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Finite size scaling of meson propagators with isospin chemical potential

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Abstract

We determine the volume and mass dependence of scalar and pseudoscalar two-point functions in N_f-flavour QCD, in the presence of an isospin chemical potential and at fixed gauge-field topology. We obtain these results at second order in the ϵ-expansion of Chiral Perturbation Theory and evaluate all relevant zero-mode group integrals analytically. The virtue of working with a non-vanishing chemical potential is that it provides the correlation functions with a dependence on both the chiral condensate, Σ, and the pion decay constant, F, already at leading order. Our results may therefore be useful for improving the determination of these constants from lattice QCD calculations. As a side product, we rectify an earlier calculation of the $\mathcal{O}(\epsilon^2)$ finite-volume correction to the decay constant appearing in the partition function. We also compute a generalised partition function which is useful for evaluating $U(N_f)$ group integrals.

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1 Introduction

With the possibility of performing unquenched lattice gauge theory calculations at increasingly small quark masses, new opportunities for understanding the low-energy properties of QCD are emerging. The theoretical framework for such analyses is provided by Chiral Perturbation Theory (ChPT), which describes the long wavelength excitations of QCD above the vacuum, in terms of physical parameters known as low-energy constants (LECs). While there exist a number of ways to approach the chiral limit, we will consider here the ϵ-regime of [1]. This regime is well suited to numerical lattice calculations, since it is reached by lowering the quark masses in such a way that the Compton wavelength of the corresponding pseudo-Goldstone bosons is kept much larger than the typical linear extent of the finite (simulated) four-volume V. Low energy constants can be determined in this regime, by comparing lattice calculations of a given observable with analytical expressions for this observable obtained in ChPT. In the present work we will focus on ChPT calculations which allow the determination of the two leading LECs, the chiral condensate Σ and the pion decay constant F, and we refer to [2] for a recent review of lattice results for these as well as higher order LECs.

Two-point correlation functions, of either scalar and pseudoscalar densities, or of vector and axial vector currents, are relatively straightfoward quantities to compute with lattice QCD methods. They are also calculable in the ϵ-regime of ChPT, and have been studied up to and including second order corrections, in the quenched, partially quenched or full theory, with or without fixed gauge-field topology [3, 4, 5, 6, 7]. Comparisons of these analytical predictions with lattice results have been reported in [8]. At second order, these correlation functions couple to both LECs, Σ and F. However, in scalar and pseudoscalar two-point functions, F only appears at sub-leading $\mathcal{O}(\epsilon^2) \sim 1/\sqrt{V}$. In vector and axial correlation functions, it is present at leading non-vanishing order, but these functions are $\mathcal{O}(\epsilon^4)$-suppressed compared to the scalar and pseudoscalar correlators. All of these suppression factors are clearly a handicap for determining F from lattice calculations of the above two-point functions.

For that reason, we propose to consider two-point scalar and pseudoscalar correlation functions in the presence of a non-zero imaginary isospin chemical potential μ, leaving vector and axial-vector correlation functions for future work. The interest of adding a chemical potential is that it couples the decay constant, F, to the zero modes in the leading order Lagrangian, through the combination $\mu^2 V F^2$ [9]. This implies that the scalar and pseudoscalar correlation functions now depend on F, already at leading order. In turn, the increased sensitivity to F should facilitate the extraction of this LEC from lattice calculations. Since all of our results depend only on μ^2, they hold true for both real and imaginary isospin chemical potentials for any number of flavours N_f. For real μ, only the even flavour case has a real action.

The proposal to use a chemical potential to facilitate the determination of F was originally made in the context of spectral correlators [10, 11]. In [10], the spectral two-point density was singled out as particularly sensitive to F. More recently, all spectral correlators [12] and individual eigenvalue correlations [13] with imaginary isospin chemical potential were computed in chiral Random Two-Matrix Theory, which is equivalent to leading order ChPT in the epsilon regime [14]. This setup includes the partially quenched theory, where μ-dependent valence quarks propagate in a background of μ-independent sea quarks, allowing for the use of standard, zero-density lattice configurations. In fact, a lattice calculation of F has already been performed in this framework [15].

The remainder of the article is organised as follows. In Sec. 2, we derive the $\mathcal{O}(\epsilon^2)$ improved partition function and two-point meson correlation functions in terms of expectation values over the pion zero-mode. In particular, we correct an earlier calculation [16] of the one-loop finite-volume corrections to the partition function. In Sec. 3 we present our results for both flavoured and unflavoured scalar and pseudoscalar two-point correlation functions in N_f-flavour QCD, given in terms of the finite-volume propagator, the mass dependent condensate and other generating functions. These general expressions are spelled out most explicitly in the case of two flavours. Our conclusions are given in Sec. 4. Two appendices, A and B, provide technical details for group integral identities, including a new result for a generalised partition function.

2 The epsilon expansion in ChPT with a chemical potential

Since imaginary chemical potentials couple to fields like the time component of a constant vector current, they can be accounted for in ChPT through the covariant derivative on the pion fields, $U(x) \in SU(N_f)$:

$$\nabla_\rho U(x) \equiv \partial_\rho U(x) - i\,\delta_{\rho,0}[C, U(x)]\ , \quad \nabla_\rho U(x)^\dagger \ = \ \partial_\rho U(x)^\dagger - i\,\delta_{\rho,0}[C, U(x)^\dagger]\ , \tag{2.1}$$

where C is a matrix proportional to the chemical potential μ, whose form depends on the charge μ couples to. Moreover, it was shown in [1], for calculations up to and including second order in the ϵ-expansion, that the usual leading order chiral Lagrangian is sufficient. Thus, we consider here the Lagrangian:

$$\mathcal{L}_2 = \frac{F^2}{4}\mathrm{Tr}\left[\nabla_\rho U(x)^\dagger \nabla_\rho U(x)\right] - \ \frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}^\dagger U(x) + U(x)^\dagger \mathcal{M}\right]\ , \tag{2.2}$$

where $\mathcal{M}$ is the mass matrix. We will take $\mathcal{M} = \mathrm{diag}(m_1, \ldots, m_{N_f})$ and throughout this section the external vector current, C, will be kept general. In Sec. 3 we will specify this current to be the isospin charge matrix, $C = \mathrm{diag}(\mu \mathbf{1}_{N_1}, -\mu \mathbf{1}_{N_2}) \equiv \mu\Gamma$. The simplest and interesting case of $N_f = N_1 + N_2 = 1 + 1$ flavours of equal mass $m_{1,2} = m$ will be displayed in great detail in Sec. 3. As our final results will depend only on μ^2, we can also rotate back to real chemical potential by $\mu^2 \to -\mu^2$ at the end of our calculations, for any number of flavours $N_f = N_1 + N_2$.

We use the standard parametrisation of the Goldstone fields for the ϵ-expansion:

$$U(x) \equiv U_0 \exp\left[i\frac{\sqrt{2}}{F}\xi(x)\right]\ . \tag{2.3}$$

These fields live on the Goldstone group manifold, $U(x) \in\ SU(N_f)$, whereas the propagating Hermitean fields $\xi(x) = \xi(x)^\dagger$ are members of the corresponding Lie algebra, e.g. for two flavours, $N_f = 2$, they are given by $\xi(x) = \frac{1}{2}\sigma_a \xi_a(x)$, in terms of the Pauli matrices. In particular, we have split off the zero momentum mode, $U_0 \in\ SU(N_f)$, explicitly, and will treat its effects exactly. The power counting in the epsilon expansion in the presence of a chemical potential is given by [1]:

$$V \sim \epsilon^{-4}, \quad \mathcal{M} \sim \epsilon^4, \quad \mu \sim \epsilon^2, \quad \partial_\rho \sim \epsilon, \quad \xi(x) \sim \epsilon\ . \tag{2.4}$$

All other quantities, such as Σ and F, are of $\mathrm{O}(\epsilon^0)$.

With this power counting, it is straightforward to expand the action, $\mathcal{S}_2 \equiv \int d^4x\, \mathcal{L}_2$ to $\mathrm{O}(\epsilon^2)$. With the notation $\mathcal{S}_2 = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \mathcal{S}^{(2)} + \mathcal{O}(\epsilon^3)$, we obtain:

- $\mathcal{O}(1)$: $\quad \mathcal{S}^{(0)} = \mathcal{S}^{(0)}_{U_0} + \mathcal{S}^{(0)}_{\partial^2}$

$$\mathcal{S}^{(0)}_{U_0}(\Sigma, F) \ = \ -V\frac{F^2}{4}\mathrm{Tr}\Big[[C, U_0^\dagger][C, U_0]\Big] - V\frac{\Sigma}{2}\mathrm{Tr}[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}] \tag{2.5}$$

$$\mathcal{S}^{(0)}_{\partial^2} \ = \ \frac{1}{2}\int d^4x\ \mathrm{Tr}[\partial_\rho \xi(x)\partial_\rho \xi(x)] \tag{2.6}$$

- $\mathcal{O}(\epsilon)$:

$$\mathcal{S}^{(1)} = -\frac{i}{2}\int d^4x\ \mathrm{Tr}\Big[\partial_0 \xi(x)[C + U_0^\dagger C U_0, \xi(x)]\Big] \tag{2.7}$$

- $\mathcal{O}(\epsilon^2)$: $\quad \mathcal{S}^{(2)} = \mathcal{S}^{(2)}_{\partial^2} + \mathcal{S}^{(2)}_{\xi, U_0}$

$$\mathcal{S}^{(2)}_{\partial^2} \ = \ \frac{1}{12F^2}\int d^4x\ \mathrm{Tr}\Big[[\partial_\rho \xi(x), \xi(x)][\partial_\rho \xi(x), \xi(x)]\Big] \tag{2.8}$$

$$\begin{aligned}\mathcal{S}^{(2)}_{\xi, U_0} \ &= \ -\frac{1}{\sqrt{2}F}\int d^4x\ \mathrm{Tr}\Big[\xi(x)\partial_0 \xi(x)\,\xi(x) U_0^\dagger [C, U_0]\Big] - \frac{1}{2}\int d^4x\ \mathrm{Tr}\Big[U_0^\dagger C U_0[\xi(x), [C, \xi(x)]]\Big] \\ &\quad + \frac{\Sigma}{2F^2}\int d^4x\ \mathrm{Tr}[\mathcal{M}^\dagger U_0 \xi(x)^2 + \xi(x)^2 U_0^\dagger \mathcal{M}]\end{aligned} \tag{2.9}$$

where the superscripts specify the order in the ϵ-expansion. Note that all terms containing C vanish for $C \propto \mathbf{1}_{N_f}$, i.e. pions do not couple to baryon chemical potential.

2.1 Partition function at $\mathcal{O}(\epsilon^2)$ and effective couplings

The partition function for the N_f-flavour chiral theory is given by

$$\begin{aligned} \mathcal{Z} &\equiv \int_{SU(N_f)} [d_H U(x)] \, \exp[-\mathcal{S}] \\ &= \int_{SU(N_f)} d_H U_0 \exp[-\mathcal{S}^{(0)}_{U_0}(\Sigma, F)] \, \mathcal{Z}_\xi(U_0) \ , \end{aligned} \tag{2.10}$$

where

$$\mathcal{Z}_\xi(U_0) \equiv \int [d\xi(x)] \, J(\xi) \, \exp[\mathcal{S}^{(0)}_{U_0}(\Sigma, F) - \mathcal{S}] \tag{2.11}$$

and where $\mathcal{S}$ is the chiral action at an, as of yet, unspecified order. In Eq. (2.10) we integrate over the Goldstone manifold with Haar measure $d_H U$ and in Eq. (2.11), $J(\xi) = \{1 - \frac{N_f}{3F^2 V} \int d^4x \mathrm{Tr}[\xi(x)^2] + \mathcal{O}(\epsilon^3)\}$ is the Jacobian corresponding to the change of variables of Eq. (2.3) [1].

In this theory, expectation values are given by

$$\langle O \rangle_{U(x)} \equiv \frac{1}{\mathcal{Z}} \int_{SU(N_f)} [d_H U(x)] \, O \, \exp[-\mathcal{S}] \ , \tag{2.12}$$

As we are interested in evaluating observables up to $\mathcal{O}(\epsilon^2)$, we need the partition function at that order. Since $\mathcal{S}^{(0)}_{U_0}(\Sigma, F)$ is $\mathcal{O}(1)$, we must compute $Z_\xi(U_0)$ to $\mathcal{O}(\epsilon^2)$. Expanding $J(\xi) \exp[\mathcal{S}^{(0)}_{U_0}(\Sigma, F) - S]$ to second order and performing the resulting Gaussian integrals with the propagator,

$$\frac{1}{\int [d\xi(x)] e^{-\mathcal{S}^{(0)}_{\partial^2}}} \int [d\xi(x)] e^{-\mathcal{S}^{(0)}_{\partial^2}} \, \xi(x)_{ij} \xi(y)_{kl} = \Big(\delta_{il}\delta_{jk} - \frac{1}{N_f} \delta_{ij}\delta_{kl} \Big) \bar{\Delta}(x-y) \ , \tag{2.13}$$

we obtain

$$\begin{aligned} \mathcal{Z}_\xi(U_0) &= \mathcal{N} \Bigg\{ 1 - \frac{V\Sigma}{2F^2} \frac{(N_f^2 - 1)}{N_f} \bar{\Delta}(0) \, \mathrm{Tr}[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M})] \\ &\quad - \frac{V N_f}{2} \Big(\bar{\Delta}(0) - \int d^4x [\partial_0 \bar{\Delta}(x)]^2 \Big) \, \mathrm{Tr}\Big[[C, U_0^\dagger][C, U_0] \Big] \Bigg\} + \mathcal{O}(\epsilon^3) \ , \end{aligned} \tag{2.14}$$

where $\mathcal{N}$ is an overall normalisation factor which does not contribute to the expectation values defined above. Now, re-exponentiating the corrections in Eq. (2.14), we obtain for the partition function:

$$\mathcal{Z} = \mathcal{N} \int_{SU(N_f)} d_H U_0 \, \exp[-\mathcal{S}^{(0)}_{U_0}(\Sigma_{eff}, F_{eff})] + \mathcal{O}(\epsilon^3) \ , \tag{2.15}$$

with Σ and F replaced in the argument of $\mathcal{S}^{(0)}_{U_0}$ by the 1-loop corrected couplings:

$$\Sigma_{eff} = \Sigma \left(1 - \frac{(N_f^2 - 1)}{N_f F^2} \bar{\Delta}(0) \right), \tag{2.16}$$

$$F_{eff} = F \left(1 - \frac{N_f}{F^2} \Big(\bar{\Delta}(0) - \int d^4x [\partial_0 \bar{\Delta}(x)]^2 \Big) \right). \tag{2.17}$$

While the correction to the condensate Σ Eq. (2.16) has been known a long time [1], the correction to F was computed only very recently [16], apart from the second term in Eq. (2.17) which is new and seems to have been omitted in [16]. This term arises from the contribution proportional to $(\mathcal{S}^{(1)})^2$ in the computation of $\mathcal{Z}_\xi(U_0)$ to $\mathcal{O}(\epsilon^2)$.

In dimensional regularisation, the propagator $\bar{\Delta}(0)$ is finite and is given by [17]:

$$\bar{\Delta}(0) = -\beta_1 / \sqrt{V} \ . \tag{2.18}$$

Moreover [3],

$$\int d^4x\,[\partial_0\bar{\Delta}(x)]^2 \;=\; -\frac{1}{2\sqrt{V}}\left[\beta_1-\frac{T^2}{\sqrt{V}}k_{00}\right]\;, \tag{2.19}$$

where T is the time extent of the box in which the system is enclosed. In Eqs. (2.18) and (2.19), β_1 and k_{00} are numerical constants which depend on the geometry of the box considered [17, 3]. Together with Eqs. (2.16) and (2.17), these equations imply:

$$\begin{aligned}\Sigma_{eff} &= \Sigma\left(1+\beta_1\frac{(N_f^2-1)}{N_f F^2\sqrt{V}}\right),\\ F_{eff} &= F\left(1+\left[\beta_1+\frac{T^2}{\sqrt{V}}k_{00}\right]\frac{N_f}{2F^2\sqrt{V}}\right),\end{aligned}$$

In the particular case of hypercube, i.e. a box with sides $T{=}L_1{=}L_2{=}L_3{=}V^{1/4}$, $k_{00}=\beta_1/2$.

2.2 Two-point correlation functions

We consider here two-point correlators of the scalar and pseudoscalar quark bilinears,

$$\begin{aligned}S_0(x) &\equiv \bar{\psi}(x)\mathbf{1}_{N_f}\psi(x)\;,\quad S_b(x)\equiv\bar{\psi}(x)t_b\mathbf{1}_{N_f}\psi(x)\;,\\ P_0(x) &\equiv \bar{\psi}(x)i\gamma_5\mathbf{1}_{N_f}\psi(x)\;,\quad P_b(x)\equiv\bar{\psi}(x)t_b i\gamma_5\mathbf{1}_{N_f}\psi(x)\;,\end{aligned} \tag{2.20}$$

where the t_a denote the $SU(N_f)$ generators, and we have normalised $\mathrm{Tr}[(t_a)^2]=\frac{1}{2}$. For $N_f=2$ we have $t_a=\frac{1}{2}\sigma_a$ in Eq. (2.20), the Pauli matrices for $a=1,2,3$, and $\frac{1}{2}$ times the identity for $t_{a=0}$.

In the effective theory, the scalar and pseudoscalar densities are most easily obtained by introducing Hermitean sources, $s(x)=s_a(x)t_a$ and $p(x)=p_a(x)t_a$, which have the same spurionic transformation properties as in QCD, i.e. through the replacement [18]:

$$\mathcal{M}\to\mathcal{M}+s(x)+i\,p(x)\;. \tag{2.21}$$

The ϵ-expansion counting for the sources is thus the same as that of the quark masses, i.e. $s(x)$, $p(x)\sim\epsilon^2$.

Once this replacement is made, the partition function depends on the sources, and the two-point functions are obtained, as usual, by taking adequate functional derivatives:

$$\langle S_a(x)S_b(0)\rangle_{U(x)} \;=\; \frac{1}{\mathcal{Z}}\,\frac{\delta^2}{\delta s_a(x)\delta s_b(0)}\mathcal{Z}[s,p]\Big|_{s=p=0} \tag{2.22}$$

and likewise for the pseudoscalar correlators. Expanding the observables, the action and the Jacobian to $\mathcal{O}(\epsilon^2)$, a calculation analogous to the one performed for the partition function in the preceeding section yields:

$$\begin{aligned}\langle S_0(x)S_0(0)\rangle_{U(x)} &= \frac{\Sigma_{eff}^2}{4}\left\langle(\mathrm{Tr}[U_0+U_0^\dagger])^2\right\rangle_{U_0}-\frac{\Sigma_{eff}^2}{2F^2}\left[\langle\mathrm{Tr}\left[(U_0-U_0^\dagger)^2\right]\rangle_{U_0}-\frac{1}{N_f}\left\langle(\mathrm{Tr}[U_0-U_0^\dagger])^2\right\rangle_{U_0}\right]\bar{\Delta}(x)\\ &+\;\mathcal{O}(\epsilon^3),\end{aligned} \tag{2.23}$$

where all that remains are expectation values with respect to the zero mode, given by:

$$\langle O\rangle_{U_0}\equiv\frac{1}{\int d_HU_0\;e^{-\mathcal{S}^{(0)}_{U_0}(\Sigma_{eff},F_{eff})}}\int_{SU(N_f)}d_HU_0\;O\;e^{-\mathcal{S}^{(0)}_{U_0}(\Sigma_{eff},F_{eff})}\;. \tag{2.24}$$

In the second term of Eq. (2.23), since $\bar{\Delta}(x)$ is $\mathcal{O}(\epsilon^2)$, the expectation values with respect to U_0 can be calculated with $\mathcal{S}^{(0)}_{U_0}(\Sigma,F)$ instead of $\mathcal{S}^{(0)}_{U_0}(\Sigma_{eff},F_{eff})$.

Along the same lines, one can derive a similar expression for the pseudoscalar correlator:

$$\begin{aligned}\langle P_0(x)P_0(0)\rangle_{U(x)} &= -\frac{\Sigma_{eff}^2}{4}\left\langle(\mathrm{Tr}[U_0-U_0^\dagger])^2\right\rangle_{U_0}+\frac{\Sigma_{eff}^2}{2F^2}\left[\langle\mathrm{Tr}\left[(U_0+U_0^\dagger)^2\right]\rangle_{U_0}-\frac{1}{N_f}\left\langle(\mathrm{Tr}[U_0+U_0^\dagger])^2\right\rangle_{U_0}\right]\bar{\Delta}(x)\\ &+\;\mathcal{O}(\epsilon^3)\;.\end{aligned} \tag{2.25}$$

Both results Eqs. (2.23) and (2.25) have the same form as the corresponding expressions at zero chemical potential, $\mu = 0$ [4]. However, when inserting the group integral averages over U_0 below, they will differ explicitly by μ-dependent terms.

Flavoured two-point functions can be computed exactly in the same way. But there is an important difference to the calculation in [4]: while for $\mu = 0$ and equal quark masses one has $\langle S_a(x)S_b(0)\rangle_{U(x)} \sim \delta_{ab}$, this is explicitly broken by the chemical potential $\mu \neq 0$ (as well as by non-degenerate masses of course).

For that reason we will only compute the following diagonal sum over flavoured combinations, where we can use the $SU(N_f)$ completeness relation $\sum_a (t_a)_{ij}(t_a)_{kl} = \frac{1}{2}(\delta_{il}\delta_{jk} - \frac{1}{N_f}\delta_{ij}\delta_{kl})$:

$$\begin{aligned}\sum_a \langle S_a(x)S_a(0)\rangle_{U(x)} &= \frac{1}{8}\Sigma_{eff}^2 \left[\langle \mathrm{Tr}\left[(U_0+U_0^\dagger)^2\right]\rangle_{U_0} - \frac{1}{N_f}\left\langle (\mathrm{Tr}[U_0+U_0^\dagger])^2\right\rangle_{U_0}\right] \\ &+ \bar{\Delta}(x)\frac{\Sigma_{eff}^2}{4F^2}\left[-\frac{1}{2}\left\langle (\mathrm{Tr}[U_0+U_0^\dagger])^2\right\rangle_{U_0} - \frac{1}{2N_f^2}(N_f^2+2)\left\langle (\mathrm{Tr}[U_0-U_0^\dagger])^2\right\rangle_{U_0}\right. \\ &\left. +2N_f^2 + \frac{2}{N_f}\langle \mathrm{Tr}\left[(U_0-U_0^\dagger)^2\right]\rangle_{U_0}\right] + \mathcal{O}(\epsilon^3)\ . \end{aligned} \tag{2.26}$$

For clarity we have made the sum over flavour indices explicit (we always use summation conventions unless otherwise stated). In complete analogy, one obtains for the flavoured pseudoscalar correlation function,

$$\begin{aligned}\sum_a \langle P_a(x)P_a(0)\rangle_{U(x)} &= -\frac{1}{8}\Sigma_{eff}^2 \left[\langle \mathrm{Tr}\left[(U_0-U_0^\dagger)^2\right]\rangle_{U_0} - \frac{1}{N_f}\left\langle (\mathrm{Tr}[U_0-U_0^\dagger])^2\right\rangle_{U_0}\right] \\ &-\bar{\Delta}(x)\frac{\Sigma_{eff}^2}{4F^2}\left[-\frac{1}{2}\left\langle (\mathrm{Tr}[U_0-U_0^\dagger])^2\right\rangle_{U_0} - \frac{1}{2N_f^2}(N_f^2+2)\left\langle (\mathrm{Tr}[U_0+U_0^\dagger])^2\right\rangle_{U_0}\right. \\ &\left. -2N_f^2 + \frac{2}{N_f}\langle \mathrm{Tr}\left[(U_0+U_0^\dagger)^2\right]\rangle_{U_0}\right] + \mathcal{O}(\epsilon^3)\ . \end{aligned} \tag{2.27}$$

Both results agree again with [4] at $\mu = 0$, when normalised by $(N_f^2 - 1)$. The resulting group averages differ however, both by explicitly μ-dependent factors and by a functional change of the mass dependent condensate. This is the subject of the next section.

Before turning to the evaluation of the relevant group integrals, it is useful to make a comment on the spacetime dependence of the above correlation functions. For comparisons with lattice QCD calculations, it is useful to consider the corresponding zero-momentum correlation functions, which are functions only of the Euclidean time t, i.e.

$$C_S(t) \equiv \frac{T}{V}\int d^3x\, \langle S_a(x)S_a(0)\rangle_{U(x)} \qquad \text{and} \qquad C_P(t) \equiv \frac{T}{V}\int d^3x\, \langle P_a(x)P_a(0)\rangle_{U(x)}\ , \tag{2.28}$$

where either $a = 0$, for the singlet case, or there is an implicit sum over the $SU(N_f)$ adjoint index a, for the flavoured case. As above, in Eq. (2.28) T is the time extent of the box in which the system is enclosed and V/T is its spatial volume. For the massless propagtor [17],

$$\bar{\Delta}(x) \equiv \frac{1}{V}\sum_p{}' \frac{e^{ip\cdot x}}{p^2} \qquad \Rightarrow \qquad h_1(\tau) \equiv \frac{1}{T}\int d^3x\, \bar{\Delta}(x) = \frac{1}{2}\left[(\tau - \frac{1}{2})^2 - \frac{1}{12}\right]\ , \tag{2.29}$$

with $0 < \tau < 1$ and where $\tau \equiv t/T$. Thus, the zero-momentum correlation functions $C_{S,P}(t)$ are simply obtained from the expressions of Eqs. (2.23), (2.25), (2.26) and (2.27) above, and those of Eqs. (3.15), (3.16), (3.24) and (3.25) below, by making the replacement:

$$\bar{\Delta}(x) \longrightarrow \frac{T^2}{V}\, h_1(\tau)\ . \tag{2.30}$$

3 Results for scalar and pseudoscalar correlators at fixed topology

In order to make the group integrals, which appear in the results of the previous section, tractable analytically, we choose to work in sectors of fixed gauge-field topology. This is done by introducing the theta vacuum angle, θ, of QCD into the effective theory, through the replacement [18]

$$\mathcal{M} + s(x) + i\,p(x) \to \big(\mathcal{M} + s(x) + i\,p(x)\big)\, e^{i\theta/N_f} \tag{3.1}$$

and Fourier transforming the results with respect to this angle. Thus for instance, the partition function in the sector of topology ν is given by:

$$\mathcal{Z}_\nu \equiv \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-i\theta\nu}\, \mathcal{Z}[\theta]\ . \tag{3.2}$$

It is easy to convince oneself that such a transformation has for effect to replace all $SU(N_f)$ group integrals by integrals over the much simpler group manifold of $U(N_f)$. Thus, to $\mathcal{O}(\epsilon^2)$:

$$\mathcal{Z}_\nu = \int_{U(N_f)} d_H U_0 \det[U_0^\nu] \exp\left[\frac{1}{2} V F_{eff}^2 \mu^2 \mathrm{Tr}[\Gamma U_0^\dagger \Gamma U_0] + \frac{1}{2} V \Sigma_{eff} m \mathrm{Tr}[U_0 + U_0^\dagger]\right] + \mathcal{O}(\epsilon^3)\ , \tag{3.3}$$

$$\langle\, O\,\rangle_{U_0}^\nu \equiv \frac{1}{\mathcal{Z}_\nu} \int_{U(N_f)} d_H U_0\, O \det[U_0^\nu] \exp\left[\frac{1}{2} V F_{eff}^2 \mu^2 \mathrm{Tr}[\Gamma U_0^\dagger \Gamma U_0] + \frac{1}{2} V \Sigma_{eff} m \mathrm{Tr}[U_0 + U_0^\dagger]\right] + \mathcal{O}(\epsilon^3)\ , \tag{3.4}$$

where we have absorbed the constant $\mathcal{N}$ of Eq. (2.15) into the measure $d_H U_0$, as well as the constant term $\frac{1}{4} V F_{eff}^2 \mu^2 \mathrm{Tr}[\Gamma^2]$ from $\mathcal{S}_{U_0}^{(0)}(\Sigma_{eff}, F_{eff})$.

In the following, we only consider the theory with equal masses,

$$\mathcal{M} = m \mathbf{1}_{N_f}\ . \tag{3.5}$$

This includes the interesting case of $N_f = 2$ with degenerate up and down quarks. Furthermore, we specify the external vector current to be

$$C = \mu\Gamma\ , \quad \text{with} \quad \Gamma \equiv \mathrm{diag}(\mathbf{1}_{N_1}, -\mathbf{1}_{N_2})\ , \tag{3.6}$$

such that $\Gamma^2 = \mathbf{1}_{N_f}$ and we will also consider more general partition functions, containing higher powers $\mathrm{Tr}[(\Gamma U_0^\dagger \Gamma U_0)^k]$ as generating functionals, for computing all group integrals.

Correlation functions at fixed topology are obtained by replacing the averages $\langle\, O\,\rangle_{U_0} \to \langle\, O\,\rangle_{U_0}^\nu$ in Eqs. (2.23) and (2.25). The building blocks that we have to compute in Eqs. (2.23) and (2.25) are $\langle(\mathrm{Tr}[U_0])^2 + (\mathrm{Tr}[U_0^\dagger])^2\rangle_{U_0}^\nu$, $\langle \mathrm{Tr}[U_0]\mathrm{Tr}[U_0^\dagger]\rangle_{U_0}^\nu$, and $\langle \mathrm{Tr}[U_0^2] + \mathrm{Tr}[U_0^{\dagger\,2}]\rangle_{U_0}^\nu$. With the help of some $U(N_f)$ group integral identities derived in Appendix A, these can be expressed in terms of the following known quantities: the partition function and its derivatives with respect to mass, chemical potential and an additional external field. The equations that we need are:

$$\Big\langle (\mathrm{Tr}[U_0 + U_0^\dagger])^2 \Big\rangle_{U_0}^\nu = 4N_f \frac{\partial}{\partial \eta_{eff}} \frac{\Sigma_\nu(\eta_{eff}, \alpha_{eff})}{\Sigma} + 4N_f^2 \frac{\Sigma_\nu(\eta_{eff}, \alpha_{eff})^2}{\Sigma^2}\ , \tag{3.7}$$

$$\Big\langle (\mathrm{Tr}[U_0 - U_0^\dagger])^2 \Big\rangle_{U_0}^\nu = -4 \frac{N_f}{\eta_{eff}} \frac{\Sigma_\nu(\eta_{eff}, \alpha_{eff})}{\Sigma} + \frac{4\nu^2 N_f^2}{\eta_{eff}^2}\ , \tag{3.8}$$

for the squared traces, and for single traces of the squared matrices

$$\begin{aligned}\langle \mathrm{Tr}\left[(U_0 \pm U_0^\dagger)^2\right]\rangle_{U_0}^\nu &= -4 \frac{N_f^2}{\eta_{eff}} \frac{\Sigma_\nu(\eta_{eff}, \alpha_{eff})}{\Sigma} + \frac{4\nu^2 N_f}{\eta_{eff}^2} + (2 \pm 2) N_f \\ &\quad + \frac{4\alpha_{eff}^2}{\eta_{eff}^2}\Big(N_f \mathcal{Y}_\nu(\eta_{eff}, \alpha_{eff}) - (\mathrm{Tr}[\Gamma])^2 + \frac{1}{2}\alpha_{eff}^2 (\mathcal{X}_\nu(\eta_{eff}, \alpha_{eff}) - N_f)\Big)\ .\end{aligned} \tag{3.9}$$

The generating functionals on the right hand sides are as follows. The mass dependent condensate is given by

$$\frac{\Sigma_\nu(\eta_{eff}, \alpha_{eff})}{\Sigma} \equiv \frac{1}{N_f} \frac{\partial}{\partial \eta_{eff}} \ln[\mathcal{Z}_\nu] = \frac{1}{2N_f} \langle \mathrm{Tr}[U_0 + U_0^\dagger]\rangle_{U_0}^\nu\ . \tag{3.10}$$

It depends both on the rescaled mass

$$\eta_{eff} \equiv m\Sigma_{eff}V \ , \tag{3.11}$$

and the rescaled chemical potential,

$$\alpha_{eff}^2 \equiv \mu^2 F_{eff}^2 V \ . \tag{3.12}$$

We will also need the derivative of the condensate to generate the expectation value $\langle(\mathrm{Tr}[U_0+U_0^\dagger])^2\rangle_{U_0}^\nu$. Whether or not to consider the $\mathcal{O}(\epsilon^2)$ correction is trivially achieved by keeping or dropping the subscript "eff" in the couplings in Eqs. (3.10) and (3.13). For example, the partition function $\mathcal{Z}_\nu$ to leading order is given by Eq. (3.3) with leading order couplings Σ and F.

We define a second functional, which is the derivative of the partition function with respect to the rescaled chemical potential, to generate the following average:

$$\mathcal{Y}_\nu(\eta_{eff},\alpha_{eff}) \equiv 2\frac{\partial}{\partial\alpha_{eff}^2}\ln[\mathcal{Z}_\nu] = \langle\mathrm{Tr}[\Gamma U_0^\dagger\Gamma U_0]\rangle_{U_0}^\nu \ . \tag{3.13}$$

We will also need a third generating functional

$$\mathcal{X}_\nu(\eta_{eff},\alpha_{eff}) \equiv \langle\mathrm{Tr}\left[(\Gamma U_0^\dagger\Gamma U_0)^2\right]\rangle_{U_0}^\nu \ , \tag{3.14}$$

to be given below in Eq. (3.22) in order to complete the calculation.

3.1 Unflavoured correlation functions

We begin with the scalar two-point function, inserting Eqs. (3.7) - (3.9) into Eq. (2.26) to obtain

$$\begin{aligned}\langle S_0(x)S_0(0)\rangle_{U(x)}^\nu &= \Sigma_{eff}^2 N_f\left(\frac{\partial}{\partial\eta_{eff}}\frac{\Sigma_\nu(\eta_{eff},\alpha_{eff})}{\Sigma}+N_f\frac{\Sigma_\nu(\eta_{eff},\alpha_{eff})^2}{\Sigma^2}\right)\\ &+ \bar\Delta(x)\frac{2\Sigma^2}{\eta^2F^2}\left[(N_f^2-1)\eta\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}-\alpha^2\Big(N_f\mathcal{Y}_\nu(\eta,\alpha)-(\mathrm{Tr}[\Gamma])^2+\frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha)-N_f)\Big)\right]\\ &+ \mathcal{O}(\epsilon^3)\ .\end{aligned} \tag{3.15}$$

Here $\langle\cdots\rangle_{U(x)}^\nu$ denotes the expectation value with respect to the full field $U(x)$ at fixed topology ν. In the term proportional to $\bar\Delta(x)$, the subscripts eff can be dropped since $\bar\Delta(x)$ is already of $\mathcal{O}(\epsilon^2)$. The form of the first three terms is identical to the result for $\mu=0$ in [4]. The only difference is that here, the resolvent $\Sigma_\nu(\eta,\alpha)$ also depends on the rescaled chemical potential. The remaining term proportional to α^2 is an explicit correction to the result of [4], at non-vanishing chemical potential.

For the pseudoscalar two-point function, we obtain in the same way

$$\begin{aligned}\langle P_0(x)P_0(0)\rangle_{U(x)}^\nu &= \frac{\Sigma_{eff}^2N_f}{\eta_{eff}}\left(\frac{\Sigma_\nu(\eta_{eff},\alpha_{eff})}{\Sigma}-\frac{\nu^2N_f}{\eta_{eff}}\right)\\ &+ \bar\Delta(x)\frac{2\Sigma^2}{F^2}\left[-\frac{\partial}{\partial\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}-N_f\frac{\Sigma_\nu(\eta,\alpha)^2}{\Sigma^2}-\frac{N_f^2}{\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}+\frac{\nu^2N_f}{\eta^2}+N_f\right.\\ &\left.+\frac{\alpha^2}{\eta^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha)-(\mathrm{Tr}[\Gamma])^2+\frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha)-N_f)\Big)\right]+\mathcal{O}(\epsilon^3)\ .\end{aligned} \tag{3.16}$$

As before, all terms but the last are of the same form as for $\mu=0$, but now depend on the rescaled chemical potential α and thus F, also at leading order. The term in the last line proportional to α^2 is again an explicit correction term.

We now explicitly give the three functions appearing in the results for the two-point correlations above in the particular case of $N_f=2$ flavours. The results for more flavours follow easily using Appendix B.

The $N_f=2$ flavour partition function reads [19, 20]

$$\underline{N_f=2}: \quad \mathcal{Z}_\nu(\eta_{eff}) = \int_0^1 d\lambda\lambda\ e^{\frac{1}{2}\alpha_{eff}^2(4\lambda^2-2)}I_\nu(\lambda\,\eta_{eff})^2\ , \tag{3.17}$$

where I_ν denotes the modified I-Bessel function.

From the above equations, the condensate of Eq. (3.10) easily follows, and for two flavours we arrive at

$$\underline{N_f=2:}\ \frac{\Sigma_\nu(\eta_{eff},\alpha_{eff})}{\Sigma} = \frac{1}{\mathcal{Z}_\nu(\eta_{eff})}\int_0^1 d\lambda\lambda\ e^{\frac{1}{2}\alpha_{eff}^2(4\lambda^2-2)}I_\nu(\lambda\,\eta_{eff})\Big(\frac{\nu}{\eta_{eff}}I_\nu(\lambda\,\eta_{eff})+\lambda I_{\nu+1}(\lambda\,\eta_{eff})\Big). \quad (3.18)$$

The second quantity $\mathcal{Y}_\nu(\eta,\alpha)$ equally follows from its definition (3.13) by differentiation. Using Eq. (3.17) we obtain for two flavours

$$\underline{N_f=2:}\ \mathcal{Y}_\nu(\eta_{eff},\alpha_{eff}) = \frac{2}{\mathcal{Z}_\nu(\eta_{eff})}\int_0^1 d\lambda\lambda\frac{1}{2}(4\lambda^2-2)\,e^{\frac{1}{2}\alpha_{eff}^2(4\lambda^2-2)}I_\nu(\lambda\,\eta_{eff})^2\ , \quad (3.19)$$

given here to $\mathcal{O}(\epsilon^2)$, which we will need for the flavoured correlators below.

The third function is obtained by differentiation of the following generalised partition function computed in Appendix B

$$\mathcal{Z}_{gen} \equiv \int_{U(N_f)} d_H U_0 \det[U_0^\nu] e^{\frac{1}{2}VF_{eff}^2\mu^2\mathrm{Tr}[\Gamma U_0^\dagger\Gamma U_0]+\frac{1}{2}V\Sigma_{eff}m\mathrm{Tr}[U_0+U_0^\dagger]\ +\ \omega\mathrm{Tr}\left[(\Gamma U_0^\dagger\Gamma U_0)^2\right]}\ . \quad (3.20)$$

It is remarkable that this group integral can be calculated, even when adding any power $\mathrm{Tr}\left[(\Gamma U_0^\dagger\Gamma U_0)^k\right]$ to the partition function of Eq. (3.3), as was pointed out in [21]. We generalise the result given in [21] in Appendix B to arbitrary masses $\mathcal{M}\neq m\mathbf{1}_{N_f}$, with $\mathrm{Tr}[\Gamma]\neq 0$. For two flavours this generalised partition function reads

$$\mathcal{Z}_{gen}(\eta_{eff}) = \int_0^1 d\lambda\lambda\ e^{\frac{1}{2}\alpha_{eff}^2(4\lambda^2-2)+\omega\left(16(\lambda^4-\lambda^2)+2\right)}I_\nu(\lambda\,\eta_{eff})^2 \quad (3.21)$$

and for more flavours the result is given by Eq. (B.8). We then obtain

$$\mathcal{X}_\nu(\eta_{eff},\alpha_{eff}) = \frac{\partial}{\partial\omega}\ln[\mathcal{Z}_{gen}]\Big|_{\omega=0} = \langle\mathrm{Tr}\left[(\Gamma U_0^\dagger\Gamma U_0)^2\right]\rangle_{U_0}^\nu\ , \quad (3.22)$$

making only the ordinary partition function of Eq. (3.17) appear in the denominator. We again display the two-flavour result at equal mass,

$$\underline{N_f=2:}\ \mathcal{X}_\nu(\eta_{eff},\alpha_{eff}) = \frac{1}{\mathcal{Z}_\nu(\eta_{eff})}\int_0^1 d\lambda\lambda\left(16(\lambda^4-\lambda^2)+2\right)e^{\frac{1}{2}\alpha_{eff}^2(4\lambda^2-2)}I_\nu(\lambda\,\eta_{eff})^2\ . \quad (3.23)$$

3.2 Flavoured correlation functions

We now give the sums over flavoured two-point correlations, which are easier to compute on the lattice, since they do not involve disconnected contributions. This is just a matter of inserting Eqs. (3.7) - (3.9) into the results of Eqs. (2.26) and (2.27) at fixed topology. Here it is useful to observe that we have already computed the combinations $\langle\mathrm{Tr}\left[(U_0\pm U_0^\dagger)^2\right]\rangle_{U_0}-\frac{1}{N_f}\left\langle(\mathrm{Tr}[U_0\pm U_0^\dagger])^2\right\rangle_{U_0}$ in the unflavoured cases. We obtain for the scalar two-point function,

$$\begin{aligned}\sum_a\langle S_a(x)S_a(0)\rangle_{U(x)}^\nu &= \frac{\Sigma_{eff}^2}{2}\Bigg[-\frac{\partial}{\partial\eta_{eff}}\frac{\Sigma_\nu(\eta_{eff},\alpha_{eff})}{\Sigma}-N_f\frac{\Sigma_\nu(\eta_{eff},\alpha_{eff})^2}{\Sigma^2}-\frac{N_f^2}{\eta_{eff}}\frac{\Sigma_\nu(\eta_{eff},\alpha_{eff})}{\Sigma}+\frac{\nu^2N_f}{\eta_{eff}^2}\\ &\quad +N_f+\frac{\alpha_{eff}^2}{\eta_{eff}^2}\Big(N_f\mathcal{Y}_\nu(\eta_{eff},\alpha_{eff})-(\mathrm{Tr}[\Gamma])^2+\frac{\alpha_{eff}^2}{2}(\mathcal{X}_\nu(\eta_{eff},\alpha_{eff})-N_f)\Big)\Bigg]\\ &+\ \bar\Delta(x)\frac{\Sigma^2}{2F^2}\Bigg[-N_f\frac{\partial}{\partial\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}-N_f^2\frac{\Sigma_\nu(\eta,\alpha)^2}{\Sigma^2}+\frac{(-3N_f^2+2)}{N_f\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}-(N_f^2-2)\frac{\nu^2}{\eta^2}\\ &\quad +N_f^2+\frac{4\alpha^2}{N_f\eta^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha)-(\mathrm{Tr}[\Gamma])^2+\frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha)-N_f)\Big)\Bigg]\ +\mathcal{O}(\epsilon^3)\ ,\end{aligned} \quad (3.24)$$

and for the pseudoscalar correlator,

$$\begin{aligned}
\sum_a \langle P_a(x)P_a(0)\rangle^{\nu}_{U(x)} &= \frac{\Sigma_{eff}^2}{2}\Bigg[\frac{(N_f^2-1)}{\eta_{eff}}\frac{\Sigma_\nu(\eta_{eff},\alpha_{eff})}{\Sigma} \\
&\qquad -\frac{\alpha_{eff}^2}{\eta_{eff}^2}\Big(N_f\mathcal{Y}_\nu(\eta_{eff},\alpha_{eff})-(\mathrm{Tr}[\Gamma])^2+\frac{\alpha_{eff}^2}{2}(\mathcal{X}_\nu(\eta_{eff},\alpha_{eff})-N_f)\Big)\Bigg] \\
&+ \bar{\Delta}(x)\frac{\Sigma^2}{2F^2}\Bigg[\frac{(N_f^2+2)}{N_f}\Big(\frac{\partial}{\partial\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}+N_f\frac{\Sigma_\nu(\eta,\alpha)^2}{\Sigma^2}\Big)+3\frac{N_f}{\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma}+(N_f^2-4)\frac{\nu^2}{\eta^2} \\
&\qquad +N_f^2-4-\frac{4\alpha^2}{N_f\eta^2}\Big(N_f\mathcal{Y}_\nu(\eta,\alpha)-(\mathrm{Tr}[\Gamma])^2+\frac{\alpha^2}{2}(\mathcal{X}_\nu(\eta,\alpha)-N_f)\Big)\Bigg] \\
&+ \mathcal{O}(\epsilon^3)\ , \qquad (3.25)
\end{aligned}$$

where we now have explicit μ-dependent corrections at leading order, which is not the case for the other correlation functions.

4 Conclusions

We have computed scalar and pseudoscalar two-point correlation functions in the epsilon regime of Chiral Perturbation Theory, up to and including corrections of $\mathcal{O}(\epsilon^2)$. The main new feature of our results is the inclusion of an isospin chemical potential of real or imaginary type. This leads to the appearance of the low energy constant F already at leading $\mathcal{O}(\epsilon^0)$. We find corrections to the meson correlation functions obtained at $\mu = 0$, which contain both explicit terms proportional to $\alpha^2 = \mu^2F^2V$ and α^4, as well as implicit corrections which arise through those to the partition function and its derivatives. All of our results are obtained for N_f-flavour QCD. Possible extensions of our work include the calculation of correlations functions in quenched or partially quenched QCD, which would require computing supersymmetric extensions of the group integrals. Axial or vector current two-point functions are also feasible.

Our results provide alternate means of extracting the low energy constants Σ and F from lattice calculations. In particular, the additional parameter μ and the fact that F appears at leading order should help improve the precision in the determination of this low energy constant. Moreover, we have obtained a new one-loop result for F_{eff}, which includes one of the $\mathcal{O}(\epsilon^2)$ corrections to the partition function.

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A Derivation of zero-mode group integral identities for $\mu \neq 0$

In this appendix we will derive a set of $U(N_f)$ group identities among expectation values of averages over various traces. Those relations are needed to arrive at Eqs. (3.7) - (3.9) in the expressions for the two-point functions.

In particular we have to specify here

$$C = \mu\Gamma = \mu\ \mathrm{diag}(\mathbf{1}_{N_1}, -\mathbf{1}_{N_2})\ , \qquad (A.1)$$

with $N_1 \neq N_2$ in general. Furthermore let us stress that our identities hold for degenerate masses only,

$$V\Sigma\mathcal{M} = \eta\,\mathbf{1}_{N_f} \qquad (A.2)$$

where η is the rescaled mass. In order to simplify notation we will use

$$\mathcal{Z}_\nu \equiv \int_{U(N_f)} d_H U \det[U^\nu] \exp\left[\frac{1}{2}\alpha^2 \mathrm{Tr}[\Gamma U^\dagger \Gamma U] + \frac{1}{2}\eta \mathrm{Tr}[U+U^\dagger]\right], \tag{A.3}$$

$$\langle\, O \,\rangle \equiv \frac{1}{\mathcal{Z}_\nu}\int_{U(N_f)} d_H U \; O \; \det[U^\nu] \exp\left[\frac{1}{2}\alpha^2 \mathrm{Tr}[\Gamma U^\dagger \Gamma U] + \frac{1}{2}\eta \mathrm{Tr}[U+U^\dagger]]\right], \tag{A.4}$$

dropping most indices from the main text. In particular here $U = U_0$ denotes the constant $U(N_f)$ matrix. The full $\mathcal{O}(\epsilon^2)$ improved expectation values Eqs. (3.3) and (3.4) are trivially obtained by shifting $\alpha \to \alpha_{eff}$ and $\eta \to \eta_{eff}$.

Following [4] we introduce the explicit representation of the left differentiation with respect to group elements U_{kl} of $U(N_f)$

$$\nabla_a \equiv i(t_a U)_{kl} \frac{\partial}{\partial U_{kl}} \; . \tag{A.5}$$

Throughout this appendix, the t_a denote the generators of the algebra of $u(N_f)$. They satisfy the $U(N_f)$ completeness relation

$$(t_a)_{ij}(t_a)_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} \tag{A.6}$$

in the normalisation $\mathrm{Tr}[t_a t_b] = \frac{1}{2}\delta_{ab}$. For example, we obtain

$$\begin{aligned} \nabla_a U &= +i t_a U \; , \quad \nabla_a U^\dagger = -i U^\dagger t_a \; , \\ \nabla_a \det[U] &= i\mathrm{Tr}[t_a]\det[U] \; . \end{aligned} \tag{A.7}$$

Due to the left invariance of the Haar measure, integrals over total derivatives with respect to ∇_a vanish:

$$0 = \int_{U(N_f)} d_H U \; \nabla_a \mathrm{Tr}\left[t_a G(U) \det[U^\nu] \exp\left[\frac{1}{2}\alpha^2 \mathrm{Tr}[\Gamma U^\dagger \Gamma U] + \frac{1}{2}\eta \mathrm{Tr}[U+U^\dagger]\right]\right], \tag{A.8}$$

for any function $G(U)$. By choosing a suitable set of functions, we will generate a closed set of equations for averages that can be solved for in terms of known generating functions.

The simplest choice in Eq. (A.8) is $G(U) = \mathbf{1}$ and $a = 0$, without summing over a, which leads to the following identity

$$0 = \nu N_f + \frac{\eta}{2}\langle \mathrm{Tr}[U - U^\dagger]\rangle \; . \tag{A.9}$$

This equation is invariant under complex conjugation when simultaneously changing $\nu \to -\nu$ (see the measure in Eq. (A.4)). Next, in choosing $G(U) = U - U^\dagger$ and keeping $a = 0$, we obtain:

$$0 = \frac{\eta}{2}\left\langle (\mathrm{Tr}[U-U^\dagger])^2 \right\rangle + \nu N_f \langle \mathrm{Tr}[U-U^\dagger]\rangle + \langle \mathrm{Tr}[U+U^\dagger]\rangle \; . \tag{A.10}$$

Next, we sum over a in Eq. (A.8) is $G(U) = U$, leading to:

$$0 = (N_f + \nu)\langle \mathrm{Tr}[U]\rangle + \frac{\eta}{2}\left(\langle \mathrm{Tr}\left[U^2\right]\rangle - N_f\right) + \frac{\alpha^2}{2}\left(\langle \mathrm{Tr}\left[U^2 \Gamma U^\dagger \Gamma\right]\rangle - \langle \mathrm{Tr}[U]\rangle\right) \; . \tag{A.11}$$

Either by complex conjugation and changing $\nu \to -\nu$, or simply choosing $G(U) = -U^\dagger$ we obtain

$$0 = (N_f - \nu)\langle \mathrm{Tr}[U^\dagger]\rangle + \frac{\eta}{2}\left(\langle \mathrm{Tr}\left[U^{\dagger\,2}\right]\rangle - N_f\right) + \frac{\alpha^2}{2}\left(\langle \mathrm{Tr}\left[U \Gamma U^{\dagger\,2} \Gamma\right]\rangle - \langle \mathrm{Tr}[U^\dagger]\rangle\right) \; . \tag{A.12}$$

Taking the sum of Eqs. (A.11) and (A.12) we obtain an equation for $\langle \mathrm{Tr}\left[(U-U^\dagger)^2\right]\rangle$.

The quantity $\langle \mathrm{Tr}\left[U^2 \Gamma U^\dagger \Gamma\right]\rangle$ and its conjugate which are new for $\alpha \neq 0$ cannot be easily derived from a known generating functional. Therefore we need an additional set of equations compared to [4], which is generated by choosing $G(U) = U\Gamma U^\dagger \Gamma$

$$0 = \frac{\eta}{2}\left(\langle \mathrm{Tr}\left[U^2 \Gamma U^\dagger \Gamma\right]\rangle - \langle \mathrm{Tr}[U^\dagger]\rangle\right) + (N_f + \nu)\langle \mathrm{Tr}[U \Gamma U^\dagger \Gamma]\rangle - \left\langle (\mathrm{Tr}[\Gamma])^2 \right\rangle + \frac{\alpha^2}{2}\left(\langle \mathrm{Tr}\left[(U \Gamma U^\dagger \Gamma)^2\right]\rangle - N_f\right) \; , \tag{A.13}$$

and its complex conjugate

$$0 \;=\; \frac{\eta}{2}\left(\langle \mathrm{Tr}\Big[U\Gamma U^{\dagger\,2}\Gamma\Big]\rangle - \langle \mathrm{Tr}[U]\rangle\right) + (N_f-\nu)\langle \mathrm{Tr}[U\Gamma U^\dagger\Gamma]\rangle - \Big\langle (\mathrm{Tr}[\Gamma])^2\Big\rangle + \frac{\alpha^2}{2}\left(\langle \mathrm{Tr}\Big[(U\Gamma U^\dagger\Gamma)^2\Big]\rangle - N_f\right) \;. \quad \text{(A.14)}$$

We can now eliminate $\langle \mathrm{Tr}\Big[U^2\Gamma U^\dagger\Gamma\Big]\rangle$ and its conjugate by using the sum of Eqs. (A.13) and (A.14) to obtain

$$\begin{aligned}\langle \mathrm{Tr}\Big[(U-U^\dagger)^2\Big]\rangle \;&=\; -\frac{2N_f}{\eta}\langle \mathrm{Tr}[U+U^\dagger]\rangle - \frac{2\nu}{\eta}\langle \mathrm{Tr}[U-U^\dagger]\rangle \\ &\quad + \frac{2\alpha^2}{\eta^2}\left[2N_f\langle \mathrm{Tr}[U\Gamma U^\dagger\Gamma]\rangle - 2\Big\langle(\mathrm{Tr}[\Gamma])^2\Big\rangle + \alpha^2\Big(\langle \mathrm{Tr}\Big[(U\Gamma U^\dagger\Gamma)^2\Big]\rangle - N_f\Big)\right] \;. \quad \text{(A.15)}\end{aligned}$$

All objects on the right hand side can now be generated. Differentiating the partition function with respect to the mass we have

$$\langle \mathrm{Tr}[U+U^\dagger]\rangle \;=\; 2\frac{\partial}{\partial\eta}\ln[\mathcal{Z}_\nu] \;=\; 2N_f\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma} \;, \quad \text{(A.16)}$$

using the definition Eq. (3.10). Eq. (A.9) provides the difference

$$\langle \mathrm{Tr}[U-U^\dagger]\rangle \;=-\frac{2\nu N_f}{\eta} \;. \quad \text{(A.17)}$$

Differentiating the partition function with respect to α^2 we obtain (see the definition Eq. (3.13))

$$\langle \mathrm{Tr}[U\Gamma U^\dagger\Gamma]\rangle \;=\; 2\frac{\partial}{\partial\alpha^2}\ln[\mathcal{Z}_\nu] \;=\; \mathcal{Y}_\nu(\eta,\alpha) \;. \quad \text{(A.18)}$$

The trace of the square is obtained from the generalised partition function $\mathcal{Z}_{gen}$ derived in Appendix B below,

$$\langle \mathrm{Tr}\Big[(U\Gamma U^\dagger\Gamma)^2\Big]\rangle \;=\; \frac{\partial}{\partial_\omega}\ln[\mathcal{Z}_{gen}]\Big|_{\omega=0} \;=\; \mathcal{X}_\nu(\eta,\alpha) \;, \quad \text{(A.19)}$$

see the definition Eq. (3.14). Inserting all these into Eq. (A.15) we arrive at Eq. (3.9), where $\langle \mathrm{Tr}\Big[(U+U^\dagger)^2\Big]\rangle$ is obtained trivially by adding $+4N_f$.

The missing squares of traces in Eqs. (2.23) and (2.25) follow by differentiating twice with respect to the mass,

$$\Big\langle(\mathrm{Tr}[U+U^\dagger])^2\Big\rangle \;=\; 4\frac{1}{\mathcal{Z}_\nu}\frac{\partial^2}{\partial_\eta^2}\mathcal{Z}_\nu \;=\; 4N_f\frac{\partial}{\partial\eta}\frac{\Sigma_\nu(\eta,\alpha)}{\Sigma} + 4N_f^2\frac{\Sigma_\nu(\eta,\alpha)^2}{\Sigma^2} \;, \quad \text{(A.20)}$$

leading to Eq. (3.7). Finally, a last equation is needed that contains the square of the trace of the difference, which appears in Eq. (A.10) above. Using Eq. (A.9) and the generator of Eq. (A.16), we obtain the Eq. (3.8) in Sec. 3, after replacing the couplings α and η by their effective counterparts, α_{eff} and η_{eff}.

B Calculation of the generalised partition function

In this appendix we compute the following generalisation of the partition function of Eq. (3.20)

$$\mathcal{Z}_{gen}(\{\eta\}) \;\equiv\; \int_{U(N_f)} d_H U_0 \det[U_0]^\nu \exp\left[\frac{1}{2}\mathrm{Tr}[\mathcal{M}(U_0+U_0^\dagger)] + \sum_p a_p \mathrm{Tr}\left[(U_0\Gamma U_0^\dagger\Gamma)^p\right]\right] \;, \quad \text{(B.1)}$$

where $\mathcal{M} = \mathrm{diag}(\eta_{f1=1},\cdots,\eta_{N_1},\eta_{f2=1},\cdots,\eta_{N_2})$ contains the rescaled masses which may now be different. As in the previous appendix, the inclusion of effective couplings is trivial. Without loss of generality we choose $N_2 \geq N_1$ in $N_f = N_1 + N_2$. The volume V and higher order coupling constants are all absorbed into the coefficients a_p, where the sum can have a finite or an infinite number of terms. We only require that the integrals converge.

The result for this generalised partition function was given in [21] for degenerate masses and $N_1 = N_2$. Our generalisation follows [20] closely. Because we differentiate the generalised partition function with respect to

the couplings a_p in order to generate expectation values, we have to keep track of all constants that depend on the a_p. The unitary matrix U_0 can be parametrised as follows [19, 20]

$$
\begin{aligned}
U_0 &= \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \Lambda \begin{pmatrix} v_1^\dagger & 0 \\ 0 & v_2^\dagger \end{pmatrix} , \\
\Lambda &\equiv \begin{pmatrix} \hat{\lambda} & \sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & 0 \\ \sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & -\hat{\lambda} & 0 \\ 0 & 0 & -\mathbf{1}_{N_2 - N_1} \end{pmatrix} .
\end{aligned} \tag{B.2}
$$

Here, we denote by the matrix $\hat{\lambda} \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_{N_1})$ containing the real numbers $\lambda_k \in [0,1]$ for $k = 1, \ldots, N_1$. The unitary submatrices are $u_1, v_1 \in U(N_1)$, $u_2 \in U(N_2)$ and $v_2 \in \tilde{U}(N_2) \equiv U(N_2)/(U(1)^{N_1} \times U(N_2 - N_1))$. The matrix Λ is Hermitean and we observe that

$$
\mathrm{Tr}\left[(U_0 \Gamma U_0^\dagger \Gamma)^p\right] = \mathrm{Tr}\Big[(\Lambda \Gamma \Lambda \Gamma)^p\Big] , \tag{B.3}
$$

so that all unitary degrees of freedom drop out. From

$$
(\Lambda\Gamma)^2 = \begin{pmatrix} 2\hat{\lambda}^2 - \mathbf{1}_{N_1} & -2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & 0 \\ 2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & 2\hat{\lambda}^2 - \mathbf{1}_{N_1} & 0 \\ 0 & 0 & \mathbf{1}_{N_2 - N_1} \end{pmatrix} , \tag{B.4}
$$

we obtain

$$
\mathrm{Tr}\left[(U_0 \Gamma U_0^\dagger \Gamma)^p\right] = \mathrm{Tr}\left[\begin{pmatrix} 2\hat{\lambda}^2 - \mathbf{1}_{N_1} & -2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} \\ 2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} & 2\hat{\lambda}^2 - \mathbf{1}_{N_1} \end{pmatrix}^p \right] - \mathrm{Tr}[\mathbf{1}_{N_2 - N_1}] . \tag{B.5}
$$

The $2N_1$ eigenvalues can be written in diagonal matrix form as

$$
X_\pm \equiv 2\hat{\lambda}^2 - \mathbf{1}_{N_1} \pm i\, 2\hat{\lambda}\sqrt{\mathbf{1}_{N_1} - \hat{\lambda}^2} . \tag{B.6}
$$

We thus arrive at

$$
\begin{aligned}
\mathrm{Tr}\left[(U_0 \Gamma U_0^\dagger \Gamma)^p\right] + (N_2 - N_1) &= \mathrm{Tr}[X_+^p + X_-^p] = \sum_{i=1}^{N_1} 2 \sum_{q=0}^{[\frac{p}{2}]} \binom{p}{2q} \Big(2\lambda_i^2 - 1\Big)^{p-2q} \Big(- 4\lambda_i^2 (1 - \lambda_i^2)\Big)^q \\
&= \sum_{i=1}^{N_1} 2 T_{2p}(\lambda_i) .
\end{aligned} \tag{B.7}
$$

The real polynomials $T_{2p}(\lambda_i)$ of degree $2p$ that we obtain are the Chebyshev polynomials of the first kind[1].

Coming back to the integral in Eq. (B.1), we can now go to an eigenvalue basis using the parametrisation of Eq. (B.2). Because of the decoupling of the unitary degrees of freedom in Eq. (B.3), the calculation is identical to the one presented in [20] Sec. 3, to which we refer the reader for details. In particular integrating out all unitary degrees of freedom cancels the Jacobian from the parametrisation eq. (B.2). Collecting all of these results, we obtain the following expression for N_f flavours:

$$
\mathcal{Z}_{gen}(\{\eta\}) = \frac{\mathrm{N}}{\Delta_{N_1}(\{\eta_{f1}\})\Delta_{N_2}(\{\eta_{f2}\})} \begin{vmatrix} \mathcal{Z}_{gen}(\eta_{f1=1}, \eta_{f2=1}) & \cdots & \mathcal{Z}_{gen}(\eta_{f1=1}, \eta_{N_2}) \\ \cdots & \cdots & \cdots \\ \mathcal{Z}_{gen}(\eta_{N_1}, \eta_{f2=1}) & \cdots & \mathcal{Z}_{gen}(\eta_{N_1}, \eta_{N_2}) \\ I_\nu(\eta_{f2=1}) & \cdots & I_\nu(\eta_{N_2}) \\ \eta_{f2=1}^{N_2 - N_1 - 1} I_\nu^{(N_2 - N_1 - 1)}(\eta_{f2=1}) & \cdots & \eta_{N_2}^{N_2 - N_1 - 1} I_\nu^{(N_2 - N_1 - 1)}(\eta_{N_2}) \end{vmatrix} , \tag{B.8}
$$

[1] In [21] the polynomial was given in the form $\cos(2p \cos^{-1}(\lambda))$.

where $\Delta_{N_1}(\{\eta_{f1}\} = \prod_{j>i}^{N_1}(\eta_j^2 - \eta_i^2)$ is the Vandermonde determinant within each flavour. The generalised $N_f = 2$ flavour partition function, $\mathcal{Z}_{gen}(\eta_1, \eta_2)$, that is used inside the determinant above is given by

$$\mathcal{Z}_{gen}(\eta_1, \eta_2) \;\equiv\; \int_0^1 d\lambda\, \lambda\, \exp\left[\sum_p a_p \Big(2T_{2p}(\lambda) - (N_2 - N_1)\Big)\right] I_\nu(\lambda\eta_1) I_\nu(\lambda\eta_2)\ . \tag{B.9}$$

With $I_\nu^{(k)}(\eta_{f1})$ we denote the k-th derivative of the I-Bessel function. The I-Bessel function itself, in fact, is a one-flavour partition function, $\mathcal{Z}_\nu = I_\nu(\eta)$, up to a trivial a_p dependent constant. The constant N in Eq. (B.8) is an irrelevant normalisation factor that does not contribute in expectation values. Eq. (B.8) is the main new result of this appendix. It is the solution of the generalised partition function given by the group integral (B.1). It extends the previous result of [21] to nondegenerate masses and $\mathrm{Tr}[\Gamma] \neq 0$.

As an example, for $p = 1$, we get

$$2T_2(\lambda) \;=\; 4\lambda^2 - 2\ , \tag{B.10}$$

which gives back the standard partition function of Eq. (3.17), for $a_1 = \frac{1}{2}\alpha_{eff}^2$ (and mass η_{eff}). It is explicitly given in Eq. (3.17), up to a constant prefactor. For $p = 2$, the second polynomial is

$$2T_4(\lambda) \;=\; 16(\lambda^4 - \lambda^2) + 2\ , \tag{B.11}$$

leading to the generalised partition function with $a_2 = \omega$ that is needed in Eq. (3.21).

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