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# AUTOMORPHISMS OF COXETER GROUPS AND LUSZTIG'S CONJECTURES FOR HECKE ALGEBRAS WITH UNEQUAL PARAMETERS

CÉDRIC BONNAFÉ

ABSTRACT. Let  $(W, S)$  be a Coxeter system, let  $G$  be a finite solvable group of automorphisms of  $(W, S)$  and let  $\varphi$  be a weight function which is invariant under  $G$ . Let  $\varphi_G$  denote the weight function on  $W^G$  obtained by restriction from  $\varphi$ . The aim of this paper is to compare the  $\mathbf{a}$ -function, the set of Duflo involutions and the Kazhdan-Lusztig cells associated with  $(W, \varphi)$  and with  $(W^G, \varphi_G)$ , provided that Lusztig's Conjectures hold.

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Let  $(W, S)$  be a Coxeter system, with  $S$  finite, let  $\Gamma$  be a totally ordered abelian group and let  $\varphi : W \rightarrow \Gamma$  be a *weight function* such that  $\varphi(s) > 0$  for all  $s \in S$ .

Let  $G$  be a group of automorphisms of  $W$  stabilizing  $S$  and  $\varphi$ . We denote by  $\varphi_G$  the restriction of  $\varphi$  to the fixed points subgroup  $W^G$ . If  $\omega \in S/G$  (the orbit set) is

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such that  $W_\omega (= \langle \omega \rangle)$  is finite, we denote by  $s_\omega$  the longest element of the standard parabolic subgroup  $W_\omega$  and we set  $S_G = \{s_\omega \mid \omega \in S/G \text{ and } W_\omega \text{ is finite}\}$ . Then it is well-known that  $(W^G, S_G)$  is a Coxeter system and that  $\varphi_G : W^G \rightarrow \Gamma$  is a weight function (such that  $\varphi_G(s_\omega) > 0$  for all  $\omega \in S/G$ ).

With the datum  $(W, S, \Gamma, \varphi)$  are associated a Hecke algebra  $\mathcal{H}(W, S, \Gamma, \varphi)$  over the ring  $\mathbb{Z}[\Gamma]$ , a Kazhdan-Lusztig basis  $(C_w)_{w \in W}$  of  $\mathcal{H}(W, S, \Gamma, \varphi)$ , equivalence relations  $\sim_{\mathcal{L}}$ ,  $\sim_{\mathcal{R}}$  and  $\sim_{\mathcal{LR}}$  and two functions  $\mathbf{a} : W \rightarrow \Gamma$  and  $\Delta : W \rightarrow \Gamma$  (see [L]). We set  $\mathcal{D} = \{w \in W \mid \mathbf{a}(w) = \Delta(w)\}$ . With the datum  $(W^G, S_G, \Gamma, \varphi_G)$ , we associate similarly  $\sim_{\mathcal{L}}^G$ ,  $\sim_{\mathcal{R}}^G$ ,  $\sim_{\mathcal{LR}}^G$ ,  $\mathbf{a}_G$ ,  $\Delta_G$  and  $\mathcal{D}_G$ . The main result of this paper is the following:

**Theorem A.** *Assume that  $G$  is a finite solvable group and that Lusztig's conjectures  $(P_1)$ ,  $(P_2)$ ,  $(P_3)$ ,  $(P_4)$  in [L, Chapter 14] hold for the datum  $(W^H, S_H, \Gamma, \varphi_H)$  for all subgroups  $H$  of  $G$ . Let  $x, y \in W^G$ . Then:*

- (a)  $\mathbf{a}_G(x) = \mathbf{a}(x)$ .
- (b)  $\mathcal{D}_G = \mathcal{D} \cap W^G$ .
- (c) *Assume moreover that Lusztig's Conjecture  $(P_{13})$  in [L, Chapter 14] hold for the datum  $(W^H, S_H, \Gamma, \varphi_H)$  for all subgroups  $H$  of  $G$ . If  $? \in \{\mathcal{L}, \mathcal{R}\}$ , then  $x \sim_? y$  if and only if  $x \sim_?^G y$ .*
- (d) *Assume moreover that Lusztig's Conjectures  $(P_9)$  and  $(P_{13})$  in [L, Chapter 14] hold for the datum  $(W^H, S_H, \Gamma, \varphi_H)$  for all subgroups  $H$  of  $G$ . Then  $x \sim_{\mathcal{LR}} y$  if and only if  $x \sim_{\mathcal{LR}}^G y$ .*

REMARK - If  $G$  is not solvable and if we assume moreover that Lusztig's conjecture  $(P_{12})$  in [L, Chapter 14] holds, then the statements (a), (b) and (c) of Theorem A hold. It is probable that (d) also holds, but the proof should rely on a really different argument than the one presented here. Indeed, using  $(P_{12})$  and a theorem of Meinolf Geck [1], one can reduce the problem to the case where  $W_\omega$  is finite for all  $G$ -orbits  $\omega$  in  $S$ . Then, since the automorphism groups of irreducible finite Coxeter systems are solvable, one can assume that  $G$  is solvable and apply Theorem A above.  $\square$

The proof of this Theorem makes essential use of reduction modulo  $p$ . Indeed, an easy induction argument reduces immediately the problem to the case where  $G$  is a  $p$ -group for some prime number  $p$ . The main ingredient is then the following: the natural stupid map  $\mathcal{H}(W^G, S_G, \Gamma, \varphi_G) \rightarrow \mathcal{H}(W, S, \Gamma, \varphi)^G$  is not a morphism of algebras in general. However, if we denote by  $\text{Br}_G(\mathcal{H}(W, S, \Gamma, \varphi))$  the quotient of  $\mathcal{H}(W, S, \Gamma, \varphi)^G$  by the two-sided ideal  $\sum_{H < G} \text{Tr}_H^G(\mathcal{H}(W, S, \Gamma, \varphi)^H)$  (Brauer's quotient, see for instance [T, Page 91]), then:

**Proposition B.** *Assume that  $G$  is a finite  $p$ -group. Then the natural linear map  $\mathcal{H}(W^G, S_G, \Gamma, \varphi_G) \rightarrow \text{Br}_G(\mathcal{H}(W, S, \Gamma, \varphi)^G)$  is a morphism of algebras whose kernel is generated by  $p$ . Moreover, it preserves the Kazhdan-Lusztig basis.*

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## 1. THE SET-UP

**1.A. The group  $(W, S)$ .** Let  $(W, S)$  be a Coxeter system (with  $S$  finite), let  $\ell : W \rightarrow \mathbb{N}$  denote the length function, let  $\Gamma$  be a totally ordered abelian group and let  $\varphi : W \rightarrow \Gamma$  be a *weight function* [L, §3.1] that is, a map such that  $\varphi(ww') = \varphi(w) + \varphi(w')$  whenever  $\ell(ww') = \ell(w) + \ell(w')$ .

Let  $A$  be the group algebra  $\mathbb{Z}[\Gamma]$ : we will use an exponential notation for  $A$ , namely  $A = \bigoplus_{\gamma \in \Gamma} \mathbb{Z}e^\gamma$ , where  $e^\gamma \cdot e^{\gamma'} = e^{\gamma+\gamma'}$  for all  $\gamma, \gamma' \in \Gamma$ . If  $a = \sum_{\gamma \in \Gamma} a_\gamma e^\gamma \in A$ , we denote by  $\deg a$  (resp.  $\text{val } a$ ) the *degree* (resp. the *valuation*) of  $a$ , that is, the element  $\gamma$  of  $\Gamma$  such that  $a_\gamma \neq 0$  and which is maximal (resp. minimal) for this condition (by convention,  $\deg 0 = -\infty$  and  $\text{val } 0 = +\infty$ ).

We shall denote by  $\mathcal{H}$  the Hecke algebra associated with the datum  $(W, S, \Gamma, \varphi)$ . It is a free  $A$ -module, with standard basis  $(T_w)_{w \in W}$ , and the multiplication is entirely determined by the following rules:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'); \\ (T_s - e^{\varphi(s)})(T_s + e^{-\varphi(s)}) = 0 & \text{if } s \in S. \end{cases}$$

Note that this implies that  $T_w$  is invertible in  $\mathcal{H}$  for all  $w \in W$ . This algebra is endowed with an  $A$ -anti-linear involution  $\bar{\phantom{x}} : \mathcal{H} \rightarrow \mathcal{H}$  which is determined by the following properties:

$$\begin{cases} \overline{e^\gamma} = e^{-\gamma} & \text{if } \gamma \in \Gamma, \\ \overline{T_w} = T_{w^{-1}} & \text{if } w \in W. \end{cases}$$

By [L, Theorem 5.2], there exists a unique element  $C_w \in \mathcal{H}$  such that

$$\begin{cases} \overline{C_w} = C_w, \\ C_w \equiv T_w \pmod{\mathcal{H}_{<0}}, \end{cases}$$

where  $\mathcal{H}_{<0} = \bigoplus_{w \in W} A_{<0} T_w$ , and where  $A_{<0} = \bigoplus_{\gamma < 0} \mathbb{Z}e^\gamma$ .

Let  $\tau : \mathcal{H} \rightarrow A$  be the unique  $A$ -linear map such that

$$\tau(T_w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If  $w \in W$ , we set

$$\Delta(w) = -\deg \tau(C_w),$$

and we denote by  $n_w$  the coefficient of  $e^{-\Delta(w)}$  in  $\tau(C_w)$ . Finally, if  $x, y \in W$ , we write

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z,$$

where the  $h_{x,y,z}$ 's are in  $A$  and satisfy  $\overline{h_{x,y,z}} = h_{x,y,z}$ .

**1.B. The group  $(W^G, S_G)$ .** Let  $G$  be a group of automorphisms of  $W$  such that, for all  $\sigma \in G$ , we have

$$\sigma(S) = S \quad \text{and} \quad \varphi \circ \sigma = \varphi.$$

If  $I$  is a subset of  $S$ , we denote by  $W_I$  the (standard parabolic) subgroup of  $W$  generated by  $I$ . If  $\omega \in S/G$  is such that  $W_\omega$  is finite, we denote by  $s_\omega$  the longest element of  $W_\omega$ . We denote by  $S_G$  the set of  $s_\omega$ , where  $\omega$  runs over the set of  $G$ -orbits in  $S$  such that  $W_\omega$  is finite. Recall the following proposition [H, Corollary 3.5 and Proof of Proposition 3.4]:

**Proposition 1.1.**  *$(W^G, S_G)$  is a Coxeter system. Let  $\ell_G : W^G \rightarrow \mathbb{N}$  denote the corresponding length function and let  $x, y \in W^G$ . Then  $\ell(xy) = \ell(x) + \ell(y)$  if and only if  $\ell_G(xy) = \ell_G(x) + \ell_G(y)$ .*

Let

$$\begin{aligned} \varphi_G : W^G &\longrightarrow \Gamma \\ w &\longmapsto \varphi(w) \end{aligned}$$

denote the restriction of  $\varphi$  to  $W^G$ . Then, by Proposition 1.1,

**(1.2)**  $\varphi_G$  is a weight function.

Therefore, we can define  $\mathcal{H}_G, \mathcal{H}_{G,<0}, T_w^G, C_w^G, \tau_G, \Delta_G, n_z^G$  and  $h_{x,y,z}^G$  with respect to  $(W^G, S_G, \Gamma, \varphi_G)$  in a similar way as  $\mathcal{H}, \mathcal{H}_{<0}, T_w, C_w, \tau, \Delta, n_z$  and  $h_{x,y,z}$  were defined with respect to  $(W, S, \Gamma, \varphi)$ .

## 2. BRAUER QUOTIENT

**Hypothesis and notation.** *From now on, and until the end of section 3, we fix a prime number  $p$  and we assume that  $G$  is a finite  $p$ -group.*

**2.A. Definition.** For all the facts contained in this subsection, the reader may refer to [T, §11]: even though this reference deals only with  $\mathcal{O}$ -algebras (where  $\mathcal{O}$  is a commutative complete local noetherian  $\mathbb{Z}_p$ -algebra) which are  $\mathcal{O}$ -modules of finite type, the proofs can be applied almost word by word to our slightly more general situation.

Let  $R$  be a commutative ring and let  $M$  be an  $RG$ -module. If  $H$  is a subgroup of  $G$ , we set

$$\begin{aligned} \mathrm{Tr}_H^G : M^H &\longrightarrow M^G \\ m &\longmapsto \sum_{\sigma \in [G/H]} \sigma(m). \end{aligned}$$

Here,  $[G/H]$  denotes a set of representatives classes in  $G/H$ . We also define

$$\mathrm{Tr}(M) = \sum_{H < G} \mathrm{Tr}_H^G(M^H).$$

This is an  $R$ -submodule of  $M^G$ , containing  $pM^G$ . The *Brauer quotient*  $\mathrm{Br}_G(M)$  is then defined by

$$\mathrm{Br}_G(M) = M^G / \mathrm{Tr}(M)$$

and we denote by  $\mathrm{br}_G : M^G \rightarrow \mathrm{Br}_G(M)$  the canonical map.

**Lemma 2.1.** *Assume that  $pR \neq R$  and that  $M$  admits an  $R$ -basis  $\mathcal{B}$  which is permuted by the action of  $G$ . Then  $\mathrm{Br}_G(M)$  is a free  $R/pR$ -module with basis  $(\mathrm{br}_G(b))_{b \in \mathcal{B}^G}$ .*

If  $M$  is an  $R$ -algebra and if  $G$  acts on  $M$  by automorphisms of algebra, then  $\mathrm{Tr}(M)$  is a two-sided ideal of  $M^G$  and so  $\mathrm{Br}_G(M)$  is an  $R$ -algebra. Of course,  $\mathrm{br}_G$  is a morphism of algebras in this case. We recall the following result:

**Lemma 2.2.** *Assume that  $pR \neq R$ , that  $M$  is an  $R$ -algebra, that  $G$  acts on  $M$  by automorphisms of algebra, that  $M$  admits an  $R$ -basis  $\mathcal{B}$  which is permuted by  $G$  and let us write  $ab = \sum_{c \in \mathcal{B}} \lambda_{a,b,c} c$  for  $a, b \in \mathcal{B}$ . If  $a, b \in \mathcal{B}^G$ , then*

$$\mathrm{br}_G(a) \mathrm{br}_G(b) = \sum_{c \in \mathcal{B}^G} \pi(\lambda_{a,b,c}) \mathrm{br}_G(c),$$

where  $\pi : R \rightarrow R/pR$  is the canonical morphism.

**2.B. Applications to Hecke algebras.** Since  $G$  stabilizes  $S$  and  $\varphi$ , it also acts on  $\mathcal{H}$  by automorphisms of  $A$ -algebra (by  $\sigma(T_w) = T_{\sigma(w)}$  for all  $w \in W$ ). Moreover, it permutes the standard basis  $(T_w)_{w \in W}$ , so it follows from Lemma 2.1 that:

**Corollary 2.3.**  *$(\mathrm{br}_G(T_w))_{w \in W^G}$  is an  $\mathbb{F}_p[\Gamma]$ -basis of the  $\mathbb{F}_p[\Gamma]$ -algebra  $\mathrm{Br}_G(\mathcal{H})$ .*

Now, let

$$\text{can}_G : \mathcal{H}_G \longrightarrow \text{Br}_G(\mathcal{H})$$

be the unique  $A$ -linear map such that

$$\text{can}_G(T_w^G) = \text{br}_G(T_w)$$

for all  $w \in W^G$ . The main result of this subsection is the following:

**Proposition 2.4.** *The map  $\text{can}_G : \mathcal{H}_G \longrightarrow \text{Br}_G(\mathcal{H})$  is a surjective morphism of  $A$ -algebras whose kernel is  $p\mathcal{H}_G$ .*

*Proof.* It follows from Corollary 2.3 that  $\text{can}_G$  is surjective and that  $\text{Ker}(\text{can}_G) = p\mathcal{H}_G$ . It remains to show that  $\text{can}_G$  is a morphism of algebras. First, note that if  $x, y \in W^G$  satisfy  $\ell_G(xy) = \ell_G(x) + \ell_G(y)$ , then  $\ell(xy) = \ell(x) + \ell(y)$  (by Proposition 1.1) and so

$$\begin{aligned} \text{can}_G(T_x^G T_y^G) &= \text{can}_G(T_{xy}^G) = \text{br}_G(T_{xy}) \\ &= \text{br}_G(T_x T_y) = \text{br}_G(T_x) \text{br}_G(T_y) = \text{can}_G(T_x^G) \text{can}_G(T_y^G). \end{aligned}$$

So it remains to show that, if  $\omega$  is a  $G$ -orbit in  $S$  such that  $W_\omega$  is finite, then

$$(\text{?}) \quad \text{br}_G((T_{s_\omega} - e^{\varphi(s_\omega)})(T_{s_\omega} + e^{-\varphi(s_\omega)})) = 0.$$

Since  $s_\omega$  is the longest element of  $W_\omega$ , we have [L, Corollary 12.2]

$$C_{s_\omega} = \sum_{w \in W_\omega} e^{\varphi(w) - \varphi(s_\omega)} T_w$$

and [L, Theorem 6.6 (b)]

$$(T_{s_\omega} - e^{\varphi(s_\omega)})C_{s_\omega} = 0.$$

But  $(W_\omega)^G = \{1, s_\omega\}$ . Since  $\varphi(w) = \varphi(\sigma(w))$  for all  $w \in W_\omega$  and all  $\sigma \in G$ , we have

$$C_{s_\omega} \equiv T_{s_\omega} + e^{-\varphi(s_\omega)} \pmod{\text{Tr}(\mathcal{H})}.$$

This completes the proof of (?). □

**Corollary 2.5.**  $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{H}_G \simeq \text{Br}_G(\mathcal{H})$ .

**Corollary 2.6.** *If  $h \in \mathcal{H}_G$  and  $h' \in \mathcal{H}^G$  are such that  $\text{can}_G(h) = \text{br}_G(h')$ , then  $\tau_G(h) \equiv \tau(h') \pmod{pA}$ .*

**Proposition 2.7.** *If  $w \in W^G$ , then  $\text{can}_G(C_w^G) = \text{br}_G(C_w)$ .*

*Proof.* Let  $C = \text{can}_G(C_w^G) - \text{br}_G(C_w)$ . Then

$$\overline{C} = C.$$

where  $\overline{\cdot} : \text{Br}_G(\mathcal{H}) \rightarrow \text{Br}_G(\mathcal{H})$  is defined by  $\overline{\text{br}_G(h)} = \text{br}_G(\overline{h})$  for all  $h \in \mathcal{H}^G$ . Moreover, there exists a family  $(\alpha_w)_{w \in W^G}$  of elements of  $\mathbb{F}_p \otimes_{\mathbb{Z}} A_{<0}$  such that

$$C = \sum_{w \in W^G} \alpha_w \text{br}_G(T_w).$$

Assume that  $C \neq 0$  and let  $w$  be maximal (for the Bruhat order) such that  $\alpha_w \neq 0$ . Then

$$\overline{C} = \overline{\alpha}_w \text{br}_G(T_{w^{-1}}) + \sum_{\substack{x \in W^G \\ x \neq w}} \overline{\alpha}_x \text{br}_G(T_{x^{-1}}).$$

Therefore, the coefficient of  $\text{br}_G(T_w)$  in  $\overline{C}$  is equal to  $\overline{\alpha}_w$ . But  $C = \overline{C}$ , so  $\alpha_w = \overline{\alpha}_w$ . Since  $\alpha_w \neq 0$  and  $\alpha_w \in \mathbb{F}_p \otimes_{\mathbb{Z}} A_{<0}$ , we get a contradiction. So  $C = 0$ , as desired.  $\square$

**Corollary 2.8.** *If  $x, y, z \in W^G$ , then  $h_{x,y,z} \equiv h_{x,y,z}^G \pmod{pA}$  and  $\tau(C_z) \equiv \tau_G(C_z^G) \pmod{pA}$ .*

*Proof.* This follows immediately from Proposition 2.7, from Lemma 2.2 and from Corollary 2.6.  $\square$

### 3. LUSZTIG'S CONJECTURES

**3.A. Cells.** With  $(W, S, \Gamma, \varphi)$  are associated preorder relations  $\leq_{\mathcal{L}}$ ,  $\leq_{\mathcal{R}}$  and  $\leq_{\mathcal{LR}}$  on  $W$  as defined in [L, §8.1]. The associated equivalence relations are denoted by  $\sim_{\mathcal{L}}$ ,  $\sim_{\mathcal{R}}$  and  $\sim_{\mathcal{LR}}$  respectively. The equivalence classes for the relation  $\sim_{\mathcal{L}}$  (respectively  $\sim_{\mathcal{R}}$ , respectively  $\sim_{\mathcal{LR}}$ ) are called left (respectively right, respectively two-sided) cells of  $W$  (or for  $(W, S, \Gamma, \varphi)$  if it is necessary to emphasize the weight function).

Similarly, with  $(W^G, S_G, \Gamma, \varphi_G)$  are associated preorder relations  $\leq_{\mathcal{L}}^G$ ,  $\leq_{\mathcal{R}}^G$  and  $\leq_{\mathcal{LR}}^G$  on  $W$ . The associated equivalence relations are denoted by  $\sim_{\mathcal{L}}^G$ ,  $\sim_{\mathcal{R}}^G$  and  $\sim_{\mathcal{LR}}^G$  respectively. We shall compare in this section the (left, right or two-sided) cells of  $W$  and the ones of  $W^G$ .

**3.B. Boundedness.** Following Lusztig [L, §13.2], we say that  $(W, S, \Gamma, \varphi)$  is *bounded* if there exists  $\gamma_0 \in \Gamma$  such that  $\deg \tau(T_x T_y T_z) \leq \gamma_0$  for all  $x, y$  and  $z \in W$ . Lusztig has conjectured [L, Conjecture 13.4] that  $(W, S, \Gamma, \varphi)$  is always bounded.

**Hypothesis.** *From now on, and until the end of this paper, we assume that  $(W, S, \Gamma, \varphi)$  and  $(W^G, S_G, \Gamma, \varphi_G)$  are bounded. Recall that  $p$  is a prime number and that  $G$  is a finite  $p$ -group.*



REMARK - A finite group is of course bounded. An affine Weyl group is also bounded [L, §13.2].  $\square$

By [L, Lemma 13.5 (b)], this hypothesis allows us to define Lusztig's function  $\mathbf{a} : W \rightarrow \Gamma$  by

$$\mathbf{a}(z) = \max_{x,y \in W} (\deg h_{x,y,z}).$$

If  $x, y, z \in W$ , we shall denote by  $\gamma_{x,y,z^{-1}}$  the unique element of  $\mathbb{Z}$  such that

$$h_{x,y,z} \equiv \gamma_{x,y,z^{-1}} e^{\mathbf{a}(z)} \pmod{\left(\bigoplus_{\gamma < \mathbf{a}(z)} \mathbb{Z} e^\gamma\right)}.$$

Similarly, we define a function  $\mathbf{a}_G : W^G \rightarrow \Gamma$  and elements  $\gamma_{x,y,z^{-1}}^G$  of  $\mathbb{Z}$  (for  $x, y, z \in W^G$ ).

Let  $\mathcal{D} = \{z \in W \mid \mathbf{a}(z) = \Delta(z)\}$ . If  $I \subseteq S$ , we denote by  $\mathbf{a}_I$  the analogue of the function  $\mathbf{a}$  but defined for  $W_I$  instead of  $W$ : if  $z \in W_I$ , then

$$\mathbf{a}_I(z) = \max_{x,y \in W_I} \deg h_{x,y,z}.$$

**Lusztig's Conjectures for  $(W, S, \Gamma, \varphi)$ .** *With the above notation, we have:*

- P<sub>1</sub>.** *If  $z \in W$ , then  $\mathbf{a}(z) \leq \Delta(z)$ .*
- P<sub>2</sub>.** *If  $d \in \mathcal{D}$  and if  $x, y \in W$  satisfy  $\gamma_{x,y,d} \neq 0$ , then  $x = y^{-1}$ .*
- P<sub>3</sub>.** *If  $y \in W$ , then there exists a unique  $d \in \mathcal{D}$  such that  $\gamma_{y^{-1},y,d} \neq 0$ .*
- P<sub>4</sub>.** *If  $z' \leq_{\mathcal{LR}} z$ , then  $\mathbf{a}(z) \leq \mathbf{a}(z')$ . Therefore, if  $z \sim_{\mathcal{LR}} z'$ , then  $\mathbf{a}(z) = \mathbf{a}(z')$ .*
- P<sub>5</sub>.** *If  $d \in \mathcal{D}$  and  $y \in W$  satisfy  $\gamma_{y^{-1},y,d} \neq 0$ , then  $\gamma_{y^{-1},y,d} = n_d = \pm 1$ .*
- P<sub>6</sub>.** *If  $d \in \mathcal{D}$ , then  $d^2 = 1$ .*
- P<sub>7</sub>.** *If  $x, y, z \in W$ , then  $\gamma_{x,y,z} = \gamma_{y,z,x}$ .*
- P<sub>8</sub>.** *If  $x, y, z \in W$  satisfy  $\gamma_{x,y,z} \neq 0$ , then  $x \sim_{\mathcal{L}} y^{-1}$ ,  $y \sim_{\mathcal{L}} z^{-1}$  and  $z \sim_{\mathcal{L}} x^{-1}$ .*
- P<sub>9</sub>.** *If  $z' \leq_{\mathcal{L}} z$  and  $\mathbf{a}(z') = \mathbf{a}(z)$ , then  $z' \sim_{\mathcal{L}} z$ .*
- P<sub>10</sub>.** *If  $z' \leq_{\mathcal{R}} z$  and  $\mathbf{a}(z') = \mathbf{a}(z)$ , then  $z' \sim_{\mathcal{R}} z$ .*
- P<sub>11</sub>.** *If  $z' \leq_{\mathcal{LR}} z$  and  $\mathbf{a}(z') = \mathbf{a}(z)$ , then  $z' \sim_{\mathcal{LR}} z$ .*
- P<sub>12</sub>.** *If  $I \subset S$  and  $z \in W_I$ , then  $\mathbf{a}_I(z) = \mathbf{a}(z)$ .*
- P<sub>13</sub>.** *Every left cell  $\mathcal{C}$  of  $W$  contains a unique element  $d \in \mathcal{D}$ . If  $y \in \mathcal{C}$ , then  $\gamma_{y^{-1},y,d} \neq 0$ .*
- P<sub>14</sub>.** *If  $z \in W$ , then  $z \sim_{\mathcal{LR}} z^{-1}$ .*
- P<sub>15</sub>.** *If  $x, x', y, w \in W$  are such that  $\mathbf{a}(y) = \mathbf{a}(w)$ , then*

$$\sum_{y' \in W} h_{w,x',y'} \otimes_{\mathbb{Z}} h_{x,y',y} = \sum_{y' \in W} h_{y',x',y} \otimes_{\mathbb{Z}} h_{x,w,y'}$$

in  $A \otimes_{\mathbb{Z}} A$ .

Let us recall the following result:

**Lemma 3.1.** *Assume that Lusztig's Conjectures  $(P_1)$ ,  $(P_2)$ ,  $(P_3)$  and  $(P_4)$  hold for  $(W, S, \Gamma, \varphi)$ . Then:*

- (a) *Lusztig's Conjectures  $(P_5)$ ,  $(P_6)$ ,  $(P_7)$  and  $(P_8)$  hold for  $(W, S, \Gamma, \varphi)$ .*
- (b) *If  $d \in \mathcal{D}$ , then  $\gamma_{d,d,d} = n_d = \pm 1$ .*
- (c) *If  $x \in W$  and if  $d \in \mathcal{D}$  is the unique element of  $W$  such that  $\gamma_{x^{-1},x,d} \neq 0$ , then  $\gamma_{x,d,x^{-1}} = \pm 1$ .*

*Proof.* (a) is proved in [L, Chapter 14].

(b) By  $(P_6)$ , we get that  $d^2 = 1$ . By  $(P_3)$ , there exists a unique  $e \in \mathcal{D}$  such that  $\gamma_{d,d,e} \neq 0$ . By  $(P_5)$ , this implies that  $\gamma_{d,d,e} = n_e = \pm 1$ . By  $(P_7)$ , this implies that  $\gamma_{e,d,d} = \pm 1$ . By  $(P_2)$ , we get that  $e = d^{-1} = d$ .

(c) If  $x \in W$  and if  $d \in \mathcal{D}$  is the unique element of  $W$  such that  $\gamma_{x^{-1},x,d} \neq 0$ , then  $\gamma_{x,d,x^{-1}} = \gamma_{x^{-1},x,d} = \pm 1$  by  $(P_7)$  and  $(P_5)$ .  $\square$

We can now state the main result of this paper (from which the Theorem A in the introduction follows easily by an induction argument on the order of  $G$ ):

**Theorem 3.2.** *Recall that  $G$  is a finite  $p$ -group. Assume that Lusztig's conjectures  $(P_1)$ ,  $(P_2)$ ,  $(P_3)$  and  $(P_4)$  hold for both  $(W, S, \Gamma, \varphi)$  and  $(W^G, S_G, \Gamma, \varphi_G)$ . Let  $x$  and  $y$  be two elements of  $W^G$ . Then:*

- (a)  $\mathbf{a}_G(x) = \mathbf{a}(x)$ .
- (b)  $\mathcal{D}_G = \mathcal{D} \cap W^G (= \mathcal{D}^G)$ .
- (c) *Assume moreover that Lusztig's Conjecture  $(P_{13})$  holds for both  $(W, S, \Gamma, \varphi)$  and  $(W^G, S_G, \Gamma, \varphi_G)$ . Then  $x \sim_{\mathcal{L}}^G y$  (respectively  $x \sim_{\mathcal{R}}^G y$ ) if and only if  $x \sim_{\mathcal{L}} y$  (respectively  $x \sim_{\mathcal{R}} y$ ).*
- (d) *Assume moreover that Lusztig's Conjectures  $(P_9)$  and  $(P_{13})$  hold for both  $(W, S, \Gamma, \varphi)$  and  $(W^G, S_G, \Gamma, \varphi_G)$ . Then  $x \sim_{\mathcal{LR}}^G y$  if and only if  $x \sim_{\mathcal{LR}} y$ .*

*Proof.* (a) By Corollary 2.8, we have, for all  $x, y, z \in W^G$ :

- (1) If  $\gamma_{x,y,z^{-1}} \not\equiv 0 \pmod{p}$ , then  $\mathbf{a}(z) \leq \mathbf{a}_G(z)$ .
- (2) If  $\gamma_{x,y,z^{-1}}^G \not\equiv 0 \pmod{p}$ , then  $\mathbf{a}_G(z) \leq \mathbf{a}(z)$ .

Now let  $z \in W^G$ . By  $(P_3)$ , there exists a unique  $d \in \mathcal{D}$  such that  $\gamma_{z^{-1},z,d} \neq 0$ . From the uniqueness, we get that  $d \in \mathcal{D}^G \subseteq W^G$ . By Lemma 3.1 (c), we get that  $\gamma_{z,d,z^{-1}} = \pm 1$ . So  $\mathbf{a}(z) \leq \mathbf{a}_G(z)$  by (1).

The same argument shows that there exists  $d \in \mathcal{D}_G$  such that  $\gamma_{z,d,z^{-1}}^G = \pm 1$ , so (2) can be applied to get that  $\mathbf{a}_G(z) \leq \mathbf{a}(z)$ . The proof of (a) is complete.

Before going further, let us state the following consequence of (a):

**Corollary 3.3.** *If  $x, y, z \in W^G$ , then  $\gamma_{x,y,z} \equiv \gamma_{x,y,z}^G \pmod{p}$ .*

*Proof.* This follows easily from Theorem 3.2 (a) and Corollary 2.8.  $\square$

(b) Let  $d \in \mathcal{D}^G$ . By Lemma 3.1 (b), we have  $n_d = \pm 1$ . Moreover, by Corollary 2.8, we have

$$\tau(C_d) \equiv \tau_G(C_d^G) \pmod{pA}.$$

This shows that the coefficient of  $e^{-\Delta(d)}$  in  $\tau_G(C_d^G)$  is non-zero. So  $\Delta_G(d) \leq \Delta(d)$ . But, by  $(P_1)$ ,

$$\mathbf{a}_G(d) \leq \Delta_G(d) \leq \Delta(d) = \mathbf{a}(d).$$

So  $\mathbf{a}_G(d) = \Delta_G(d) = \Delta(d) = \mathbf{a}(d)$  by (a). In particular,  $d \in \mathcal{D}_G$ .

The same argument shows that, if  $d \in \mathcal{D}_G$ , then  $\Delta(d) \leq \Delta_G(d)$  and again we get similarly that  $d \in \mathcal{D}$ . The proof of (b) is complete.

(c) Let  $d$  (respectively  $e$ ) be the unique element of  $\mathcal{D}$  such that  $\gamma_{x^{-1},x,d} = \pm 1$  (respectively  $\gamma_{y^{-1},y,e} = \pm 1$ ). By uniqueness, we have  $d, e \in \mathcal{D}^G = \mathcal{D}_G$ . By Corollary 3.3, we also get  $\gamma_{x^{-1},x,d}^G \neq 0$  and  $\gamma_{y^{-1},y,e}^G \neq 0$ . Therefore, by  $(P_8)$ , we have

$$x \sim_{\mathcal{L}} d, \quad x \sim_{\mathcal{L}}^G d, \quad y \sim_{\mathcal{L}} e \quad \text{and} \quad y \sim_{\mathcal{L}}^G e.$$

But, by  $(P_{13})$ , we have  $x \sim_{\mathcal{L}} y$  (respectively  $x \sim_{\mathcal{L}}^G y$ ) if and only if  $d = e$ . This proves (c).

(d) Recall that  $(P_9)$  implies  $(P_{10})$ . Moreover, it follows easily from  $(P_4)$ ,  $(P_9)$  and  $(P_{10})$  that  $\sim_{\mathcal{LR}}$  (respectively  $\sim_{\mathcal{LR}}^G$ ) is the equivalence relation generated by  $\sim_{\mathcal{L}}$  and  $\sim_{\mathcal{R}}$  (respectively  $\sim_{\mathcal{L}}^G$  and  $\sim_{\mathcal{R}}^G$ ). So (d) follows from (c).  $\square$

**3.C. Asymptotic algebra.** Let  $J$  (respectively  $J_G$ ) be the free abelian group with basis  $(t_w)_{w \in W}$  (respectively  $(t_w^G)_{w \in W}$ ).

**Hypothesis.** *In this subsection, and only in this subsection, we assume moreover that Lusztig's Conjectures  $(P_1), (P_2), \dots, (P_{15})$  hold for  $(W, S, \Gamma, \varphi)$  and  $(W^G, S_G, \Gamma, \varphi_G)$ .*

By [L, §18.3],  $J$  (respectively  $J_G$ ) can be endowed with a structure of associative ring, the multiplication being defined by  $t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$  (respectively  $t_x^G t_y^G = \sum_{z \in W^G} \gamma_{x,y,z^{-1}}^G t_z^G$ ). Then it follows immediately from Corollary 3.3 and from Lemma 2.2 that:

**Theorem 3.4.** *Assume that  $G$  is a finite  $p$ -group and that Lusztig's Conjectures  $(P_1), (P_2), \dots, (P_{15})$  hold for  $(W, S, \Gamma, \varphi)$  and  $(W^G, S_G, \Gamma, \varphi_G)$ . Then*

$$\mathbb{F}_p \otimes_{\mathbb{Z}} J_G \simeq \text{Br}_G(J).$$

#### 4. OPEN QUESTIONS

The results of this paper should be compared with [L, Chapter 14], where the *quasi-split case* is considered: more particularly, see [L, Lemmas 16.5, 16.6 and 16.14]. This leads to the following questions:

- Does Theorem A (d) hold if  $G$  is not solvable? It is probably the case, but a proof should rely on completely different arguments. For the statements (a), (b) and (c), see the remark after Theorem A in the introduction.
- Let  $z \in W^G$ . Is it true that  $\Delta(z) \leq \Delta_G(z)$ ? See [L, Lemma 16.5] for the quasi-split case.
- Let  $x, y \in W^G$  be such that  $x \leq_{\mathcal{L}}^G y$ . Is it true that  $x \leq_{\mathcal{L}} y$ ? See [L, 16.13 (a)] for the quasi-split case.

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