1$$

Now, since K is given by (4.1), by linearity, Eq. (4.3), and consequently Theorem 4.3, follow from the following

Lemma 4.5. *Let β_j, θ_j be continuous functions with compact support in $\mathbb{R}^N$ and $\alpha_j \in L^\infty(\mathcal{D})$, $j = 1, 2$. Then, there is some $\alpha > -\sigma$ such that*

$$\lim_{|\beta| \to \infty} \|K_1(\alpha + i\beta - T)^{-1} K_2\| = 0$$

where

$$K_j\varphi(x,v)=\alpha_j(x)\beta_j(v)\int_{\mathbb{R}^N}\theta_j(w)\varphi(x,w)\mathrm{d}w,\qquad j=1,2;\quad \varphi\in X_2.$$

Proof. Let $\alpha>-\sigma$ be fixed. According to Eq. (2.2)

$$(\lambda-T)^{-1}=B_\lambda H(I-M_\lambda H)^{-1}G_\lambda+C_\lambda.$$

Moreover, it is well-known from [4] that

$$\lim_{|\beta|\to\infty}\|K_1C_{\alpha+i\beta}K_2\|=0,\qquad\forall\alpha>-\sigma. \tag{4.4}$$

since C_λ is the resolvent of the transport operator with no-reentry boundary conditions. Therefore, one has to prove that $\lim_{|\mathrm{Im}\lambda|\to\infty}\|K_1B_\lambda H(I-M_\lambda H)^{-1}G_\lambda K_2\|=0$ where $\mathrm{Re}\lambda>-\sigma$. According to Eq. (3.4), it suffices to establish that

$$\lim_{|\beta|\to\infty}\|K_1\mathscr{J}_n(\alpha+i\beta)K_2\|=0,\qquad\forall n\in\mathbb{N},\qquad\alpha>-\sigma.$$

Let us prove the result for $\mathscr{J}_{2n+1}(\alpha+i\beta)$, $n\in\mathbb{N}$. Let $\lambda=\alpha+i\beta$, $\alpha>-\sigma$ and let $n\in\mathbb{N}$ be fixed. Technical calculations show that

$$K_1\mathscr{J}_{2n+1}(\lambda)K_2\varphi=\mathcal{A}_3\mathcal{A}_2(\lambda)\mathcal{A}_1\varphi$$

where

$$\mathcal{A}_1\ :\ \varphi\in L^2(\mathcal{D}\times\mathbb{R}^N)\mapsto\mathcal{A}_1\varphi(x)=\alpha_2(x)\int_{\mathbb{R}^N}\varphi(x,w)\theta_2(w)\mathrm{d}w\in L^2(\mathcal{D}),$$

$$\mathcal{A}_3\ :\ \psi\in L^2(\mathcal{D})\mapsto\mathcal{A}_3\psi(x,v)=\alpha_1(x)\beta_1(v)\psi(x)\in L^2(\mathcal{D}\times\mathbb{R}^N)$$

and $\mathcal{A}_2(\lambda)\ :L^2(\mathcal{D})\to L^2(\mathcal{D})$ is given by

$$\mathcal{A}_2(\lambda)\varphi(x)=\int_{\mathbb{R}^N}\exp\Big\{-2n(\lambda+\sigma)\tau(x,v')-(\lambda+\sigma)t_-(x,v')\Big\}\beta_2(-v')\theta_1(v')\mathrm{d}v'$$
$$\int_0^{\tau(x,v')}\exp(-(\lambda+\sigma)s)\varphi(x-t_-(x,v')v'+sv')\mathrm{d}s.$$

Therefore, it is sufficient to prove that

$$\lim_{|\beta|\to\infty}\|\mathcal{A}_2(\alpha+i\beta)\|_{\mathscr{B}(L^2(\mathcal{D}))}=0,\qquad\alpha>-\sigma. \tag{4.5}$$

To do so, we adopt the approach of [4] and [1]. Precisely, setting $\mu=\lambda+\sigma$ and $h(v)=\theta_1(v)\beta(-v)$, $v\in\mathbb{R}^N$, the change of variable $s\mapsto t=s-\tau(x,v')$ leads to

$$\mathcal{A}_2(\lambda)\varphi(x)=\int_{\mathbb{R}^N}h(v')\exp\{-2n\mu\tau(x,v')\}\mathrm{d}v'\int_{-t_-(x,v')}^{t_+(x,v')}\varphi(x+tv')\exp(-\mu t)\mathrm{d}t.$$

Since $\mathcal{D}$ is convex, given $(x,v')\in\mathcal{D}\times\mathbb{R}^N$,

$$t\in(-t_-(x,v'),t_+(x,v'))\Longleftrightarrow y=x+tv'\in\mathcal{D}$$

The change of variable $y = x + tv'$ shows that $\mathcal{A}_2(\lambda)$ is an integral operator :

$$\mathcal{A}_2(\lambda)\varphi(x) = \int_{\mathcal{D}} \kappa(\lambda, x, y)\varphi(y)\mathrm{d}y$$

where

$$\begin{aligned}\kappa(\lambda, x, y) &= \int_{\mathbb{R}} h\left(\frac{y-x}{t}\right) \exp\left\{-\mu t - 2n\mu\tau\left(x, \frac{y-x}{t}\right)\right\}\frac{\mathrm{d}t}{t^N} \\ &= \int_{\mathbb{R}} h\left(\frac{y-x}{t}\right) \exp\left\{ -\mu t - 2n\mu\frac{|t|}{|x|}\tau\left(x+z, -\frac{x}{|x|}\right)\right\}\frac{\mathrm{d}t}{t^N},\end{aligned}$$

where we used the know property $\tau(x, \frac{v}{s}) = |s|\tau(x, v)$ for any $(x, v) \in \mathcal{D} \times \mathbb{R}^N$ and any $s \in \mathbb{R}$. Notice that the very rough estimate

$$\|\mathcal{A}_2(\lambda)\|_{\mathscr{B}(L^2(\mathcal{D}))} \leqslant \left(\int_{\mathcal{D}\times\mathcal{D}} |\kappa(\lambda, x, y)|^2 \mathrm{d}x\mathrm{d}y\right)^{1/2}$$

apparently does not lead to (4.5). We have to estimate the norm of $\mathcal{A}_2(\lambda)$ more carefully. With respect to [4], one of the difficulty in estimating $\|\mathcal{A}_2(\lambda)\|$ is that $\mathcal{A}_2(\lambda)$ is not a convolution operator because of the dependence in x of $\tau(x, \cdot)$. To overcome this difficulty, we follow the approach of [1]. Precisely, set

$$N_\lambda(x, z) = \int_{\mathbb{R}} h\left(-\frac{x}{t}\right) \exp\left\{-\mu t - 2n\mu\frac{|t|}{|x|}\tau\left(x+z, -\frac{x}{|x|}\right)\right\}\frac{\mathrm{d}t}{t^N},$$

where $(x, z) \in \mathcal{D} \times \mathcal{D}$ with $x + z \in \mathcal{D}$. Let us point out that, from assumption, there is no loss of generality assuming that there exist two constants $a, b > 0$ such that

$$\mathrm{Supp}\,(h) \subset \left\{v \in \mathbb{R}^N\,;\, a \leqslant |v| \leqslant b\right\}.$$

In this case, in the above integral, one sees that $t \in \mathbb{R}$ is such that

$$a \leqslant \left|\frac{x}{t}\right| \leqslant b$$

which implies that $|t| \leqslant |x|/a$. This means that the above integral over $\mathbb{R}$ reduces actually to an integral over $\left[-\frac{d}{a}, \frac{d}{a}\right]$ where d is the diameter of $\mathcal{D}$. Then,

$$\kappa(\lambda, x, y) = N_\lambda(x - y, y), \qquad \text{for any } (x, y) \in \mathcal{D} \times \mathcal{D}$$

and, setting

$$G_\lambda(x) = \sup_{z\in\mathcal{D}-x} |N_\lambda(x, z)|\,, \qquad x \in \mathcal{D}$$

one has

$$|\mathcal{A}_2(\lambda)\varphi(x)| \leqslant \int_{\mathcal{D}} G_\lambda(x - y)|\varphi(y)|\mathrm{d}y, \qquad \forall x \in \mathcal{D},\ \varphi \in L^2(\mathcal{D}).$$

Consequently,

$$\|\mathcal{A}_2(\lambda)\|_{\mathcal{B}(L^2(\mathcal{D}))} \leqslant \int_{\mathcal{D}} G_\lambda(x)\mathrm{d}x.$$

To prove (4.5), one has then to show that

$$\lim_{|\beta|\to\infty}\int_{\mathcal{D}} G_{\alpha+i\beta}(x)\mathrm{d}x = 0, \qquad \forall \alpha > -\sigma.$$

First, one sees that for any $(x,z)\in\mathcal{D}\times\mathcal{D}$ with $x+z\in\mathcal{D}$ and any $\lambda=\alpha+i\beta$, one has

$$|N_\lambda(x,z)| \leqslant \int_{\mathbb{R}} \left|h\left(-\frac{x}{t}\right)\right| \exp\big\{-(\alpha+\sigma)t\big\}\frac{dt}{t^N}$$

where we used the fact that $\tau(\cdot,\cdot)\geqslant 0$. Then,

$$\begin{aligned}\int_{\mathcal{D}}\sup_{\lambda=\alpha+i\beta}|G_\lambda(x)|\mathrm{d}x &\leqslant \int_{\mathbb{R}}\exp\big\{-(\alpha+\sigma)t\big\}\frac{\mathrm{d}t}{t^N}\int_{\mathcal{D}}\left|h\left(-\frac{x}{t}\right)\right|\mathrm{d}x\\ &\leqslant \int_{-\frac{d}{a}}^{\frac{d}{a}}\exp\big\{-(\alpha+\sigma)t\big\}\mathrm{d}t\int_{\mathbb{R}^N}|h(v)|\mathrm{d}v<\infty,\end{aligned}$$

where we performed the change of variables $v=\frac{x}{t}$ in the x integral. Therefore, from the dominated convergence theorem, it suffices to prove that

$$\lim_{|\beta|\to\infty} G_{\alpha+i\beta}(x) = 0, \qquad \text{a. e. } x\in\mathcal{D}.$$

Using the fact that, for any fixed $x\in\mathcal{D}$, the mapping $z\mapsto\tau\left(x+z,-\frac{x}{|x|}\right)$ is bounded, this can be done as in [1] thanks to the Riemann-Lebesgue's Lemma. This achieves to prove that

$$\lim_{|\beta|\to\infty}\|K_1\mathscr{J}_{2n+1}(\alpha+i\beta)K_2\| = 0, \qquad \forall n\in\mathbb{N}, \qquad \alpha>-\sigma, \qquad \forall n\in\mathbb{N}.$$

One proves in the same way that

$$\lim_{|\beta|\to\infty}\|K_1\mathscr{J}_{2n}(\alpha+i\beta)K_2\| = 0, \qquad \forall n\in\mathbb{N}, \qquad \alpha>-\sigma, \quad \forall n\in\mathbb{N}$$

and, combined with (4.4) and (3.4), yield the result. □

Remark 4.6. *Let us observe that Lemma 4.5 allows to describe the asymptotic spectrum of the transport operator $T+K$. Indeed, combining Lemma 4.5 with the compactness of $(\lambda-T)^{-1}K$ [32] and* [4, Lemma 1.1] *we infer that*

- *i) $\sigma(T+K)\cap\{\lambda\in\mathbb{C}\,:\,\mathrm{Re}\lambda>-\lambda^*\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicity;*
- *ii) for any $\eta>0$, the set $\sigma(T+K)\cap\{\lambda\in\mathbb{C}\,:\,\mathrm{Re}\lambda>-\lambda^*+\eta\}$ is finite or empty.*

Clearly, this result may be also derived from Theorem 4.3 using the fact that the spectral mapping theorem holds true for the point spectrum.

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