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Abstract. The electromagnetic waves propagation in a non linear medium can be described by the Kerr model in case of instantaneous response of the material, or by the Kerr-Debye model if the material exhibits a finite response time. Both models are quasilinear hyperbolic and are endowed with a dissipative entropy. Initial-boundary value problem with the maximal dissipative impedance boundary condition is considered. When the response time is fixed, in the one dimensional and the two dimensional transverse electric cases, the global existence of smooth solutions for the Kerr-Debye system is established. When the response time tends to zero, the convergence of the Kerr-Debye model to the Kerr model is proved in the general case: the Kerr model is the zero relaxation limit of the Kerr-Debye model.

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1 Introduction

Nonlinear Maxwell's equations are used for modelling nonlinear optical phenomena. The electromagnetic wave propagation is described by Maxwell's system:

$$\begin{aligned} &\partial_t D - \operatorname{curl} H = 0, \\ &\partial_t B + \operatorname{curl} E = 0, \\ &\operatorname{div} D = \operatorname{div} B = 0. \end{aligned}$$

The field quantities E and H represent the electric and magnetic fields, D and B the electric and magnetic displacements. The constitutive relations are given by

$$\begin{aligned} B &= \mu_0 H, \\ D &= \varepsilon_0 E + P, \end{aligned}$$

where P is the polarization.
We consider a homogeneous isotropic nonlinear medium (a cristal for example), so the polarization is nonlinear. The Kerr model describes an instantaneous response of the medium, in this case, P is given by:

$$P = P_K = \varepsilon_0 \varepsilon_r |E|^2 E.$$

If the medium exhibits a finite response time τ we consider a Kerr-Debye model in which P is given by:

$$P = P_{KD} = \varepsilon_0 \chi E,$$

where

$$\partial_t \chi + \frac{1}{\tau}\chi = \frac{1}{\tau}\varepsilon_r |E|^2$$

(see for example [17] or [20]).

So the Kerr-Debye model is a relaxation approximation of the Kerr model and τ is the relaxation parameter (for a general presentation of relaxation problems, see [15]). Formally, when τ tends to 0, χ converges to $\varepsilon_r |E|^2$ and P_{KD} converges to P_K.

In order to study realistic physical situations we study the initial-boundary value problem (IBVP) in the simplest geometry. We denote by $\Omega = I\!R^+ \times I\!R^2$ the domain in which the nonlinear material is confined, and by $\Gamma = \{0\} \times I\!R^2$ its boundary. We consider the Kerr and the Kerr-Debye models in the domain $I\!R_t^+ \times \Omega$ with the impedance boundary condition on $I\!R_t^+ \times \Gamma$ and with null initial data.
Once nondimensionalized the IBVP for the Kerr model writes:
for $(t,x) \in I\!R^+ \times \Omega$

$$\begin{cases} \partial_t D - \text{curl } H = 0, \\ \partial_t H + \text{curl } E = 0, \end{cases} \tag{1}$$

with the constitutive relation :

$$D = (1 + |E|^2)E. \tag{2}$$

We suppose that the initial data vanish:

$$D(0,x) = H(0,x) = 0 \text{ for } x \in \Omega \tag{3}$$

so that we obtain the conservative relations

$$\text{div } D = \text{div } H = 0 \text{ for } t \geq 0. \tag{4}$$

We denote by $n = {}^t(-1,0,0)$ the outer unit normal on Γ. We consider the impedance boundary condition

$$H \times n + a((E \times n) \times n) = \varphi \text{ for } (t,x) \in I\!R^+ \times \Gamma, \tag{5}$$

where a is a positive endomorphism acting on Γ. If $a = Id$, (5) is the classical ingoing wave condition, if $a = 0$, (5) is a Dirichlet boundary condition. The system is resting for $t \leq 0$. It is only excited by the source term φ which is localized in the variable (t, x_2, x_3) and takes its values in Γ.

In the same way the IBVP for the Kerr-Debye model (in which τ is replaced by ε) writes:
for $(t,x) \in I\!R^+ \times \Omega$

$$\begin{cases} \partial_t D_\varepsilon - \text{curl } H_\varepsilon = 0, \\ \partial_t H_\varepsilon + \text{curl } E_\varepsilon = 0, \\ \partial_t \chi_\varepsilon = \dfrac{1}{\varepsilon}(|E_\varepsilon|^2 - \chi_\varepsilon), \end{cases} \tag{6}$$

with the constitutive relation :

$$D_\varepsilon = (1 + \chi_\varepsilon)E_\varepsilon. \tag{7}$$

We suppose that the initial data vanishes :

$$D_\varepsilon(0,x) = H_\varepsilon(0,x) = 0,\ \chi_\varepsilon(0,x) = 0 \text{ for } x \in \Omega, \tag{8}$$

and we have also

$$\text{div } D_\varepsilon = \text{div } H_\varepsilon = 0 \text{ for } t > 0. \tag{9}$$

In addition we suppose that we have the same impedance boundary condition

$$H_\varepsilon \times n + a((E_\varepsilon \times n) \times n) = \varphi \text{ for } (t,x) \in I\!R^+ \times \Gamma. \tag{10}$$

Two dimensional models

Following [19] we can also introduce the two-dimensional transverse magnetic (TM) and transverse electric (TE) models.
For the transverse magnetic case we assume that

$$\begin{aligned} H(x_1,x_2,x_3) &= {}^t(0, H_2(x_1,x_3), 0), \\ E(x_1,x_2,x_3) &= {}^t(E_1(x_1,x_3), 0, E_3(x_1,x_3)), \end{aligned} \tag{11}$$

in the domain $(x_1, x_3) \in \{x_1 > 0\} \times I\!R$. The Maxwell system becomes

$$\begin{cases} \partial_t D_1 + \partial_3 H_2 = 0, \\ \partial_t D_3 - \partial_1 H_2 = 0, \\ \partial_t H_2 + \partial_3 E_1 - \partial_1 E_3 = 0, \end{cases} \tag{12}$$

with the divergence conservation condition

$$\partial_1 D_1 + \partial_3 D_3 = 0 \tag{13}$$

(in this case the divergence condition for H is irrelevant). The impedance boundary condition writes

$$H_2 - aE_3 = \varphi \text{ with } a \geq 0. \tag{14}$$

This system is coupled with (2) for the Kerr model and with (7) and the third equation in (6) for the Kerr-Debye model.

In the transverse electric case, we assume that

$$\begin{array}{l} E(x_1, x_2, x_3) = {}^t(0, E_2(x_1, x_3), 0), \\ H(x_1, x_2, x_3) = {}^t(H_1(x_1, x_3), 0, H_3(x_1, x_3)). \end{array} \tag{15}$$

We obtain

$$\begin{cases} \partial_t D_2 - \partial_3 H_1 + \partial_1 H_3 = 0, \\ \partial_t H_1 - \partial_3 E_2 = 0, \\ \partial_t H_3 + \partial_1 E_2 = 0, \end{cases} \tag{16}$$

with the divergence conservation condition

$$\partial_1 H_1 + \partial_3 H_3 = 0, \tag{17}$$

and the impedance boundary condition becomes

$$H_3 + aE_2 = \varphi \text{ with } a \geq 0. \tag{18}$$

For the Kerr model, (16)-(18) is coupled with

$$D_2 = (1 + (E_2)^2),$$

and for the Kerr-Debye model, (16)-(18) is coupled with

$$D_{2,\varepsilon} = (1 + \chi_\varepsilon) E_{2,\varepsilon},$$

$$\partial_t \chi_\varepsilon = \frac{1}{\varepsilon}((E_{2,\varepsilon})^2 - \chi_\varepsilon).$$

In the case of a fixed finite response time, numerical simulations are obtained for these two-dimensional models by finite-difference methods in [19] and by finite-element methods in [13].

One dimensional model

In [3] the one dimensionnal model is introduced:

$$\begin{array}{l} E(x_1, x_2, x_3) = {}^t(0, e(x_1), 0), \\ H(x_1, x_2, x_3) = {}^t(0, 0, h(x_1)). \end{array} \tag{19}$$

In this case, the IBVP for the Kerr model becomes

$$\begin{cases} \partial_t d + \partial_1 h = 0, \\ \partial_t h + \partial_1 e = 0, \\ d = (1 + e^2) e, \end{cases} \tag{20}$$

with the impedance boundary condition

$$h(t,0) + ae(t,0) = \varphi(t),\ t \geq 0, \text{where}\ \ a \geq 0, \tag{21}$$

and the null initial data

$$e(0,x_1) = h(0,x_1) = 0,\ x_1 \geq 0. \tag{22}$$

The one-dimensional IBVP for the Kerr-Debye model is

$$\begin{cases} \partial_t d_\varepsilon + \partial_1 h_\varepsilon = 0, \\ \partial_t h_\varepsilon + \partial_1 e_\varepsilon = 0, \\ \partial_t \chi_\varepsilon = \dfrac{1}{\varepsilon}((e_\varepsilon)^2 - \chi_\varepsilon), \\ d_\varepsilon = (1+\chi_\varepsilon)e_\varepsilon, \end{cases} \tag{23}$$

with the impedance boundary condition

$$h_\varepsilon(t,0) + ae_\varepsilon(t,0) = \varphi(t),\ t \geq 0, \text{where}\ \ a \geq 0, \tag{24}$$

and the null initial data

$$e_\varepsilon(0,x_1) = h_\varepsilon(0,x_1) = \chi_\varepsilon(0,x_1) = 0,\ x_1 \geq 0. \tag{25}$$

We can also remark that the divergence conditions on h and d are irrelevant for both models.

Mathematical properties and main results

First at all Kerr and Kerr-Debye models are endowed with strictly convex entropies so they are symmetrizable hyperbolic. Furthermore the boundary is characteristic of constant multiplicity (except for the system (20)). So as the boundary conditions are maximal dissipative, for smooth data, the existence of regular solution is a consequence of general results in [10]. For Kerr model a more precise local existence result is proved in [16]. An analogous result for the Kerr-Debye model is established in the Appendix.

We denote by T^* and T^*_ε the lifespan of such regular solutions for Kerr and Kerr-Debye models. The Kerr model is a homogeneous quasilinear hyperbolic system. In the one dimensional case it is a p-system which is genuinely nonlinear for $d \neq 0$. Generally, using the results of [14], the lifespan T^* is finite with formation of shock waves (see also [4]). On the other hand we prove that the Kerr-Debye model is totally linearly degenerated. So we can expect that, if the lifespan T^*_ε is finite, the behaviour of the smooth solution is analogous to the semilinear case. Indeed we obtain this result in the 1d case in [5]: if T^*_ε is finite then the solution and its gradient explode, so no shock wave can appear. In fact, using more precise dissipative properties for the Kerr-Debye model we prove in [7] that $T^*_\varepsilon = +\infty$ for the one dimensionnal Cauchy problem. In the present paper, we prove a global existence result for the IBVP in the 1d case and the 2dTE case.
The general stability problem is considered for relaxation models in [11] and [1]. In these papers, for the Kerr-Debye model it is proved that a constant equilibrium state $(\bar{e}, \bar{h}, \bar{\chi} = (\bar{e})^2)$ is stable if $\bar{e} \neq 0$. Our global existence results do not show the stability property of the state $(0,0,0)$ which remains an open problem.

Kerr-Debye model is a relaxation approximation of Kerr model in the sense developed in [8] . The stability conditions in [8] and [18] are satisfied so it is natural to study the behaviour of the smooth solutions of the Kerr-Debye model when the relaxation coefficient ε tends to 0. Concerning the Cauchy problem with initial data (D_0, H_0, χ_0) satisfying div $D_0 =$ div $H_0 = 0$, $\chi_0 \geq 0$, the convergence for the smooth solutions is proved by [12] using the results of [18]. Generally a boundary layer in time appears because of the non compatibility of the initial data with the equilibrium condition $\chi = |E|^2$. In the present paper we study the convergence for the IBVP. In this case no boundary layer appears: in the time variable the null initial data fits with the equilibrium condition, in the space variables, the boundary condition is the same for both IBVP. In the one dimensional case, we presented in [3] a first convergence result. In fact we obtained the convergence of (23)-(24)-(25) to (20)-(21)-(22) on some interval $[0, \tilde{T}] \subset [0, T^*[$. The same kind of convergence result is announced for the 3d-case in [6]. Here we improve these results in all cases since the convergence is obtained on each interval $[0, T] \subset [0, T^*[$.

In the present paper we chose to establish the main results for the impedance IBVP (see Theorem 1 and Theorem 3 below). In the proofs we must take into account a new difficulty which does not appear for the Cauchy problem: the boundary Γ of the domain Ω is characteristic. Our paper is organized as follows. In Section 2 we exhibit general properties of Kerr and Kerr-Debye models. For a fixed ε we establish the global existence result of the solutions for the Kerr Debye 2dTE model. Section 3 is devoted to convergence results when the relaxation parameter tends to zero.

2 General Properties

2.1 Properties of the Kerr model

We recall the initial-boundary value problem for the general Kerr model.

$$\begin{cases} \partial_t D - \text{curl } H = 0, \\ \partial_t H + \text{curl } E = 0, \\ D = (1 + |E|^2)E, \end{cases} \tag{26}$$

for $(t, x) \in I\!R^+ \times \Omega$,

$$D(t = 0) = H(t = 0) = 0 \text{ for } x \in \Omega, \tag{27}$$

$$H \times n + a((E \times n) \times n) = \varphi \text{ for } (t, x) \in I\!R^+ \times \Gamma. \tag{28}$$

The energy density given by

$$\mathcal{E}_K(D, H) = \frac{1}{2}(|E|^2 + |H|^2 + \frac{3}{2}|E|^4) \tag{29}$$

is a strictly convex entropy (the associated flux function is $E \times H$), so (26) is a quasilinear hyperbolic symmetrizable system.
In the three dimensional case, the eigenvalues are, for $\xi \neq 0$,

$$\lambda_1(E, \xi) \leq \lambda_2(E, \xi) < \lambda_3 = \lambda_4 = 0 < \lambda_5 = -\lambda_2 \leq \lambda_6 = -\lambda_1,$$

so the boundary $I\!R^+ \times \Gamma$ is characteristic of constant multiplicity equal to two. By direct calculations we obtain

$$\lambda_1(E, \xi) = -(1 + |E|^2)^{-\frac{1}{2}}|\xi|,$$

$$\lambda_2(E, \xi) = -(1 + |E|^2)^{-\frac{1}{2}}(1 + 3|E|^2)^{-\frac{1}{2}}((1 + |E|^2)|\xi|^2 + 2(E \cdot \xi)^2)^{\frac{1}{2}}.$$

In the two dimensional cases, TM and TE, the eigenvalues are of the form:

$$\lambda_1(E, \xi) < \lambda_2 = 0 < \lambda_3 = -\lambda_1,$$

so the boundary $I\!R^+ \times \Gamma$ is characteristic of constant multiplicity equal to one. By direct calculations we obtain for the TM model:

$$\lambda_1^{TM}(E, \xi) = -(1 + |E|^2)^{-\frac{1}{2}}(1 + 3|E|^2)^{-\frac{1}{2}}((1 + |E|^2)|\xi|^2 + 2(E \cdot \xi)^2)^{\frac{1}{2}},$$

and for the TE model:

$$\lambda_1^{TE}(E, \xi) = -(1 + 3|E_2|^2)^{-\frac{1}{2}}|\xi|.$$

In the one dimensional case, the system is strictly hyperbolic and the boundary is non characteristic. We have

$$\lambda_1(E) < 0 < \lambda_2 = -\lambda_1,$$

with

$$\lambda_1(E) = -(1 + 3e^2)^{-\frac{1}{2}}.$$

We remark that the impedance boundary condition (28) is maximal dissipative. Generally speaking local existence results of smooth solutions to IBVP for quasilinear hyperbolic systems with characteristic boundary conditions are available in [10]. For the special Maxwell system (26)-(27)-(28) we can also apply the more adapted result in [16]. We precise these results in the 3-d case. We assume that the source term φ is compactly supported in $\mathbb{R}_t^+ \times \Gamma$. We denote by H^s the classical Sobolev spaces and we suppose that φ belongs to $H^s(\mathbb{R}_t \times \Gamma)$ for s great enough. So the boundary condition (28) and the initial data (27) are compatible and by [16] we obtain smooth local solutions.

Proposition 1 *Under the previous assumptions there exists a maximal smooth solution (E, H) to the IBVP (26)-(27)-(28) which lifespan is denoted by T^* and such that*

$$\partial_t^i(D, H) \in \mathcal{C}^0([0, T^*[; H^{3-i}(\Omega)) \text{ for } i = 0, 1, 2, 3. \tag{30}$$

If we apply the results in [10] we obtain the same proposition in which we replace (30) by

$$\partial_t^i(D, H) \in \mathcal{C}^0([0, T^*[; H^{4-i}(\Omega)) \text{ for } i = 0, 1, 2, 3, 4.$$

We have analogous results in the 2-d and the 1-d cases.

2.2 Properties of the Kerr-Debye models.

These models write, for $(t, x) \in \mathbb{R}^+ \times \Omega$

$$\begin{cases} \partial_t D_\varepsilon - \text{curl } H_\varepsilon = 0, \\ \partial_t H_\varepsilon + \text{curl } E_\varepsilon = 0, \\ \partial_t \chi_\varepsilon = \dfrac{1}{\varepsilon}(|E_\varepsilon|^2 - \chi_\varepsilon), \\ D_\varepsilon = (1 + \chi_\varepsilon) E_\varepsilon. \end{cases} \tag{31}$$

$$D_\varepsilon(0, x) = H_\varepsilon(0, x) = 0,\ \chi_\varepsilon(0, x) = 0 \text{ for } x \in \Omega, \tag{32}$$

$$H_\varepsilon \times n + a((E_\varepsilon \times n) \times n) = \varphi \text{ for } (t, x) \in \mathbb{R}^+ \times \Gamma. \tag{33}$$

The divergence free conditions are preserved by the system:

$$\text{div } H_\varepsilon = \text{div } (1 + \chi_\varepsilon) E_\varepsilon = 0. \tag{34}$$

By the third equation in (31) we observe that we have

$$\chi_\varepsilon \geq 0. \tag{35}$$

The energy density given by

$$\mathcal{E}_{KD}(D, H, \chi) = \frac{1}{2}(1 + \chi)^{-1}|D|^2 + \frac{1}{2}|H|^2 + \frac{1}{4}\chi^2 \tag{36}$$

is a strictly convex entropy in the domain $\{\chi \geq 0\}$ (with associated flux $E \times H = (1 + \chi)^{-1} D \times H$). So (31) is a quasilinear symmetrizable hyperbolic system.

In the three dimensional case the eigenvalues are, for $\xi \neq 0$,

$$\mu_1(\chi, \xi) = \mu_2 < \mu_3 = \mu_4 = \mu_5 = 0 < \mu_6 = \mu_7 = -\mu_1,$$

where $\mu_1 = -(1 + \chi)^{-\frac{1}{2}}|\xi|$. So the boundary is characteristic of constant multiplicity equal to three.

In the two dimensional cases we obtain

$$\mu_1(\chi, \xi) < \mu_2 = \mu_3 = 0 < \mu_4 = -\mu_1,$$

where $\mu_1 = -(1+\chi)^{-\frac{1}{2}}|\xi|$. So the boundary is characteristic of constant multiplicity equal to two.

In the one dimensional case, the system is strictly hyperbolic and the boundary is characteristic of constant multiplicity one. The eigenvalues are

$$\mu_1(\chi) < \mu_2 = 0 < \mu_3 = -\mu_1,$$

where $\mu_1 = -(1+\chi)^{-\frac{1}{2}}$.

We remark that each characteristic field of the quasi linear hyperbolic system (31) is linearly degenerated. It suffices to prove this property for the eigenvalue $\mu_1(\chi,\xi) = -(1+\chi)^{-\frac{1}{2}}|\xi|$. The last component of the corresponding eigenvector vanishes so we have:

$$\nabla\mu_1 \cdot r_1 \equiv 0.$$

In the 3d case this property is also obtained by the general result in [2].

As for the Kerr model we could apply the existence results in [10] and we should obtain local smooth solutions to (31)-(32)-(33) such that

$$\partial_t^i(D_\varepsilon, H_\varepsilon, \chi_\varepsilon) \in \mathcal{C}^0([0, T_\varepsilon^*[; H^{4-i}(\Omega)) \text{ for } i = 0,1,2,3,4.$$

In fact, using the special structure of Kerr-Debye model, we obtain the following existence result:

Proposition 2 *Under the assumptions in Proposition 1, there exists a maximal smooth solution $(D_\varepsilon, H_\varepsilon, \chi_\varepsilon)$ to the IBVP (31)-(32)-(33) which lifespan is denoted by T_ε^* and such that*

$$\partial_t^i(D_\varepsilon, H_\varepsilon, \chi_\varepsilon) \in \mathcal{C}^0([0, T_\varepsilon^*[; H^{3-i}(\Omega)) \text{ for } i = 0,1,2,3. \tag{37}$$

We prove this result in the Appendix.

2.3 Dissipative properties of the Kerr-Debye models.

For the Cauchy problem, it is well known (see [12]) that the Kerr-Debye system is dissipative and we have:

$$\frac{d}{dt}\int_{\mathbb{R}^3} \mathcal{E}_{KD}(D,H,\chi)dx = -\frac{\varepsilon}{2}\int_{\mathbb{R}^3} |\partial_t\chi|^2 dx. \tag{38}$$

We generalize this dissipative formula for the IBVP (31)-(32)-(33). In addition we exhibit another dissipative property for the time derivatives. With this object, we lift the boundary condition (33) in the following way: we replace H_2 by $H_2 + \varphi_3(t,x_2,x_3)\eta(x_1)$, H_3 by $H_3 - \varphi_2(t,x_2,x_3)\eta(x_1)$, where η is a cut off function which is equal to 1 in a neighbourdhood of 0, compactly supported in the interval $[0,1[$. Furthermore, to preserve the divergence free condition (34) we replace H_1 by

$$H_1 + (\partial_2\varphi_3 - \partial_3\varphi_2)(t,x_2,x_3)\int_{x_1}^{+\infty}\eta(s)ds$$

and we denote

$$\mathcal{R}(t,x) = \begin{pmatrix} (\partial_2\varphi_3 - \partial_3\varphi_2)(t,x_2,x_3)\displaystyle\int_{x_1}^{+\infty}\eta(s)ds \\ \\ \varphi_3(t,x_2,x_3)\eta(x_1) \\ \\ -\varphi_2(t,x_2,x_3)\eta(x_1) \end{pmatrix}$$

Then the system (31)-(34) becomes in the variable $V = (U,\chi) = (E,H,\chi)$:

$$\begin{aligned} &(i) \quad (1+\chi)\partial_t E + (\partial_t\chi)E - \text{curl } H = G_1, \\ &(ii) \quad \partial_t H + \text{curl } E = G_2, \\ &(iii) \quad \partial_t\chi = \frac{1}{\varepsilon}(|E|^2 - \chi), \end{aligned} \tag{39}$$

where $G_1 = \text{curl}\,\mathcal{R}$ and $G_2 = -\partial_t \mathcal{R}$, with the null initial and boundary conditions:

$$V(0,x) = 0, x \in \Omega, \tag{40}$$

$$H \times n + a((E \times n) \times n) = 0, (t,x) \in I\!R^+ \times \Gamma. \tag{41}$$

and the divergence free conditions

$$\text{div}\, H = \text{div}\,((1+\chi)E) = 0. \tag{42}$$

For the IBVP (39)-(42) the dissipation propoerties are described by the following result:

Proposition 3 *Let V the regular solution for (39)-(42) given by Proposition 2. Then on the interval $[0, T^*[$ we have, with $E_T = (0, E_2, E_3)$,*

$$\frac{d}{dt}\int_\Omega \mathcal{E}_{KD}(D,H,\chi)dx + \frac{\varepsilon}{2}\int_\Omega |\partial_t \chi|^2 dx + \int_\Gamma a(E_T)\cdot E_T dx_2 dx_3 = \int_\Omega (G_1 \cdot E + G_2 \cdot H)dx, \tag{43}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi)|\partial_t E|^2 + |\partial_t H|^2 + \frac{1}{2}(\partial_t \chi)^2\right)dx + \frac{3}{2\varepsilon}\int_\Omega |E|^2|\partial_t E|^2 dx + \frac{\varepsilon}{2}\int_{I\!R^3}|\partial_{tt}\chi|^2 dx \\ + \int_\Gamma a(\partial_t E_T)\cdot \partial_t E_T dx_2 dx_3 = \frac{3}{2\varepsilon}\int_\Omega \chi|\partial_t E|^2 dx + \int_\Omega (\partial_t G_1 \cdot \partial_t E + \partial_t G_2 \cdot \partial_t H)dx.\end{aligned} \tag{44}$$

Proof. Taking the inner product of (39.i) by E and (39.ii) by H we obtain

$$\int_\Omega (1+\chi)\partial_t E\cdot E dx + \int_\Omega |E|^2 \partial_t \chi dx - \int_\Omega \text{curl}\, H\cdot E dx + \int_\Omega \partial_t H \cdot H dx + \int_\Omega \text{curl}\, E \cdot H dx = \int_\Omega (G_1\cdot E + G_2 \cdot H)dx. \tag{45}$$

Integrating by parts we obtain that

$$\begin{aligned}-\int_\Omega \text{curl}\, H\cdot E dx + \int_\Omega \text{curl}\, E\cdot H dx &= \int_\Gamma (H\times n)\cdot E dx_2 dx_3 \\ &= -\int_\Gamma a((E\times n)\times n)\cdot E dx_2 dx_3 \\ &= \int_\Gamma a(E_T)\cdot E_T dx_2 dx_3.\end{aligned}$$

In addition, we have

$$\int_\Omega (1+\chi)\partial_t E\cdot E dx = \frac{1}{2}\frac{d}{dt}\int_\Omega (1+\chi)|E|^2 dx - \frac{1}{2}\int_\Omega |E|^2 \partial_t \chi dx.$$

Using the last equation in (39), we ibtain

$$\frac{1}{2}\int_\Omega |E|^2 \partial_t \chi dx = \frac{1}{4}\frac{d}{dt}\int_\Omega |\chi|^2 dx + \frac{\varepsilon}{2}\int_\Omega (\partial_t \chi)^2 dx.$$

So replacing in (45) we obtain (43).

We can derivate (39) and (41) with respect to t:

$$\begin{aligned}&(i) \quad (1+\chi)\partial_{tt}E + 2\partial_t\chi\partial_t E + \partial_{tt}\chi E - \text{curl}\,\partial_t H = \partial_t G_1,\\ &(ii) \quad \partial_{tt}H + \text{curl}\,\partial_t E = \partial_t G_2,\\ &(iii) \quad \partial_{tt}\chi = \frac{2}{\varepsilon}E\cdot\partial_t E - \frac{1}{\varepsilon}\partial_t\chi,\end{aligned} \tag{46}$$

$$\partial_t H \times n + a((\partial_t E \times n) \times n) = 0 \text{ in } I\!R_t^+ \times \Gamma. \tag{47}$$

As before we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi)|\partial_t E|^2 + |\partial_t H|^2\right) dx + \frac{3}{2}\int_\Omega \partial_t\chi|\partial_t E|^2 dx + \int_\Omega \partial_{tt}\chi E\cdot\partial_t E dx + \int_\Gamma a(\partial_t E_T)\cdot\partial_t E_T dx_2 dx_3$$
$$= \int_\Omega (\partial_t G_1\cdot\partial_t E + \partial_t G_2\cdot\partial_t H) dx.$$

From the last equation in (39) :

$$\int_\Omega \partial_{tt}\chi E\cdot\partial_t E dx = \frac{1}{4}\frac{d}{dt}\int_\Omega (\partial_t\chi)^2 dx + \frac{\varepsilon}{2}\int_\Omega (\partial_{tt}\chi)^2 dx$$

so (44) holds.

Remark 1 *It is clear that (44) holds also for the Cauchy problem.*

2.4 Global existence for the 2d TE Kerr-Debye system.

For the two dimensional TE model, the system is the following: for $t>0$, for $(x_1,x_3)\in I\!R^+\times I\!R$,

$$\begin{cases} \partial_t D_2 - \partial_3 H_1 + \partial_1 H_3 = 0, \\ \partial_t H_1 - \partial_3 E_2 = 0, \\ \partial_t H_3 + \partial_1 E_2 = 0, \\ \partial_t\chi = \dfrac{1}{\varepsilon}(|E_2|^2 - \chi), \end{cases} \tag{48}$$

with null initial data:

$$D_2(0,x_1,x_3) = H_1(0,x_1,x_3) = H_3(0,x_1,x_3) = \chi(0,x_1,x_3) = 0 \text{ for } x_1>0, x_3\in I\!R, \tag{49}$$

and the impedance boundary condition:

$$H_3(t,0,x_3) + aE_2(t,0,x_3) = \varphi(t,x_3) \text{ for } t>0, x_3\in I\!R. \tag{50}$$

Recall that we have $\chi\geq 0$ and

$$\partial_1 H_1 + \partial_3 H_3 = 0. \tag{51}$$

Theorem 1 *Let $\varphi\in H^s(I\!R_t\times I\!R_{x_3})$ (s great enough) with supp $\varphi\subset[0,+\infty[\times I\!R_{x_3}$. Let $\varepsilon>0$. Then the regular solution $W=(D_2,H_1,H_3,\chi)$ to the IBVP (48)-(49)-(50) is defined on $[0,+\infty[$ ($T_\varepsilon^*=+\infty$).*

Proof of Theorem 1.
We fix $\varepsilon>0$ and we assume that $T_\varepsilon^*<+\infty$. Then from Proposition 2, if $T_\varepsilon^*<+\infty$, then

$$\lim_{T\to T_\varepsilon^*} \|W\|_{\mathcal{H}^3(\Omega_T)} = +\infty, \tag{52}$$

where

$$\mathcal{H}^p(\Omega_T) = \left\{ W \text{ such that } \|W\|_{\mathcal{H}^p(\Omega_T)} := \sum_{i=0}^p \|\partial_t^i W\|_{L^\infty(0,T;H^{p-i}(\Omega))} < +\infty \right\}. \tag{53}$$

By variational estimates we will prove uniform bounds on $\|W\|_{\mathcal{H}^3(\Omega_T)}$ for $T<T_\varepsilon^*$ and so we will obtain a contradiction with (52), which prove that $T_\varepsilon^*=+\infty$.

With this object, as in the 3d case, we lift the boundary condition (52): we replace $H_3(t,x_1,x_3)$ by $H_3(t,x_1,x_3)+\varphi(t,x_3)\eta(x_1)$ where η is a smooth function compactly supported in $\mathbb{R}^+$, equal to 1 in a neighbourhood of 0. Furthermore, to preserve the divergence free condition (51) we replace $H_1(t,x_1,x_3)$ by $H_1(t,x_1,x_3)+\partial_3\varphi(t,x_3)\displaystyle\int_{x_1}^{+\infty}\eta(s)ds$. Then the system (48)-(49)-(50) becomes, in the variable $V=(E_2,H_1,H_3,\chi)$:

$$\begin{cases} (1+\chi)\partial_t E_2-\partial_3 H_1+\partial_1 H_3=-\partial_t\chi E_2+G_1,\\ \partial_t H_1-\partial_3 E_2=G_2,\\ \partial_t H_3+\partial_1 E_2=G_3, \end{cases} \tag{54}$$

$$\partial_t\chi=\frac{1}{\varepsilon}(|E_2|^2-\chi), \tag{55}$$

with the null initial and boundary conditions:

$$V(0,x_1,x_3)=0, \tag{56}$$

$$H_3(t,0,x_3)+aE_2(t,0,x_3)=0, \tag{57}$$

where

$$G_1(t,x_1,x_3)=\partial_3^2\varphi(t,x_3)\int_{x_1}^{+\infty}\eta(s)ds-\varphi(t,x_3)\eta'(x_1),$$

$$G_2(t,x_1,x_3)=\partial_t\partial_3\varphi(t,x_3)\int_{x_1}^{+\infty}\eta(s)ds,$$

$$G_3(t,x_1,x_3)=-\partial_t\varphi(t,x_3)\eta(x_1).$$

We recall that the field H remains divergence free:

$$\partial_1 H_1+\partial_3 H_3=0. \tag{58}$$

In this case the dissipation formulae (43)-(44) write:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left((1+\chi)|E_2|^2+|H|^2+\frac{1}{2}|\chi|^2\right)dx_1dx_3+\frac{\varepsilon}{2}\int_\Omega|\partial_t\chi|^2dx+\int_\Gamma a|E_2|^2dx_3=\int_\Omega G\cdot Udx. \tag{59}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega\left((1+\chi)|\partial_t E_2|^2+|\partial_t H|^2+\frac{1}{2}|\partial_t\chi|^2\right)dx_1dx_3+\frac{3}{2\varepsilon}\int_\Omega|E_2|^2|\partial_t E_2|^2dx+\frac{\varepsilon}{2}\int_\Omega|\partial_{tt}\chi|^2dx\\ +\int_\Gamma a|\partial_t E_2|^2dx_3=\frac{3}{2\varepsilon}\int_\Omega\chi|\partial_t E|^2dx+\int_\Omega\partial_t G\cdot\partial_t Udx\end{aligned} \tag{60}$$

where $G=(G_1,G_2,G_3)$ and $U=(E_2,H_1,H_3)$.
From (59), using Gronwall lemma (recall that $\chi\geq 0$), there exists a constant C such that

$$\|V\|_{\mathcal{H}^0(\Omega_{T_\varepsilon^*})}=\|V\|_{L^\infty(0,T_\varepsilon^*;L^2(\Omega))}\leq C. \tag{61}$$

Estimates on the first order time derivatives.
We recall the derivatives of (54)-(55)-(57) with respect to t:

$$\begin{cases} (1+\chi)\partial_t^2 E_2+2\partial_t\chi\partial_t E_2+\partial_t^2\chi E_2-\partial_3\partial_t H_1+\partial_1\partial_t H_3=\partial_t G_1,\\ \partial_t^2 H_1-\partial_3\partial_t E_2=\partial_t G_2,\\ \partial_t^2 H_3+\partial_1\partial_t E_2=\partial_t G_3, \end{cases} \tag{62}$$

$$\partial_t^2 \chi = \frac{1}{\varepsilon}(2E_2 \partial_t E_2 - \partial_t \chi), \tag{63}$$

$$\partial_t H_3(t,0,x_3) + a\partial_t E_2(t,0,x_3) = 0, \tag{64}$$

with the null initial condition obtained by the equations:

$$\partial_t V(0,x_1,x_3) = 0, \tag{65}$$

From these equations, we obtained (60), and by Gronwall lemma, there exists $C > 0$ such that

$$\|\partial_t V\|_{L^\infty(0,T_\varepsilon^*;L^2(\Omega))} \leq C. \tag{66}$$

From (ii) and (iii) in (54) we obtain

$$\|E_2\|_{L^\infty(0,T_\varepsilon^*;H^1(\Omega))} \leq C. \tag{67}$$

Solving (55) we obtain:

$$\chi(t,x) = \frac{1}{\varepsilon}\int_0^t \exp(\frac{s-t}{\varepsilon})|E_2(s,x)|^2 ds. \tag{68}$$

From Sobolev theorem and (66) for all $p \in [2,+\infty[$ there exists $C > 0$ such that

$$\|E_2\|_{L^\infty(0,T_\varepsilon^*;L^p(\Omega))} \leq C, \tag{69}$$

and so with (68), for all p with $1 \leq p < +\infty$, there exists C such that

$$\|\chi\|_{L^\infty(0,T_\varepsilon^*;L^p(\Omega))} \leq C. \tag{70}$$

So by (55) we obtain that for all $p \in [1,+\infty[$,

$$\|\partial_t \chi\|_{L^\infty(0,T_\varepsilon^*;L^p(\Omega))} \leq C. \tag{71}$$

Estimates on the second order times derivatives.
We derivate (62)-(63)-(64) with respect to t. We obtain:

$$\begin{cases} (1+\chi)\partial_t^3 E_2 + 3\partial_t\chi\partial_t^2 E_2 + 3\partial_t^2\chi\partial_t E_2 + \partial_t^3\chi E_2 - \partial_3\partial_t^2 H_1 + \partial_1\partial_t^2 H_3 = \partial_t^2 G_1, \\ \partial_t^3 H_1 - \partial_3\partial_t^2 E_2 = \partial_t^2 G_2, \\ \partial_t^3 H_3 + \partial_1\partial_t^2 E_2 = \partial_t^2 G_3, \end{cases} \tag{72}$$

$$\partial_t^3\chi = \frac{1}{\varepsilon}(2|\partial_t E_2|^2 + 2E_2\partial_t^2 E_2 - \partial_t^2\chi), \tag{73}$$

with the null boundary condition:

$$\partial_t^2 H_3(t,0,x_3) + a\partial_t^2 E_2(t,0,x_3) = 0, \tag{74}$$

and by the equations, we preserve the null initial condition:

$$\partial_t^2 V(0,x_1,x_3) = 0. \tag{75}$$

Taking the inner product of (72) with $\partial_t^2 U$ we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi)|\partial_t^2 E_2|^2 + |\partial_t^2 H|^2\right) dx_1 dx_3 + \int_\Gamma a|\partial_t^2 E_2|^2 dx_3$$

$$+\frac{5}{2}\int_\Omega \partial_t\chi|\partial_t^2 E_2|^2 dx + 3\int_\Omega \partial_t^2\chi\partial_t E_2\partial_t^2 E_2 + \int_\Omega \partial_t^3\chi E_2\partial_t^2 E_2 = \int_\Omega \partial_t^2 G\cdot\partial_t^2 U dx.$$

From (55) we have:

$$\int_\Omega \partial_t\chi|\partial_t^2 E_2|^2dx = \frac{1}{\varepsilon}\int_\Omega |E_2|^2|\partial_t^2 E_2|^2dx - \frac{1}{\varepsilon}\int_\Omega \chi|\partial_t^2 E_2|^2dx.$$

From (63) we have:

$$\int_\Omega \partial_t^2\chi\partial_t E_2\partial_t^2 E_2dx = \frac{2}{\varepsilon}\int_\Omega E_2|\partial_t E_2|^2\partial_t^2 E_2dx - \frac{1}{\varepsilon}\int_\Omega \partial_t\chi\partial_t E_2\partial_t^2 E_2dx.$$

From (73) we have:

$$\int_\Omega \partial_t^3\chi E_2\partial_t^2 E_2dx = \frac{2}{\varepsilon}\int_\Omega E_2|\partial_t E_2|^2\partial_t^2 E_2dx + \frac{2}{\varepsilon}\int_\Omega |E_2|^2|\partial_t^2 E_2|^2dx - \frac{2}{\varepsilon^2}\int_\Omega |E_2|^2\partial_t E_2\partial_t^2 E_2dx$$
$$+\frac{1}{\varepsilon^2}\int_\Omega \partial_t\chi E_2\partial_t^2 E_2dx.$$

So we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi)|\partial_t^2 E_2|^2 + |\partial_t^2 H|^2\right)dx_1dx_3 + \int_\Gamma a|\partial_t^2 E_2|^2dx_3 + \frac{9}{2\varepsilon}\int_\Omega |E_2|^2|\partial_t^2 E_2|^2dx =$$
$$\int_\Omega \partial_t^2 G\cdot\partial_t^2 Udx + \frac{5}{2\varepsilon}\int_\Omega \chi|\partial_t^2 E_2|^2dx - \frac{8}{\varepsilon}\int_\Omega E_2\partial_t^2 E_2|\partial_t E_2|^2dx + \frac{3}{\varepsilon}\int_\Omega \partial_t\chi\partial_t E_2\partial_t^2 E_2dx \tag{76}$$
$$+\frac{2}{\varepsilon^2}\int_\Omega |E_2|^2\partial_t E_2\partial_t^2 E_2dx - \frac{1}{\varepsilon^2}\int_\Omega \partial_t\chi E_2\partial_t^2 E_2dx.$$

By (62) and (66) we have

$$\|\partial_t E_2\|_{H^1(\Omega))} \le C(1+\|\partial_t^2 H\|_{L^2(\Omega)}). \tag{77}$$

We estimate the right hand side terms in (76).

$$\left|\frac{8}{\varepsilon}\int_\Omega E_2\partial_t^2 E_2|\partial_t E_2|^2dx\right| \le \frac{9}{4\varepsilon}\int_\Omega |E_2|^2|\partial_t^2 E_2|^2dx + K(\varepsilon)\int_\Omega |\partial_t E_2|^4dx.$$

By interpolation inequalities and Sobolev theorem, we have:

$$\|u\|_{L^4(\Omega)} \le C\|u\|_{H^{\frac{1}{2}}(\Omega)} \le \|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{H^1(\Omega)}.$$

So by (66) and (77) we obtain:

$$\left|\frac{8}{\varepsilon}\int_\Omega E_2\partial_t^2 E_2|\partial_t E_2|^2dx\right| \le \frac{9}{4\varepsilon}\int_\Omega |E_2|^2|\partial_t^2 E_2|^2dx + K(\varepsilon)(1+\|\partial_t^2 H\|^2_{L^2(\Omega)}). \tag{78}$$

By (71) and (77) we have

$$\begin{aligned}\left|\int_\Omega \partial_t\chi\partial_t E_2\partial_t^2 E_2dx\right| \le\ & C\|\partial_t\chi\|_{L^4(\Omega)}\|\partial_t E_2\|_{L^4(\Omega)}\|\partial_t^2 E_2\|_{L^2(\Omega)}\\ \le\ & C\|\partial_t E_2\|_{H^1(\Omega)}\|\partial_t^2 E_2\|_{L^2(\Omega)}\\ \le\ & C(1+\|\partial_t^2 U\|^2_{L^2(\Omega)}).\end{aligned} \tag{79}$$

In the same way, by (69) and (77) we have

$$\left|\int_\Omega |E_2|^2\partial_t E_2\partial_t^2 E_2dx\right| \le C(1+\|\partial_t^2 U\|^2_{L^2(\Omega)}), \tag{80}$$

and by (69) and (71) we have

$$\left|\int_\Omega \partial_t\chi E_2\partial_t^2 E_2 dx\right| \leq C(1+\|\partial_t^2 U\|^2_{L^2(\Omega)}). \tag{81}$$

Using (78), (79), (80), (81) in (76), we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi)|\partial_t^2 E_2|^2+|\partial_t^2 H|^2\right)dx_1dx_3 \leq C(1+\|\partial_t^2 U\|^2_{L^2(\Omega)})+\frac{5}{2\varepsilon}\int_\Omega \chi|\partial_t^2 E_2|^2 dx,$$

so by Gronwall lemma there exists $C=C(T_\varepsilon^*)$ such that

$$\|\partial_t^2 U\|_{L^\infty(0,T_\varepsilon^*;L^2(\Omega))}\leq C. \tag{82}$$

So by (77),

$$\|\partial_t E_2\|_{L^\infty(0,T_\varepsilon^*;H^1(\Omega))}\leq C, \tag{83}$$

and by Sobolev theorem, for all p, $2\leq p<+\infty$,

$$\|\partial_t E_2\|_{L^\infty(0,T_\varepsilon^*;L^p(\Omega))}\leq C. \tag{84}$$

As $\partial_t^2\chi=\frac{2}{\varepsilon}E_2\partial_t E_2-\frac{1}{\varepsilon}\partial_t\chi$, by (69), (71), (84) we have for $1\leq p<+\infty$:

$$\|\partial_t^2\chi\|_{L^\infty(0,T_\varepsilon^*;L^p(\Omega))}\leq C. \tag{85}$$

Estimates on the third order time derivatives.
We derivate (72)-(73)-(74) with respect to t:

$$\begin{cases} (1+\chi)\partial_t^4 E_2+4\partial_t\chi\partial_t^3 E_2+6\partial_t^2\chi\partial_t^2 E_2+4\partial_t^3\chi\partial_t E_2+\partial_t^4\chi E_2-\partial_3\partial_t^3 H_1+\partial_1\partial_t^3 H_3=\partial_t^3 G_1,\\ \partial_t^4 H_1-\partial_3\partial_t^3 E_2=\partial_t^3 G_2,\\ \partial_t^4 H_3+\partial_1\partial_t^3 E_2=\partial_t^3 G_3, \end{cases} \tag{86}$$

$$\partial_t^4\chi=\frac{6}{\varepsilon}\partial_t E_2\partial_t^2 E_2+\frac{2}{\varepsilon}E_2\partial_t^3 E_2-\frac{2}{\varepsilon^2}|\partial_t E_2|^2-\frac{2}{\varepsilon^2}E_2\partial_t^2 E_2+\frac{1}{\varepsilon^2}\partial_t^2\chi, \tag{87}$$

with the null boundary condition:

$$\partial_t^3 H_3(t,0,x_3)+a\partial_t^3 E_2(t,0,x_3)=0, \tag{88}$$

and by the equations, we preserve the null initial condition:

$$\partial_t^3 V(0,x_1,x_3)=0. \tag{89}$$

We take the inner product of (86) by $\partial_t^3 U$ and we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi)|\partial_t^3 E_2|^2+|\partial_t^3 H|^2\right)dx_1dx_3+\int_\Gamma a|\partial_t^3 E_2|^2dx_3+\frac{7}{2}\int_\Omega \partial_t\chi|\partial_t^3 E_2|^2dx+6\int_\Omega \partial_t^2\chi\partial_t^2 E_2\partial_t^3 E_2$$

$$+4\int_\Omega \partial_t^3\chi\partial_t E_2\partial_t^3 E_2+\int_\Omega \partial_t^4\chi E_2\partial_t^3 E_2=\int_\Omega \partial_t^3 G\cdot\partial_t^3 U dx.$$

From (55) we have

$$\int_\Omega \partial_t\chi|\partial_t^3 E_2|^2dx=\frac{1}{\varepsilon}\int_\Omega |E_2|^2|\partial_t^3 E_2|^2dx-\frac{1}{\varepsilon}\int_\Omega \chi|\partial_t^3 E_2|^2dx.$$

From (73) we have:

$$\int_\Omega \partial_t^3\chi\partial_t E_2\partial_t^3 E_2=\frac{2}{\varepsilon}\int_\Omega |\partial_t E_2|^3\partial_t^3 E_2dx+\frac{2}{\varepsilon}\int_\Omega E_2\partial_t E_2\partial_t^2 E_2\partial_t^3 E_2dx-\int_\Omega \partial_t^2\chi\partial_t E_2\partial_t^3 E_2.$$

From (87), we have:

$$\int_\Omega \partial_t^4\chi E_2\partial_t^3E_2 = \frac{6}{\varepsilon}\int_\Omega E_2\partial_tE_2\partial_t^2E_2\partial_t^3E_2dx + \frac{2}{\varepsilon}\int_\Omega |E_2|^2|\partial_t^3E_2|^2dx - \frac{2}{\varepsilon^2}\int_\Omega E_2|\partial_tE_2|^2\partial_t^3E_2dx$$
$$-\frac{2}{\varepsilon^2}\int_\Omega |E_2|^2\partial_t^2E_2\partial_t^3E_2dx + \frac{1}{\varepsilon^2}\int_\Omega \partial_t^2\chi E_2\partial_t^3E_2.$$

So we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi)|\partial_t^3E_2|^2 + |\partial_t^3H|^2\right)dx_1dx_3 + \int_\Gamma a|\partial_t^3E_2|^2dx_3 + \frac{11}{2\varepsilon}\int_\Omega |E_2|^2|\partial_t^3E_2|^2dx \\ = \int_\Omega \partial_t^3G\cdot\partial_t^3Udx + \frac{7}{2\varepsilon}\int_\Omega \chi|\partial_t^3E_2|^2dx + I_1 + I_2\end{aligned} \tag{90}$$

where

$$\begin{aligned}I_1 = \quad & -6\int_\Omega \partial_t^2\chi\partial_t^2E_2\partial_t^3E_2dx - \frac{14}{\varepsilon}\int_\Omega E_2\partial_tE_2\partial_t^2E_2\partial_t^3E_2dx + \frac{2}{\varepsilon}\int_\Omega |E_2|^2\partial_t^2E_2\partial_t^3E_2dx\\ I_2 = \quad & -\frac{8}{\varepsilon}\int_\Omega |\partial_tE_2|^3\partial_t^3E_2dx + \frac{4}{\varepsilon}\int_\Omega \partial_t^2\chi\partial_tE_2\partial_t^3E_2dx + \frac{2}{\varepsilon^2}\int_\Omega E_2|\partial_tE_2|^2\partial_t^3E_2dx\\ & -\frac{1}{\varepsilon^2}\int_\Omega \partial_t^2\chi E_2\partial_t^3E_2dx.\end{aligned}$$

From (72) and (82) we have

$$\|\partial_t^2E_2\|_{H^1(\Omega]} \le K(1+\|\partial_t^3H\|_{L^2(\Omega)}). \tag{91}$$

So by Sobolev theorem and (69), (84), (85) and (91) we have

$$|I_1| \le C(1+\|\partial_t^3U\|^2_{L^2(\Omega)}). \tag{92}$$

By (69), (84), (85), we have

$$|I_2| \le C\|\partial_t^3E_2\|_{L^2(\Omega)}. \tag{93}$$

We carry back estimates (92) and (93) in (90) and we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi)|\partial_t^3E_2|^2 + |\partial_t^3H|^2\right)dx_1dx_3 \le C(1+\|\partial_t^3U\|^2_{L^2(\Omega)}) + \frac{7}{2\varepsilon}\int_\Omega \chi|\partial_t^3E_2|^2dx.$$

So there exists a constant $C = C(T_\varepsilon^*) > 0$ such that

$$\|\partial_t^3U\|_{L^\infty(0,T_\varepsilon^*;L^2(\Omega))} \le C. \tag{94}$$

By (91) we have:

$$\|\partial_t^2E_2\|_{L^\infty(0,T_\varepsilon^*;H^1(\Omega))} \le C, \tag{95}$$

and, for all p, $2 \le p < +\infty$,

$$\|\partial_t^2E_2\|_{L^\infty(0,T_\varepsilon^*;L^p(\Omega))} \le C. \tag{96}$$

As $\partial_t^3\chi = \frac{2}{\varepsilon}|\partial_tE_2|^2 + \frac{2}{\varepsilon}E_2\partial_t^2E_2 - \frac{1}{\varepsilon}\partial_t^2\chi$, using (69), (84), (85) and (96), for $1 \le p < +\infty$,

$$\|\partial_t^3\chi\|_{L^\infty(0,T_\varepsilon^*;L^p(\Omega))} \le C. \tag{97}$$

Estimates in $\mathcal{H}^3(\Omega_{T_\varepsilon^*})$.

In order to estimate the space derivatives of H we recall the following div-curl lemma (see [9]).

Lemma 1 *Let $\Omega = \{(x_1, x_3), x_1 > 0\}$. We denote $\Gamma = \partial\Omega$. Let $H = (H_1, H_3) \in H^k(\Omega)$ such that* div $H = \partial_1 H_1 + \partial_3 H_3 \in H^k(\Omega)$, curl $H = -\partial_1 H_3 + \partial_3 H_1 \in H^k(\Omega)$, $H_3 \in H^{k+\frac{1}{2}}(\Gamma)$. *Then $H \in H^{k+1}(\Omega)$ and we have*

$$\|H\|_{H^{k+1}(\Omega)} \leq C \left(\|H\|_{H^k(\Omega)} + \|\text{div } H\|_{H^k(\Omega)} + \|\text{curl } H\|_{H^k(\Omega)} + \|H\|_{H^{k+\frac{1}{2}}(\Gamma)} \right).$$

By the first equation in (54), using (69), (70), 71) and (84), we obtain

$$\|\text{curl } H\|_{L^\infty(0,T_\varepsilon^*;L^2(\Omega))} \leq C.$$

Recall that div $H = 0$ and on the boundary Γ, $H_3(t, 0, x_3) = -aE_2(t, 0, x_3)$, so by (67),

$$\|H_3(t, 0, .)\|_{L^\infty(0,T_\varepsilon^*;H^{\frac{1}{2}}(\Gamma))} \leq C.$$

Using lemma 1, we obtain

$$\|H\|_{L^\infty(0,T_\varepsilon^*;H^1(\Omega))} \leq C. \tag{98}$$

By the first equation in (62), using (69), (70), 71), (84), (85) and (96), we have:

$$\|\text{curl } \partial_t H\|_{L^\infty(0,T_\varepsilon^*;L^2(\Omega))} \leq C.$$

By (57), using (83), we have:

$$\|\partial_t H_3(t, 0, .)\|_{L^\infty(0,T_\varepsilon^*;H^{\frac{1}{2}}(\Gamma))} \leq C,$$

thus

$$\|\partial_t H\|_{L^\infty(0,T_\varepsilon^*;H^1(\Omega))} \leq C. \tag{99}$$

By the second and the third equations in (54), by (67) and (99),

$$\|E_2\|_{L^\infty(0,T_\varepsilon^*;H^2(\Omega))} \leq C, \tag{100}$$

and by Sobolev theorem,

$$\|E_2\|_{L^\infty([0,T_\varepsilon^*[\times\Omega)} \leq C. \tag{101}$$

As $H^2(\Omega)$ is an algebra, using (68), we obtain

$$\|\chi\|_{L^\infty(0,T_\varepsilon^*;H^2(\Omega))} \leq C, \tag{102}$$

and by Sobolev theorem:

$$\|\chi\|_{L^\infty([0,T_\varepsilon^*[\times\Omega)} \leq C, \tag{103}$$

and since $\partial_t \chi = \dfrac{1}{\varepsilon}(|E_2|^2 - \chi)$,

$$\|\partial_t \chi\|_{L^\infty(0,T_\varepsilon^*;H^2(\Omega))} \leq C, \tag{104}$$

and by Sobolev theorem:

$$\|\partial_t \chi\|_{L^\infty([0,T_\varepsilon^*[\times\Omega)} \leq C. \tag{105}$$

In addition with (73) we have

$$\|\partial_t^2 \chi\|_{L^\infty(0,T_\varepsilon^*;H^1(\Omega))} \leq C. \tag{106}$$

By the first equation in (54), using (83), (100), (102), (104), we obtain

$$\|\text{curl } H\|_{L^\infty(0,T_\varepsilon^*;H^1(\Omega))} \leq C.$$

By (57), using (100), we have

$$\|H_3(t, 0, .)\|_{L^\infty(0,T_\varepsilon^*;H^{\frac{3}{2}}(\Gamma))} \leq C,$$

so by Lemma 1,

$$\|H\|_{L^\infty(0,T_\varepsilon^*;H^2(\Omega))} \leq C. \tag{107}$$

By analogous arguments, we prove successively that

$$\|\partial_t^2 H\|_{L^\infty(0,T_\varepsilon^*;H^1(\Omega))} \leq C,$$

$$\|\partial_t E_2\|_{L^\infty(0,T_\varepsilon^*;H^2(\Omega))} \leq C,$$

$$\|\partial_t^2 \chi\|_{L^\infty(0,T_\varepsilon^*;H^2(\Omega))} \leq C,$$

$$\|\partial_t H\|_{L^\infty(0,T_\varepsilon^*;H^2(\Omega))} \leq C,$$

$$\|E_2\|_{L^\infty(0,T_\varepsilon^*;H^3(\Omega))} \leq C,$$

$$\|\chi\|_{L^\infty(0,T_\varepsilon^*;H^3(\Omega))} \leq C,$$

$$\|H\|_{L^\infty(0,T_\varepsilon^*;H^3(\Omega))} \leq C.$$

Finally,

$$\|V\|_{\mathcal{H}^3(\Omega_{T_\varepsilon^*})} \leq C. \tag{108}$$

This estimate contradicts (52), so $T_\varepsilon^* = +\infty$.

2.5 Global existence for the 1d Kerr-Debye system.

The one dimensionnal Kerr Debye IBVP writes: for $(t,x) \in I\!R^+ \times I\!R^+$,

$$\begin{cases} \partial_t d + \partial_x h = 0, \\ \partial_t h + \partial_x e = 0, \\ \partial_t \chi = \dfrac{1}{\varepsilon}(e^2 - \chi), \\ d = (1+\chi)e, \end{cases} \tag{109}$$

with the initial condition

$$(d,h,\chi)(0,x) = 0 \text{ for } x \geq 0, \tag{110}$$

and with the boundary condition:

$$h(t,0) + ae(t,0 = \varphi(t) \text{ for } t \geq 0. \tag{111}$$

Theorem 2 *Let $\varphi \in H^s(I\!R_t)$ (s great enough) with supp $\varphi \subset [0,+\infty[$. Let $\varepsilon > 0$. Then the regular solution (d,h,χ) to the IBVP (109)-(110)-(111) is defined on $[0,+\infty[$.*

The proof of Theorem 2 is analogous to the proof of Theorem 1. The one dimensional case is easier: the divergence free condition is irrelevant and the space regularity is obtained directely by the equations. In addition, by Sobolev theorem, the H^1-estimates implies bounds in the L^∞-norm. By another way, the detailled proof of the global existence for the Cauchy problem is given in [7].

Remark 2 *The obtention of the global existence for the 2dTM and the 3d models seems more difficult. First we lack for an adapted div-curl lemma to obtain the space regularity. Indeed, in these cases, the divergence free condition (34) is non linear in (χ, E). In addition, the Sobolev embeddings used in the 2d case don't work in the 3d case.*

3 Convergence result

We replace Kerr-Debye model in the general framework of [8]. The equilibrium manifold in (31) is defined by:

$$\mathcal{V} = \left\{(D, H, \chi) \in I\!R^7, \chi = |E|^2 = (1+\chi)^{-2}|D|^2\right\}.$$

So the reduced system associated with the Kerr-Debye model (31) is the Kerr model (26).
The strictly convex entropy $\mathcal{E}_{KD}$ satisfies the stability condition in the definition 2.1 of [8] and, on the equilibrium manifold $\mathcal{V}$, we have the relation:

$$\mathcal{E}_K(D, H) = \mathcal{E}_{KD}(D, H, \chi(D)).$$

Furthermore characteristic speeds associated to (31) and (26) are interlaced on the equilibrium manifold $\mathcal{V}$. In our case, by the previous calculations, in the 3d model we have:

$$\mu_1(|E|^2, \xi) = \lambda_1(E, \xi) = \mu_2(|E|^2, \xi) \leq \lambda_2(E, \xi) \leq 0 = \mu_3 = \lambda_3 = \mu_4 = \lambda_4 = \mu_5.$$

In order to prove the convergence results it is more convenient to use the entropic variables which are introduced in [11]. These variables are obtained taking the gradient of the convex entropy (36).

$$\begin{cases} \partial_D \mathcal{E}_{KD} = (1+\chi)^{-1} D = E, \\ \\ \partial_H \mathcal{E}_{KD} = H, \\ \\ \partial_\chi \mathcal{E}_{KD} = \dfrac{1}{2}(\chi - |E|^2) := v, \end{cases}$$

The IBVP (31) (32) (33) becomes

$$A_0(W_\varepsilon)\partial_t W_\varepsilon + \sum_{j=1}^{3} A_j \partial_j W_\varepsilon = \frac{1}{\varepsilon} Q(W_\varepsilon) \tag{112}$$

for $(t, x) \in I\!R^+ \times \Omega$, where

- $W_\varepsilon = \begin{pmatrix} E_\varepsilon \\ H_\varepsilon \\ v_\varepsilon \end{pmatrix}, \qquad Q(W_\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ -2v_\varepsilon \end{pmatrix},$
- $A_0(W_\varepsilon) = \begin{pmatrix} (|E_\varepsilon|^2 + 2v_\varepsilon + 1)I_3 + 2E_\varepsilon\, {}^t E_\varepsilon & 0 & 2E_\varepsilon \\ 0 & I_3 & 0 \\ 2\, {}^t E_\varepsilon & 0 & 2 \end{pmatrix},$
- $\sum_{j=1}^{3} A_j \partial_j = \begin{pmatrix} 0 & -\mathrm{curl} & 0 \\ \mathrm{curl} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$

with the initial data

$$E_\varepsilon(0, x) = H_\varepsilon(0, x) = 0,\ v_\varepsilon(0, x) = 0 \text{ for } x \in \Omega, \tag{113}$$

and with the boundary condition

$$H_\varepsilon \times n + a((E_\varepsilon \times n) \times n) = \varphi \text{ for } (t, x) \in I\!R^+ \times \Gamma. \tag{114}$$

We observe that the boundary condition is linear for the variables (E, H). In addition the equilibrium manifold $\left\{(D, H, \chi),\ \chi = |E|^2\right\}$ is linearized as $\left\{(E, H, v),\ v = 0\right\}$ and the relaxation term is linear.
We rewrite the divergence free condition in the entropic variables:

$$\mathrm{div}\ ((1 + |E_\varepsilon|^2 + 2v_\varepsilon)E_\varepsilon) = \mathrm{div}\ H_\varepsilon = 0. \tag{115}$$

In the same way we can write the Kerr model in its entropic variables :

$$\begin{cases} \partial_D \mathcal{E}_K = E, \\ \\ \partial_H \mathcal{E}_K = H. \end{cases}$$

The IBVP (26)-(27)-(28) becomes:

$$\begin{aligned} &((1+|E_0|^2)I_3 + 2E_0^t E_0)\partial_t E_0 - \text{curl } H_0 = 0, \\ &\partial_t H_0 + \text{curl } H_0 = 0, \end{aligned} \tag{116}$$

for $(t,x) \in I\!R^+ \times \Omega$, with null initial data

$$E_0(0,x) = H_0(0,x) = 0 \text{ for } x \in \Omega, \tag{117}$$

and with the impedance boundary condition

$$H_0 \times n + a((E_0 \times n) \times n) = \varphi \text{ for } (t,x) \in I\!R + \times \Gamma. \tag{118}$$

We remark that the divergence free condition is satisfied

$$\text{div } ((1+|E_0|^2)E_0) = \text{div } H_0 = 0. \tag{119}$$

In these entropic variables the IBVP (116)-(117)-(118) is the reduced system of the IBVP (112)-(113)-(114). Let us recall the assumptions on the source term φ in both problems:

$$\begin{aligned} &\varphi \text{ is compactly supported in } I\!R_t^+ \times \Gamma, \\ &\varphi \in H^s(I\!R_t \times \Gamma), \end{aligned} \tag{120}$$

where s is chosen great enough to ensure a sufficiant regularity on the profile (E_0, H_0).

As remarked in the introduction, we don't expect boundary layer formation in the IBVP (112)-(113)-(114) near the profile $(E_0, H_0, 0)$. Hence we use a Hilbert expansion to describe the behaviour of the solution when ε tends to zero.
As in the previous section we introduce for $T > 0$ and $k \in I\!N$:

$$\mathcal{H}^k(\Omega_T) = \left\{ W \text{ such that } \|W\|_{\mathcal{H}^k(\Omega_T)} := \sum_{i=0}^{k} \|\partial_t^i W\|_{L^\infty(0,T;H^{k-i}(\Omega))} < +\infty \right\}. \tag{121}$$

We obtain the following convergence result.

Theorem 3 *Let φ satisfying (120). Let (E_0, H_0) be the solution of (116)-(117)-(118) which lifespan is denoted by T^*. For $\varepsilon > 0$, let $(E_\varepsilon, H_\varepsilon, v_\varepsilon)$ be the solution of (112)-(113)-(114) which lifespan is denoted by T_ε^*. We fix $T < T^*$. There exists $\varepsilon_0 > 0$ such that*

$$\forall\, \varepsilon < \varepsilon_0, T_\varepsilon^* \geq T, \tag{122}$$

and there exists a constant C such that for all $\varepsilon < \varepsilon_0$,

$$\|E_\varepsilon - E_0\|_{\mathcal{H}^3(\Omega_T)} + \|H_\varepsilon - H_0\|_{\mathcal{H}^3(\Omega_T)} + \|v_\varepsilon\|_{\mathcal{H}^3(\Omega_T)} \leq C\varepsilon. \tag{123}$$

Proof of Theorem 3.

We denote by $\rho_\varepsilon =^t (R_\varepsilon, S_\varepsilon, s_\varepsilon)$ the remainder term in the Hilbert expansion of W_ε:

$$\begin{aligned} E_\varepsilon &= E_0 + \varepsilon R_\varepsilon, \\ H_\varepsilon &= H_0 + \varepsilon S_\varepsilon, \\ v_\varepsilon &= \varepsilon s_1 + \varepsilon s_\varepsilon, \end{aligned} \tag{124}$$

with $s_1 = -E_0\partial_t E_0$. For the convenience of the reader we omit the dependance on the index ε. Using (116)-(117)-(118) and (112)-(113)-(114), the rest term ρ satisfies the following system:

$$A_0(t,x)\partial_t\rho + \sum_{j=1}^{3} A_j\partial_j\rho + L(t,x)\rho + B(t,x) + G(t,x,R) + F(t,x,\tilde{\rho})\partial_t\tilde{\rho} = -{}^t(0,0,\frac{2}{\varepsilon}s), \tag{125}$$

where

- $\tilde{\rho} = \begin{pmatrix} R \\ 0 \\ s \end{pmatrix}$
- $A_0(t,x) = \begin{pmatrix} (1+|E_0|^2)I_3 + 2E_0\,{}^tE_0 & 0 & 2E_0 \\ 0 & I_3 & 0 \\ 2\,{}^tE_0 & 0 & 2 \end{pmatrix}$
- $L(t,x)\rho = \begin{pmatrix} 2s\partial_t E_0 + 2E_0\cdot R\partial_t E_0 + 2(E_0\,{}^tR + R\,{}^tE_0)\partial_t E_0 + 2\varepsilon\partial_t s_1 R \\ 0 \\ 2\,{}^tR\partial_t E_0 \end{pmatrix}$
- $B(t,x) = \begin{pmatrix} 2s_1\partial_t E_0 + 2\partial_t s_1 E_0 \\ 0 \\ 2\partial_t s_1 \end{pmatrix}$
- $G(t,x,R) = \begin{pmatrix} \varepsilon|R|^2\partial_t E_0 + \varepsilon R\,{}^tR\partial_t E_0 \\ 0 \\ 0 \end{pmatrix}$
- $F(t,x,\tilde{\rho})\partial_t\tilde{\rho} = \begin{pmatrix} \varepsilon(R\,{}^tE_0 + E_0\,{}^tR)\partial_t R + \varepsilon s\partial_t R + \varepsilon^2 R\,{}^tR\partial_t R + 2\varepsilon R\partial_t s + \varepsilon^2|R|^2\partial_t R \\ 0 \\ 2\varepsilon R\cdot\partial_t R \end{pmatrix}$

with the null initial data:

$$\rho(0,x) = 0 \text{ for } x\in\Omega, \tag{126}$$

and the homogeneous boundary condition

$$S\times n + a((R\times n)\times n) = 0 \text{ for } (t,x)\in I\!R^+\times\Gamma. \tag{127}$$

We remark that we introduced the one order term εs_1 in the expansion of v_ε to avoid a singular source term in the last equation of System (125). We fix $T < T^*$ and we define T_ε by

$$T_\varepsilon = \sup\left\{t\le T, \|\rho_\varepsilon\|_{\mathcal{H}^3(\Omega_t)} \le \frac{1}{\sqrt{\varepsilon}}\right\}, \tag{128}$$

so by Proposition 2, $T_\varepsilon \le T_\varepsilon^*$.

For $t\le T_\varepsilon$ we define φ_ε and ψ_ε by

$$\varphi_\varepsilon(t) = \left(\|\rho_\varepsilon(t)\|^2_{L^2(\Omega)} + \sum_{i\neq 1}\|\partial_i\rho_\varepsilon(t)\|^2_{L^2(\Omega)} + \sum_{i,j\neq 1}\|\partial_{ij}\rho_\varepsilon(t)\|^2_{L^2(\Omega)} + \sum_{i,j,k\neq 1}\|\partial_{ijk}\rho_\varepsilon(t)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \tag{129}$$

$$\psi_\varepsilon(t) = \left(\|\partial_1\rho_\varepsilon(t)\|^2_{L^2(\Omega)} + \sum_{i}\|\partial_{1i}\rho_\varepsilon(t)\|^2_{L^2(\Omega)} + \sum_{i,j}\|\partial_{1ij}\rho_\varepsilon(t)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{130}$$

Let us remark that φ_ε measures the tangential derivatives $\partial_0 = \partial_t$, ∂_2, ∂_3 and we have

$$(\varphi_\varepsilon(t))^2 + (\psi_\varepsilon(t))^2 = \sum_{i=0}^{3} \|\partial_t^i \rho_\varepsilon(t)\|_{H^{3-i}(\Omega)}.$$

We define also

$$\Phi_\varepsilon(t) = \sup_{s\in[0,t]} \varphi_\varepsilon(s), \tag{131}$$

$$\Psi(t) = \sup_{s\in[0,t]} \psi_\varepsilon(s), \tag{132}$$

so we have

$$\Phi_\varepsilon(T) + \Psi_\varepsilon(T) \equiv \|\rho_\varepsilon\|_{\mathcal{H}^3(\Omega_T)}.$$

The proof of Theorem 3 is organized as follows. In the first step, by variational methods on the system (125)-(126)-(127) we estimate the tangential derivatives of ρ_ε. In the second step we bound the normal derivatives of ρ_ε by solving the last equation in (125) and using the divergence free conditions (115) and (119). We will use the following classical lemma:

Lemma 2 *For $k \geq 2$, $\mathcal{H}^k(\Omega_T)$ is an algebra.*

First step: estimates on φ_ε.

Let us establish the following lemma:

Lemma 3 *There exists a constant C such that for $t \in [0, T_\varepsilon]$,*

$$\begin{aligned}\frac{d}{dt}\Bigg[\int_\Omega \Bigg(A_0\rho\cdot\rho + \sum_{i\neq 1} A_0\partial_i\rho\cdot\partial_i\rho + \sum_{i,j\neq 1} A_0\partial_{ij}\rho\cdot\partial_{ij}\rho + \sum_{i,j,k\neq 1} A_0\partial_{ijk}\rho\cdot\partial_{ijk}\rho\Bigg)dx \\ + \int_\Omega \Bigg(\sum_{i,j,k\neq 1} F(t,x,\tilde\rho)\partial_{ijk}\tilde\rho\cdot\partial_{ijk}\tilde\rho\Bigg)dx\Bigg] \leq C(1+\varphi^2).\end{aligned} \tag{133}$$

Proof: the nonlinear terms in (125) are bounded thanks to the following estimates, which are a straightforward consequence of Lemma 2 and (128): there exists a constant C such that for all $\varepsilon > 0$,

$$\begin{aligned}&\|G(t,x,R)\|_{\mathcal{H}^3(\Omega_{T_\varepsilon})} \leq C,\\ &\|F(t,x,\tilde\rho)\partial_t\tilde\rho\|_{\mathcal{H}^2(\Omega_{T_\varepsilon})} \leq C,\\ &\|F(t,x,\tilde\rho)\|_{\mathcal{H}^3(\Omega_{T_\varepsilon})} \leq C\sqrt{\varepsilon}.\end{aligned} \tag{134}$$

Taking the inner product of (125) with ρ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega A_0\rho\cdot\rho dx - \int_\Gamma a((R\times n)\times n)\cdot R dx_2dx_3 + \frac{2}{\varepsilon}\int_\Omega |s|^2dx =$$

$$\frac{1}{2}\int_\Omega \partial_t A_0\rho\cdot\rho dx - \int_\Omega L\rho\cdot\rho dx - \int_\Omega B\cdot\rho dx - \int_\Omega G(t,x,R)\cdot\rho dx - \int_\Omega F(t,x,\tilde\rho)\partial_t\tilde\rho\cdot\rho dx.$$

By assumption on the operator a, we have

$$\int_\Gamma a((R\times n)\times n)\cdot R dx_2dx_3 \leq 0.$$

Since E_0 is smooth enough we have immediately

$$\left|\frac{1}{2}\int_\Omega \partial_t A_0\rho\cdot\rho dx - \int_\Omega L\rho\cdot\rho dx\right| \leq C(\varphi(t))^2,$$

and

$$\left|\int_\Omega B\cdot\rho dx\right|\leq C\varphi(t).$$

From (134),

$$\left|\int_\Omega G(t,x,R)\cdot\rho dx\right|\leq \|G\|_{L^2(\Omega)}\|\rho\|_{L^2(\Omega)}\leq \|G\|_{\mathcal{H}^3(\Omega_{T_\varepsilon})}\varphi(t)\leq C\varphi(t),$$

and

$$\left|\int_\Omega F(t,x,\tilde\rho)\partial_t\tilde\rho\cdot\rho dx\right|\leq \|F\partial_t\tilde\rho\|_{L^2(\Omega)}\|\rho\|_{L^2(\Omega)}\leq \|F\partial_t\tilde\rho\|_{\mathcal{H}^2(\Omega_{T_\varepsilon})}\|\rho\|_{L^2(\Omega)}\leq C\varphi(t).$$

Thus we obtain the L^2 estimate:

$$\frac{d}{dt}\int_\Omega A_0\rho\cdot\rho dx\leq C(1+(\varphi(t))^2). \tag{135}$$

In the same way, derivating (125) with respect to the tangential variables, we obtain

$$\frac{d}{dt}\left[\int_\Omega\sum_{i\neq 1}A_0\partial_i\rho\cdot\partial_i\rho dx+\int_\Omega\sum_{i,j\neq 1}A_0\partial_{ij}\rho\cdot\partial_{ij}\rho dx\right]\leq C(1+(\varphi(t))^2).$$

For the third order derivate, we use also the estimate : for $i,j,k\neq 1$,

$$\left|\int_\Omega\partial_{ijk}(F(t,x,\tilde\rho)\partial_t\tilde\rho)\cdot\partial_{ijk}\tilde\rho dx-\frac{1}{2}\frac{d}{dt}\int_\Omega F(t,x,\tilde\rho)\partial_{ijk}\tilde\rho\cdot\partial_{ijk}\tilde\rho dx\right|\leq C(1+(\varphi(t))^2),$$

and we conclude the proof of Lemma 3.

Let us recall that $A_0(t,x)=A_0(E_0)$ (see (125)), so, since E_0 is smooth on $[0,T]\times\overline{\Omega}$, there exists a constant $\alpha>0$ such that for all $\xi\in I\!R^3$ and $(t,x)\in[0,T]\times\overline{\Omega}$,

$$A_0(t,x)\xi\cdot\xi\geq\alpha|\xi|^2. \tag{136}$$

In addition, from the last estimate in (134) we remark that, for all $t\in[0,T_\varepsilon]$,

$$\left|\int_\Omega\sum_{i,j,k\neq 1}F(t,x,\tilde\rho)\partial_{ijk}\tilde\rho\cdot\partial_{ijk}\tilde\rho dx\right|\leq C_1\sqrt{\varepsilon}(\varphi(t))^2. \tag{137}$$

So, integrating (133) and using (136) and (137) we obtain that, for $t\in[0,T_\varepsilon]$,

$$\alpha\varphi^2(t)-C_1\sqrt{\varepsilon}\varphi^2(t)\leq C_2\int_0^t(1+\varphi^2(s))ds,$$

so for ε small enough,

$$\varphi^2(t)\leq C\int_0^t(1+\varphi^2(s))ds,$$

so by Gronwall lemma, we obtain that there exists C such that for all $t\in[0,T_\varepsilon]$

$$\varphi^2(t)\leq C. \tag{138}$$

Second step: estimates on ψ.

In order to estimate the derivatives with respect to the normal variable ∂_1 we proceed as follows:

1. We rewrite (125) isolating $\partial_1 R_3$, $\partial_1 R_2$ $\partial_1 S_3$ and $\partial_1 S_2$ in curl R and curl S:

$$\begin{cases} \partial_1 R_3 = \partial_3 R_1 + \partial_t S_2 \\ \partial_1 R_2 = \partial_2 R_1 - \partial_t S_3 \end{cases} \tag{139}$$

so from (138),

$$\|\partial_1 R_3\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} + \|\partial_1 R_2\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{140}$$

$$\begin{cases} \partial_1 S_3 = \partial_3 S_1 - (A_0\partial_t\rho)_2 - (L\rho)_2 + M_2(t,x,\varepsilon,\tilde\rho,\partial_t\tilde\rho), \\ \partial_1 S_2 = \partial_2 S_1 + (A_0\partial_t\rho)_3 + (L\rho)_3 + M_3(t,x,\varepsilon,\tilde\rho,\partial_t\tilde\rho), \end{cases} \tag{141}$$

where, by (134),

$$\|M_2\|_{\mathcal{H}^2(\Omega_{T_\varepsilon})} + \|M_3\|_{\mathcal{H}^2(\Omega_{T_\varepsilon})} \leq C. \tag{142}$$

Using also (138), we obtain that

$$\|\partial_1 S_3\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} + \|\partial_1 S_2\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{143}$$

2. In order to estimate $\partial_1 S_1$, $\partial_1 R_1$ and $\partial_1 s$, let us rewrite the divergence free conditions. From (115) and (119), we obtain

$$\partial_1 S_1 = -\partial_2 S_2 - \partial_3 S_3, \tag{144}$$

so we have

$$\|\partial_1 S_1\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{145}$$

It remains to estimate $\partial_1 R_1$ and $\partial_1 s$. From (115) and (119), we have

$$\begin{aligned} \text{div } \big((1+|E_0|^2R + 2(E_0\cdot R)E_0 + 2sE_0\big) = \quad & \tilde N(t,x,\varepsilon\tilde\rho,\partial_1\tilde\rho,\partial_2\tilde\rho,\partial_3\tilde\rho) \\ := -\text{div } \big(2s_1E_0 \ \ & +\varepsilon|R|^2E_0 + 2\varepsilon(s_1+s+(E_0\cdot R)R) + \varepsilon^2|R|^2R\big). \end{aligned}$$

We expend the left hand side term and we obtain:

$$\begin{aligned} (1+|E_0|^2+2(E_{0,1})^2)\partial_1 R_1 + 2E_{0,1}\partial_1 s + \tilde\Lambda(t,x)\rho + \tilde\Lambda_2(t,x)\partial_2\rho + \tilde\Lambda_3(t,x)\partial_3\rho \\ +2E_{0,1}E_{0,2}\partial_1 R_2 + 2E_{0,1}E_{0,3}\partial_1 R_3 = \tilde N \end{aligned}$$

where $\tilde\Lambda(t,x)$, $\tilde\Lambda_2(t,x)$ and $\tilde\Lambda_3(t,x)$ are linear operators.
Using (139) we obtain:

$$\partial_1 R_1 + 2(1+|E_0|^2+2(E_{0,1})^2)^{-1}E_{0,1}\partial_1 s = \Lambda(t,x)\rho + \sum_{i\in\{0,2,3\}} \Lambda_i(t,x)\partial_i\rho + N(t,x,\varepsilon,\tilde\rho,\partial_1\tilde\rho,\partial_2\tilde\rho,\partial_3\tilde\rho), \tag{146}$$

where $\Lambda(t,x)$ and the $\Lambda_i(t,x)$ are linear operators, and $N = (1+|E_0|^2+2(E_{0,1})^2)^{-1}\tilde N$. By (134) we have

$$\|N\|_{\mathcal{H}^2(\Omega_{T_\varepsilon})} \leq C. \tag{147}$$

3. Les us consider the last equation in (125):

$$\partial_t s + \frac{1}{\varepsilon}s = -\partial_t(E_0\cdot R + s_1 + \frac{\varepsilon}{2}|R|^2). \tag{148}$$

We derivate this equation with respect to x_1:

$$\partial_t\partial_1 s + \frac{1}{\varepsilon}\partial_1 s = -\partial_t(E_{0,1}\partial_1 R_1) - \partial_t(E_{0,2}\partial_1 R_2 + E_{0,3}\partial_1 R_3 + \partial_1 E_0\cdot R + \partial_1 s_1 + \frac{\varepsilon}{2}\partial_1|R|^2),$$

and using (146), we obtain

$$\partial_t(h\partial_1 s + b) + \frac{1}{\varepsilon}\partial_1 s = 0, \tag{149}$$

where

$$h = (1 + |E_0|^2 + 2(E_{0,1})^2)^{-1}(1 + |E_0|^2),$$

$$b = E_{0,1}(\Lambda\rho + \sum_{i\neq 1} \Lambda_i \partial_i \rho + N) + E_{0,2}\partial_1 R_2 + E_{0,3}\partial_1 R_3 + \partial_1 E_0 \cdot R + \partial_1 s_1 + \frac{\varepsilon}{2}\partial_1 |R|^2.$$

We have

$$\frac{1}{3} \leq h \leq 1, \tag{150}$$

and, from (138), (147), (140) and (128),

$$\|b\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{151}$$

We remark that $h\partial_1 s + b = 0$ at $t = 0$ and solving (149) :

$$\partial_1 s(t) = -\frac{b(t)}{h(t)} + \frac{1}{h(t)} \int_0^t \frac{1}{\varepsilon} \exp\left(-\int_\sigma^t \frac{1}{\varepsilon h(\tau)} d\tau\right) \frac{b(\sigma)}{h(\sigma)} d\sigma,$$

so, by (150) and (151),

$$\|\partial_1 s(t)\|_{L^\infty(0;T_\varepsilon;L^2(\Omega))} \leq C, \tag{152}$$

and by (146),

$$\|\partial_1 R_1\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{153}$$

Therefore we obtain the $\mathcal{H}^1$ estimate:

$$\|\rho\|_{\mathcal{H}^1(\Omega_{T_\varepsilon})} \leq C. \tag{154}$$

In order to obtain the $\mathcal{H}^2$ estimate on the remainder term ρ, first we deal with the derivative $\partial_{1i}\rho$, $i \in \{0, 2, 3\}$. We derivate (139) and (144) with respect to ∂_i and use (138) to obtain

$$\|\partial_{1i} R_2\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} + \|\partial_{1i} R_3\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} + \|\partial_{1i} S_1\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{155}$$

We derivate (149) with respect to ∂_i:

$$\partial_t(h\partial_{1,i} s + (\partial_i h \partial_1 s + \partial_i b)) + \frac{1}{\varepsilon}\partial_{1i} s = 0,$$

from (138), (155), (128) we remark that

$$\|\partial_i h \partial_1 s + \partial_i b\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C,$$

so, in the previous process, we can replace b by $\partial_i h \partial_1 s + \partial_i b$, and we obtain

$$\|\partial_{1i} s\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{156}$$

We derivate (146) with respect to ∂_i and we use (147), (138), (156) to obtain

$$\|\partial_{1i} R_1\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{157}$$

Therefore, by (155)-(157) we have for $i \neq 1$,

$$\|\partial_{1i}\rho\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{158}$$

Now we estimate $\partial_{11}\rho$. Derivating (139), (144) and (141) with respect to ∂_1, using (138), (158), (142) and (154) we obtain that

$$\|\partial_{11} R_2\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} + \|\partial_{11} R_3\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} + \|\partial_{11} S\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C. \tag{159}$$

Derivating (149) and (146) with respect to ∂_1 and using in particular (158) we obtain by the same method that

$$\|\partial_{11} s\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} + \|\partial_{11} R_1\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))} \leq C, \tag{160}$$

which conclude the $\mathcal{H}^2$ estimate:

$$\|\rho\|_{\mathcal{H}^2(\Omega_{T_\varepsilon})} \leq C. \tag{161}$$

For the $\mathcal{H}^3$ estimate, we bound successively $\|\partial_{1ij}\rho\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))}$ for $i,j \neq 1$, $\|\partial_{11i}\rho\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))}$ for $i \neq 1$ and $\|\partial_{111}\rho\|_{L^\infty(0,T_\varepsilon;L^2(\Omega))}$.So we obtain that there exists a constant C such that

$$\|\rho\|_{\mathcal{H}^3(\Omega_{T_\varepsilon})} \leq C. \tag{162}$$

Last step of the proof of Theorem 3.

We recall that we fixed $T < T^*$, and that T_ε is defined by (128) so, either $T = T_\varepsilon$ or $\|\rho_\varepsilon\|_{\mathcal{H}^3(\Omega_{T_\varepsilon})} = \dfrac{1}{\sqrt{\varepsilon}}$, which is contradictory with (162) for ε small enough. So there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $T^*_\varepsilon \geq T_\varepsilon = T$ and by (162), $\|\rho_\varepsilon\|_{\mathcal{H}^3(\Omega_T)} \leq C$, so we obtain (122) and (123) in Theorem 3.

Remark 3 *The same conclusions hold in the two and one dimensional cases. In the 1d case we can replace (123) by:*

$$\|e_\varepsilon - e_0\|_{\mathcal{H}^2(\Omega_T)} + \|h_\varepsilon - h_0\|_{\mathcal{H}^2(\Omega_T)} + \|v_\varepsilon\|_{\mathcal{H}^2(\Omega_T)} \leq C\varepsilon.$$

For the 1d and 2dTE cases, (122) is irrelevant since $T^*_\varepsilon = +\infty$ by Theorems 1 and 2.

4 Appendix: Proof of Proposition 2

As in Section 2.3 the IBVP (31)-(34) is equivalent to the following system in the variable $V = (U, \chi) = (E, H, \chi)$:

$$\begin{cases} (i) & (1+\chi)\partial_t E + (\partial_t \chi)E - \text{curl } H = G_1, \\ (ii) & \partial_t H + \text{curl } E = G_2, \\ (iii) & \partial_t \chi = \dfrac{1}{\varepsilon}(|E|^2 - \chi), \end{cases} \tag{163}$$

with the null initial and boundary conditions:

$$V(0,x) = 0, x \in \Omega, \tag{164}$$

$$H \times n + a((E \times n) \times n) = 0, (t,x) \in I\!R^+ \times \Gamma. \tag{165}$$

We recall the divergence free conditions

$$\text{div } H = \text{div } ((1+\chi)E) = 0, \tag{166}$$

and the positiveness property

$$\chi \geq 0. \tag{167}$$

The proof is based on the following iteration scheme:
First, $\chi_k \geq 0$ being given, we define $U_{k+1} = (E_{k+1}, H_{k+1})$ by

$$\begin{cases} (1+\chi_k)\partial_t E_{k+1} + (\partial_t \chi_k)E_{k+1} - \text{curl } H_{k+1} = G_1, \\ \partial_t H_{k+1} + \text{curl } E_{k+1} = G_2, \end{cases} \tag{168}$$

with the null initial and boundary conditions:

$$U_{k+1}(0,x) = 0, x \in \Omega, \tag{169}$$

$$H_{k+1} \times n + a((E_{k+1} \times n) \times n) = 0, (t,x) \in I\!R^+ \times \Gamma, \tag{170}$$

so the divergence free conditions hold:

$$\text{div } H_{k+1} = \text{div } ((1+\chi_k)E_{k+1}) = 0. \tag{171}$$

Afterwards we define χ_{k+1} solving the differential equation

$$\partial_t \chi_{k+1} + \frac{1}{\varepsilon}\chi_{k+1} = \frac{1}{\varepsilon}|E_{k+1}|^2, \tag{172}$$

with the initial condition

$$\chi_{k+1}(0,x) = 0, x \in \Omega. \tag{173}$$

We remark that

$$\chi_{k+1} \geq 0. \tag{174}$$

We initialize the iteration scheme by taking $V_0 = 0$. Assuming that for any $T > 0$, $V_k \in \mathcal{H}^3(\Omega_T)$ with $\partial_t \chi_k \in \mathcal{H}^3(\Omega_T)$, using the existence results for linear problems in [16], the system (168)-(169)-(170) admits as unique solution $U_{k+1} = (E_{k+1}, H_{k+1}) \in \mathcal{H}^3(\Omega_T)$ which satisfies the divergence free condition (171). So (172)-(173) admits an unique solution $\chi_{k+1} \in \mathcal{H}^3(\Omega_T)$ and by these equations $\partial_t \chi_{k+1} \in \mathcal{H}^3(\Omega_T)$ and $\chi_{k+1} \geq 0$.

4.1 High-norm boundedness

We denote

$$\xi_k(t) = \sum_{i=0}^{3} \|\partial_t^i U_k(t)\|_{H^{3-i}(\Omega)}, \tag{175}$$

and

$$\varphi_k(t) = \sum_{i=0}^{3} \|\partial_t^i V_k(t)\|_{H^{3-i}(\Omega)} + \sum_{i=0}^{3} \|\partial_t^{i+1} \chi_k(t)\|_{H^{3-i}(\Omega)}. \tag{176}$$

For $\alpha = (\alpha_0, \alpha_2, \alpha_3) \in I\!N^3$ we define the derivative operator $\partial^\alpha = \partial_t^{\alpha_0} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ and we denote

$$\eta_k(t) = \sum_{\alpha=(\alpha_0,\alpha_2,\alpha_3), |\alpha| \leq 3} \|\partial^\alpha U_k(t)\|_{L^2(\Omega)}, \tag{177}$$

that is, $\eta_k(t)$ measures the L^2-norm of the tangential derivatives of U_k.
Finally the source terms are estimated by

$$\Gamma(t) = \|G\|_{\mathcal{H}^3(\Omega_t)}. \tag{178}$$

We introduce the time T_k defined by

$$T_k = \max \left\{ t \geq 0, \sup_{s \in [0,t]} \varphi_k(s) \leq 1, \Gamma(t) \leq 1 \right\}. \tag{179}$$

To obtain the boundedness, we shall prove that there exists $\tilde{T} > 0$ such that

$$\forall\, k \geq 0, T_k \geq \tilde{T}. \tag{180}$$

In order to estimate the tangential derivatives of U_k we first establish the following

Lemma 4 *There exists a constant $K > 0$ such that for all k we have:*

$$\forall\, t \leq T_k, \ \frac{d}{dt} \int_\Omega \sum_{\alpha=(\alpha_0,\alpha_2,\alpha_3), |\alpha| \leq 3} \left((1+\chi_k)|\partial^\alpha E_{k+1}|^2 + |\partial^\alpha H_{k+1}|^2 \right) dx \leq K(1 + (\xi_{k+1})^2). \tag{181}$$

Proof: taking the inner product of (168) with U_{k+1} we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi_k)|E_{k+1}|^2+|H_{k+1}|^2\right)dx - \int_{\partial\Omega} a((E_{k+1}\times n)\times n)\cdot E_{k+1}dx_2dx_3$$
$$= -\frac{1}{2}\int_\Omega \partial_t\chi_k|E_{k+1}|^2dx + \int_\Omega G\cdot U_{k+1}dx.$$

Since $\|\partial_t\chi_k\|_{L^\infty(\Omega)}\leq\varphi_k$ we get the L^2-estimate:

$$\forall\, t\leq T_k,\ \ \frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi_k)|E_{k+1}|^2+|H_{k+1}|^2\right)dx \leq K(1+(\xi_{k+1})^2).$$

By the same method we obtain similar estimates concerning the tangential derivatives. Let us describe the order 3 derivatives estimates: for $\alpha=(\alpha_0,\alpha_2,\alpha_3)$ with $|\alpha|=\alpha_0+\alpha_2+\alpha_3=3$ we take the inner product of $\partial^\alpha(168)$ with $\partial^\alpha U_{k+1}$, so we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((1+\chi_k)|\partial^\alpha E_{k+1}|^2+|\partial^\alpha H_{k+1}|^2\right)dx \leq I_1+\ldots+I_5$$

with

$$I_1 = K\sum_{\substack{|\beta|=|(\beta_0,\beta_2,\beta_3)|=1\\ |\gamma|=|(\gamma_0,\gamma_2,\gamma_3)|=3}}\int_\Omega |\partial^\beta\chi_k||\partial^\gamma E_{k+1}||\partial^\alpha E_{k+1}|dx$$
$$\leq K\sum_{|\beta|=1,|\gamma|=3}\|\partial^\beta\chi_k\|_{L^\infty(\Omega)}\|\partial^\gamma E_{k+1}\|_{L^2(\Omega)}\|\partial^\alpha E_{k+1}\|_{L^2(\Omega)},$$

$$I_2 = K\sum_{|\beta|=2,|\gamma|=2}\int_\Omega |\partial^\beta\chi_k||\partial^\gamma E_{k+1}||\partial^\alpha E_{k+1}|dx$$
$$\leq K\sum_{|\beta|=2,|\gamma|=2}\|\partial^\beta\chi_k\|_{L^4(\Omega)}\|\partial^\gamma E_{k+1}\|_{L^4(\Omega)}\|\partial^\alpha E_{k+1}\|_{L^2(\Omega)},$$

$$I_3 = K\sum_{|\beta|=3,|\gamma|=1}\int_\Omega |\partial^\beta\chi_k||\partial^\gamma E_{k+1}||\partial^\alpha E_{k+1}|dx$$
$$\leq K\sum_{|\beta|=3,|\gamma|=1}\|\partial^\beta\chi_k\|_{L^2(\Omega)}\|\partial^\gamma E_{k+1}\|_{L^\infty(\Omega)}\|\partial^\alpha E_{k+1}\|_{L^2(\Omega)},$$

$$I_4 = \int_\Omega |\partial_t\partial^\alpha\chi_k||E_{k+1}||\partial^\alpha E_{k+1}|dx$$
$$\leq K\|\partial_t\partial^\alpha\chi_k\|_{L^2(\Omega)}\|E_{k+1}\|_{L^\infty(\Omega)}\|\partial^\alpha E_{k+1}\|_{L^2(\Omega)}$$

$$I_5 = \int_\Omega |\partial^\alpha G||\partial^\alpha U_{k+1}|dx$$
$$\leq \|\partial^\alpha G\|_{L^2(\Omega)}\|\partial^\alpha U_{k+1}\|_{L^2(\Omega)}.$$

So using Sobolev inequalities we obtain that for all $t\leq T_k$,

$$I_1+\ldots+I_5\leq K(1+(\xi_{k+1})^2).$$

Now we control all the derivatives of U_{k+1} by the tangential derivatives with the following:

Lemma 5 *There exists $K > 0$ such that*

$$\forall\, k, \forall\, t \leq T_k,\ \xi_{k+1}(t) \leq K(\eta_{k+1}(t) + \Gamma(t)). \tag{182}$$

Proof: in order to estimate $\partial_1 U_{k+1}$ we rewrite (168) and (171) to obtain

$$\begin{aligned}
\partial_1 H_{k+1,3} &= -\partial_t \chi_k E_{k+1,2} - (1+\chi_k)\partial_t E_{k+1,2} + \partial_3 H_{k+1,1} + G_{1,2} \\
\partial_1 H_{k+1,2} &= +\partial_t \chi_k E_{k+1,3} + (1+\chi_k)\partial_t E_{k+1,3} + \partial_2 H_{k+1,1} - G_{1,3} \\
\partial_1 E_{k+1,3} &= \partial_3 E_{k+1,1} + \partial_t H_{k+1,2} - G_{2,2} \\
\partial_1 E_{k+1,2} &= \partial_2 E_{k+1,1} - \partial_t H_{k+1,3} + G_{2,3} \\
\partial_1 H_{k+1,1} &= -\partial_2 H_{k+1,2} - \partial_3 H_{k+1,3} \\
\partial_1 E_{k+1,1} &= -\partial_2 E_{k+1,2} - \partial_3 E_{k+1,3} - (1+\chi_k)^{-1}(\nabla \chi_k \cdot E_{k+1})
\end{aligned} \tag{183}$$

Using (175), (176), (177), (178) and (179), we obtain the estimate

$$\forall\, t \leq T_k, \|\partial_1 U_{k+1}\|_{L^2(\Omega)} \leq K(\eta_{k+1}(t) + \Gamma(t)).$$

In order to estimate $\partial_{1i} U_{k+1}$ we derivate (183) with respect to i. For $i \neq 1$, we obtain directly that

$$\forall\, t \leq T_k,\ \|\partial_{1i} U_{k+1}\|_{L^2(\Omega)} \leq K(\eta_{k+1}(t) + \Gamma(t)), \tag{184}$$

and using (184) we obtain

$$\forall\, t \leq T_k,\ \|\partial_{11} U_{k+1}\|_{L^2(\Omega)} \leq K(\eta_{k+1}(t) + \Gamma(t)).$$

With the same arguments we estimate successively $\|\partial_{1ij} U_{k+1}\|_{L^2(\Omega)}$, $\|\partial_{11j} U_{k+1}\|_{L^2(\Omega)}$ and $\|\partial_{111} U_{k+1}\|_{L^2(\Omega)}$ where $i \neq 1$ and $j \neq 1$, so we conclude the proof of Lemma 5.

Using (181) and (182), by Gronwall lemma, we obtain that there exists $K > 0$ such that

$$\forall\, k, \forall\, t \leq T_k, (\eta_{k+1}(t))^2 \leq e^{Kt} - 1,$$

so by (182), there exists $K > 0$ such that

$$\forall\, k, \forall\, t \leq T_k, \xi_{k+1}(t) \leq K\left((e^{Kt} - 1)^{\frac{1}{2}} + \Gamma(t)\right). \tag{185}$$

Then, solving (172) we obtain that

$$\|\chi_{k+1}(t)\|_{H^3(\Omega)} \leq K \sup_{s \in [0,t]} (\xi_{k+1}(s))^2.$$

Using (172), $\partial_t(172)$, $\partial_t^2(172)$ and $\partial_t^3(172)$ we obtain that

$$\sum_{i=1}^{4} \|\partial_t^i \chi_{k+1}\|_{H^{4-i}(\Omega)} \leq K \sup_{s \in [0,t]} (\xi_{k+1}(s))^2.$$

So there exists $K > 0$ such that

$$\forall\, k,\ \varphi_{k+1}(t) \leq \xi_{k+1}(t) + K \sup_{s \in [0,t]} (\xi_{k+1}(s))^2.$$

Therefore with (185) there exists $K > 0$ such that:

$$\forall\, k, \forall\, t \leq T_k, \varphi_{k+1}(t) \leq g(t) := K\left((e^{Kt} - 1)^{\frac{1}{2}} + \Gamma(t) + ((e^{Kt} - 1)^{\frac{1}{2}} + \Gamma(t))^2\right).$$

Since $g(0) = 0$ there exists $\tilde{T} > 0$ such that for $t \leq \tilde{T}$, $g(t) \leq \frac{3}{4}$ and $\Gamma(t) \leq \frac{3}{4}$, so we conclude the proof of (180). In addition we have obtained the high-norm boundedness:

$$\forall\, k,\ sup_{[0,\tilde{T}]} \varphi_k(t) \leq 1. \tag{186}$$

4.2 Low-norm contraction

Substracting (168) for $k-1$ to the one for k we have:

$$\left\{\begin{array}{rcl}(1+\chi_k)\partial_t(E_{k+1}-E_k)-\mathrm{curl}\,(H_{k+1}-H_k) &=& -\partial_t\chi_k(E_{k+1}-E_k)+(\chi_{k-1}-\chi_k)\partial_t E_k \\ & & +(\partial_t\chi_{k-1}-\partial_t\chi_k)E_k, \\ \partial_t(H_{k+1}-H_k)+\mathrm{curl}\,(E_{k+1}-E_k) &=& 0,\end{array}\right. \tag{187}$$

with the null initial and boundary conditions:

$$(E_{k+1}-E_k)(0,x)=(H_{k+1}-H_k)(0,x)=0,\ x\in\Omega, \tag{188}$$

$$(H_{k+1}-H_k)\times n+a(((E_{k+1}-E_k)\times n)\times n)=0,\ (t,x)\in I\!R^+\times\Gamma. \tag{189}$$

Taking the inner product of (187) by $U_{k+1}-U_k$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega((1+\chi_k)|E_{k+1}-E_k|^2+|H_{k+1}-H_k|^2)dx\le-\frac{1}{2}\int_\Omega\partial_t\chi_k|E_{k+1}-E_k|^2dx$$

$$+\int_\Omega(\chi_{k-1}-\chi_k)\partial_t E_k(E_{k+1}-E_k)dx+\int_\Omega(\partial_t\chi_{k-1}-\partial_t\chi_k)E_k\cdot(E_{k+1}-E_k)dx$$

so using (186)

$$\begin{array}{c}\displaystyle\frac{d}{dt}\int_\Omega((1+\chi_k)|E_{k+1}-E_k|^2+|H_{k+1}-H_k|^2)dx \\ \le K\left(\|E_{k+1}-E_k\|^2_{L^2(\Omega)}+\|\chi_k-\chi_{k-1}\|^2_{L^2(\Omega)}+\|\partial_t\chi_k-\partial_t\chi_{k-1}\|^2_{L^2(\Omega)}\right).\end{array} \tag{190}$$

Substracting (172) for $k-2$ to the one for $k-1$ we have

$$\partial_t(\chi_k-\chi_{k-1})+\frac{1}{\varepsilon}(\chi_k-\chi_{k-1})=\frac{1}{\varepsilon}(E_k+E_{k-1})\cdot(E_k-E_{k-1}), \tag{191}$$

so

$$\chi_k-\chi_{k-1}=\int_0^t\frac{1}{\varepsilon}\exp\left(\frac{t-s}{\varepsilon}\right)(E_k+E_{k-1})(s)\cdot(E_k-E_{k-1})(s)ds. \tag{192}$$

We introduce

$$u_k(t)=\|U_{k+1}-U_k\|_{L^\infty(0,t;L^2(\Omega))},$$

and from (192) and (186) we have

$$\|(\chi_k-\chi_{k-1})(t)\|_{L^2(\Omega)}\le Ku_{k-1}(t), \tag{193}$$

and from (191)

$$\|(\partial_t\chi_k-\partial_t\chi_{k-1})(t)\|_{L^2(\Omega)}\le Ku_{k-1}(t). \tag{194}$$

Integrating (190) and using (193) and (194) we have

$$(u_k(t))^2\le Kt(u_k(t))^2+Kt(u_{k-1}(t))^2. \tag{195}$$

We fix $\bar T\le\tilde T$ such that in (195) $K\bar T\le\frac{1}{4}$ so we have

$$\forall\, k,\ \forall\, t\le\bar T,\ u_k(t)\le\frac{1}{\sqrt{3}}u_{k-1}(t).$$

Hence U_k is a Cauchy sequence in $\mathcal{C}(0,\bar T;L^2(\Omega))$ and by (193), χ_k is also a Cauchy sequence in $\mathcal{C}(0,\bar T;L^2(\Omega))$. By standard arguments we obtain a regular local solution to (163)-(164)-(165). The end of the proof of Proposition 2 is classical.

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