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# Stiffest Elastic Networks

Gérald Gurtner and Marc Durand

*Matière et Systèmes Complexes (MSC),*

*UMR 7057 CNRS & Université Paris Diderot,*

*10 rue Alice Domon et Léonie Duquet, 75205 Paris Cedex 13, France*

## Abstract

Elastic networks of beams can deform either by stretching or bending of their elements. Rigidity of such networks crucially depends on the dominant mechanism of deformation: under the same loading conditions, stretch-dominated structures are much stiffer than bending-dominated structures. We prove the existence of stiffest networks which deform by extension (or compression) of their elements only, *under any loading conditions*. Indeed, we provide a convenient set of necessary and sufficient geometrical conditions to identify such networks. In addition, values of the effective elastic moduli of these structures are derived. Our results shed light on the relationship between microscopic geometry and macroscopic mechanics of elastic networks, and may be of interest for structural applications.

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Various elastic systems can be understood as networks of interconnected struts which deform by a combination of bending, stretching, twisting and shearing mechanisms. Examples include polymer gels, protein networks and cytoskeletal structures [1, 2, 3, 4, 5], crystal atomic lattices and granular materials [7, 8], or foams, wood, and bones [9, 10, 11, 12, 13]. Moreover, the pairwise interaction potentials used in standard elastic percolation models can also be identified with the strain energy of elastic struts [1, 7, 14]. Despite extensive research [3, 4, 5, 6], the connection between the mechanical properties of such networks on a macroscopic level and the description of their structures on a microscopic level has not been completely elucidated yet. Interestingly, under identical loading conditions, some structures appear to deform primarily through the local stretching of the struts, while in other structures the elastic energy is stored via local bending [9, 10] (twisting and shearing contributions usually are neglected). A transition between these two regimes has been reported, both experimentally and numerically, when the mean node connectivity is increased [5].

In most cases, the apparent stiffness of an elastic network is also dramatically affected by the loading conditions: for instance, the square lattice is bending-dominated under shear stress, but is stretch-dominated when loaded uniaxially along one of its principal axis. On dimensional grounds [10], the strain energy density associated with a stretch-dominated deformation varies linearly with the relative density  $\phi$ , defined as the volume of struts per unit volume of material, while it scales as  $\phi^2$  for a bending-dominated deformation. Thus, for the low-density materials considered here ( $\phi \ll 1$ ), a structure deforming primarily through the stretching of the beams is much stiffer, and hence much more efficient for structural applications.

In this letter, we analyze the structural conditions under which a network of beams deforms exclusively by extension/compression of its elements, *for any loading conditions*. We show that such structures do exist, and we provide a convenient set of criteria on the local geometry to identify them. For this purpose, we proceed in two steps: we first establish bounds on the elastic moduli of any network of beams, using a variational approach. We then show that these bounds are reached for some specific network geometries, determined by a set of necessary and sufficient conditions. As expected, these networks deform by axial extension or compression of the beams only. We will limit our study to isotropic structures, though the reasoning can be transposed without difficulty to materials with lower symmetries.

For small strains, the strain energy per unit volume of an isotropic body can be written in general as [15]:

$$\varepsilon = \frac{\lambda}{2} \left( \sum_l u_{ll} \right)^2 + \mu \sum_{kl} u_{kl}^2, \quad (1)$$

where  $\lambda$  is the Lamé's first parameter,  $\mu$  the shear modulus (or Lamé's second parameter), and  $u_{kl} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$  are the components of the strain tensor ( $u_k$  are the components of the displacement field  $\mathbf{u}(\mathbf{r})$ ). The elastic properties of any isotropic body are then determined by two independent elastic moduli. In what follows we will use the Young's modulus  $E$  rather than  $\lambda$ , which measures the stiffness of a body under uniaxial loading. This elastic modulus is defined as  $E = 4\mu(\lambda + \mu) / (\lambda + 2\mu)$  and  $E = \mu(3\lambda + 2\mu) / (\lambda + \mu)$  for two-dimensional (2d) and three-dimensional (3d) materials, respectively. According to Eq. 1, the strain energy is an increasing function of  $E$  and  $\mu$ .

We first establish bounds on these two elastic moduli using the Principle of Minimum Potential Energy (PMPE): consider a body with volume  $V$  and prescribed displacements on its boundaries; the PMPE states that, among all kinematically admissible displacement fields (i.e. all continuous displacement fields satisfying the displacement constraints on the boundary), the actual displacement (i.e. the one satisfying the equations of mechanical equilibrium) is the one that makes the potential energy  $\mathcal{E} = \int_V \varepsilon dV$  an absolute minimum.

Suppose first that a uniform shear strain is applied on the boundary of an homogeneous isotropic body in the  $xy$  plan. The macroscopic displacement field associated with such a strain is  $\mathbf{u}(\mathbf{r}) = (\delta X/Y)y \mathbf{e}_x$ , where  $X$  and  $Y$  are the dimensions of the sample in the  $x$  and  $y$  directions, respectively, and  $\delta X$  the maximal displacement in the  $x$  direction. The corresponding strain energy is  $\mathcal{E} = \mu(\delta X/Y)^2 V/2$ . On a microscopic level, the body is comprised of beams that can have natural curvatures and non-uniform cross-sections. Let us denote  $(i, j)$  the beam linking the nodes  $i$  and  $j$ . We note  $s_{ij}(l)$ ,  $\mathbf{t}_{ij}(l)$ , and  $\mathbf{r}_{ij}(l)$  the local cross-sectional area, the local tangent unit vector and the position vector originated from node  $i$ , respectively, where  $l$  refers to the curvilinear coordinate along the beam (see Fig. 1). We also note  $L_{ij}$  the length of the line segment linking nodes  $i$  and  $j$ ,  $\mathbf{e}_{ij}$  the unit vector along this line segment, and  $\cos \theta_{ij}^\alpha = \mathbf{e}_\alpha \cdot \mathbf{e}_{ij}$  the director cosine of this line segment with the  $\alpha$  axis ( $\alpha \in \{x, y\}$  for 2d materials, and  $\alpha \in \{x, y, z\}$  for 3d materials). Let us consider the trial displacement field defined as:  $\mathbf{u}_i = (\delta X/Y)y_i \mathbf{e}_x$  for every node  $i$  ( $y_i$  is the

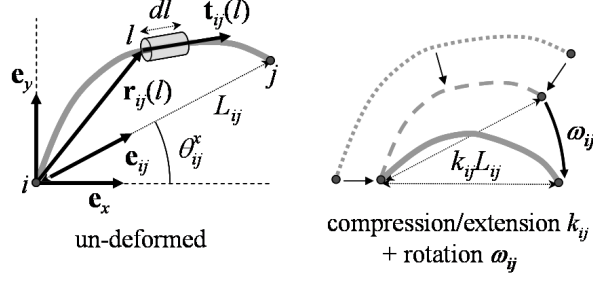


FIG. 1: Geometry of a typical beam  $(i, j)$ . In the initial configuration, the beam can have a natural curvature and a non-uniform cross-section. Under the trial displacement field 2, the beam is subjected to a translation  $\mathbf{u}_i$ , a uniform compression/extension  $k_{ij}$  and a uniform rotation  $\boldsymbol{\omega}_{ij}$ .

coordinate of node  $i$  along the  $y$  axis), and:

$$\mathbf{u}_{ij}(l) = \mathbf{u}_i + k_{ij}\mathbf{r}_{ij}(l) + \boldsymbol{\omega}_{ij} \times \mathbf{r}_{ij}(l) \quad (2)$$

for the point of the central line of beam  $(i, j)$  located at position  $l$ , where  $k_{ij} = \mathbf{e}_{ij} \cdot (\mathbf{u}_j - \mathbf{u}_i) / L_{ij} = (\delta X / Y) \cos \theta_{ij}^x \cos \theta_{ij}^y$  and  $\boldsymbol{\omega}_{ij} = \mathbf{e}_{ij} \times (\mathbf{u}_j - \mathbf{u}_i) / L_{ij} = -(\delta X / Y) \cos^2 \theta_{ij}^y \mathbf{e}_z$ . Thus, each node follows the macroscopic displacement, while each infinitesimal piece of beam is subject to a translation  $\mathbf{u}_i$ , an axial compression/extension  $k_{ij}$  and a rotation  $\boldsymbol{\omega}_{ij}$ . Since the rotation vector  $\boldsymbol{\omega}_{ij}$  is constant, there is no bending or twisting of the beam. Moreover, we can complete the definition of the displacement field by assuming that the cross-sections remain perpendicular to the central line. Then, the only contribution to the strain energy of a piece of beam with infinitesimal length  $dl$  is the stretching term  $(E_0/2) s_{ij}(l) k_{ij}^2 dl$ , where  $E_0$  is the Young's modulus of the beam material. For low-density structures, the node contribution to the strain energy is negligible. Therefore, the energy of the whole structure associated with this trial displacement field is  $\mathcal{E}_{trial} = E_0/2(\delta X/Y)^2 \sum_{(i,j)} v_{ij} \cos^2 \theta_{ij}^x \cos^2 \theta_{ij}^y$ , where the summation is over all the beams that constitute the network, and  $v_{ij}$  is the volume of beam  $(i, j)$ . By comparing  $\mathcal{E}$  with  $\mathcal{E}_{trial}$ , and according to the PMPE, it finally comes that the effective shear modulus is bounded as:

$$\frac{\mu}{E_0} \leq \phi \langle \cos^2 \theta^x \cos^2 \theta^y \rangle, \quad (3)$$

where the brackets denote the average defined, for any quantity  $q_{ij}$ , as:  $\langle q \rangle = \sum_{(i,j)} v_{ij} q_{ij} / \sum_{(i,j)} v_{ij}$ .

We can use the same argumentation to obtain a bound on the Young's modulus: suppose that a uniform elongational strain is applied on the boundary of a homogeneous isotropic

body along the  $x$  axis. The macroscopic displacement field associated with such a strain is  $\mathbf{u}(\mathbf{r}) = (\delta X/X)(x\mathbf{e}_x - \nu y\mathbf{e}_y)$  and  $\mathbf{u}(\mathbf{r}) = (\delta X/X)(x\mathbf{e}_x - \nu y\mathbf{e}_y - \nu z\mathbf{e}_z)$  for 2d and 3d materials, respectively, where  $\nu$  is the effective Poisson's ratio of the network. In both cases, the strain energy associated with such a displacement field is  $\mathcal{E} = E_0/2(\delta X/X)^2V$ . We can define a trial displacement field similar to Eq. 2, with here the node displacement defined as  $\mathbf{u}_i = (\delta X/X)(x_i\mathbf{e}_x - \nu y_i\mathbf{e}_y)$  for 2d materials, and  $\mathbf{u}_i = (\delta X/X)(x_i\mathbf{e}_x - \nu y_i\mathbf{e}_y - \nu z_i\mathbf{e}_z)$  for 3d materials (where  $x_i, y_i, z_i$  are the coordinates of node  $i$ ). We can evaluate the parameters  $k_{ij}, \boldsymbol{\omega}_{ij}$ , and verify that the only contribution to the associated strain energy comes from the axial extension/compression of every infinitesimal piece of beam. Then, the total strain energy  $\mathcal{E}_{trial}$  associated with such a field can be easily evaluated. Using again the PMPE, we finally obtain that the effective Young's modulus is bounded as:

$$\frac{E}{E_0} \leq \phi \left\langle (\cos^2 \theta^x - \nu \cos^2 \theta^y)^2 \right\rangle \quad (2d) \quad (4)$$

$$\frac{E}{E_0} \leq \phi \left\langle (\cos^2 \theta^x - \nu \cos^2 \theta^y - \nu \cos^2 \theta^z)^2 \right\rangle \quad (3d). \quad (4)$$

The expressions for the bounds 3 and 4 involve average quantities which can be evaluated using symmetry arguments. For isotropic structures, the expressions of the bounds must be identical for any orientation of the applied strain. That is, for 2d structures, the transformation  $\theta^x \rightarrow \theta^x + \varphi, \theta^y \rightarrow \theta^y - \varphi$  must leave the expressions unchanged, for any rotation angle  $\varphi$ . Similarly for 3d structures, the bound expressions must remain unchanged by rotations around the three perpendicular axes. Furthermore, bound expressions must be invariant under permutation of the axis labels. After a little algebra, these invariance properties lead to a set of relations which can be summarized as:

$$\left\{ \begin{array}{l} \langle \cos^2 \theta^\alpha \rangle = \frac{1}{d} \quad \langle \cos^4 \theta^\alpha \rangle = \frac{3}{d(d+2)} \\ \langle \cos^2 \theta^\alpha \cos \theta^\beta \cos \theta^\gamma \rangle = 0 \quad (\beta \neq \gamma) \end{array} \right. \quad (5)$$

with  $\alpha, \beta, \gamma \in \{x, y\}$  (resp.  $\{x, y, z\}$ ), and  $d = 2$  (resp.  $d = 3$ ) for 2d (resp. 3d) structures [16]. The relations 5 imply in particular that  $\langle \cos^2 \theta^\beta \cos^2 \theta^\gamma \rangle = 1/(d(d+2))$  and

$\langle \cos \theta^\beta \cos \theta^\gamma \rangle = 0$  (with  $\beta \neq \gamma$ ) . Accordingly, the inequalities 3 and 4 simplify to:

$$\frac{\mu}{E_0} \leq \frac{\phi}{8} \qquad \frac{E}{E_0} \leq \frac{\phi}{8}(3\nu^2 - 2\nu + 3) \qquad (2d) \qquad (6)$$

$$\frac{\mu}{E_0} \leq \frac{\phi}{15} \qquad \frac{E}{E_0} \leq \frac{\phi}{15}(8\nu^2 - 4\nu + 3) \qquad (3d)$$

It must be noted that the bounds on  $E$  are implicit, since they depend on the Poisson's ratio, which in turns is related to  $E$  and  $\mu$  by:  $\nu = E/2\mu - 1$  (for both two- and three-dimensional bodies).

We now show that inequalities 6 become strict equalities for some specific network geometry determined by the following set of rules (along with the isotropy conditions 5):

- a) All the beams are straight.
- b) All the beams have uniform cross-sectional areas:  $s_{ij}(l) = s_{ij}$ .
- c) At every junction  $i$  of the 2d (resp. 3d) network, and for all  $\mathbf{e}_\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma \in \{\mathbf{e}_x, \mathbf{e}_y\}$  (resp.  $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ ), the following equality is satisfied:

$$\sum_j s_{ij} (\mathbf{e}_{ij} \cdot \mathbf{e}_\alpha) (\mathbf{e}_{ij} \cdot \mathbf{e}_\beta) (\mathbf{e}_{ij} \cdot \mathbf{e}_\gamma) = 0, \qquad (7)$$

where the summation is over all the junctions that are connected to the junction  $i$ . This condition corresponds to a set of 4 (resp. 10) equations per node for 2d (resp. 3d) structures.

The demonstration is straightforward: according to the PMPE, each inequality in Eq. 6 become a strict equality if and only if the respective trial displacement field is the (unique) displacement field that satisfies the equations of mechanical equilibrium. Inspection of force and moment balances along each beam and at each junction leads to the three necessary and sufficient conditions stated above. Let us make this precise. Consider a specific beam  $(i, j)$ ; we already mentioned that with the trial displacements we chose, each infinitesimal piece of beam is only subjected to an axial deformation. Therefore, the force and local moment deriving from the trial displacement fields are  $\mathbf{F}_{ij}(l) = -E_0 s_{ij}(l) k_{ij} \mathbf{t}_{ij}(l)$  and  $\mathbf{M}_{ij}(l) = \mathbf{0}$ , respectively ( $\mathbf{F}_{ij}(l)$  and  $\mathbf{M}_{ij}(l)$  are defined as the force and local torque exerted at position  $l$  by the  $i$ -side of the bar to the  $j$ -side of the bar). The force balance equation  $d\mathbf{F}_{ij}(l)/dl = \mathbf{0}$  along the beam  $(i, j)$  implies that the direction of  $\mathbf{F}_{ij}$  does not change with  $l$ , i.e.  $\mathbf{t}_{ij}(l) = \mathbf{e}_{ij}$  for all  $l$ . Hence, one also has  $s_{ij}(l) = const$ . Therefore, the force balance along every beam of the network leads to condition a) and b). The forces acting on any junction  $i$  must

	$\mu/E_0$	$E/E_0$	$\lambda/E_0$	$\kappa/E_0$	$\nu$
<b>2d</b>	$\phi/8$	$\phi/3$	$\phi/8$	$\phi/4$	$1/3$
<b>3d</b>	$\phi/15$	$\phi/6$	$\phi/15$	$\phi/9$	$1/4$

TABLE I: Elastic moduli of stiffest isotropic structures normalized by the Young’s modulus of the beam material  $E_0$ : ( $\mu$ ), Young’s modulus ( $E$ ), bulk modulus ( $\kappa$ ), Lamé’s first parameter ( $\lambda$ ), and Poisson’s ratio ( $\nu$ ).

balance as well. The force exerted by the straight and uniform beam ( $i, j$ ) on this junction is  $E_0 s_{ij} k_{ij} \mathbf{e}_{ij}$ . Thus, mechanical equilibrium at every junction  $i$  implies:  $\sum_j s_{ij} k_{ij} \mathbf{e}_{ij} = \mathbf{0}$ . This relation must hold for any orientation of the strain field. By replacing  $k_{ij}$  with its expression and using the same invariance argument as for the bound expressions, we finally obtain, for shear as for elongational strain, that condition c) is also required. Obviously the moment balance equation  $\mathbf{r}_{ij}(l) \times \mathbf{F}_{ij}(l) = \mathbf{0}$  along every beam ( $i, j$ ) is also satisfied for structures satisfying conditions a) and b).

Let us discuss some of the implications of the conditions a)-c). First, as long as the isotropy conditions (5) are satisfied, the two elastic moduli attain their respective bounds under the very same conditions. We can thus eliminate  $\nu$  and express the value of the Young’s modulus:  $E = E_0 \phi/3$  (resp.  $E = E_0 \phi/6$ ) for 2d (resp. 3d) networks. For isotropic structures, any other elastic modulus is related to  $E$  and  $\mu$ , and hence can be easily evaluated. Values of most common moduli are reported on table I.

Interestingly, the Poisson’s ratio of such structures is less than the one expected for an incompressible body (1 for 2d body, 1/2 for 3d body). It can also be noticed that the two Lamé’s parameters  $\lambda$  and  $\mu$  are equal. Values of the shear, bulk, and Young’s moduli reported on table I coincide with the Hashin-Shtrikman bounds [17] for low-density ( $\phi \ll 1$ ) 2d structures, but are strictly lower (i.e. sharper) for 3d structures [18].

It is worth mentioning that condition c) implies in particular that the vectorial equality  $\sum_j s_{ij} \mathbf{e}_{ij} = \mathbf{0}$  is satisfied in stiffest structures. Therefore, these structures have maximal bulk modulus [20] and maximal electrical (or thermal) conductivity [21, 22, 23] as well, for a given value of  $\phi$ .

Our results also reveal why open-cell foams are almost always bending-dominated. Foams are particular cellular materials: their structures result from a surface minimization process,



leading to geometrical and topological rules, known as Plateau’s laws. Assuming that edges have identical cross-sections, Plateau’s laws state that edges meet in threefold (resp. fourfold) junctions with equal angles of  $120^\circ$  (resp.  $109.5^\circ$ ) in 2d (resp. 3d) foams. One can easily check that such node configurations cannot satisfy condition c) [24]. Therefore, open-cell foams will rather deform by bending of their edges.

Some examples of stiffest structures can be given. Let us inspect the particular case of periodic structures. 2d isotropic structures with periodic arrangement must have hexagonal symmetry [15]. If we restrict our attention to networks with identical beams, only two candidates are left: the hexagonal and triangular lattices. Although the hexagonal lattice satisfy the isotropy conditions 5, it does not satisfy the condition c), and therefore is not stretch-dominated, as it has been already reported [9, 19] [actually, it can be shown that a three-fold junction of beams with equal sections cannot satisfy condition c)]. The triangular lattice, on the other hand, satisfies all the conditions. Indeed, the elastic moduli of this lattice match the values expected for stiffest structures [13, 19].

The 3d case is less obvious. The structure depicted on Fig. 2a (where the bars join the center of the faces of a regular dodecahedron to its center) is an example of 3d isotropic stiffest structure: it satisfies the isotropy conditions 5 and the set of criteria a)-c). However, it cannot be repeated periodically because the dodecahedron does not perfectly tile the space. In fact, we could not identify any periodic and isotropic rigid structures with one single kind of bar. However, we found stiff structures with two types of bars. For instance, the structure depicted on 2b satisfies all the required conditions and can be repeated periodically: the bars join the centers of the faces of a Kelvin cell (which is known for tiling the space). The length ratio of the two kinds of bars is imposed by the Kelvin cell geometry:  $l_2/l_1 = \sqrt{3}/2$ . We then adjust the section ratio for the isotropy conditions 5 to be satisfied. We obtain:  $s_2/s_1 = 3\sqrt{3}/4$ .

In summary, we established a convenient set of conditions on the local geometry of isotropic networks of beams to be stiff under any loading conditions. The elastic moduli of these networks are also derived, and can be simply expressed in terms of the Young’s modulus of the beam material and the relative density of the structure. Examples of two- and three-dimensional stiff structures are also given. These results explain why most cellular structures, and foams in particular, are bending-dominated. We hope these results are of interest for structural applications as well as for our understanding of biological systems.

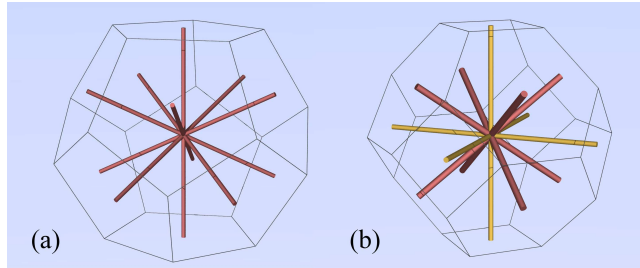


FIG. 2: (color online) stiff three-dimensional structures: **(a)**: structure with identical bars: the bars join the center of the faces of a regular dodecahedron to its center. This structure satisfies the isotropy conditions 5 and the set of criteria a)-c) but cannot be repeated periodically. **(b)**: structure with two types of bars. The bars join the centers of the faces of a Kelvin cell. Such structure satisfies both the isotropy conditions and the set of criteria a)-c) and can be repeated periodically.

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