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SELF-ADJOINTNESS VIA PARTIAL HARDY-LIKE INEQUALITIES

MARIA J. ESTEBAN¹ AND MICHAEL LOSS²

ABSTRACT. Distinguished selfadjoint extensions of operators which are not semibounded can be deduced from the positivity of the Schur Complement (as a quadratic form). In practical applications this amounts to proving a Hardy-like inequality. Particular cases are Dirac-Coulomb operators where distinguished selfadjoint extensions are obtained for the optimal range of coupling constants.

Keywords: Relativistic quantum mechanics, Dirac operator, self-adjoint operator, self-adjoint extension, Schur complement.

1. INTRODUCTION.

In [4] we defined distinguished self-adjoint extensions of Dirac-Coulomb operators in the optimal range for the coupling constant. This was done by using a Hardy-like inequality which allowed the extension of one component of the operator by using the Friedrichs extension. Then, the remaining component could be extended by choosing the right domain for the whole operator. The method of proof used simple arguments of distributional differentiation. This work was the sequel of a series of papers where distinguished self-adjoint extensions of Dirac-Coulomb like operators were defined by different methods almost in the optimal range, without reaching the limit case (see [10, 9, 12, 13, 14, 7, 6]).

Here we present an abstract version of the method introduced in [4]. We believe that this will clarify the precise structure and hypotheses necessary to define distinguished self-adjoint extensions by this method.

The main idea in our method is that Hardy-like inequalities are fundamental to define distinguished (physically relevant) self-adjoint extensions *even* for operators that are not bounded below.

We are going to apply our method to operators H defined on \mathcal{D}_0^2 , where \mathcal{D}_0 is some dense subspace of a Hilbert space \mathcal{H}_0 . The general structure taken into account here is:

$$(1) \quad H = \begin{pmatrix} P & Q \\ T & -S \end{pmatrix},$$

where all the above operators satisfy $Q = T^*$, $P = P^*$, $S = S^*$ and $S \geq c_1 I > 0$. Moreover we assume that $P, Q, S, T, S^{-1}T$ and $QS^{-1}T$ send \mathcal{D}_0 into \mathcal{H}_0 .

In the Dirac-Coulomb case our choice was $\mathcal{H}_0 = L^2(\mathbb{R}^3, \mathbb{C}^2)$ and

$$P = V + 2 - \gamma, \quad Q = T = -i\sigma \cdot \nabla, \quad S = \gamma - V,$$

where V is a potential bounded from above satisfying

$$(2) \quad \sup_{x \neq 0} |x| |V(x)| \leq 1.$$

Moreover, σ_i , $i = 1, 2, 3$, are the Pauli matrices (see [4]) and γ is a constant slightly above $\max_{\mathbb{R}^3} V(x)$. For \mathcal{D}_0 we chose $C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$. Note that in our paper [4], where we deal with Dirac-Coulomb like operators, there is an omission. We forgot to specify the conditions on the potential V so that $QS^{-1}T$ is a symmetric operator on $C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$. The natural condition is that each component of

$$(3) \quad (\gamma - V)^{-2} \nabla V$$

is locally square integrable. This is easily seen to be true for the Coulomb-type potentials.

In the general context of the operator H , as defined in (1), our main assumption is that there exists a constant $c_2 > 0$ such that for all $u \in \mathcal{D}_0$,

$$(4) \quad q_{c_2}(u, u) := ((S + c_2)^{-1}Tu, Tu) + ((P - c_2)u, u) \geq 0.$$

Note that since $\frac{d}{d\alpha}q_\alpha(u, u) \leq -(u, u)$, (4) implies in fact that for all $0 \leq \alpha \leq c_2$ and for all $u \in \mathcal{D}_0$,

$$(5) \quad q_\alpha(u, u) := ((S + \alpha)^{-1}Tu, Tu) + ((P - \alpha)u, u) \geq 0.$$

Another consequence of assumption (4) is that the quadratic form

$$(6) \quad q_0(u, u) = (S^{-1}Tu, Tu) + (Pu, u),$$

defined for $u \in \mathcal{D}_0$, is positive definite:

$$(7) \quad q_0(u, u) = (S^{-1}Tu, Tu) + (Pu, u) \geq c_2(u, u).$$

Note that the operator $P + QS^{-1}T$ which is associated with the quadratic form q_0 is actually the Schur complement of $-S$. Note also that by our assumptions on P, Q, TS and by (4), for any $0 \leq \alpha \leq c_2$, q_α is the quadratic form associated with a positive symmetric operator. Therefore, by Thm. X.23 in [8]), it is closable and we denote its closure by \widehat{q}_α and its form domain, which is easily seen to be independent of α (see [4]) by \mathcal{H}_{+1} . Our main result states the following:

Theorem 1. *Assume the above hypotheses on the operators P, Q, T, S and (4). Then there is a unique self-adjoint extension of H such that the domain of the operator is contained in $\mathcal{H}_{+1} \times \mathcal{H}_0$.*

Remark. Note that what this theorem says that “in some sense” the Schur complement of $-S$ is positive, and therefore has a natural self-adjoint extension, then one can define a distinguished self-adjoint extension of the operator H which is unique among those whose domain is contained in the form domain of the Schur complement of $-S$ times \mathcal{H}_0 .

2. INTERMEDIATE RESULTS AND PROOFS.

We denote by R the unique selfadjoint operator associated with \widehat{q}_0 : for all $u \in D(R) \subset \mathcal{H}_{+1}$,

$$(8) \quad \widehat{q}_0(u, u) = (u, Ru).$$

R is an isometric isomorphism from \mathcal{H}_{+1} to its dual \mathcal{H}_{-1} . Using the second representation theorem in [5], Theorem 2.23, we know that \mathcal{H}_{+1} is the operator domain of $R^{1/2}$, and

$$(9) \quad \widehat{q}_0(u, u) = (R^{1/2}u, R^{1/2}u),$$

for all $u \in \mathcal{H}_{+1}$.

Definition 2. *We define the domain \mathcal{D} of H as the collection of all pairs $u \in \mathcal{H}_{+1}, v \in \mathcal{H}_0$ such that*

$$(10) \quad Pu + Qv, \quad Tu - Sv \in \mathcal{H}_0.$$

The meaning of these two expressions is in the weak (distributional) sense, i.e., the linear functional $(P\eta, u) + (Q^*\eta, v)$, which is defined for all test functions $\eta \in \mathcal{D}_0$, extends uniquely to a bounded linear functional on \mathcal{H}_0 . Likewise the same for $(-S\eta, v) + (T^*\eta, v)$.

On the domain \mathcal{D} , we define the operator H as

$$(11) \quad H \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Pu + Qv \\ Tu - Sv \end{pmatrix}.$$

Note that for all vectors $(u, v) \in \mathcal{D}$ the expected total energy is finite.

The following two results are important in the proof of Theorem 1.

Proposition 3. *Under the assumptions of Theorem 1*

$$(12) \quad \mathcal{H}_{+1} \subset \left\{ u \in \mathcal{H}_0 : S^{-1}Tu \in \mathcal{H}_0 \right\},$$

where the embedding holds in the continuous sense. Therefore, we have the ‘scale of spaces’ $\mathcal{H}_{+1} \subset \mathcal{H}_0 \subset \mathcal{H}_{-1}$.

Proof. Choose $c_2 \geq \alpha > 0$. Since $S \geq c_1 I$, we have for all $0 < \delta \leq \frac{c_1 \alpha}{c_1 + \alpha}$

$$(13) \quad S^{-1} - (S + \alpha)^{-1} \geq \delta S^{-2},$$

and so, for all $u \in \mathcal{D}_0$,

$$(14) \quad q_0(u, u) \geq q_\alpha(u, u) + \alpha(u, u) + \delta(S^{-1}Tu, S^{-1}Tu) \geq \delta(u, u) + \delta(S^{-1}Tu, S^{-1}Tu).$$

The proof can be finished by density arguments. \square

Lemma 4. For any F in \mathcal{H}_0 ,

$$(15) \quad QS^{-1}F \in \mathcal{H}_{-1}.$$

Proof. By our assumptions on H and by Proposition 3, for every $\eta \in \mathcal{D}_0$,

$$(16) \quad |(S^{-1}T\eta, F)| \leq \delta^{-1/2} \|\eta\|_{\mathcal{H}_{+1}} \|F\|_2.$$

Hence, the linear functional

$$(17) \quad \eta \rightarrow (Q^*\eta, S^{-1}F)$$

extends uniquely to a bounded linear functional on \mathcal{H}_{+1} . \square

Proof of Theorem 1. We shall prove Theorem 1 by showing that H is symmetric and a bijection from its domain \mathcal{D} onto \mathcal{H}_0 . To prove the symmetry we have to show that for both pairs (u, v) , (\tilde{u}, \tilde{v}) in the domain \mathcal{D} ,

$$(18) \quad \left(H \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right) = (Pu + Qv, \tilde{u}) + (Tu - Sv, \tilde{v})$$

equals

$$(19) \quad (u, P\tilde{u} + Q\tilde{v}) + (v, T\tilde{u} - S\tilde{v}) = \left(\begin{pmatrix} u \\ v \end{pmatrix}, H \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right).$$

First, note that since (u, v) is in the domain,

$$(20) \quad S(v - S^{-1}Tu) \in \mathcal{H}_0.$$

We now claim that

$$(21) \quad (Pu + Qv, \tilde{u}) = (Ru, \tilde{u}) + (S(v - S^{-1}Tu), S^{-1}T\tilde{u}).$$

Note that each term makes sense. The one on the left, by definition of the domain and the first on the right, because both u, \tilde{u} are in \mathcal{H}_{+1} . The second term on the right side makes sense because of (20) above and Proposition 3. Moreover both sides coincide for \tilde{u} chosen to be a test function and both are continuous in \tilde{u} with respect to the \mathcal{H}_{+1} -norm. Hence the two expressions coincide on the domain. Thus we get that

$$(22) \quad \left(H \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right)$$

equals

$$(23) \quad (Ru, \tilde{u}) - (S(v - S^{-1}Tu), \tilde{v} - S^{-1}T\tilde{u}),$$

an expression which is symmetric in (u, v) and (\tilde{u}, \tilde{v}) . To show that the operator is onto, pick any F_1, F_2 in \mathcal{H}_0 . Since R is an isomorphism, there exists a unique u in \mathcal{H}_{+1} such that

$$(24) \quad Ru = F_1 + QS^{-1}F_2.$$

Indeed, F_1 is in \mathcal{H}_0 and therefore in \mathcal{H}_{-1} . Moreover the second term is also in \mathcal{H}_{-1} by Lemma 4.

Now define v by

$$(25) \quad v = S^{-1}(Tu - F_2),$$

which by Proposition 3 is in \mathcal{H}_0 .

Now for any test function η we have that

$$(26) \quad (P\eta, u) + (Q^*\eta, v) = (P\eta, u) + (T\eta, v) = (P\eta, u) + (T\eta, S^{-1}Tu) + (T\eta, (v - S^{-1}Tu))$$

which equals

$$(27) \quad (\eta, Ru) + (T\eta, (v - S^{-1}Tu)) = (\eta, F_1)$$

This holds for all test functions η , but since F_1 is in \mathcal{H}_0 , the functional $\eta \rightarrow (P\eta, u) + (T\eta, v)$ extends uniquely to a linear continuous functional on \mathcal{H}_0 which implies that

$$(28) \quad Pu + Qv = F_1 .$$

Hence (u, v) is in the domain \mathcal{D} and the operator H applied to (u, v) yields (F_1, F_2) .

Let us now prove the injectivity of H . Assuming that

$$(29) \quad H \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

we find by (24) and (25),

$$v = S^{-1}Tu , \quad Ru = 0 .$$

Since R is an isomorphism, this implies that $u = v = 0$.

It remains to show the uniqueness part in our theorem. By the bijectivity result proved above, for all $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \in \mathcal{H}_0^2$, there exists a unique pair $(\hat{u}, \hat{v}) \in \mathcal{H}_{+1} \times \mathcal{H}_0$ such that $H \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$. Let us now pick any other self-adjoint extension with domain \mathcal{D}' included in $\mathcal{H}_{+1} \times \mathcal{H}_0$. Then for all $(u, v) \in \mathcal{D}'$, $H \begin{pmatrix} u \\ v \end{pmatrix}$ belongs to \mathcal{H}_0^2 . Hence there exist a unique pair $(\hat{u}, \hat{v}) \in \mathcal{H}_{+1} \times \mathcal{H}_0$ such that $H \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = H \begin{pmatrix} u \\ v \end{pmatrix}$. But, by the above considerations on injectivity, $u = \hat{u}$ and $v = \hat{v}$. Therefore, $\mathcal{D}' \subset \mathcal{D}$ and so necessarily, $\mathcal{D}' = \mathcal{D}$. \square

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REFERENCES

- [1] J. Dolbeault, M. J. Esteban, M. Loss. Relativistic hydrogenic atoms in strong magnetic fields. To appear in Ann. H. Poincaré.
- [2] J. Dolbeault, M. J. Esteban, M. Loss, and L. Vega. An analytical proof of Hardy-like inequalities related to the Dirac operator, J. Funct. Anal., **216** (2004), p. 1-21.
- [3] J. Dolbeault, M. J. Esteban, and E. Séré. On the eigenvalues of operators with gaps. Application to Dirac operators, J. Funct. Anal., **174** (2000), pp. 208–226.
- [4] M. J. Esteban, M. Loss. Self-adjointness for Dirac operators via Hardy-Dirac inequalities. To appear in J. Math. Physics.
- [5] T. Kato. *Perturbation Theory for Linear Operators*, Springer Verlag New York, 1966.
- [6] M. Klaus, R. Wüst. Characterization and uniqueness of distinguished self-adjoint extensions of Dirac operators. Comm. Math. Phys. **64(2)** (1978-79), p. 171-176.
- [7] G. Nenciu. Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms. Comm. Math. Phys. **48** (1976), p. 235-247.
- [8] M. Reed, B. Simon. *Methods of modern mathematical physics, Vol. 4*. Academic Press, New York. 1978.
- [9] U.-W. Schmincke. Essential self-adjointness of Dirac operators with strongly singular potential.. Math. Z. **126** (1972), p. 71-81.
- [10] U.-W. Schmincke. Distinguished self-adjoint extensions of Dirac operators. Math. Z. **129** (1972), p. 335-349.
- [11] B. Thaller. *The Dirac Equation*. Springer-Verlag, 1992.
- [12] R. Wüst. A convergence theorem for self-adjoint operators applicable to Dirac operators with cut-off potentials. Math. Z. **131** (1973), p. 339-349.
- [13] R. Wüst. Distinguished self-adjoint extensions of Dirac operators constructed by means of cut-off potentials. Math. Z. **141** (1975), p. 93-98.
- [14] R. Wüst. Dirac operators with strongly singular potentials. Math. Z. **152** (1977), p. 259-271.

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