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Abstract

We prove that any positive solution of $\partial_t u - \Delta u + u^q = 0$ $(q > 1)$ in $\mathbb{R}^N \times (0, \infty)$ with initial trace $(F, 0)$, where F is a closed subset of $\mathbb{R}^N$ can be represented, up to two universal multiplicative constants, by a series involving the Bessel capacity $C_{2/q,q'}$. As a consequence we prove that there exists a unique positive solution of the equation with such an initial trace. We also characterize the blow-up set of $u(x,t)$ when $t \downarrow 0$, by using the "density" of F expressed in terms of the $C_{2/q,q'}$-Bessel capacity.

2000 Mathematics Subject Classification. 35K05;35K55; 31C15; 31B10; 31C40.

Key words. Heat equation; singularities; Borel measures; Besov spaces; real interpolation; Bessel capacities; quasi-additivity; capacitary measures; Wiener type test; initial trace.

Contents

1 Introduction

Let $T \in (0,\infty]$ and $Q_T = \mathbb{R}^N \times (0,T]$ ($N \geq 1$). If $q > 1$ and $u \in C^2(Q_T)$ is nonnegative and verifies

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T, \tag{1.1}$$

it has been proven by Marcus and Véron [28] that there exists a unique outer-regular positive Borel measure ν in $\mathbb{R}^N$ such that

$$\lim_{t\to 0} u(.,t) = \nu, \tag{1.2}$$

in the sense of Borel measures; the set of such measures is denoted by $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$. To each of its element ν is associated a unique couple $(\mathcal{S}_\nu, \mu_\nu)$ (we write $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$) where $\mathcal{S}_\nu$, *the singular part* of ν, is a closed subset of $\mathbb{R}^N$ and μ_ν, *the regular part* is a nonnegative Radon measure on $\mathcal{R}_\nu = \mathbb{R}^N \setminus \mathcal{S}_\nu$. In this setting, relation (1.2) has the following meaning :

$$\begin{array}{lll} (i) & \lim_{t\to 0} \int_{\mathcal{R}_\nu} u(.,t)\zeta dx = \displaystyle\int_{\mathcal{R}_\nu} \zeta d\mu_\nu, & \forall \zeta \in C_0(\mathcal{R}_\nu), \\ (ii) & \lim_{t\to 0} \displaystyle\int_{\mathcal{O}} u(.,t)dx = \infty, & \forall \mathcal{O} \subset \mathbb{R}^N \text{ open}, \ \mathcal{O} \cap \mathcal{S}_\nu \neq \emptyset. \end{array} \tag{1.3}$$

The measure ν is by definition *the initial trace* of u and denoted by $Tr_{\mathbb{R}^N}(u)$. It is wellknown that equation (1.1) admits a critical exponent

$$1 < q < q_c = 1 + \frac{N}{2}.$$

This is due to the fact, proven by Brezis and Friedman [7], that if $q \geq q_c$, isolated singularities of solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ are removable. Conversely, if $1 < q < q_c$, it is proven by the same authors that for any $k > 0$, equation (1.1) admits a unique solution $u_{k\delta_0}$ with initial data $k\delta_0$. This existence and uniqueness results extends in a simple way if the initial data $k\delta_0$ is replaced by any Radon measure μ in $\mathbb{R}^N$ (see [6]). Furthermore, if $k \to \infty$, $u_{k\delta_0}$ increases and converges to a positive, radial and self-similar solution u_∞ of (1.1). Writing it under the form $u_\infty(x,t) = t^{-\frac{1}{q-1}} f(|x|/\sqrt{t})$, f is a positive solution of

$$\begin{cases} \Delta f + \frac{1}{2} y.Df + \frac{1}{q-1} f - f^q = 0 \quad \text{in } \mathbb{R}^N \\ \qquad \lim_{|y|\to\infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \tag{1.4}$$

The existence, uniqueness and the expression of the asymptotics of f has been studied thoroughly by Brezis, Peletier and Terman in [8]. Later on, Marcus and Véron proved in [28] that in the same range of exponents, for any $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 \qquad \text{in } Q_\infty, \\ \qquad Tr_{\mathbb{R}^N}(u) = \nu, \end{cases} \tag{1.5}$$

admits a unique positive solution. This result means that the initial trace establishes a one to one correspondence between the set of positive solutions of (1.1) and $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$. A key step for proving the uniqueness is the following inequalities

$$t^{-\frac{1}{q-1}} f(|x-a|/\sqrt{t}) \leq u(x,t) \leq ((q-1)t)^{-\frac{1}{q-1}} \qquad \forall (x,t) \in Q_\infty, \tag{1.6}$$

valid for any $a \in \mathcal{S}_\nu$. As a consequence of Brezis and Friedman's result, if $q \geq q_c$, i.e. in the *supercritical range*, Problem (1.5) may admit no solution at all. If $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$, the necessary and sufficient conditions for the existence of a maximal solution $u = \overline{u}_\nu$ to Problem (1.5) are obtained in [28] and expressed in terms of the the Bessel capacity $C_{2/q,q'}$, (with $q' = q/(q-1)$). Furthermore, uniqueness does not hold in general as it was pointed out by Le Gall [23]. In the particular case where $\mathcal{S}_\nu = \emptyset$ and ν is simply the Radon measure μ_ν, the necessary and sufficient condition for solvability is that μ_ν does not charge Borel subsets with $C_{2/q,q'}$-capacity zero. This result was already proven by Baras and Pierre [5] in the particular case of bounded measures and extended by Marcus and Véron [28] to the general case. We denote by $\mathfrak{M}_+^q(\mathbb{R}^N)$ the positive cone of the space $\mathfrak{M}^q(\mathbb{R}^N)$ of Radon measures which do not charge Borel subsets with zero $C_{2/q,q'}$-capacity. Notice that $W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ is a subset of $\mathfrak{M}_+^q(\mathbb{R}^N)$ where $\mathfrak{M}_+^b(\mathbb{R}^N)$ is the cone of positive bounded Radon mesures in $\mathbb{R}^N$. For such measures, uniqueness always holds and we denote $\overline{u}_{\mu_\nu} = u_{\mu_\nu}$.

In view of the already known results concerning the parabolic equation, it is useful to recall the main advanced results previously obtained for the stationary equation

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{1.7}$$

in a smooth bounded domain Ω of $\mathbb{R}^N$. This equation has been intensively studied since 1993, both by probabilists (Le Gall, Dynkin, Kuznetsov) and by analysts (Marcus, Véron). The existence of a *boundary trace* for positive solutions, in the class of outer-regular positive Borel measures on $\partial\Omega$, is proven by Le Gall [22], [23] in the case $q = N = 2$, by probabilistic methods, and by Marcus and Véron in [26], [27] in the general case $q > 1$, $N > 1$. The existence of a critical exponent $q_e = (N+1)/(N-1)$ is due to Gmira and Véron [14] who shew that, if $q \geq q_e$ boundary isolated singularities of solutions of (1.7) are removable, which is not the case if $1 < q < q_e$. In this *subcritical case* Le Gall and Marcus and Véron proved that the boundary trace establishes a one to one correspondence between positive solutions of (1.7) in Ω and outer regular positive Borel measures on $\partial\Omega$. This fundamental result does not hold in the *supercritical case* $q \geq q_e$. In [12] Dynkin and Kuznetsov introduced the notion of σ-moderate solution which means that u is a positive solution of (1.7) such that there exists an increasing sequence of positive Radon measures on $\partial\Omega$ $\{\mu_n\}$ belonging to $W^{-2/q,q'}(\partial\Omega)$ such that the corresponding solutions $v = v_{\mu_n}$ of

$$\begin{cases} -\Delta v + v^q = 0 & \text{in } \Omega \\ v = \mu_n & \text{in } \partial\Omega \end{cases} \tag{1.8}$$

converges to u locally uniformly in Ω. This class of solutions plays a fundamental role since Dynkin and Kuznetsov proved that a σ-moderate solution of (1.7) is uniquely determined by its *fine trace*, a new notion of trace introduced in order to avoid the non-uniqueness phenomena. Later on, it is proved by Mselati (if $q = 2$) [36], then by Dynkin (if $q_e \leq q \leq 2$) [10] and finally by Marcus with no restriction on q [25], that all the positive solutions of (1.7) are σ-moderate. One of the key-stones element in their proof (partially probabilistic) is the fact that the maximal solution $\overline{u}_K$ of (1.7) with a boundary trace vanishing outside a compact subset $K \subset \partial\Omega$ is indeed σ-moderate. This deep result was obtained by a combination of probabilistic and analytic methods by Mselati [36] in the case $q = 2$ and by purely analytic tools by Marcus and Véron [31], [32] in the case $q \geq q_e$. Defining $\underline{u}_K$ as the largest σ-moderate solution of

(1.7) with a boundary trace concentrated on K, the crucial step in Marcus-Véron's proof (non probabilistic) is the bilateral estimate satisfied by $\overline{u}_K$ and $\underline{u}_K$

$$C^{-1}\rho(x)W_K(x) \le \underline{u}_K(x) \le \overline{u}_K(x) \le C\rho(x)W_K(x). \tag{1.9}$$

In this expression $C = C(\Omega, q)$, $\rho(x) = \text{dist}\,(x, \partial\Omega)$ and $W_F(x)$ is the *elliptic capacitary potential* of K defined by

$$W_K(x) = \sum_{-\infty}^{\infty} 2^{-\frac{m(q+1)}{q-1}} C_{2/q,q'}(2^m K_m(x)), \tag{1.10}$$

where $K_m(x) = K \cap \{z : 2^{-m-1} \le |z-x| \le 2^{-m}\}$, the Bessel capacity being relative to $\mathbb{R}^{N-1}$. Note that, using a technique introduced in [27], inequality $\overline{u}_K \le C^2 \underline{u}_K$ implies $\underline{u}_K = \overline{u}_K$.

The aim of this article is to initiate the fine study of the complete initial trace problem for positive solutions of (1.1) in the supercritical case $q \ge q_c$ and to give in particular the parabolic counterparts of the results of [36], [31] and [32]. Extending Dynkin's ideas to the parabolic case, we introduce the following notion

Definition 1.1 *A positive solution u of (1.1) is called σ-moderate if their exists an increasing sequence $\{\mu_n\} \subset W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ such that the corresponding solution $u := u_{\mu_n}$ of*

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_\infty \\ \qquad\qquad u(x,0) = \mu_n \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.11}$$

converges to u locally uniformly in Q_∞.

If F is a closed subset of $\mathbb{R}^N$, we denote by $\overline{u}_F$ the maximal solution of (1.1) with an initial trace vanishing on F^c, and by $\underline{u}_F$ the maximal σ-moderate solution of (1.1) with an initial trace vanishing on F^c. Thus $\underline{u}_F$ is defined by

$$\underline{u}_F = \sup\{u_\mu : \mu \in W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N), \mu(F^c) = 0\}, \tag{1.12}$$

(and clearly $W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ can be replaced by $\mathfrak{M}^q_+(\mathbb{R}^N)$). One of the main goal of this article is to prove that $\overline{u}_F$ is σ-moderate and more precisely,

Theorem 1.2 *For any $q > 1$ and any closed subset F of $\mathbb{R}^N$, $\overline{u}_F = \underline{u}_F$.*

We define below a set function which will play a fundamental role in the sequel.

Definition 1.3 *Let F be a closed subset of $\mathbb{R}^N$. The Bessel parabolic capacitary potential W_F of F is defined by*

$$W_F(x,t) = \frac{1}{t^{\frac{N}{2}}} \sum_{n=0}^{\infty} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \qquad \forall (x,t) \in Q_\infty, \tag{1.13}$$

where $C_{2/q,q'}$ is the N-dimensional Bessel capacity, $d_n = \sqrt{nt}$ and $F_n = \{y \in F : d_n \le |x-y| \le d_{n+1}\}$.

In our study, it is useful to introduce a variant of W_F with the help of the Besov capacity: if $\Omega \subset \mathbb{R}^N$ is a bounded domain, we set

$$\|\phi\|_{B_{2/q,q'}} = \left(\iint_{\Omega\times\Omega} \frac{|\phi(x)-\phi(y)|^{q'}}{|x-y|^{N+\frac{2}{q-1}}} dxdy \right)^{1/q'}, \tag{1.14}$$

if $1 < 2/q < 1$, and $\|\phi\|_{B_{1,2}} = \|\nabla\phi\|_{L^2}$ if $2/q = 1$ (i.e. $N = 2$ and $q = 2$). The Besov capacity of a compact set $K \subset \Omega$ relative to Ω is expressed by

$$R^{\Omega}_{2/q,q'} = \inf \left\{ \|\phi\|^{q'}_{B_{2/q,q'}} : \phi \in C^{\infty}_0(\Omega), 0 \le \phi \le 1, \eta = 1 \text{ on K} \right\}. \tag{1.15}$$

The Besov-parabolic capacitary potential $\tilde{W}_F$ of F is defined by

$$\tilde{W}_F(x,t) = t^{-\frac{N}{2}} \sum_{n=0}^{\infty} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} R^{\Gamma_n}_{2/q,q'} \left(\frac{F_n}{d_{n+1}} \right) \qquad \forall (x,t) \in Q_\infty, \tag{1.16}$$

where $\Gamma_n = B_{d_{n+1}} \setminus \overline{B_{d_n}}$. The Besov-parabolic capacitary potential is equivariant with respect to the same scaling transformation which let (1.1) invariant in the sense that, for any $\ell > 0$,

$$\ell^{\frac{1}{q-1}} \tilde{W}_F(\sqrt{\ell}x, \ell t) = \tilde{W}_{F/\sqrt{\ell}}(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.17}$$

and we prove that there exists $c = c(N,q) > 0$ such that

$$c^{-1}\tilde{W}_F(x,t) \le W_F(x,t) \le c\tilde{W}_F(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.18}$$

One of the tool for proving Theorem 1.2 is the following bilateral estimate which is only meaningful in the supercritical case, otherwhile it reduces to (1.6);

Theorem 1.4 *For any $q \ge q_c$ there exist two positive constants $C_1 \ge C_2 > 0$, depending only on N and q such that for any closed subset F of $\mathbb{R}^N$, there holds*

$$C_2 W_F(x,t) \le \underline{u}_F(x,t) \le \overline{u}_F(x,t) \le C_1 W_F(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.19}$$

Actually our result is more general since the upper estimate in (1.19) is valid for *any* positive solution of

$$\partial_t u - \Delta u + u^q \le 0 \qquad \text{in } Q_T \tag{1.20}$$

satisfying

$$\lim_{t\to 0} u(x,t) = 0 \qquad \text{locally uniformly in } F^c. \tag{1.21}$$

Extension to positive solutions of

$$\partial_t u - \Delta u + f(u) = 0 \qquad \text{in } Q_T \tag{1.22}$$

where f is continuous from $\mathbb{R}^+$ to $\mathbb{R}^+$ and satisfies

$$c_2 r^q \le f(r) \le c_1 r^q \qquad \forall r \ge 0 \tag{1.23}$$

for some $0 < c_2 \leq c_1$ is straightforward.

This *quasi representation*, up to uniformly upper and lower bounded functions, is also interesting in the sense that it indicates precisely *how to characterize the blow-up points* of $\overline{u}_F = \underline{u}_F := u_F$. Introducing an integral expression comparable to W_F, we show in particular the following results

$$\lim_{\tau\to 0} \tau^{\frac{2}{q-1}-N} C_{2/q,q'}\left(F \cap B_\tau(x)\right) = \gamma \in [0,\infty) \Longrightarrow \lim_{t\to 0} t^{\frac{1}{q-1}} u_F(x,t) = C\gamma \tag{1.24}$$

for some $C_\gamma = C(N,q,\gamma) > 0$, and

$$\limsup_{\tau\to 0} \tau^{\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) < \infty \Longrightarrow \limsup_{t\to 0} u_F(x,t) < \infty. \tag{1.25}$$

Our paper is organized as follows. In Section1 we recall some properties of the Besov spaces with fractional derivatives $B^{s,p}$ and their links with heat equation. In Section 2 we obtain estimates from above on $\overline{u}_F$. In Section 3 we give estimates from below on $\underline{u}_F$. In Section 4 we prove the main theorems and expose various consequences. In Appendix we derive a series of sharp integral inequalities.

Aknowledgements The authors are grateful to the European RTN Contract N° HPRN-CT-2002-00274 for the support provided in the realization of this work. The authors are grateful to Luc Tartar for providing them the proof of the sharp Poincaré inequality Proposition 2.5 and related references.

2 Estimates from above

Some notations. Let Ω be a domain in $\mathbb{R}^N$ with a compact C^2 boundary and $T > 0$. Set $B_r(a)$ the open ball of radius $r > 0$ and center a (and $B_r(0) := B_r$) and

$$Q_T^\Omega := \Omega \times (0,T), \quad \partial_\ell Q_T^\Omega = \partial\Omega \times (0,T), \quad Q_T := Q_T^{\mathbb{R}^N}, \quad Q_\infty := Q_\infty^{\mathbb{R}^N}.$$

Let $\mathbb{H}^\Omega[.]$ (resp. $\mathbb{H}[.]$) denote the heat potential in Ω with zero lateral boundary data (resp. the heat potential in $\mathbb{R}^N$) with corresponding kernel

$$(x,y,t) \mapsto H^\Omega(x,y,t) \quad (\text{resp.}(x,y,t) \mapsto H(x,y,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}).$$

We denote by $q_c := 1 + \frac{N}{2}$, the Brezis-Friedman critical exponent.

Theorem 2.1 *Let $q \geq q_c$. Then there exists a positive constant $C_1 = C_1(N,q)$ such that for any closed subset F of $\mathbb{R}^N$ and any $u \in C^2(Q_\infty) \cap C(\overline{Q_\infty} \setminus F)$ satisfying*

$$\begin{aligned} \partial_t u - \Delta u + u^q = 0 \quad & \text{in } Q_\infty \\ \lim_{t\to 0} u(x,t) = 0 \quad & \text{locally uniformly in } F^c, \end{aligned} \tag{2.1}$$

there holds

$$u(x,t) \leq C_1 W_F(x,t) \qquad \forall (x,t) \in Q_\infty, \tag{2.2}$$

where W_F is the $(2/q,q')$-parabolic capacitary potential of F defined by (1.13).

First we consider the case where $F = K$ is compact and

$$K \subset B_r \subset \overline{B}_r, \tag{2.3}$$

and then we extend to the general case by a covering argument.

2.1 Capacities and Besov spaces

2.1.1 L^p regularity

Throughout this paper C will denote a generic positive constant, depending only on N, q and sometimes T, the value of which may vary from one occurrence to another. We also use sometimes the notation $A \approx B$ for meaning that there exists a constant $C > 0$ independent of the data such that $C^{-1}A \leq B \leq CA$.

We recall some classical results dealing with L^p capacities as they are developed in [5]: if $1 < p < \infty$ we denote

$$W_p^{2,1}(\mathbb{R}^{N+1}) := \{\phi \in L^p(\mathbb{R}^{N+1}) : \partial_t\phi, \nabla\phi, D^2\phi \in L^p(\mathbb{R}^{N+1})\}, \tag{2.4}$$

with the associated norm

$$\|\phi\|_{W_p^{2,1}} = \|\phi\|_{L^p} + \|\nabla\phi\|_{L^p} + \|\partial_t\phi\|_{L^p} + \left\|D^2\phi\right\|_{L^p}. \tag{2.5}$$

We define a corresponding capacity on compact sets, that we extend it classicaly on capacitable sets.

$$C_{2,1,p}(E) = \inf\{\|\phi\|_{W_p^{2,1}} : \phi \in C_0^\infty(\mathbb{R}^{N+1}) : \phi \geq 1 \text{ in a neighborhood of } E\}, \tag{2.6}$$

We extend the heat kernel H in $\mathbb{R}^{N+1} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}\}$ by assigning the value 0 for $t < 0$. Then, for any $\eta \in C_0(\mathbb{R}^N)$,

$$\mathbb{H}[\eta](x,t) = \begin{cases} 0 & \text{if } t < 0 \\ H * (\eta \otimes \delta_0)(x,t) & \text{if } t > 0, \end{cases} \tag{2.7}$$

where δ_0 has to be understood as the Dirac measure on $\mathbb{R}$ at $t = 0$. For any subset $E \in \mathbb{R}^{N+1}$

$$C_{H,p}(E) = \inf\{\|f\|_{L^p} : f \in L^p(\mathbb{R}^{N+1}), H * f \geq 1 \text{ on } E\}. \tag{2.8}$$

The following result is proved in [5, Prop 2.1].

Proposition 2.2 *For any $T > 0$, there exists $c = c(T,p,N)$ such that*

$$c^{-1}C_{H,p}(E) \leq C_{2,1,p}(E) \leq cC_{H,p}(E) \qquad \forall E \subset \mathbb{R}^N \times]-T,T[,\ E \text{ Borel.} \tag{2.9}$$

We recall the Gagliardo Nirenberg inequality valid for any $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\|\nabla\phi\|_{L^{2p}}^{2p} \leq c_{d,p} \|\phi\|_{L^\infty}^p \left\|D^2\phi\right\|_{L^p}^p. \tag{2.10}$$

Furthermore, the trace at $t = 0$ of functions in $W_p^{2,1}$ belongs to the Besov space $B^{2-\frac{2}{p},p}(\mathbb{R}^N)$. However, in our range of exponents $B^{2-\frac{2}{p},p}(\mathbb{R}^N) = W^{2-\frac{2}{p},p}(\mathbb{R}^N)$. The reason for this is that $2 - \frac{2}{p}$ is not an integer except if $p = 2$, in which case equality holds also. If we set

$$c_{2-\frac{2}{p},p}(K) = \inf\{\|\phi\|_{W^{2-\frac{2}{p},p}} : \phi \in C_0^\infty(\mathbb{R}^N), \phi \geq 1 \text{ in a neighborhood of } K\}. \tag{2.11}$$

then [5, Prop 2.3].

Proposition 2.3 *There exist $c = c(N,p) > 0$ such that*

$$c^{-1}c_{2-\frac{2}{p},p}(E) \leq C_{2,1,p}(E \times \{0\}) \leq cc_{2-\frac{2}{p},p}(E) \qquad \forall E \subset \mathbb{R}^N, \; E \text{ Borel.} \tag{2.12}$$

The $c_{2-\frac{2}{p},p}$-capacity is equivalent to the Bessel capacity $C_{2-\frac{2}{p},p}$ defined by

$$C_{2-\frac{2}{p},p}(E) = \inf\{\|f\|_{L^p} : f \in L^p(\mathbb{R}^N), G_{2-\frac{2}{p}} * f \geq 1 \text{ on } E\} \tag{2.13}$$

where $G_{2-\frac{2}{p}} = \mathcal{F}[(1+|\xi|^2)^{\frac{1}{p}-1}]$ denotes the Bessel kernel associated to the operator $(-\Delta+I)^{1-\frac{1}{p}}$.

2.1.2 The Aronszajn-Slobodeckij integral

If Ω is a domain in $\mathbb{R}^N$ and $0 < s < 1$, we denote by $\|.\|_{\dot{B}^{s,p}(\Omega)}$ the Aronszajn-Slobodeckij norm defined on $C_0^\infty(\Omega)$ by

$$\|\eta\|_{\dot{B}^{s,p}} = \left(\iint_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p} \qquad \forall \eta \in C_0^\infty(\Omega). \tag{2.14}$$

In the case $1 < s < 2$, all the results which are presented still holds by replacing the function by its gradient. We also consider the case $s = 1$, but in our range of exponents the corresponding exponent for p is 2, in which case the space under consideration is just $H_0^1(\Omega)$. Since the imbedding of $W^{1,p}(\Omega)$ is compact, it follows the imbedding of $B^{s,p}(\Omega)$ into $L^p(\Omega)$ is compact too. Therefore the following Poincaré type inequality holds [39, p. 134]. Actually, the proof, obtained by contradiction, is given with $W^{1,p}(\Omega)$ instead of $B^{s,p}(\Omega)$, but it depends only on the compactness of the imbedding.

Proposition 2.4 *Let Ω be a bounded domain and, $p \in (1,\infty)$ and $0 < s \leq 1$ such that $sp \leq N$. Then there exists $\lambda = \lambda(\Omega,N,p) > 0$ such that*

$$\iint_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy \geq \lambda \int_\Omega |\eta(x)|^p dx \quad \forall \eta \in C_0^\infty(\Omega). \tag{2.15}$$

Remark. If $sp > N$, the same proof re holds for all $\eta \in C_0^\infty(\Omega)$ (see the proof of [9, Th 8.2])

$$\left(\iint_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p} \geq C\frac{|\eta(z)-\eta(z')|}{|z-z'|^\alpha} \quad \forall (z,z') \in \Omega\times\Omega, \; z \neq z', \tag{2.16}$$

with $\alpha = s - N/p$ and $C = C(s, N, p)$. This estimate implies

$$\left(\iint_{\Omega\times\Omega} \frac{|\eta(x) - \eta(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p} \geq C d^{-\alpha} \|u\|_{L^\infty}, \tag{2.17}$$

where d is the width of Ω, i.e. the smallest of $\delta > 0$ such that there exists an isometry $\mathcal{R}$ such that $\mathcal{R}(\Omega) \subset D_\delta := \{x = (x_1, x') : 0 < x_1 < \delta\}$.

The related unpublished result due to L. Tartar [40] will be useful in the sequel. We reproduce its proof for the sake of completeness.

Proposition 2.5 *Assume $b > a$ and $\Omega \subset \Gamma_{a,b} := \{x = (x_1, x') : a < x_1 < b\}$ is a domain. If $sp \leq N$ there exists $C = C(s, p, N, b/a) > 0$ such that that*

$$\iint_{\Omega\times\Omega} \frac{|\eta(x) - \eta(y)|^p}{|x-y|^{N+sp}} dxdy \geq \lambda (b-a)^{sp} \int_\Omega |\eta(x)|^p dx \quad \forall \eta \in C_0^\infty(\Omega). \tag{2.18}$$

Proof. Using the notation of [24], $W^{s,p}(\mathbb{R}^N)$ is the interpolation space $[W^{1,p}(\mathbb{R}^N), L^p(\mathbb{R}^N)]_{s,p}$ and subset of $L^p(\mathbb{R}^{N-1}; [W^{1,p}(\mathbb{R}), L^p\mathbb{R}]_{s,p}) = L^p(\mathbb{R}^{N-1}; W^{s,p}(\mathbb{R}))$, with continuous imbedding. Thus there exist $C > 0$ such that

$$\begin{aligned} \|\eta\|_{L^p}^p + \int_{\mathbb{R}^{N-1}} \iint_{\mathbb{R}\times\mathbb{R}} \frac{|\eta(x_1, x') - \eta(y_1, x')|^p}{|x_1 - y_1|^{1+sp}} dx_1 dy_1 dx' \\ \leq C \left(\|\eta\|_{L^p}^p + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^p}{|x-y|^{N+sp}} dxdy \right) \end{aligned} \tag{2.19}$$

for all $\eta \in C_0^\infty(\mathbb{R}^N)$. This inequality is valid if η is replaced by η_τ where $\eta_\tau(x) = \eta(\tau x)$ and $\tau > 0$. This gives

$$\begin{aligned} \|\eta\|_{L^p}^p + \tau^{sp-N} \int_{\mathbb{R}^{N-1}} \iint_{\mathbb{R}\times\mathbb{R}} \frac{|\eta(x_1, x') - \eta(y_1, x')|^p}{|x_1 - y_1|^{1+sp}} dx_1 dy_1 dx' \\ \leq C \left(\|\eta\|_{L^p}^p + \tau^{sp-N} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^p}{|x-y|^{N+sp}} dxdy \right). \end{aligned}$$

Letting $\tau \to 0$, we obtained

$$\int_{\mathbb{R}^{N-1}} \iint_{\mathbb{R}\times\mathbb{R}} \frac{|\eta(x_1) - \eta(y_1)|^p}{|x_1 - y_1|^{1+sp}} dx_1 dy_1 dx' \leq C \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^p}{|x-y|^{N+sp}} dxdy \qquad \forall \eta \in C_0^\infty(\mathbb{R}^N). \tag{2.20}$$

Using Proposition 2.4 with $N = 1$ we get

$$\int_0^1 \int_0^1 \frac{|\eta(x_1, x') - \eta(y_1, x')|^p}{|x_1 - y_1|^{1+sp}} |dx_1 dy_1 \geq \lambda \int_0^1 |\eta(x_1, x')|^p dx_1 \quad \forall \eta \in C_0^\infty\left((0,1) \times \mathbb{R}^{N-1}\right)$$

for all $x' \in \mathbb{R}^{N-1}$. Using a standard change of scale, it transforms into

$$\int_a^b \int_a^b \frac{|\eta(x_1, x') - \eta(y_1, x')|^p}{|x_1 - y_1|^{1+sp}} |dx_1 dy_1 \geq \lambda (b-a)^{sp} \int_a^b |\eta(x_1, x')|^p dx_1 \quad \forall \eta \in C_0^\infty\left((a,b) \times \mathbb{R}^{N-1}\right)$$

Integrating over $\mathbb{R}^{N-1}$ and using (2.20), we derive (2.18). □

Definition 2.6 *Assume $s \in (0,1)$ and $sp < 1$ or $s = 1$ and $p = 2$. If Ω is any domain in $\mathbb{R}^N$, the Besov space $B_0^{s,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm*

$$\|\eta\|_{B^{s,p}} = \|\eta\|_{\dot{B}^{s,p}} + \|\eta\|_{L^p}. \tag{2.21}$$

The following result is derived from Proposition 2.5.

Corollary 2.7 *Let $b > a > 0$ and Ω be an open domain of $\mathbb{R}^N$ such that $\Omega \subset B_b \setminus \overline{B_a}$. Then there exists a constant $C = C(s,p,N) > 0$ such that for any $\eta \in C_0^\infty(\Omega)$*

$$\|\eta\|_{\dot{B}^{s,p}} \le \|\eta\|_{B^{s,p}} \le C(b-a)^{sp} \|\eta\|_{\dot{B}^{s,p}}. \tag{2.22}$$

2.1.3 Heat potential and Besov space

If $\eta \in C_0^\infty(\Omega)$, we extend it by 0 outside Ω and set

$$\|\eta\|_{\tilde{B}^{s,p}} = \left(\iint_{Q_\infty} \left| t^{1-s/2} \partial_t \mathbb{H}[\eta] \right|^p dx \frac{dt}{t} \right)^{1/p} \tag{2.23}$$

It is well known (see e.g. [3]) that the Besov space $B^{s,p}(\Omega)$ can be defined directly as the space of $\eta \in L^p(\Omega)$ functions such that $\|\eta\|_{\dot{B}^{s,p}} < \infty$ or or such that $\|\eta\|_{\tilde{B}^{s,p}} < \infty$. It coincides with the the interpolation space $\left[W^{2,p}(\Omega), L^p(\Omega)\right]_{s/2,p}$ (see [24]). Furthermore, there exists $C = C(s,p,N) > 0$ such that

$$C^{-1} \left(\|\eta\|_{L^p} + \|\eta\|_{\dot{B}^{s,p}} \right) \le \|\eta\|_{L^p} + \|\eta\|_{\tilde{B}^{s,p}} \le C \left(\|\eta\|_{L^p} + \|\eta\|_{\dot{B}^{s,p}} \right) \quad \forall \eta \in B^{s,p}(\Omega). \tag{2.24}$$

Lemma 2.8 *Assume $0 < s < 1$ and $1 < p < \infty$ or $s = 1$ and $p = 2$. Then there exists a positive constant C, depending only on s, p, N, such that for any domain Ω, there holds*

$$C^{-1} \|\eta\|_{\dot{B}^{s,p}} \le \|\eta\|_{\tilde{B}^{s,p}} \le C \|\eta\|_{\dot{B}^{s,p}} \quad \forall \eta \in C_0^\infty(\Omega). \tag{2.25}$$

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ and $\tau > 0$. Set $\eta_\tau(x) = \eta(\tau x)$, then (2.25) applied to η_τ yields to

$$C^{-1} \left(\|\eta\|_{L^p} + \tau^s \|\eta\|_{\dot{B}^{s,p}} \right) \le \left(\|\eta\|_{L^p} + \tau^s \|\eta\|_{\tilde{B}^{s,p}} \right) \le C \left(\|\eta\|_{L^p} + \tau^s \|\eta\|_{\dot{B}^{s,p}} \right).$$

Since it holds for any arbitrary large τ and $\eta \in C_0^\infty(\mathbb{R}^N)$, (2.25) follows. □

We denote by $\mathcal{T}_\Omega(K)$ the set of functions $\eta \in C_0^\infty(\Omega)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on K. If Ω is a bounded subset of $\mathbb{R}^N$, we define the Besov capacity of a compact set $K \subset \Omega \subset \mathbb{R}^N$ by

$$R_{s,p}^\Omega(K) = \inf\{\|\eta\|_{\dot{B}^{s,p}}^p : \eta \in \mathcal{T}_\Omega(K)\}, \tag{2.26}$$

and the Bessel capacity relative to Ω by

$$C_{s,p}^\Omega(K) = \inf\{\|\eta\|_{B^{s,p}}^p : \eta \in \mathcal{T}_\Omega(K)\}. \tag{2.27}$$

We extend classicaly this capacity to any capacitable set $K \subset \Omega$. This capacity has the following scaling property.

Lemma 2.9 *For any $\tau > 0$ and any capacitable set $K \subset \Omega$, there holds*

$$R^{\Omega}_{s,p}(K) = \tau^{N-sp} R^{\tau^{-1}\Omega}_{s,p}(\tau^{-1}K). \tag{2.28}$$

Furthermore, if $\Omega \subset B_b \setminus \overline{B_a}$, there exists $c = c(b-a, b/a, N, s, p) > 0$ such that

$$c^{-1} C^{\Omega}_{s,p}(K) \le R^{\Omega}_{s,p}(K) \le c C^{\Omega}_{s,p}(K). \tag{2.29}$$

Finally, if $K \subset \Omega' \subset \overline{\Omega'} \subset \Omega$, there exists $c = c(N, s, p, \operatorname{dist}(\Omega', \Omega^c))$ such that

$$C_{s,p}(K) \le C^{\Omega}_{s,p}(K) \le c C_{s,p}(K). \tag{2.30}$$

Proof. The scaling property (2.28) is clear by change of variable. Estimate (2.29) is a consequence of Definition 2.6 and Proposition 2.5. For the last statement, the left-hand side is obvious. For the right-hand side, consider a smooth nonnegative cut-off function ζ which is 1 on $\overline{\Omega'}$, has value between 0 and 1 and has compact support in Ω. If $\eta \in \mathcal{T}_{\mathbb{R}^N}(K)$, $\zeta\eta \in \mathcal{T}_{\Omega}(K)$ and

$$\begin{aligned} \|\zeta\eta\|^p_{B^{s,p}} &= \|\zeta\eta\|^p_{L^p(\Omega)} + \|\zeta\eta\|^p_{\dot{B}^{s,p}} \\ &\le \|\eta\|^p_{L^p(\Omega)} + \|\eta\|^p_{\dot{B}^{s,p}} + \|\zeta\|^p_{\dot{B}^{s,\infty}} \|\eta\|^p_{L^p} \\ &\le c \|\eta\|^p_{B^{s,p}}, \end{aligned}$$

where

$$\|\zeta\|_{\dot{B}^{s,\infty}} = \sup_{x \neq y} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^s}$$

and $c \approx 1 + (\operatorname{dist}(\Omega', \Omega^c))^{-s}$. The proof follows. □

In the sequel we assume that $q \ge q_c$ and we take $p = q'$ and $s = 2/q$. If $K \subset \Omega$, Ω is bounded and $\eta \in \mathcal{T}_{\Omega}(K)$, we set

$$R[\eta] = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta]|. \tag{2.31}$$

Lemma 2.10 *There exists $C = C(N, q) > 0$ such that for every $\eta \in \mathcal{T}_{\Omega}(K)$*

$$\|\eta\|^{q'}_{\tilde{B}^{2/q,q'}} \le \iint_{Q_\infty} (R[\eta])^{q'} \, dx\, dt := \|R[\eta]\|^{q'}_{L^{q'}} \le C \|\eta\|^{q'}_{\tilde{B}^{2/q,q'}} \tag{2.32}$$

Proof. Using (2.23) and Lemma 2.8, it follows from Corollary 2.7 that

$$\|\eta\|^{q'}_{\tilde{B}^{2/q,q'}} \approx \iint_{Q_\infty} |\partial_t \mathbb{H}[\eta]|^{q'} \, dxdt.$$

Using the Gagliardo-Nirenberg inequality in $\mathbb{R}^N$, an elementary elliptic estimate and the fact that $0 \le \mathbb{H}[\eta] \le 1$, we see that

$$\int_{\mathbb{R}^N} |\nabla(\mathbb{H}[\eta](.,t))|^{2q'} \, dx \le C \left\| D^2 \mathbb{H}[\eta](.,t) \right\|^{q'}_{L^{q'}} \|\mathbb{H}[\eta](.,t)\|^{q'}_{L^\infty} \le C \|\Delta \mathbb{H}[\eta](.,t)\|^{q'}_{L^{q'}}, \tag{2.33}$$

for all $t > 0$. Since $\partial_t \mathbb{H}[\eta] = \Delta \mathbb{H}[\eta]$, it implies (2.32). □

The dual space $B^{-2/q,q}(\Omega)$ of $B^{2/q,q'}(\Omega)$ is naturally endowed with the norm

$$\|\mu\|_{B^{-2/q,q}} = \sup\left\{\mu(\eta) : \eta \in B^{2/q,q'}(\Omega), \|\eta\|_{B^{2/q,q'}} \leq 1\right\}.$$

The following result is may be already known, but we have not found it in the literature. If μ is a bounded measure in $\mathbb{R}^N$, we denote by $\mathbb{H}[\mu]$ the solution of heat equation in Q_∞ with initial data μ.

Lemma 2.11 *Assume $q \geq q_c$. For any $T > 0$, there exist a constant $c > 0$ such that, for any bounded measure μ belonging to $B^{-2/q,q}(\mathbb{R}^N)$, there holds*

$$c^{-1}\|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}[\mu]\|_{L^q(Q_T)} \leq c\|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)}. \tag{2.34}$$

Furthermore, if $q > q_c$ there holds

$$c^{-1}\|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}[\mu]\|_{L^q(Q_\infty)} \leq c\|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} + c\|\mu\|_{\mathfrak{M}(\mathbb{R}^N)}. \tag{2.35}$$

.

Proof. If $\mu \in B^{-2/q,q}(\mathbb{R}^N)$, there exists a unique $\omega \in B^{2-2/q,q}(\mathbb{R}^N)$ such that $\mu = (I-\Delta)\omega$, and $\|\mu\|_{B^{-2/q,q}} \approx \|\omega\|_{B^{2-2/q,q}}$. Applying standard interpolation methods to the analytic semi-group $e^{-t(I-\Delta)} = e^{-t}e^{t\Delta}$ (see e.g. [3], [41]) we obtain,

$$\begin{aligned}\left(\iint_{Q_\infty} \left|t^{1/q}(I-\Delta)\mathbb{H}[\omega]\right|^q dx\frac{e^{-qt}dt}{t}\right)^{1/q} &= \left(\iint_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t}\right)^{1/q} \\ &\approx \|\omega\|_{B^{2-2/q,q}} \\ &\approx \|\mu\|_{B^{-2/q,q}}.\end{aligned} \tag{2.36}$$

Clearly

$$e^{-qT}\iint_{Q_T} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} \leq \iint_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t},$$

and

$$\begin{aligned}\iint_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t} &= \sum_{n=0}^{\infty}\iint_{Q_{T+n+1}\setminus Q_{T+n}} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t} \\ &= \sum_{n=0}^{\infty}\iint_{Q_T} |\mathbb{H}[\mu](s+n)|^q\, e^{-q(s+n)}ds \\ &\leq \left(\sum_{n=0}^{\infty} e^{-qn}\right)\iint_{Q_T} \left|t^{1/q}\mathbb{H}[\mu]\right|^q \frac{dt}{t}.\end{aligned}$$

This implies (2.34). Furthermore, $\||\mathbb{H}[\mu](.,t)|\|_{L^q}^q \leq ct^{-N(q-1)/2}\|\mu\|_{\mathfrak{M}}^q$, thus $\mathbb{H}[\mu] \in L^q(Q_\infty)$ if $q > q_c$ (but this does not hold if $q = q_c$). If $q > q_c$ (equivalently $N(q-1)/2 > 1$),

$$\begin{aligned}\iint_{Q_\infty}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} &= \sum_{n=0}^{\infty}\iint_{Q_{T+n+1}\setminus Q_{T+n}}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t}\\ &= \iint_{Q_T}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} + \iint_{Q_T}\sum_{n=1}^{\infty}|\mathbb{H}[\mu](s+n)|^q\,dxds\\ &\le \iint_{Q_T}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} + C\left(\sum_{n=1}^{\infty}n^{-N(q-1)/2}\right)\|\mu\|^q_{\mathfrak{M}}\,.\end{aligned}$$

Thus we obtain (2.35). □

2.2 Global L^q-estimates

Let $\rho > 0$, we assume (2.3) holds. With the previous notations, $\mathcal{T}_{r,r+\rho}(K)$ denotes the set of functions $\eta \in C_0^\infty(B_{r+\rho})$, such that $0 \le \eta \le 1$ and value 1 on K. If $\eta \in \mathcal{T}_{r,\rho}(K)$, we set

$$\eta^* = 1 - \eta \ \text{ and } \ \zeta = \mathbb{H}[\eta^*]^{2q'}.$$

Lemma 2.12 *Assume u is a positive solution of (2.1) in Q_∞. There exists $C = C(N,q) > 0$ such that for every $T > 0$ and every compact set $K \subset B_r$,*

$$\iint_{Q_T} u^q\zeta dx\,dt + \int_{\mathbb{R}^N}(u\zeta)(x,T)dx \le C\|R[\eta]\|^{q'}_{L^{q'}} \qquad \forall \eta \in \mathcal{T}_{r,\rho}(K). \tag{2.37}$$

Proof. We recall that there always holds

$$0 \le u(x,t) \le \left(\frac{1}{t(q-1)}\right)^{\frac{1}{q-1}} \qquad \forall (x,t) \in Q_\infty, \tag{2.38}$$

and

$$0 \le u(x,t) \le \left(\frac{C}{t + (|x|-r)^2}\right)^{\frac{1}{q-1}} \qquad \forall (x,t) \in Q_\infty \setminus B_r \times \mathbb{R}, \tag{2.39}$$

by the Brezis-Friedman estimate [7]. Since η^* vanishes in an open neighborhood $\mathcal{N}_1$, for any open subset $\mathcal{N}_2$ such that $K \subset \mathcal{N}_2 \subset \overline{\mathcal{N}}_2 \subset \mathcal{N}_1$ there exist $c_2 = c_{\mathcal{N}_2} > 0$ and $C_2 = C_{\mathcal{N}_2} > 0$ such that

$$\mathbb{H}[\eta^*](x,t) \le C_2 e^{-\frac{c_2}{t}}, \qquad \forall (x,t) \in Q_T^{\mathcal{N}_2}.$$

Therefore

$$\lim_{t\to 0}\int_{\mathbb{R}^N}(u\zeta)(x,t)dx = 0.$$

Thus ζ is an admissible test function and one has

$$\iint_{Q_T} u^q\zeta dx\,dt + \int_{\mathbb{R}^N}(u\zeta)(x,T)dx = \iint_{Q_T} u(\partial_t\zeta + \Delta\zeta)dx\,dt. \tag{2.40}$$

Notice that the two terms on the left-hand side are nonnegative. Put $\mathbb{H}_{\eta^*} = \mathbb{H}[\eta^*]$, then

$$\begin{aligned}\partial_t\zeta + \Delta\zeta &= 2q'\mathbb{H}_{\eta^*}^{2q'-1}\left(\partial_t\mathbb{H}_{\eta^*} + \Delta\mathbb{H}_{\eta^*}\right) + 2q'(2q'-1)\mathbb{H}_{\eta^*}^{2q'-2}|\nabla\mathbb{H}_{\eta^*}|^2,\\ &= 2q'\mathbb{H}_{\eta^*}^{2q'-1}\left(\partial_t\mathbb{H}_{\eta} + \Delta\mathbb{H}_{\eta}\right) + 2q'(2q'-1)\mathbb{H}_{\eta}^{2q'-2}|\nabla\mathbb{H}_{\eta}|^2,\end{aligned}$$

because $\mathbb{H}_{\eta^*} = 1 - \mathbb{H}_\eta$, hence

$$u(\partial_t\zeta + \Delta\zeta) = u\mathbb{H}_{\eta^*}^{2q'/q}\left[2q'(2q'-1)\mathbb{H}_{\eta^*}^{2q'-2-2q'/q}|\nabla\mathbb{H}_\eta|^2 - 2q'\mathbb{H}_{\eta^*}^{2q'-1-2q'/q}(\Delta\mathbb{H}_\eta + \partial_t\mathbb{H}_\eta)\right].$$

Finally, since $2q' - 2 - 2q'/q = 0$ and $0 \le \mathbb{H}_{\eta^*} \le 1$, there holds

$$\left|\iint_{Q_T} u(\partial_t\zeta + \Delta\zeta)dx\,dt\right| \le C(q)\left(\iint_{Q_T} u^q\zeta dx\,dt\right)^{1/q}\left(\iint_{Q_T} R^{q'}(\eta)dx\,dt\right)^{1/q'},$$

where

$$R(\eta) = |\nabla\mathbb{H}_\eta|^2 + |\Delta\mathbb{H}_\eta + \partial_t\mathbb{H}_\eta|.$$

Using Lemma 2.10 one obtains (2.37). □

Proposition 2.13 *Under the assumptions of Lemma 2.12, let $r > 0$, $\rho > 0$, $T \ge (r+\rho)^2$*

$$\mathcal{E}_{r+\rho} := \{(x,t) : |x|^2 + t \le (r+\rho)^2\}$$

and $Q_{r+\rho,T} = Q_T \setminus \mathcal{E}_{r+\rho}$. There exists $C = C(N,q,T) > 0$ such that

$$\iint_{Q_{r+\rho,T}} u^q dx\,dt + \int_{\mathbb{R}^N} u(x,T)dx \le C\|R[\eta]\|_{L^{q'}}^{q'} \qquad \forall\eta \in \mathcal{T}_{r,\rho}(K). \tag{2.41}$$

Proof. In view of Lemma 2.12 we only have to show that there exists a positive constant $c(N,q)$ such that, for η as above and $T \ge (r+\rho)^2$,

$$\zeta = \mathbb{H}{\eta^*}^{2q'} > c(N,q).$$

Since, by assumption $K \subset B_r$, $\eta^* \equiv 1$ outside $B_{r+\rho}$ and $0 \le \eta^* \le 1$,

$$\begin{aligned}\mathbb{H}[\eta^*](x,t) \ge \mathbb{H}[1 - \chi_{B_{r+\rho}}](x,t) &= \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\ge r+\rho} e^{-\frac{|x-y|^2}{4t}}dy,\\ &= 1 - \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\le r+\rho} e^{-\frac{|x-y|^2}{4t}}dy.\end{aligned}$$

For $(x,t) \in Q_{r+\rho,T}$, put $x = (r+\rho)\xi$, $y = (r+\rho)\upsilon$ and $t = (r+\rho)^2\tau$. Then $(\xi,\tau) \in Q_{1,\frac{T}{(r+\rho)^2}}$ and

$$\left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\le r+\rho} e^{-\frac{|x-y|^2}{4t}}dy = \left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\le 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon.$$

We claim that

$$\max\left\{\left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\le 1}e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon : (\xi,\tau)\in Q_{1,\frac{T}{(r+\rho)^2}}\right\}=\ell, \tag{2.42}$$

for some $\ell=\ell(N,\frac{T}{(r+\rho)^2})\in(0,1]$, and ℓ is actually independent of $\frac{T}{(r+\rho)^2}$if this quantity is larger than 1. We recall that

$$\left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\le 1}e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon<1\qquad\forall\tau>0. \tag{2.43}$$

If the maximum is achieved for some $(\bar\xi,\bar\tau)\in Q_{1,\frac{T}{(r+\rho)^2}}$, it is smaller than 1 and

$$\mathbb{H}[\eta^*](x,t)\ge\mathbb{H}[1-\chi_{B_{r+\rho}}](x,t)\ge 1-\ell>0,\qquad\forall(x,t)\in Q_{r+\rho,T}. \tag{2.44}$$

Let us assume that the maximum is achieved following a sequence $\{(\xi_n,\tau_n)\}$ with $\tau_n\to 0$ and $|\xi_n|\to\alpha\ge 1$. Then

$$\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}}\int_{|\upsilon|\le 1}e^{-\frac{|\xi_n-\upsilon|^2}{4\tau_n}}d\upsilon=\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}}\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon\le\frac{1}{2}.$$

To verify this, note that $B_1(\xi_n)\cap B_1(-\xi_n)=\emptyset$, so that

$$\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon+\int_{B_1(-\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon<\int_{\mathbb{R}^N}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon<1$$

and

$$\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon=\int_{B_1(-\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon.$$

If the supremum is achieved with a sequence $\{(\xi_n,\tau_n)\}$ such that $|\xi_n|\to\infty$, the same argument applies. Finally if $\{\xi_n\}$ is bounded but $\tau_n\to\infty$ then the expression in (2.43) tends to zero. Therefore (2.43) holds. Put $C=(1-\ell)^{-1}$, then

$$\iint_{Q_{r,T}}u^q dx\,dt+\int_{\mathbb{R}^N}u(.,T)dx\le C\left\|R[\eta]\right\|_{L^{q'}}^{q'}, \tag{2.45}$$

and (2.41) follows. □

2.3 Pointwise estimates

In this subsection u is a positive solution of (2.1) in Q_∞ and the assumptions of Lemma 2.12 hold. We first derive a rough pointwise estimate.

Lemma 2.14 *There exists a constant $C=C(N,q)>0$ such that, for any $\eta\in\mathcal{T}_{r,\rho}(K)$,*

$$u(x,(r+2\rho)^2)\le\frac{C\left\|R[\eta]\right\|_{L^{q'}}^{q'}}{(\rho(r+\rho))^{\frac{N}{2}}},\qquad\forall x\in\mathbb{R}^N. \tag{2.46}$$

Proof. We recall that

$$\int_s^T \int_{\mathbb{R}^N} u^q dx\, dt + \int_{\mathbb{R}^N} u(x,T)dx = \int_{\mathbb{R}^N} u(x,s)dx \qquad \forall T > s > 0, \tag{2.47}$$

and

$$\int_{\mathbb{R}^N} u(.,s)dx \le C \left\| R[\eta] \right\|_{L^{q'}}^{q'} \qquad \forall T > s \ge (r+\rho)^2, \tag{2.48}$$

by Proposition 2.13. Using the fact that

$$u(x,\tau+s) \le \mathbb{H}[u(.,s)](x,\tau) \le \left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}} \int_{\mathbb{R}^N} u(.,s)dx,$$

(2.46) follows from (2.48) with $s = (r+\rho)^2$ and $\tau = (r+2\rho)^2 - (r+\rho)^2 \approx \rho(r+\rho)$. □

The above estimate does not take into account the fact that $u(x,0) = 0$ if $|x| \ge r$. It is mainly interesting if $|x| \le r$. In order to derive a sharper estimate which takes this fact into account, we need some lateral boundary estimates.

Lemma 2.15 *Let $\gamma \ge r + 2\rho$ and $c > 0$ and either $N = 1$ or 2 and $0 \le t \le c\gamma^2$ for some $c > 0$, or $N \ge 3$ and $t > 0$. Then, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$\int_0^t \int_{\partial B_\gamma} u dS d\tau \le C_5 \gamma \left\| R[\eta] \right\|_{L^{q'}}^{q'}. \tag{2.49}$$

where $C > 0$ depends on N, q and c if $N = 1$, 2 or depends only on N and q if $N \ge 3$.

Proof. First we assume $N = 1$ or 2. Put $G^\gamma := B_\gamma^c \times (-\infty, 0)$ and $\partial_\ell G^\gamma = \partial B_\gamma \times (-\infty, 0)$. We set

$$h_\gamma(x) = 1 - \frac{\gamma}{|x|},$$

and let ψ_γ be the solution of

$$\begin{aligned} \partial_\tau \psi_\gamma + \Delta \psi_\gamma &= 0 \qquad \text{in } G^\gamma, \\ \psi_\gamma &= 0 \qquad \text{on } \partial_\ell G^\gamma, \\ \psi_\gamma(.,0) &= h_\gamma \qquad \text{in } B_\gamma^c. \end{aligned} \tag{2.50}$$

Thus the function

$$\tilde{\psi}(x,\tau) = \psi_\gamma(\gamma x, \gamma^2 \tau)$$

satisfies

$$\begin{aligned} \partial_t \tilde{\psi} + \Delta \tilde{\psi} &= 0 \qquad \text{in } G^1 \\ \tilde{\psi} &= 0 \qquad \text{on } \partial_\ell G^1 \\ \tilde{\psi}(.,0) &= \tilde{h} \qquad \text{in } B_1^c, \end{aligned} \tag{2.51}$$

and $\tilde{h}(x) = 1 - |x|^{-1}$. By the maximum principle $0 \le \tilde{\psi} \le 1$, and by Hopf Lemma

$$-\frac{\partial \tilde{\psi}}{\partial \mathbf{n}}\Big|_{\partial B_1 \times [-c,0]} \ge \theta > 0, \tag{2.52}$$

where $\theta = \theta(N, c)$. Then $0 \le \psi_\gamma \le 1$ and

$$-\frac{\partial \psi_\gamma}{\partial \mathbf{n}}\Big|_{\partial B_\gamma \times [-\gamma^2, 0]} \ge \theta/\gamma. \tag{2.53}$$

Multiplying (1.1) by $\psi_\gamma(x, \tau - t) = \psi^*_\gamma(x, \tau)$ and integrating on $B^c_\gamma \times (0, t)$ yields to

$$\int_0^t \int_{B^c_\gamma} u^q \psi^*_r dx d\tau + \int_{B^c_\gamma} (uh_\gamma)(x, t) dx - \int_0^t \int_{\partial B_\gamma} \frac{\partial u}{\partial \mathbf{n}} \psi^*_\gamma dS d\tau = -\int_0^t \int_{\partial B_\gamma} \frac{\partial \psi^*_\gamma}{\partial \mathbf{n}} u d\sigma d\tau. \tag{2.54}$$

Since ψ^*_γ is bounded from above by 1, estimate (2.49) follows from (2.53) and Proposition 2.13 (notice that $B^c_\gamma \times (0, t) \subset \mathcal{E}^c_\gamma$), first by taking $t = T = \gamma^2 \ge (r + 2\rho)^2$, and then for any $t \le \gamma^2$.

If $N \ge 3$, we proceed as above except that we take

$$h_\gamma(x) = 1 - \left(\frac{\gamma}{|x|}\right)^{N-2}.$$

Then $\psi_\gamma(x, t) = h_\gamma(x)$ and $\theta = N - 2$ is independent of the length of the time interval. This leads to the conclusion. □

Lemma 2.16 *I- Let M, $a > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that*

$$0 \le \eta(x) \le M e^{-a|x|^2} \qquad a.e.\ in\ \mathbb{R}^N. \tag{2.55}$$

Then, for any $t > 0$,

$$0 \le \mathbb{H}[\eta](x, t) \le \frac{M}{(4at + 1)^{\frac{N}{2}}} e^{-\frac{a|x|^2}{4at+1}} \qquad \forall x \in \mathbb{R}^N. \tag{2.56}$$

II- Let M, a, $b > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that

$$0 \le \eta(x) \le M e^{-a(|x|-b)^2_+} \qquad a.e.\ in\ \mathbb{R}^N. \tag{2.57}$$

Then, for any $t > 0$,

$$0 \le \mathbb{H}[\eta](x, t) \le \frac{M e^{-\frac{a(|x|-b)^2_+}{4at+1}}}{(4at + 1)^{\frac{N}{2}}} \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0. \tag{2.58}$$

Proof. For the first statement, put $a = \frac{1}{4}s$. Then

$$0 \le \eta(x) \le M(4\pi s)^{\frac{N}{2}} \frac{1}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4s}} = C(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, s).$$

By the order property of the heat kernel,

$$0 \le \mathbb{H}[\eta](x, t) \le M(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, t + s) = M \left(\frac{s}{t + s}\right)^{\frac{N}{2}} e^{-\frac{|x|^2}{4(t+s)}},$$

and (2.56) follows by replacing s by $\frac{1}{4}a$.

For the second statement, let $\tilde{a} < a$ and $R = \max\{e^{-a(r-b)_+^2 + \tilde{a}r^2} : r \geq 0\}$. A direct computation gives $R = e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}$, and (2.58) implies

$$0 \leq \eta(x) \leq Me^{\frac{a\tilde{a}b^2}{a-\tilde{a}}} e^{-\tilde{a}|x|^2}.$$

Applying the statement I, we derive

$$0 \leq \mathbb{H}[\eta](x,t) \leq \frac{Ce^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}}{(4\tilde{a}t+1)^{\frac{N}{2}}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0. \tag{2.59}$$

Since for any $x \in \mathbb{R}^N$ and $t > 0$,

$$(4\tilde{a}t+1)^{-\frac{N}{2}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \leq e^{-\frac{a\tilde{a}b^2}{a-\tilde{a}}} (4at+1)^{-\frac{N}{2}} e^{-\frac{a(|x|-b)^2}{4at+1}},$$

(2.58) follows from (2.59). □

Lemma 2.17 *There exists a constant $C = C(N,q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$u(x,(r+2\rho)^2) \leq C \max\left\{ \frac{r+\rho}{(|x|-r-2\rho)^{N+1}}, \frac{|x|-r-2\rho}{(r+\rho)^{N+1}} \right\} e^{-\frac{(|x|-(r+2\rho))^2}{4(r+2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.60}$$

for any $x \in \mathbb{R}^N \setminus B_{r+3\rho}$.

Proof. It is classical that the Dirichlet heat kernel $H^{B_1^c}$ in the complement of B_1 satisfies, for some $C = C(N) > 0$,

$$H^{B_1^c}(x',y',t',s') \leq C_7 (t'-s')^{-(N+2)/2} (|x'|-1) e^{-\frac{|x'-y'|^2}{4(t'-s')}}, \tag{2.61}$$

for $t' > s'$. By performing the change of variable $x' \mapsto (r+2\rho)x'$, $t' \mapsto (r+2\rho)^2 t'$, for any $x \in \mathbb{R}^N \setminus B_{r+2\rho}$ and $0 \leq t \leq T$, one obtains

$$u(x,t) \leq C(|x|-r-2\rho) \int_0^t \int_{\partial B_{r+2\rho}} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{1+\frac{N}{2}}} u(y,s) d\sigma(y) ds. \tag{2.62}$$

The right-hand side term in (2.62) is smaller than

$$\max\left\{ \frac{C(|x|-r-2\rho)}{(t-s)^{1+\frac{N}{2}}} e^{-\frac{(|x|-r-2\rho)^2}{4(t-s)}} : s \in (0,t) \right\} \int_0^t \int_{\partial B_{r+2\rho}} u(y,s) d\sigma(y) ds.$$

We fix $t = (r+2\rho)^2$ and $|x| \geq r+3\rho$. Since

$$\max\left\{ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : s \in \left(0,(r+2\rho)^2\right) \right\}$$
$$= (|x|-r-2\rho)^{-2-N} \max\left\{ \frac{e^{-\frac{1}{4\sigma}}}{\sigma^{1+\frac{N}{2}}} : 0 < \sigma < \left(\frac{r+2\rho}{|x|-r-2\rho} \right)^2 \right\},$$

a direct computation gives

$$\max\left\{\frac{e^{-\frac{1}{4}\sigma}}{\sigma^{1+\frac{N}{2}}}:0<\sigma<\left(\frac{r+2\rho}{|x|-r-2\rho}\right)^2\right\}$$

$$=\begin{cases}(2N+4)^{1+\frac{N}{2}}e^{-(N+2)/2} & \text{if } r+3\rho\le|x|\le(r+2\rho)(1+\sqrt{4+2N}),\\ \left(\dfrac{|x|-r-2\rho}{r+2\rho}\right)^{2+N}e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2} & \text{if } |x|\ge(r+2\rho)(1+\sqrt{4+2N}).\end{cases}$$

Thus there exists a constant $C(N)>0$ such that

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}}:s\in\left(0,(r+2\rho)^2\right)\right\}\le C(N)\rho^{-2-N}e^{-\left(\frac{|x|-(r+2\rho)}{2r+4\rho}\right)^2}. \tag{2.63}$$

Combining this estimate with (2.49) with $\gamma=r+2\rho$ and (2.62), one derives (2.60). □

Lemma 2.18 *There exists a constant $C=C(N,q)>0$ such that*

$$0\le u(x,(r+2\rho)^2)\le C\max\left\{\frac{(r+\rho)^3}{\rho(|x|-r-2\rho)^{N+1}},\frac{1}{(r+\rho)^{N-1}\rho}\right\}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2}\|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.64}$$

for every $x\in\mathbb{R}^N\setminus B_{r+3\rho}$.

Proof. This is a direct consequence of the inequality

$$(|x|-r-2\rho)e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2}\le\frac{C(r+\rho)^2}{\rho}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2},\qquad\forall x\in B_{r+2\rho}^c, \tag{2.65}$$

and Lemma 2.17. □

Lemma 2.19 *There exists a constant $C=C(N,q)>0$ such that, for any $\eta\in\mathcal{T}_{r,\rho}(K)$, the following estimate holds*

$$u(x,t)\le\frac{C\tilde{M}e^{-\frac{(|x|-r-3\rho)_+^2}{4t}}}{t^{\frac{N}{2}}}\|R[\eta]\|_{L^{q'}}^{q'},\qquad\forall x\in\mathbb{R}^N,\ \forall t\ge(r+2\rho)^2, \tag{2.66}$$

where

$$\tilde{M}=\tilde{M}(x,r,\rho)=\begin{cases}\left(1+\frac{r}{\rho}\right)^{\frac{N}{2}} & \text{if } |x|<r+3\rho\\ \frac{(r+\rho)^{N+3}}{\rho(|x|-r-2\rho)^{N+2}} & \text{if } r+3\rho\le|x|\le c_N^*(r+2\rho)\\ 1+\frac{r}{\rho} & \text{if } |x|\ge c_N^*(r+2\rho)\end{cases} \tag{2.67}$$

with $c_N^*=1+\sqrt{4+2N}$.

Proof. It follows by the maximum principle

$$u(x,t) \le \mathbb{H}[u(.,(r+2\rho)^2)](x,t-(r+2\rho)^2).$$

for $t \ge (r+2\rho)^2$ and $x \in \mathbb{R}^N$. By Lemma 2.14 and Lemma 2.18

$$u(x,(r+2\rho)^2) \le C_{10}\tilde{M}e^{-\frac{(|x|-r-3\rho)^2}{4(r+2\rho)^2}} \left\|R[\eta]\right\|_{L^{q'}}^{q'},$$

where

$$\tilde{M} = \begin{cases} ((r+\rho)\rho)^{-\frac{N}{2}} & \text{if } |x| < r+3\rho \\ \frac{(r+\rho)^3}{\rho}\left(|x|-r-2\rho)\right)^{N+2} & \text{if } r+3\rho \le |x| \le c_N^*(r+2\rho) \\ \frac{1}{(r+\rho)^{N-1}\rho} & \text{if } |x| \ge c_N^*(r+2\rho) \end{cases}$$

Applying Lemma 2.16 with $a=(2r+4\rho)^{-2}$, $b=r+3\rho$ and t replaced by $t-(r+2\rho)^2$ implies

$$u(x,t) \le C\frac{(r+2\rho)^N\tilde{M}}{t^{\frac{N}{2}}}e^{-\frac{(|x|-r-3\rho)^2}{4t}} \left\|R[\eta]\right\|_{L^{q'}}^{q'}, \tag{2.68}$$

for all $x \in B_{r+3\rho}^c$ and $t \ge (r+2\rho)^2$, which is (2.66). □

The next estimate gives a precise upper bound for u when t is not bounded from below.

Lemma 2.20 *Assume that $0 < t \le (r+2\rho)^2$, then there exists a constant $C = C(N,q) > 0$ such that the following estimate holds*

$$u(x,t) \le C(r+\rho)\max\left\{\frac{1}{(|x|-r-2\rho)^{N+1}}, \frac{1}{\rho t^{\frac{N}{2}}}\right\} e^{-\frac{(|x|-r-3\rho)^2}{4t}} \left\|R[\eta]\right\|_{L^{q'}}^{q'}, \tag{2.69}$$

for any $(x,t) \in \mathbb{R}^N \setminus B_{r+3\rho} \times (0,(r+2\rho)^2]$.

Proof. Thanks to (2.49) the following estimate is a straightforward variant of (2.60) for any $|x| \ge r+2\rho$,

$$u(x,t) \le C_8(|x|-r-2\rho)(r+2\rho)\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \le t\right\} \left\|R[\eta]\right\|_{L^{q'}}^{q'}. \tag{2.70}$$

Clearly

$$\begin{aligned} &\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \le t\right\} \\ &= \begin{cases} (2N+4)^{1+\frac{N}{2}}(|x|-r-2\rho)^{-N-2}e^{-\frac{N+2}{2}} & \text{if } 0 < |x| \le r+2\rho+\sqrt{2t(N+2)} \\ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4t}}}{t^{1+\frac{N}{2}}} & \text{if } |x| > r+2\rho+\sqrt{2t(N+2)}. \end{cases} \end{aligned}$$

By elementary analysis, if $x \in B^c_{r+3\rho}$,

$$(|x|-r-2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \le e^{-\frac{(|x|-r-3\rho)^2}{4t}} \begin{cases} \rho e^{-\frac{\rho^2}{4t}} & \text{if } 2t < \rho^2 \\ \dfrac{2t}{\rho} e^{-1+\frac{\rho^2}{4t}} & \text{if } \rho^2 \le 2t \le 2(r+2\rho)^2. \end{cases}$$

However, since

$$\frac{\rho}{t} e^{-\frac{\rho^2}{4t}} \le \frac{4}{\rho},$$

we derive

$$(|x|-r-2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \le \frac{Ct}{\rho} e^{-\frac{(|x|-r-3\rho)^2}{4t}},$$

and (2.69) follows. □

Remark. In the subcritical case $1 < q < q_c$, it is easy to show by using Lemma 2.20, that any positive solution u of (2.1), such that $u(x,0) = 0$ for $x \neq 0$, satisfies

$$u(x,t) \le Ct^{-\frac{1}{q-1}} \min\left\{1, \left(\frac{|x|}{\sqrt{t}}\right)^{\frac{2}{q-1}-N} e^{-\frac{|x|^2}{4t}}\right\} \qquad \forall (x,t) \in Q_\infty. \tag{2.71}$$

This upper estimate corresponds to the one obtained in [8]. If $F = \overline{B}_r$ the upper estimate is less esthetic. However, it is proved in [28] by a barrier method that, if the initial trace of positive solution u of (2.1), vanishes outside F, and if $1 < q < 3$, there holds

$$u(x,t) \le t^{-\frac{1}{q-1}} f_1((|x|-r)/\sqrt{t}) \qquad \forall (x,t) \in Q_\infty,\ |x| \ge r, \tag{2.72}$$

where $f = f_1$ is the unique positive (and radial) solution of

$$\begin{cases} f'' + \dfrac{y}{2} f' + \dfrac{1}{q-1} f - f^q = 0 \quad \text{in } (0,\infty) \\ f'(0) = 0\,,\ \lim_{y\to\infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \tag{2.73}$$

Notice that the existence of f_1 follows from [8] since q belongs to the subcritical range on exponents in dimension one. Furthermore f_1 has the following asymptotic expansion

$$f_1(y) = Cy^{(3-q)/(q-1)} e^{-y^2/4t}(1 + \circ(1))) \quad \text{as } y \to \infty.$$

2.4 The upper Wiener test

Definition 2.21 We define on $\mathbb{R}^N \times \mathbb{R}$ the two *parabolic distances* δ_2 and δ_∞ by

$$\delta_2[(x,t),(y,s)] := \sqrt{|x-y|^2 + |t-s|}, \tag{2.74}$$

and

$$\delta_\infty[(x,t),(y,s)] := \max\{|x-y|, \sqrt{|t-s|}\}. \tag{2.75}$$

If $K \subset \mathbb{R}^N$ and $i = 2, \infty$,

$$\delta_i[(x,t),K] = \inf\{\delta_i[(x,t),(y,0)] : y \in K\} = \begin{cases} \max\left\{\text{dist}\,(x,K), \sqrt{|t|}\right\} & \text{if } i = \infty, \\ \sqrt{\text{dist}^{\,2}(x,K) + |t|} & \text{if } i = 2. \end{cases}$$

For $\beta > 0$ and $i = 2, \infty$, we denote by $\mathcal{B}^i_\beta(m)$ the parabolic ball of center $m = (x,t)$ and radius β in the parabolic distance δ_i.

Let K be *any* compact subset of $\mathbb{R}^N$ and $\overline{u}_K$ the maximal solution of (1.1) which blows up on K. The function $\overline{u}_K$ is constructed in [28] as being the decreasing limit of the $\overline{u}_{K_\epsilon}$ ($\epsilon > 0$) when $\epsilon \to 0$, where

$$K_\epsilon = \{x \in \mathbb{R}^N : \text{dist}\,(x,K) \le \epsilon\}$$

and $\overline{u}_{K_\epsilon} = \lim_{k\to\infty} u_{k,K_\epsilon} = \overline{u}_K$, where u_k is the solution of the classical problem,

$$\begin{cases} \partial_t u_k - \Delta u_k + u_k^q = 0 & \text{in } Q_T, \\ u_k = 0 & \text{on } \partial_\ell Q_T, \\ u_k(.,0) = k\chi_{K_\epsilon} & \text{in } \mathbb{R}^N. \end{cases} \tag{2.76}$$

If $(x,t) = m \in \mathbb{R}^N \times (0,T]$, we set $d_K = \text{dist}\,(x,K)$, $D_K = \max\{|x-y| : y \in K\}$ and $\lambda = \sqrt{d_K^2 + t} = \delta_2[m,K]$. We define a slicing of K, by setting $d_n = d_n(K,t) := \sqrt{nt}$ ($n \in \mathbb{N}$), $d_n^\pm = \left(\sqrt{nt} \pm \dfrac{\sqrt{t}}{\sqrt{n}}\right)_+$ (the positive part is only needed when $n = 0$) and

$$T_n^* = \overline{B}_{d_{n+1}^+}(x) \setminus B_{d_n^-}(x)\,,\ T_n = \overline{B}_{d_{n+1}}(x) \setminus B_{d_n}(x), \qquad \forall n \in \mathbb{N},$$

thus $T_0^* = B_{2\sqrt{t}}(x)$, $T_0 = B_{\sqrt{t}}(x)$, and

$$K_n(x,t) = K \cap T_n(x,t) \ \text{ for } n \in \mathbb{N} \text{ and } \mathcal{Q}_n(x,t) = K \cap B_{d_{n+1}}(x,t).$$

When there is no ambiguity, we will skip the (x,t) variable in the above sets. The main result of this section is the following discrete upper Wiener-type estimate.

Theorem 2.22 *Assume $q \ge q_c$. Then there exists $C = C(N,q,T) > 0$ such that*

$$\overline{u}_K(x,t) \le \frac{C}{t^{\frac{N}{2}}} \sum_{n=0}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \qquad \forall (x,t) \in Q_T, \tag{2.77}$$

where a_t is the largest integer j such that $K_j \neq \emptyset$.

With no loss of generality, we can assume that $x = 0$. Furthermore, in considering the scaling transformation $u_\ell(y,t) = \ell^{\frac{1}{q-1}} u(\sqrt{\ell}y, \ell t)$, with $\ell > 0$, we can assume $t = 1$. Thus the new compact singular set of the initial trace becomes $K/\sqrt{\ell}$, that we still denote K. We also set $a_K = a_{K,1}$ For $n \in \mathbb{N}_*$ set $\delta_n = d_{n+1} - d_n$, then $\frac{1}{2\sqrt{n+1}} \le \delta_n \le \frac{1}{2\sqrt{n}}$. By convention $\delta_0 = 1$. It

is possible to exhibit a collection Θ_n of points $a_{n,j}$ with center on the sphere $\Sigma_n = \{y \in \mathbb{R}^N : |y| = (d_{n+1} + d_n)/2\}$, such that

$$T_n \subset \bigcup_{a_{n,j} \in \Theta_n} B_{\delta_n}(a_{n,j}), \quad |a_{n,j} - a_{n,k}| \geq \delta_n \text{ and } \#\Theta_n \leq C n^{N-1},$$

for some constant $C = C(N)$. If $K_{n,j} = K_n \cap B_{\delta_n}(a_{n,j})$, there holds

$$K = \bigcup_{0 \leq n \leq a_K} \bigcup_{a_{n,j} \in \Theta_n} K_{n,j}.$$

The first intermediate step is based on the *quasi-additivity* property of capacities developed in [2].

Lemma 2.23 *Let $q \geq q_c$. There exists a constant $C = C(N, q)$ such that*

$$\sum_{a_{n,j} \in \Theta_n} R_{2/q,q'}^{B_{2\delta_n}(a_{n,j})}(K_{n,j}) \leq C d_{n+1}^{N - \frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \qquad \forall n \in \mathbb{N}_*. \tag{2.78}$$

Proof. The following result is proved in [2, Th 3]: if the spheres $B_{\rho_j^\theta}(b_j)$, $\theta = 1 - 2/N(q-1)$, are disjoint in $\mathbb{R}^N$ and G is an analytic subset of $\bigcup B_{\rho_j}(b_j)$ where the ρ_j are positive and smaller than some $\rho^* > 0$, there holds

$$C_{2/q,q'}(G) \leq \sum_j C_{2/q,q'}(G \cap B_{\rho_j}(b_j)) \leq A C_{2/q,q'}(G), \tag{2.79}$$

for some A depending on N, q and ρ^*. This property is called *quasi-additivity*. We define for $n \in \mathbb{N}_*$,

$$\tilde{T}_n = d_{n+1} T_n, \quad \tilde{K}_n = d_{n+1} K_n \quad \text{and} \quad \tilde{\mathcal{Q}}_n = d_{n+1} \mathcal{Q}_n.$$

Since $K_{n,j} \subset B_{\delta_n}(a_{n,j})$, it follows that

$$\tilde{K}_{n,j} := d_{n+1} K_{n,j} \subset B_{d_{n+1}\delta_n}(\tilde{a}_{n,j}).$$

Note that by Lemma 2.9

$$\begin{aligned} R_{2/q,q'}^{B_{2\delta_n}(a_{n,j})}(K_{n,j}) &= d_{n+1}^{\frac{2}{q-1}-N} R_{2/q,q'}^{B_{2\delta_n d_{n+1}}(d_{n+1}a_{n,j})}(\tilde{K}_{n,j}) \\ &\approx d_{n+1}^{\frac{2}{q-1}-N} C_{2/q,q'}^{B_{2\delta_n d_{n+1}}(d_{n+1}a_{n,j})}(\tilde{K}_{n,j}) \\ &\approx d_{n+1}^{\frac{2}{q-1}-N} C_{2/q,q'}(\tilde{K}_{n,j}) \end{aligned} \tag{2.80}$$

where $\tilde{K}_{n,j} = d_{n+1} K_{n,j}$. For a fixed $n > 0$ and each repartition Λ of points $\tilde{a}_{n,j} = d_{n+1}\, a_{n,j}$ such that the balls $B_{2^\theta}(\tilde{a}_{n,j})$ are disjoint, the quasi-additivity property holds: if we set

$$K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} K_{n,j}\ , \quad \tilde{K}_{n,\Lambda} = d_{n+1}\, K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} \tilde{K}_{n,j} \quad \text{and} \quad \tilde{K}_n = d_{n+1}\, K_n,$$

then

$$\sum_{a_{n,j}\in\Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \approx C_{2/q,q'}(\tilde{K}_{n,\Lambda}). \tag{2.81}$$

The maximal cardinal of any such repartition Λ is of the order of Cn^{N-1} for some positive constant $C = C(N)$, therefore, the number of repartitions needed for a full covering of the set $\tilde{T}_n$ is of finite order depending upon the dimension. Because $\tilde{K}_n$ is the union of the $\tilde{K}_{n,\Lambda}$,

$$\sum_{a_{n,j}\in\Theta_n} C_{2/q,q'}(\tilde{K}_{n,j}) = \sum_{\Lambda}\sum_{a_{n,j}\in\Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \approx C_{2/q,q'}(\tilde{K}_n). \tag{2.82}$$

By Lemma 2.9,

$$C_{2/q,q'}(\tilde{K}_n) \le C^{B_{2d_{n+1}}}_{2/q,q'}(\tilde{K}_n) \approx d_{n+1}^{N-\frac{1}{q-1}} C^{B_2}_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \approx d_{n+1}^{N-\frac{1}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right),$$

we obtain (2.78) by combining this last inequality with (2.80) and (2.82). □

Proof of Theorem 2.22. Step 1. We first notice that

$$\overline{u}_K \le \sum_{0\le n\le a_K}\sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j}}. \tag{2.83}$$

Actually, since $K = \bigcup_n \bigcup_{a_{n,j}} K_{n,j}$, for any $0 < \epsilon' < \epsilon$, there holds $\overline{K_{\epsilon'}} \subset \bigcup_n \bigcup_{a_{n,j}} K_{n,j\,\epsilon}$. Because a finite sum of positive solutions of (1.1) is a super solution,

$$\overline{u}_{K_{\epsilon'}} \le \sum_{0\le n\le a_K}\sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j\,\epsilon}}. \tag{2.84}$$

Letting successively ϵ' and ϵ go to 0 implies (2.83).

Step 2. Let $n \in \mathbb{N}$. Since $K_{n,j} \subset B_{\delta_n}(a_{n,j})$ and $|x - a_{n,j}| = (d_n + d_{n+1})/2$, we can apply the previous lemmas with $r = \delta_n$ and $\rho = r$. For $n \ge n_N$, there holds $t = 1 \ge (r + 2\rho)^2 = 9/(n + 1)$ and $|x - a_{n,j}| = (\sqrt{n+1} - \sqrt{n})/2 \ge (2 + C_N)(3/\sqrt{n+1})$ (notice that $n_N \ge 8$). Thus

$$u_{K_{n,j}}(0,1) \le Ce^{\left(\sqrt{n}-3/\sqrt{n+1}\right)^2/4} R^{B_{2\delta_n}(a_{n,j})}_{2/q,q'}(K_{n,j}) \le Ce^{3/2}e^{-\frac{n}{4}} R^{B_{2\delta_n}(a_{n,j})}_{2/q,q'}(K_{n,j}). \tag{2.85}$$

Using Lemma 2.23 we obtain, with $d_n = d_n(1) = \sqrt{n+1}$

$$\sum_{n=n_N}^{a_K}\sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) \le C\sum_{n=n_N}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{2.86}$$

Finally, we apply Lemma 2.14 if $1 \le n < n_N$ and get

$$\begin{aligned}\sum_{1}^{n_N-1}\sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) &\le C\sum_{1}^{n_N-1} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right)\\ &\le C'\sum_{1}^{n_N-1} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right).\end{aligned} \tag{2.87}$$

For $n = 0$, we proceed similarly, in splitting K_1 in a finite number of $K_{1,i}$, depending only on the dimension, such that $\operatorname{diam} K_{1,i} < 1/3$. Combining (2.86) and (2.87), we derive

$$\overline{u}_K(0,1) \le C \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \tag{2.88}$$

In order to derive the same result for any $t > 0$, we notice that

$$\overline{u}_K(y,t) = t^{-\frac{1}{q-1}} \overline{u}_{K/\sqrt{t}}(y/\sqrt{t}, 1).$$

Going back to the definition of $d_n = d_n(K,t) = \sqrt{nt} = d_n(K\sqrt{t}, 1)$, we derive from (2.88) and the fact that $a_{K,t} = a_{K\sqrt{t},1}$

$$\overline{u}_K(0,t) \le C t^{-\frac{N}{2}} \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right), \tag{2.89}$$

with $d_n = d_n(t) = \sqrt{t(n+1)}$. This is (2.77) with $x = 0$, and a space translation leads to the final result. □

Proof of Theorem 2.1. Let $m > 0$ and $F_m = F \cap \overline{B}_m$. We denote by $U_{B_m^c}$ the maximal solution of (1.1) in Q_∞ the initial trace of which vanishes on B_m. Such a solution is actually the unique solution of (2.1) which satisfies

$$\lim_{t \to 0} u(x,t) = \infty$$

uniformly on $B_{m'}^c$, for any $m' > m$: this can be easily proved by noticing that

$$U_{B_m^c\, \ell}(y,t) = \ell^{\frac{1}{q-1}} U_{B_m^c}(\sqrt{\ell} y, \ell t) = U_{B^c_{m/\sqrt{\ell}}}(y,t).$$

Furthermore

$$\lim_{m \to \infty} U_{B_m^c}(y,t) = \lim_{m \to \infty} m^{-\frac{2}{q-1}} U_{B_1^c}(y/m, t/m^2) = 0$$

uniformly on any compact subset of $\overline{Q}_\infty$. Since $\overline{u}_{F_m} + U_{B_m^c}$ is a super-solution, it is larger that $\overline{u}_F$ and therefore $\overline{u}_{F_m} \uparrow \overline{u}_F$. Because $W_{F_m}(x,t) \le W_F(x,t)$ and $\overline{u}_{F_m} \le C_1 W_{F_m}(x,t)$, the result follows. □

Remark. It is clear that Theorem 2.1 still holds if u is a positive subsolution of (1.1) satisfying the initial trace condition (1.21).

Theorem 2.1 admits the following integral expression.

Theorem 2.24 *Assume $q \ge q_c$. Then there exists a positive constant $C_1^* = C^*(N,q,T)$ such that, for any closed subset F of $\mathbb{R}^N$, there holds*

$$\overline{u}_F(x,t) \le \frac{C_1^*}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'} \left(\frac{1}{s} F \cap B_1(x) \right) s \, ds, \tag{2.90}$$

where $a_t = \min\{n : F \subset B_{\sqrt{n+1)t}}(x)\}$.

Proof. We first use

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \le C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right),$$

and we denote

$$\Phi(s) = C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \qquad \forall s > 0. \tag{2.91}$$

Step 1. The following inequality holds

$$c_1\Phi(\alpha s) \le \Phi(s) \le c_2\Phi(\beta s) \qquad \forall s > 0, \quad \forall 1/2 \le \alpha \le 1 \le \beta \le 2, \tag{2.92}$$

for some positive constants c_1, c_2 depending on N and q. See [1] and [32]. If $\beta \in [1,2]$,

$$\Phi(\beta s) = C_{2/q,q'}\left(\frac{1}{\beta}\left(\frac{F}{s} \cap B_\beta\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\beta\right) \ge c_1\Phi(s).$$

If $\alpha \in [1/2,1]$,

$$\Phi(\alpha s) = C_{2/q,q'}\left(\frac{1}{\alpha}\left(\frac{F}{s} \cap B_\alpha\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\alpha\right) \le c_2\Phi(s).$$

Step 2. By (2.92)

$$C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \le c_2 C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \qquad \forall\, s \in [d_{n+1}, d_{n+2}],$$

and $n \le a_t$. Then

$$c_2\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\,ds$$

$$\ge C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right)\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} s\,ds.$$

Using the fact that $N - \frac{2}{q-1} \ge 0$, we get,

$$\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} s\,ds \ge e^{-\frac{n+2}{4}} d_{n+1}^{N-\frac{2}{q-1}+1}(d_{n+2} - d_{n+1}) \tag{2.93}$$

$$\ge \frac{t}{4e^2} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}}. \tag{2.94}$$

Thus

$$\overline{u}_F(x,t) \le \frac{C}{t^{1+\frac{N}{2}}}\int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{1}{s}F \cap B_1\right) s\,ds, \tag{2.95}$$

which ends the proof. □

3 Estimate from below

If $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N)$, we denote by $u_\mu = u_{\mu,0}$ the solution of

$$\begin{cases} \partial_t u_\mu - \Delta u_\mu + u_\mu^q = 0 & \text{in } Q_T, \\ u_\mu(.,0) = \mu & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

The maximal σ-moderate solution of (1.1) which has an initial trace vanishing outside a closed set F is defined by

$$\underline{u}_F = \sup\left\{u_\mu : \mu \in \mathfrak{M}^q_+(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N),\ \mu(F^c) = 0\right\}. \tag{3.2}$$

The main result of this section is the next one

Theorem 3.1 *Assume $q \geq q_c$. There exists a constant $C_2 = C_2(N,q,T) > 0$ such that, for any closed subset $F \subset \mathbb{R}^N$, there holds*

$$\underline{u}_F(x,t) \geq C_2 W_F(x,t) \qquad \forall (x,t) \in Q_T. \tag{3.3}$$

We first assume that F is compact, and we denote it by K. The first observation is that if $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N)$, $u_\mu \in L^q(Q_T)$ (see lemma below) and $0 \leq u_\mu \leq \mathbb{H}[\mu] := \mathbb{H}_\mu$. Therefore

$$u_\mu \geq \mathbb{H}[\mu] - \mathbb{G}\left[\mathbb{H}[\mu]^q\right], \tag{3.4}$$

where $\mathbb{G}$ is the parabolic Green potential in Q_T defined by

$$\mathbb{G}[f](t) = \int_0^t \mathbb{H}[f(s)](t-s)ds = \int_0^t \int_{\mathbb{R}^N} H(.,y,t-s)f(y,s)dyds.$$

The main idea of the proof is as follows. For any $(x,t) \in Q_T$, construct a measure $\mu = \mu(x,t) \in \mathfrak{M}^q_+(\mathbb{R}^N)$ such that there holds

$$\mathbb{H}[\mu](x,t) \geq C W_K(x,t) \qquad \forall (x,t) \in Q_T, \tag{3.5}$$

and

$$\mathbb{G}\left(\mathbb{H}[\mu]\right)^q \leq C\, \mathbb{H}[\mu] \quad \text{in } Q_T, \tag{3.6}$$

with constants C depends only on N, q, and T. Then replace μ by $\mu_\epsilon = \epsilon\mu$ with $\epsilon = (2C)^{-\frac{1}{q-1}}$ in order to derive

$$u_{\mu_\epsilon} \geq 2^{-1}\mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \tag{3.7}$$

From this follows

$$\underline{u}_K \geq 2^{-1}\mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \tag{3.8}$$

and the proof of Theorem 3.1 with $C_2 = 2^{-1}C$. In the following sections we describe the construction of measures $\mu(x,t)$ satisfying (3.5) and (3.6).

3.1 Estimate from below of the solution of the heat equation

The purely spatial slicing used is the trace on $\mathbb{R}^N \times \{0\}$ of an *extended slicing* in Q_T which is constructed as follows: if K is a compact subset of $\mathbb{R}^N$, $m = (x,t)$, we define d_K, λ, d_n and a_t as in Section 2.3. Let $\alpha \in (0,1)$ to be fixed later on, we define $\mathcal{T}_n$ for $n \in \mathbb{Z}$ by

$$\mathcal{T}_n = \begin{cases} \mathcal{B}^2_{\sqrt{t(n+1)}}(m) \setminus \mathcal{B}^2_{\sqrt{tn}}(m) & \text{if } n \geq 1, \\ \mathcal{B}^2_{\alpha^{-n}\sqrt{t}}(m) \setminus \mathcal{B}^2_{\alpha^{1-n}\sqrt{t}}(m) & \text{if } n \leq 0, \end{cases}$$

and put

$$\mathcal{T}_n^* = \mathcal{T}_n \cap \{s : 0 \leq s \leq t\}, \quad \text{for } n \in \mathbb{Z}.$$

We recall that for $n \in \mathbb{N}_*$,

$$\mathcal{Q}_n = K \cap \mathcal{B}^2_{\sqrt{t(n+1)}}(m) = K \cap B_{d_n}(x)$$

and

$$K_n = K \cap \mathcal{T}_{n+1} = K \cap \left(B_{d_{n+1}}(x) \setminus B_{d_n}(x)\right).$$

Let $\nu_n \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap W^{-2/q,q}(\mathbb{R}^N)$ be the *q-capacitary measure* of the set K_n/d_{n+1}. See [1, Sec. 2.2]. Such a measure has support in K_n/d_{n+1} and

$$\nu_n(K_n/d_{n+1}) = C_{2/q,q'}(K_n/d_{n+1}) \text{ and } \|\nu_n\|_{W^{-2/q,q'}(\mathbb{R}^N)} = \left(C_{2/q,q'}(K_n/d_{n+1})\right)^{1/q}. \tag{3.9}$$

We define μ_n as follows

$$\mu_n(A) = d_{n+1}^{N-\frac{2}{q-1}} \nu_n(A/d_{n+1}) \qquad \forall A \subset K_n,\ A \text{ Borel}, \tag{3.10}$$

and set

$$\mu_{t,K} = \sum_{n=0}^{a_t} \mu_n,$$

and

$$\mathbb{H}_{\mu_{t,K}} = \sum_{n=0}^{a_t} \mathbb{H}_{\mu_n}. \tag{3.11}$$

Proposition 3.2 *Let $q \geq q_c$, then there holds*

$$\mathbb{H}_{\mu_{t,K}}(x,t) \geq \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{n=0}^{a_t} e^{-\frac{n+1}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{3.12}$$

in $\mathbb{R}^N \times (0,T)$.

Proof. Since

$$\mathbb{H}_{\mu_n}(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n, \tag{3.13}$$

and

$$y \in K_n \Longrightarrow |x-y| \leq d_{n+1},$$

(3.12) follows because of (3.10) and (3.11). □

3.2 Estimate from above of the nonlinear term

We write (3.4) under the form

$$\begin{aligned} u_\mu(x,t) &\geq \sum_{n\in\mathbb{Z}} \mathbb{H}_{\mu_n}(x,t) - \int_0^t \int_{\mathbb{R}^N} H(x,y,t-s) \left[\sum_{n\in A_K} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\ &= I_1 - I_2. \end{aligned} \tag{3.14}$$

since $\mu_n = 0$ if $n \notin A_K = \mathbb{N} \cap [1, a_t]$, and

$$\begin{aligned} I_2 &= \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \int_{\mathbb{R}^N} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n\in A_K} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\ &= \frac{1}{(4\pi)^{\frac{N}{2}}} (J_\ell + J'_\ell), \end{aligned} \tag{3.15}$$

for some $\ell \in \mathbb{N}^*$ to be fixed later on, where

$$J_\ell = \sum_{p\in\mathbb{Z}} \int\int_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n<p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds,$$

and

$$J'_\ell = \sum_{p\in\mathbb{Z}} \int\int_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n\geq p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds.$$

The next estimate will be used several times in the sequel.

Lemma 3.3 *Let $0 < a < b$ and $t > 0$, then,*

$$\max\left\{\sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}} : 0 \leq \sigma \leq t,\ at \leq \rho^2 + \sigma \leq bt\right\} = e^{\frac{1}{4}} \begin{cases} t^{-\frac{N}{2}} e^{-\frac{a}{4}} & \text{if } \dfrac{a}{2N} > 1, \\ \left(\dfrac{2N}{at}\right)^{\frac{N}{2}} e^{-\frac{N}{2}} & \text{if } \dfrac{a}{2N} \leq 1. \end{cases}$$

Proof. Set

$$\mathcal{J}(\rho,\sigma) = \sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}}$$

and

$$\mathcal{K}_{a,b,t} = \left\{(\rho,\sigma) \in [0,\infty) \times (0,t] :\ at \leq \rho^2 + \sigma \leq bt\right\}.$$

We first notice that, for fixed σ, the maximum of $\mathcal{J}(.,\sigma)$ is achieved for ρ minimal. If $\sigma \in [at, bt]$ the minimal value of ρ is 0, while if $\sigma \in (0, at)$, the minimum of ρ is $\sqrt{at-s}$.

- Assume first $a \geq 1$, then $\mathcal{J}(\sqrt{at-\sigma}, \sigma) = e^{\frac{1}{4}} \sigma^{-\frac{N}{4}} e^{-\frac{at}{4\sigma}}$. Thus if $1 \leq a/2N$, the minimal value of $\mathcal{J}(\sqrt{at-\sigma}, \sigma)$ is $e^{\frac{1-2N}{4}} \left(\frac{2N}{at}\right)^{\frac{N}{2}}$, while if $a/2N < 1 \leq a$, the minimum is $e^{\frac{1}{4}} t^{-\frac{N}{2}} e^{-\frac{a}{4}}$.

- Assume now $a \le 1$. Then

$$\begin{aligned}\max\{\mathcal{J}(\rho,\sigma):\ (\rho,\sigma)\in\mathcal{K}_{a,b,t}\} &= \max\left\{\max_{\sigma\in(at,t]}\mathcal{J}(0,\sigma),\ \max_{\sigma\in(0,at]}\mathcal{J}(\sqrt{at-\sigma},\sigma)\right\}\\ &= \max\left\{(at)^{-\frac{N}{2}},\ e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}\right\}\\ &= e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}.\end{aligned}$$

Combining these two estimates, we derive the result. □

Remark. The following variant of Lemma 3.3 will be useful in the sequel: *For any $\theta \ge 1/2N$ there holds*

$$\max\{\mathcal{J}(\rho,\sigma):\ (\rho,\sigma)\in\mathcal{K}(a,b,t)\} \le e^{\frac{1}{4}}\left(\frac{2N\theta}{t}\right)^{\frac{N}{2}} e^{-\frac{a}{4}} \quad \text{if } \theta a \ge 1. \tag{3.16}$$

Lemma 3.4 *There exists a positive constant $C = C(N,\ell,q)$ such that*

$$J_\ell \le Ct^{-\frac{N}{2}}\sum_{n=1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-(1+(n-\ell)_+)/4}\, C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.17}$$

Proof. The set of the p's for the summation in J_ℓ is reduced to $\mathbb{Z}\cap[-\ell+2,\infty)$, thus we write

$$J_\ell = J_{1,\ell} + J_{2,\ell}$$

where

$$J_{1,\ell} = \sum_{p=2-\ell}^{0}\iint_{\mathcal{T}_p^*}(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}}\left[\sum_{n<p+\ell}\mathbb{H}_{\mu_n}(y,s)\right]^q$$

and

$$J_{2,\ell} = \sum_{p=1}^{\infty}\iint_{\mathcal{T}_p^*}(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}}\left[\sum_{n<p+\ell}\mathbb{H}_{\mu_n}(y,s)\right]^q.$$

If $p = 2-\ell,\dots,0$,

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow t\alpha^{2-2p} \le |x-y|^2 + t - s \le t\alpha^{-2p},$$

and, if $p \ge 1$

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow pt \le |x-y|^2 + t - s \le (p+1)t.$$

By Lemma 3.3 and (3.16), there exists $C = C(N,\ell,\alpha) > 0$ such that

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y,s)\in\mathcal{T}_p^*\right\} \le Ct^{-\frac{N}{2}} e^{-\alpha^{2-2p}/4}, \tag{3.18}$$

if $p = 2-\ell,\dots,0$, and

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y,s)\in\mathcal{T}_p^*\right\} \le Ct^{-\frac{N}{2}} e^{-p/4}, \tag{3.19}$$

if $p \geq 1$. When $p = 2-\ell, \dots, 0$

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \leq C \sum_{1}^{p+\ell-1} \mathbb{H}^q_{\mu_n}(y,s), \tag{3.20}$$

for some $C = C(\ell, q) > 0$, thus

$$\begin{aligned} J_{1,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=2-\ell}^{0} e^{-\frac{\alpha^{2-2p}}{4}} \sum_{n=1}^{p+\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \sum_{p=n-\ell+1}^{0} e^{-\frac{\alpha^{2-2p}}{4}} \\ &\leq Ct^{-\frac{N}{2}} e^{-\frac{\alpha^{2\ell-2}}{4}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)}. \end{aligned} \tag{3.21}$$

If the set of p's is not upper bounded, we introduce some parameter $\delta > 0$ to be made precise later on. Then

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \leq \left[\sum_{1}^{p+\ell-1} e^{\delta q' \frac{n}{4}}\right]^{q/q'} \sum_{1}^{p+\ell-1} e^{-\frac{\delta q n}{4}} \mathbb{H}^q_{\mu_n}(y,s), \tag{3.22}$$

with $q' = q/(q-1)$. If, by convention $\mu_n = 0$ whenever $n > a_t$, we obtain, for some $C > 0$ which depends also on δ,

$$\begin{aligned} J_{2,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \sum_{n=1}^{p+\ell-1} e^{-\frac{\delta q n}{4}} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} e^{-\frac{\delta q n}{4}} \sum_{p=(n-\ell+1)\vee 1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{1+(n-\ell)_+}{4}} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)}. \end{aligned} \tag{3.23}$$

Notice that we choose δ such that $\delta \ell q < 1$. Combining (3.21) and (3.23), we derive (3.17) from Lemma 2.11, (3.9) and (3.10). □

The set of indices p for which the μ_n terms are not zero in J'_ℓ is $\mathbb{Z} \cap (-\infty, a_t - \ell]$. We write

$$J'_\ell = J'_{1,\ell} + J'_{2,\ell},$$

where

$$J'_{1,\ell} = \sum_{p=-\infty}^{0} \iint_{\mathcal{T}^*_p} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=1\vee p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds,$$

and

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s) \right]^q dyds.$$

Lemma 3.5 *There exists a constant $C = C(N,q,\ell) > 0$ such that*

$$J'_{1,\ell} \le Ct^{1-\frac{Nq}{2}} \sum_{n=0}^{a_t} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} d_{n+1}^{Nq-2q'} C_{2/q,q'}^q \left(\frac{K_n}{d_{n+1}} \right), \tag{3.24}$$

where $\beta_0 = (q-1)/4$ and $h = 2q(q+1)/(q-1)^2$.

Proof. Since

$$(y,s) \in \mathcal{T}_p^*, \text{ and } (z,0) \in K_n \Longrightarrow |y-z| \ge (\sqrt{n} - \alpha^{-p})\sqrt{t}, \tag{3.25}$$

there holds

$$\mathbb{H}_{\mu_n}(y,s) \le (4\pi s)^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2 t}{4s}} \mu_n(K_n) \le Ct^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n(K_n),$$

by Lemma 3.3. Let $\{\epsilon_n\}$ be a sequence of positive numbers such that

$$A_\epsilon = \sum_{n=1}^{\infty} \epsilon_n^{q'} < \infty,$$

then

$$\begin{aligned}
J'_{1,\ell} &\le CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{p=-\infty}^{0} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \sum_{n=1\vee(p+\ell)}^{\infty} \epsilon_n^{-q} e^{-q\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n^q(K_n) ds\, dy \\
&\le CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) \sum_{-\infty}^{p=0\wedge(n-\ell)} e^{-\frac{q(\sqrt{n}-\alpha^{-p})^2}{4}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds\, dy \\
&\le CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-\frac{q(\sqrt{n}-1)^2}{4}} \iint_{\{\cup_{p\le 0}\mathcal{T}_p^*\}} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds\, dy \\
&\le CA_\epsilon^{q/q'} t^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-\frac{q(\sqrt{n}-1)^2}{4}}.
\end{aligned} \tag{3.26}$$

Set $h = 2q(q+1)/(q-1)^2$ and $Q = (1+q)/2$, then $q(\sqrt{n}-1)^2 \ge Q(n-h)_+$ for any $n \ge 1$. If we choose $\epsilon_n = e^{-\frac{(q-1)(n-h)_+}{16q}}$, there holds $\epsilon_n^{-q} e^{-\frac{q(\sqrt{n}-1)^2}{4}} \le e^{-\frac{(q+3)(n-h)_+}{16}}$. Finally

$$J'_{1,\ell} \le Ct^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} \mu_n^q(K_n),$$

with $\beta_0 = (q-1)/4$, which yields to (3.24) by the choice of the μ_n. □

In order to make easier the obtention of the estimate of the term $J'_{2,\ell}$, we first give the proof in dimension 1.

Lemma 3.6 *Assume $N = 1$ and ℓ is an integer larger than 1. There exists a positive constant $C = C(q, \ell) > 0$ such that*

$$J'_{2,\ell} \le Ct^{-1/2} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \tag{3.27}$$

Proof. If $(y, s) \in \mathcal{T}_p^*$ and $z \in K_n$ $(p \ge 1,\ n \ge p = \ell)$, there holds $|x - y| \ge \sqrt{t}\sqrt{p}$ and $|y - z| \ge \sqrt{t}(\sqrt{n} - \sqrt{p+1})$. Therefore

$$J'_{2,\ell} \le C\sqrt{t} \sum_{p=1}^{a_t-\ell} \frac{1}{\sqrt{p}} \int_0^t e^{-\frac{pt}{4(t-s)}} \left(\sum_{n=p+\ell}^{a_t} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q.$$

If $\epsilon \in (0, q)$ is some positive parameter which will be made more precise later on, there holds

$$\left(\sum_{n=p+\ell}^{a_t} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q \le \left(\sum_{n=p+\ell}^{a_t} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \right)^{q/q'} \sum_{n=p+\ell}^{a_t} s^{-\frac{q}{2}} e^{-(q-\epsilon)\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n^q(K_n),$$

by Hölder's inequality. By comparison between series and integrals and using Gauss integral

$$\begin{aligned}
\sum_{n=p+\ell}^{a_t} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} &\le \int_{p+\ell}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\
&= 2\int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} (x + \sqrt{p+1}) dx \\
&\le \frac{4s}{\epsilon q' t} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{4s}} + 2\sqrt{p+1} \int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} dx \\
&\le C\sqrt{\frac{(p+1)s}{t}} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{2s}} \\
&\le C\sqrt{\frac{(p+1)s}{t}}.
\end{aligned}$$

If we set $q_\epsilon = q - \epsilon$, then

$$J'_{2,\ell} \le C\epsilon^{-q'/q} t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{\infty} \mu_n^q(K_n) \sum_{p=1}^{n-\ell} p^{\frac{q-2}{2}} \int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds.$$

where $C = C(\epsilon, q) > 0$. Since

$$\begin{aligned}
&\int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \\
&\qquad = \int_0^1 (1-s)^{-1/2} s^{-1/2} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}} ds,
\end{aligned}$$

we can apply Lemma A.1 with $a = 1/2$, $b = 1/2$, $A = \sqrt{p}$ and $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$. In this range of indices $B \geq \sqrt{q_\epsilon}(\sqrt{p+\ell} - \sqrt{p+1}) \geq \frac{\sqrt{q_\epsilon}(\ell-1)}{\sqrt{p}}$, thus $\kappa = \sqrt{q_\epsilon}(\ell-1)$ and

$$\sqrt{\frac{A}{A+B}}\sqrt{\frac{B}{A+B}} \leq p^{\frac{1}{4}} n^{-1/2} (\sqrt{n} - \sqrt{p})^{1/2}.$$

Therefore

$$\int_0^t (t-s)^{-1/2} s^{-\frac{q}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \leq \frac{C p^{\frac{1}{4}} (\sqrt{n} - \sqrt{p})^{1/2}}{\sqrt{n}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \quad (3.28)$$

which implies

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} \frac{\mu_n^q(K_n)}{\sqrt{n}} \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n} - \sqrt{p})^{1/2} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \quad (3.29)$$

where C depends of ϵ, q and ℓ. By Lemma A.2

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{q-3}{2}} e^{-\frac{n}{4}} \mu_n^q(K_n) \quad (3.30)$$

Because $\mu_n(K_n) = d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right)$ (remember $N = 1$) and $\operatorname{diam} \frac{K_n}{d_{n+1}} \leq n^{-1}$, there holds

$$\mu_n^q(K_n) \leq C \left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{q-3} \mu_n(K_n) = C \left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{q-3} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}(K_n/d_{n+1}) \quad (3.31)$$

and inequality (3.27) follows. □

Next we give the general proof. For this task we will use again the quasi-additivity with separated partitions.

Lemma 3.7 *Assume $N \geq 2$ and ℓ is an integer larger than 1. There exists a positive constant $C_1 = C_1(q, N, \ell) > 0$ such that*

$$J'_{2,\ell} \leq C_1 t^{-\frac{N}{2}} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \quad (3.32)$$

Proof. As in the proof of Theorem 2.22, we know that there exists a finite number J, depending only on the dimension N, of separated sub-partitions $\{\#\Theta_{t,n}^h\}_{h=1}^J$ of the rescaled sets $\tilde{T}_n = \sqrt{\frac{n+1}{t}} T_n$ by the N-dim balls $B_2(\tilde{a}_{n,j})$ where $\tilde{a}_{n,j} = \sqrt{\frac{n+1}{t}} a_{n,j}$, $|a_{n,j}| = \frac{d_{n+1} + d_n}{2}$ and $|a_{n,j} - a_{n,k}| \geq \sqrt{\frac{4t}{n+1}}$. Furthermore $\#\Theta_{t,n}^h \leq C n^{N-1}$. We denote $K_{n,j} = K_n \cap B_{\sqrt{\frac{t}{n+1}}}(a_{n,j})$.

We write $\mu_n = \sum_{h=1}^{J} \mu_n^h$, and accordingly $J'_{2,\ell} = \sum_{h=1}^{J} J'^{\,h}_{2,\ell}$, where $\mu_n^h = \sum_{j\in\Theta_{t,n}^h} \mu_{n,j}$, and $\mu_{n,j}$ are the capacitary measures of $K_{n,j}$ relative to $B_{n,j} = B_{6t/5\sqrt{n}}(a_n, j)$, which means

$$\nu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \quad \text{and} \quad \|\nu_{n,j}\|_{W^{-2/q,q'}(B_{n,j})} = \left(C_{2/q,q'}^{B_{n,j}}(K_{n,j})\right)^{1/q}. \tag{3.33}$$

Thus

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{h=1}^{J} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s)\right] dyds.$$

We denote

$$J'^{\,h}_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s)\right] dyds,$$

and clearly

$$J'_{2,\ell} \le C \sum_{h=1}^{J} J'^{\,h}_{2,\ell}, \tag{3.34}$$

where C depends only on N and q. For integers n and p such that $n \ge \ell + 1$, we set

$$\lambda_{n,j,y} = \inf\{|y-z| : z \in B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})\} = |y - a_{n,j}| - \frac{\sqrt{t}}{\sqrt{n+1}}.$$

Therefore

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} \int_{K_n} e^{-\frac{|y-z|^2}{4s}} d\mu_n^h(z) &= \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} \int_{K_{n,j}} e^{-\frac{|y-z|^2}{4s}} d\mu_{n,j}(z) \\ &\le \left(\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}}\right)^{1/q'} \left(\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} e^{-q\lambda_{n,j,y}^2 \frac{1-\epsilon}{4s}} \mu_{n,j}^q(K_{n,j})\right)^{1/q}\end{aligned}$$

where $\epsilon > 0$ will be made precise later on.

Step 1 We claim that

$$\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \le C\sqrt{\frac{ps}{t}} \tag{3.35}$$

where C depends on ϵ, q and N. If y is fixed in T_p, we denote by z_y the point of T_n which solves $|y - z_y| = \operatorname{dist}(y, T_n)$. Thus

$$\sqrt{t}(\sqrt{n} - \sqrt{p+1}) \le |y - z_y| \le t(\sqrt{n} - \sqrt{p}).$$

Let $Y = y\sqrt{t(p+1)}/|y|$. On the axis $\overrightarrow{0Y}$ we set $\mathbf{e} = Y/|Y|$, consider the points $b_k = (k\sqrt{t}/\sqrt{n})\mathbf{e}$ where $-n \le k \le n$ and denote by $G_{n,k}$ the spherical shell obtained by intersecting the spherical shell T_n with the domain $H_{n,k}$ which is the set of points in $\mathbb{R}^N$ limited by the hyperplanes orthogonal to $\overrightarrow{0Y}$ going through $((k+1)\sqrt{t}/\sqrt{n})\mathbf{e}$ and $((k-1)\sqrt{t}/\sqrt{n})\mathbf{e}$. The number of points $a_{n,j} \in G_{n,k}$ is smaller than $C(n+1-|k|)^{N-2}$, where C depends only on N, and we denote by $\Lambda_{n,k}$ the set of $j \in \Theta_{t,n}$ such that $a_{n,j} \in G_{n,k}$. Furthermore, if $a_{n,j} \in G_{n,k}$ elementary geometric considerations (Pythagora's theorem) imply that $\lambda^2_{n,j,y}$ is greater than $t(n+p+1-2k\sqrt{p+1}/\sqrt{n})$. Therefore

$$\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\epsilon q' \frac{\lambda^2_{n,j,y}}{4s}} \le C \sum_{n=p+\ell}^{a_t} \sum_{k=-n}^{n} (n+1-|k|)^{N-2} e^{-\frac{\epsilon q'(n+p+1-2k\sqrt{p+1}/)t}{4s\sqrt{n}}}. \tag{3.36}$$

Case $N = 2$. Summing a geometric series and using the inequality $\frac{e^u}{e^u-1} \le 1 + u^{-1}$ for $u > 0$, we obtain

$$\begin{aligned}\sum_{k=-n}^{n} e^{\frac{\epsilon q' (k\sqrt{p+1})t}{2s\sqrt{n}}} &\le e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \frac{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}}{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}-1} \\ &\le e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \left(1 + \frac{2s\sqrt{n}}{\epsilon q' t\sqrt{p+1}}\right).\end{aligned} \tag{3.37}$$

Thus, by comparison between series and integrals,

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q' \lambda^2_{n,j,y}}{4s}} &\le C \sum_{n=p+\ell}^{a_t} \left(1 + \frac{s\sqrt{n}}{t\sqrt{p}}\right) e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2}{4s}} \\ &\le C \int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\ &\qquad + \frac{Cs}{t\sqrt{p}} \int_{p+1}^{\infty} \sqrt{x} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx.\end{aligned} \tag{3.38}$$

Next

$$\begin{aligned}\int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}} y dy \\ &= 2\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} y dy + 2\sqrt{p+1}\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &= \frac{2s}{t}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} z dz + 2\sqrt{\frac{(p+1)s}{t}}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz,\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}\int_{p+1}^{\infty} \sqrt{x} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}} y^2 dy \\ &= 2\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} (y+\sqrt{p+1})^2 dy \\ &\le 4\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} y^2 dy + 4(p+1)\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\le 4\left(\frac{s}{t}\right)^{3/2} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} z^2 dz + 4(p+1)\sqrt{\frac{s}{t}}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz\end{aligned} \tag{3.40}$$

Jointly with (3.38), these inequalities imply

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q'\lambda_{n,j,y}^2}{4s}} \le C\sqrt{\frac{ps}{t}}. \tag{3.41}$$

Case $N>2$. Because the value of the right-hand side of (3.36) is an increasing value of N, it is sufficient to prove (3.35) when N is even, say $(N-2)/2=d\in\mathbb{N}_*$. There holds

$$\sum_{k=-n}^{n}(n+1-|k|)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}} \le 2\sum_{k=0}^{n}(n+1-k)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}}. \tag{3.42}$$

We set

$$\alpha=\epsilon q'\frac{t\sqrt{p+1}}{2s\sqrt{n}} \quad \text{and} \quad I_d=\sum_{k=0}^{n}(n+1-k)^d e^{k\alpha}.$$

Since

$$e^{k\alpha}=\frac{e^{(k+1)\alpha}-e^{k\alpha}}{e^\alpha-1},$$

we use Abel's transform to obtain

$$\begin{aligned} I_d &= \frac{1}{e^\alpha-1}\left(e^{(n+1)\alpha}-(n+1)^d+\sum_{k=1}^{n}\left((n+2-k)^d-(n+1-k)^d\right)e^{k\alpha}\right)\\ &\le \frac{1}{e^\alpha-1}\left((1-d)e^{(n+1)\alpha}-(n+1)^d+de^\alpha\sum_{k=1}^{n}\left((n+1-k)^{d-1}\right)e^{k\alpha}\right). \end{aligned}$$

Therefore the following induction holds

$$I_d\le\frac{de^\alpha}{e^\alpha-1}I_{d-1}. \tag{3.43}$$

In (3.37), we have already used the fact that

$$\frac{de^\alpha}{e^\alpha-1}\le C\left(1+\frac{s\sqrt{n}}{t\sqrt{p}}\right),$$

and

$$I_d\le C\left(1+\left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right)I_0.$$

Thus (3.38) is replaced by

$$\begin{aligned} \sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q'\lambda_{n,j,y}^2}{4s}} &\le C\sum_{n=p+\ell}^{a_t}\left(1+\left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right)e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2t}{4s}}\\ &\le C\int_{p+1}^{\infty}e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2t}{4s}}dx\\ &\qquad+\left(\frac{Cs}{t\sqrt{p}}\right)^{d+1}\int_{p+1}^{\infty}x^{(d+1)/2}e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2t}{4s}}dx. \end{aligned} \tag{3.44}$$

The first integral on the right-hand side has already been estimated in (3.39), for the second integral, there holds

$$\begin{aligned}\int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= \int_0^{\infty} (y+\sqrt{p+1}\,)^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dx \\ &\le C\int_0^{\infty} y^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dy + Cp^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\le C\left(\frac{s}{t}\right)^{2+\frac{d}{2}} \int_0^{\infty} z^{(d+1)/2} e^{-\frac{\epsilon q' z^2}{4}} dz \\ &\qquad + C\left(\frac{s}{t}\right)^{3/2} p^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz.\end{aligned} \tag{3.45}$$

Combining (3.39), (3.44) and (3.45), we derive (3.35).

Step 2. Since $\mathcal{T}_p^* \subset \Gamma_p \times [0,t]$ where $\Gamma_p = B_{d_{p+1}}(x) \setminus B_{d_{p-1}}(x)$, $(y,s) \in \mathcal{T}_p^*$ implies that $|x-y|^2 \ge (p-1)t$, thus $J'^h_{2,\ell}$ satisfies

$$\begin{aligned}J'^h_{2,\ell} &\le Ct^{\frac{1-q}{2}} \sum_{p=1}^{\infty} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} \\ &\qquad \times \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta^h_{t,n}} e^{-\frac{q\lambda^2_{n,j,y}(1-\epsilon)}{4s}} \mu^q_{n,j}(K_{n,j}) ds dy \\ &\le Ct^{\frac{1-q}{2}} \sum_{n=\ell+1}^{a_t} \sum_{j\in\Theta^h_{t,n}} \mu^q_{n,j}(K_{n,j}) \\ &\quad \times \sum_{p=1}^{n-\ell} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-|x-y|^2/4(t-s)} e^{-\frac{q\lambda^2_{n,j,y}(1-\epsilon)}{4s}} ds dy\end{aligned} \tag{3.46}$$

and the constant C depends on N, q and ϵ. Next we set $q_\epsilon = (1-\epsilon)q$. Writting

$$|y-a_{n,j}|^2 = |x-y|^2 + |x-a_{n,j}|^2 - 2\langle y-x, a_{n,j}-x\rangle \ge pt + |x-a_{n,j}|^2 - 2\langle y-x, a_{n,j}-x\rangle,$$

we get

$$\int_{\Gamma_p} e^{-\frac{q_\epsilon \left|y-a_{n,j}\right|^2}{4s}} dy = e^{-\frac{q_\epsilon \left|x-a_{n,j}\right|^2}{4s}} \int_{\sqrt{tp}}^{\sqrt{t(p+1)}} e^{-\frac{q_\epsilon r^2}{4s}} \int_{|x-y|=r} e^{2q_\epsilon \langle y-x, a_{n,j}-x\rangle/4s} dS_r(y) dr.$$

For estimating the value of the spherical integral, we can assume that $a_{n,j}-x = (0,\dots,0,|a_{n,j}-x|)$, $y=(y_1,\dots,y_N)$ and, using spherical coordinates with center at x, that the unit sphere has the representation $S^{N-1} = \{(\sin\phi.\sigma, \cos\phi) \in \mathbb{R}^{N-1}\times\mathbb{R} : \sigma \in S^{N-2},\ \phi\in[0,\pi]\}$. With this representation, $dS_r = r^{N-1}\sin^{N-2}\phi\, d\phi\, d\sigma$ and $\langle y-x, a_{n,j}-x\rangle = |a_{n,j}-x|\,|y-x|\cos\phi$. Therefore

$$\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) = r^{N-1}\left|S^{N-2}\right| \int_0^{\pi} e^{2q_\epsilon \frac{\left|a_{n,j}-x\right| r\cos\phi}{4s}} \sin^{N-2}\phi\, d\phi.$$

By Lemma A.3

$$\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) \le C \frac{r^{N-1} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}}{\left(1+\frac{r|a_{n,j}-x|}{s}\right)^{\frac{N-1}{2}}} \le C s^{\frac{N-1}{2}} \left(\frac{r}{|a_{n,j}-x|}\right)^{\frac{N-1}{2}} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}. \tag{3.47}$$

Therefore

$$\int_{\Gamma_p} e^{-q_\epsilon \frac{|y-a_{n,j}|^2}{4s}} dy \le C t^{\frac{N-1}{4}} p^{\frac{N-3}{4}} \frac{s^{\frac{N-1}{2}} e^{-q_\epsilon \frac{(|a_{n,j}-x|-\sqrt{t(p+1)})^2}{4s}}}{|a_{n,j}-x|^{\frac{N-1}{2}}}, \tag{3.48}$$

and, since $|a_{n,j}-x| \ge \sqrt{tn}$,

$$\begin{aligned}
&\int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon \frac{\lambda_{n,j,y}^2}{4s}} dy\, ds \\
&\qquad \le C \frac{\sqrt{t} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^t (t-s)^{-\frac{N}{2}} s^{-\frac{(q-1)(N-1)+1}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{tn}-\sqrt{t(p+1)})^2}{4s}} ds \\
&\qquad \le C \frac{t^{\frac{1-q(N-1)}{2}} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^1 (1-s)^{-\frac{N}{2}} s^{-\frac{(q-1)(N-1)+1}{2}} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}}.
\end{aligned} \tag{3.49}$$

We apply Lemma A.1, with $A=\sqrt{p}$, $B=\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1})$, $b=\frac{(q-1)(N-1)+1}{2}$, $a=\frac{N}{2}$ and $\kappa=\sqrt{q_\epsilon}(\ell-1)/8$ as in the case $N=1$, and noticing that, for these specific values,

$$\begin{aligned}
A^{1-a}B^{1-b}(A+B)^{a+b-2} &= p^{\frac{2-N}{4}} (\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{\frac{1-(q-1)(N-1)}{2}} \\
&\qquad \times (\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{\frac{(q-1)(N-1)+N-3}{2}} \\
&\le C \left(\frac{n}{p}\right)^{\frac{N}{4}-1/2} \left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}},
\end{aligned}$$

where C depends on N, q and κ. Therefore

$$\begin{aligned}
&\int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon |y-z|^2/4s} dy\, ds \\
&\qquad \le C \frac{t^{(1-q(N-1))/2} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \left(\frac{n}{p}\right)^{\frac{N}{4}-1/2} \left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \\
&\qquad \le C t^{\frac{1-q(N-1)}{2}} p^{-\frac{1}{4}} n^{\frac{(q-1)(N-1)-2}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned} \tag{3.50}$$

We derive from (3.46), (3.50),

$$\begin{aligned}
&J'^{\,h}_{2,\ell} \le C t^{1-\frac{Nq}{2}} \\
&\times \sum_{n=\ell+1}^{a_t} \sum_{j \in \Theta^h_{t,n}} n^{\frac{(q-1)(N-1)-2}{4}} \mu^q_{n,j}(K_{n,j}) \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned} \tag{3.51}$$

By Lemma A.2 with $\alpha = \dfrac{2q-3}{4}$, $\beta = \frac{1-(q-1)(N-1)}{2}$, $\delta = \frac{1}{4}$ and $\gamma = q_\epsilon$, we obtain

$$\sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \leq C n^{\frac{N(q-1)+q-3}{4}} e^{-\frac{n}{4}}, \tag{3.52}$$

thus

$$J'^{\,h}_{2,\ell} \leq C t^{1-\frac{Nq}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{N(q-1)}{2}-1} e^{-\frac{n}{4}} \sum_{j\in\Theta^h_{t,n}} \mu^q_{n,j}(K_{n,j}). \tag{3.53}$$

Because

$$\mu_{n,j}(K_{n,j}) = C^{B_{n,j}}_{2/q,q'}(K_{n,j}),$$

we use the rescaling procedure as in the proof of Lemma 2.23, except that the scale factor is $\sqrt{(n+1)t}$ instead of $\sqrt{n+1}$ so that the sets $\tilde{T}_n$, $\tilde{K}_n$, $\tilde{\mathcal{Q}}_n$ and $\tilde{K}_n$ remains unchanged Using again the quasi-additivity and the fact that $J'_{2,\ell} = \sum_{h=1}^{J} J'^{\,h}_{2,\ell}$, we deduce

$$J_{2,\ell} \leq C' t^{-\frac{N}{2}} \sum_{n=\ell+1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{3.54}$$

which implies (3.32). □

The proof of Theorem 3.1 follows from the previous estimates on J_1 and J_2. Furthermore the following integral expression holds

Theorem 3.8 *Assume $q \geq q_c$. Then there exists a positive constants C_2^* , depending on N,q and T, such that for any closed set F, there holds*

$$\underline{u}_F(x,t) \geq \frac{C_2^*}{t^{1+\frac{N}{2}}} \int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{s}\cap B_1(x)\right) s\, ds, \tag{3.55}$$

where a_t is the smallest integer j such that $F \subset B_{\sqrt{jt}}(x)$.

Proof. We distinguish according $q = q_c$, or $q > q_c$, and for simplicity we denote $B_r = B_r(x)$ for the various values of r.

Case 1: $q = q_c \Longleftrightarrow N - \frac{2}{q-1} = 0$. Because $F_n = F \cap (B_{d_{n+1}} \setminus B_{d_n})$ there holds

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right) - C_{2/q,q'}\left(\frac{F\cap B_{d_n}}{d_{n+1}}\right),$$

Furthermore, since $d_{n+1} \geq d_n$,

$$C_{2/q,q'}\left(\frac{F\cap B_{d_n}}{d_{n+1}}\right) = C_{2/q,q'}\left(\frac{d_n}{d_{n+1}}\frac{F\cap B_{d_n}}{d_n}\right) \leq C_{2/q,q'}\left(\frac{F}{d_n}\cap B_1\right),$$

thus

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right) - C_{2/q,q'}\left(\frac{F}{d_n}\cap B_1\right),$$

it follows

$$\begin{aligned}\sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right) \\ &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-\frac{1}{4}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \\ &\geq (1-e^{-\frac{1}{4}}) \sum_{n=1}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-\frac{1}{4}} C_{2/q,q'}\left(\frac{F}{\sqrt{t}} \cap B_1\right).\end{aligned}$$

Since, by (2.92),

$$C_{2/q,q'}\left(\frac{F}{s'} \cap B_1\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \geq C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right),$$

for any $s' \in [d_{n+1}, d_{n+2}]$ and $s \in [d_n, d_{n+1}]$, there holds

$$\begin{aligned}t e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) &\geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \int_{d_n}^{d_{n+1}} e^{-s^2/4t} s\, ds \\ &\geq \int_{d_n}^{d_{n+1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\, ds.\end{aligned}$$

This implies

$$W_F(x,t) \geq (1-e^{-\frac{1}{4}}) t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\, ds.$$

Case 2: $q > q_c \Longleftrightarrow N - \frac{2}{q-1} > 0$. In that case it follows from Lemma 2.9 that

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \approx d_{n+1}^{\frac{2}{q-1}-N} C_{2/q,q'}\left(F_n\right).$$

Thus

$$W_F(x,t) \approx t^{-1-\frac{N}{2}} \sum_{n=0}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(F_n\right).$$

Since

$$C_{2/q,q'}\left(F_n\right) \geq C_{2/q,q'}\left(F \cap B_{d_{n+1}}\right) - C_{2/q,q'}\left(F \cap B_{d_n}\right),$$

and again

$$\begin{aligned}t^{-\frac{N}{2}} \sum_{n=0}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(F_n\right) &\geq (1-e^{-\frac{1}{4}}) t^{-\frac{N}{2}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(F \cap B_{d_{n+1}}\right) \\ &\geq (1-e^{-\frac{1}{4}}) t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(F \cap B_s\right) s\, ds.\end{aligned}$$

Because $C_{2/q,q'}\left(F \cap B_s\right) \approx s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(s^{-1}F \cap B_1\right)$, (3.55) follows. □

4 Applications

The first result of this section is the following

Theorem 4.1 *Assume $N \geq 1$ and $q > 1$. Then $\overline{u}_K = \underline{u}_K$.*

Proof. If $1 < q < q_c$, the result is already proved in [28]. The proof in the super-critical case is an adaptation that we recall, for the sake of completeness. By Theorem 2.24 and Theorem 3.8 there exists a positive constant C, depending on N, q and T such that

$$\overline{u}_F(x,t) \leq C\underline{u}_F(x,t) \qquad \forall (x,t) \in Q_T.$$

By convexity $\tilde{u} = \underline{u}_F - \dfrac{1}{2C}(\overline{u}_F - \underline{u}_F)$ is a super-solution, which is smaller than $\underline{u}_F$ if we assume that $\overline{u}_F \neq \underline{u}_F$. If we set $\theta := 1/2 + 1/(2C)$, then $u_\theta = \theta \overline{u}_F$ is a subsolution. Therefore there exists a solution u_1 of (1.1) in Q_∞ such that $u_\theta \leq u_1 \leq \tilde{u} < \underline{u}_F$. If $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N)$ satisfies $\mu(F^c) = 0$, then $u_{\theta\mu}$ is the smallest solution of (1.1) which is above the subsolution θu_μ. Thus $u_{\theta\mu} \leq u_1 < \underline{u}_F$ and finally $\underline{u}_F \leq u_1 < \underline{u}_F$, a contradiction. □

If we combine Theorem 2.24 and Theorem 3.8 we derive the following integral approximation of the parabolic capacitary potential

Proposition 4.2 *Assume $q \geq q_c$. Then there exist two positive constants $C_1^\dagger$, $C_2^\dagger$, depending only on N, q and T such that*

$$\begin{aligned} C_2^\dagger t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'} \left(\frac{F}{s} \cap B_1(x)\right) s\, ds \leq W_F(x,t) \\ \leq C_1^\dagger t^{-(1+\frac{N}{2})} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'} \left(\frac{F}{s} \cap B_1(x)\right) s\, ds \end{aligned} \tag{4.56}$$

for any $(x,t) \in Q_T$.

Definition 4.3 *If F is a closed subset of $\mathbb{R}^N$, we define the $(2/q, q')$-integral parabolic capacitary potential $\mathcal{W}_F$ by*

$$\mathcal{W}_F(x,t) = t^{-1-\frac{N}{2}} \int_0^{D_F(x)} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'} \left(\frac{F}{s} \cap B_1(x)\right) s\, ds \qquad \forall (x,t) \in Q_\infty, \tag{4.57}$$

where $D_F(x) = \max\{|x-y| : y \in F\}$.

An easy computation shows that

$$\begin{aligned} 0 \leq \mathcal{W}_F(x,t) - t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'} \left(\frac{F}{s} \cap B_1(x)\right) s\, ds \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-D_F^2(x)/4t}, \end{aligned} \tag{4.58}$$

and

$$0 \le t^{-(1+\frac{N}{2})} \int_0^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds - \mathcal{W}_F(x,t) \\ \le C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-\frac{D_F^2(x)}{4t}}, \tag{4.59}$$

for some $C = C(N,q) > 0$. Furthermore

$$\mathcal{W}_F(x,t) = t^{-\frac{1}{q-1}} \int_0^{D_F(x)/\sqrt{t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4}} C_{2/q,q'}\left(\frac{F}{s\sqrt{t}} \cap B_1(x)\right) s\, ds. \tag{4.60}$$

The following result gives a sufficient condition in order that $\overline{u}_F$ does not have a strong blow-up at a point x.

Proposition 4.4 *Assume $q \ge q_c$ and F is a closed subset of $\mathbb{R}^N$. If there exists $\gamma \in [0,\infty)$ such that*

$$\lim_{\tau\to 0} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) = \gamma, \tag{4.61}$$

then

$$\lim_{t\to 0} t^{\frac{1}{q-1}} \overline{u}_F(x,t) = C\gamma, \tag{4.62}$$

for some $C = C(N,q) > 0$.

Proof. Clearly, condition (4.61) implies

$$\lim_{t\to 0} C_{2/q,q'}\left(\frac{F}{\sqrt{t}s} \cap B_1(x)\right) = \gamma$$

for any $s > 0$. Then (4.62) follows by Lebesgue's theorem. Notice also that the set of γ is bounded from above by a constant depending on N and q. □

In the next result we give a condition in order that the solution remains bounded at a point x. The proof is similar to the previous one.

Proposition 4.5 *Assume $q \ge q_c$ and F is a closed subset of $\mathbb{R}^N$. If*

$$\limsup_{\tau\to 0} \tau^{-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) < \infty, \tag{4.63}$$

then $\overline{u}_F(x,t)$ remains bounded when $t \to 0$.

Remark. If we assume that f is a convex function on $\mathbb{R}^+$ satisfying

$$c_2 r^q \le f(r) \le c_1 r^q \qquad \forall r \ge 0 \tag{4.64}$$

for some $0 < c_2 \le c_1$ we can construct in the same way as for (1.1) the solutions $\underline{u}_F$ and $\overline{u}_F$ for equation

$$\partial_t u - \Delta u + f(u) = 0 \qquad \text{in } Q_T. \tag{4.65}$$

The bilateral estimate estimate (1.19) is still valid (up to change of the C_i). Since only convexity of f is used in the proof of Theorem 4.1, there still holds $\underline{u}_F = \overline{u}_F$. Similar extensions of Proposition 4.4 and Proposition 4.5 are also clear.

A Appendix

The next estimate is crucial in our study of semilinear parabolic equations.

Lemma A.1 *Let a and b be two real numbers, $a > 0$ and $\kappa > 0$. Then there exists a constant $C = C(a, b, \kappa) > 0$ such that for any $A > 0$, $B > \kappa/A$ there holds*

$$\int_0^1 (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x} dx \le C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \tag{A.1}$$

Proof. We first notice that

$$\max\{e^{-A^2/4(1-x)} e^{-B^2/4x} : 0 \le x \le 1\} = e^{-(A+B)^2/4}, \tag{A.2}$$

and it is achieved for $x_0 = B/(A+B)$. Set $\Phi(x) = (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x}$, thus

$$\int_0^1 \Phi(x)dx = \int_0^{x_0} \Phi(x)dx + \int_{x_0}^1 \Phi(x)dx = I_{a,b} + J_{a,b}.$$

Put

$$u = \frac{A^2}{4(1-x)} + \frac{B^2}{4x}, \tag{A.3}$$

then

$$4ux^2 - (4u + B^2 - A^2)x + B^2 = 0. \tag{A.4}$$

If $0 < x < x_0$ this equation admits the solution

$$x = x(u) = \frac{1}{8u}\left(4u + B^2 - A^2 - \sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}\right)$$

$$\int_0^{x_0} (1-x)^{-a} x^{-b} e^{-A^2/4(1-x) - B^2/4x} dx = -\int_{(A+B)^2/4}^{\infty} (1-x(u))^{-a} x(u)^{-b} e^{-u} x'(u) du$$

Putting $x' = x'(u)$ and differentiating $(A.4)$,

$$4x^2 + 8uxx' - (4u + B^2 - A^2)x' - 4x = 0 \Longrightarrow -x' = \frac{4x(1-x)}{4u + B^2 - A^2 - 8ux}.$$

Thus

$$\int_0^{x_0} \Phi(x)dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1} x(u)^{-b+1} e^{-u} du}{4u + B^2 - A^2 - 8ux(u)}. \tag{A.5}$$

Using the explicit value of the root $x(u)$, we finally get

$$\int_0^{x_0} \Phi(x)dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1} x(u)^{-b+1} e^{-u} du}{\sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}}, \tag{A.6}$$

and the factorization below holds

$$16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2 = 16(u - (A+B)^2/4)(u - (A-B)^2/4).$$

We set $u = v + (A+B)^2/4$ and obtain

$$x(u) = \frac{v + (AB+B^2)/2 - \sqrt{v(v+AB)}}{2\left(v + (A+B)^2/4\right)},$$

and

$$1 - x(u) = \frac{v + (A^2+AB)/2 + \sqrt{v(v+AB)}}{2\left(v + (A+B)^2/4\right)}.$$

We introduce the relation $\approx$ linking two positive quantities depending on A and B. It means that the two sided-inequalities up to multiplicative constants independent of A and B. Therefore

$$\int_0^{x_0} \Phi(x)dx = 2^{a-b-4} e^{-(A+B)^2/4} \int_0^\infty \tilde{\Phi}(v)dv \quad \text{where}$$
$$\tilde{\Phi}(v) = \frac{\left(v + (AB+B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b} \left(v + (A^2+AB)/2 + \sqrt{v(v+AB)}\right)^{1-a}}{\left(v + (A+B)^2/4\right)^{2-a-b} \sqrt{v(v+AB)}} e^{-v}dv. \tag{A.7}$$

Case 1: $a \geq 1$, $b \geq 1$. First

$$\frac{\left(v + (A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+AB)}} \leq \frac{\left(v + (A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \approx \frac{\left(v + (A+B)^2\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \tag{A.8}$$

since $a+b-2 \geq 0$ and $AB \geq \kappa$. Next

$$\left(v + (A^2+AB)/2 + \sqrt{v(v+AB)}\right)^{1-a} \approx \left(v + A(A+B)\right)^{1-a}. \tag{A.9}$$

Furthermore

$$\begin{aligned} v + (AB+B^2)/2 - \sqrt{v(v+AB)} &= B^2 \frac{v + (A+B)^2/4}{v + B(A+B)/2 + \sqrt{v(v+AB)}} \\ &\approx B^2 \frac{v + (A+B)^2}{v + B(A+B)}. \end{aligned} \tag{A.10}$$

Then

$$\left(v + (AB+B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b} \approx B^{2-2b} \left(\frac{v + B(A+B)}{v + (A+B)^2}\right)^{b-1} \tag{A.11}$$

It follows

$$\begin{aligned} \tilde{\Phi}(v) &\leq CB^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)}\right)^{a-1} \frac{\left(v + B(A+B)\right)^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq CB^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)}\right)^{a-1} \frac{v^{b-1} + (B^2+AB)^{b-1}}{\sqrt{v(v+\kappa)}} \end{aligned} \tag{A.12}$$

where C depends on a, b and κ. The function $v \mapsto (v + (A+B)^2)/(v + A(A+B))$ is decreasing on $(0, \infty)$. If we set

$$C_1 = \int_0^\infty \frac{v^{b-1}e^{-v}dv}{\sqrt{v(v+\kappa)}} \quad \text{and} \quad C_2 = \int_0^\infty \frac{e^{-v}dv}{\sqrt{v(v+\kappa)}}$$

then

$$C_1 \leq K(B^2+AB)^{b-1}C_2$$

with $K = C_1\kappa^{1-b}/C_2$. Therefore

$$\int_0^{x_0} \Phi(x)dx \leq Ce^{-(A+B)^2/4}B^{1-b}A^{1-a}(A+B)^{a+b-2}. \tag{A.13}$$

The estimate of $J_{a,b}$ is obtained by exchanging (A,a) with (B,b) and replacing x by $1-x$. *Mutadis mutandis*, this yields directely to the same expression as in $A.13$ and finally

$$\int_0^{1} \Phi(x)dx \leq Ce^{-(A+B)^2/4}A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{A.14}$$

Case 2: $a \geq 1$, $b < 1$. Estimates $(A.7)$, $(A.8)$, $(A.9)$, $(A.10)$ and $(A.11)$ are valid. Because $v \mapsto (v+B(A+B))^{b-1}$ is decreasing, $(A.12)$ has to be replaced by

$$\tilde{\Phi}(v) \leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{\left(AB+B^2\right)^{b-1}}{\sqrt{v(v+\kappa)}}. \tag{A.15}$$

This implies $(A.13)$ directly. The estimate of $J_{a,b}$ is performed by the change of variable $x \mapsto 1-x$. If $x_1 = 1-x_0$, there holds

$$J_{a,b} = \int_0^{x_1} x^{-a}(1-x)^{-b}e^{-A^2/4x}e^{-B^2/4(1-x)}dx = \int_0^{x_1}\Psi(x)dx.$$

Then

$$\int_0^{x_1}\Psi(x)dx = 2^{b-a-4}e^{-(A+B)^2/4}\int_0^{x_1}\tilde{\Psi}(v)dv \quad \text{where}$$

$$\tilde{\Psi}(v) = \frac{\left(v+(AB+A^2)/2-\sqrt{v(v+AB)}\right)^{1-a}\left(v+(B^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-b}}{\left(v+(A+B)^2/4\right)^{2-a-b}\sqrt{v(v+AB)}}e^{-v}dv. \tag{A.16}$$

Equivalence $(A.8)$ is unchanged; $(A.9)$ is replaced by

$$\left(v+(B^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-b} \approx (v+B(A+B))^{1-b}, \tag{A.17}$$

$(A.10)$ by

$$v+(AB+A^2)/2-\sqrt{v(v+AB)} \approx A^2\frac{v+(A+B)^2}{v+A(A+B)}, \tag{A.18}$$

and $(A.11)$ by

$$\left(v+(AB+A^2)/2-\sqrt{v(v+AB)}\right)^{1-a} \approx A^{2-2a}\left(\frac{v+A(A+B)}{v+(A+B)^2}\right)^{a-1}. \tag{A.19}$$

Because $a > 1$, $(A.12)$ turns into

$$\begin{aligned}\tilde{\Psi}(v) &\leq CA^{2-2b}(v+(A+B)^2)^{b-1}\frac{(v+A^2+AB)^{a-1}(v+B^2+AB)^{1-b}}{\sqrt{v(v+\kappa)}}\\ &\leq Ce^{-(A+B)^2/4}A^{2-2b}(A+B)^{2b-2}\\ &\qquad\times\frac{v^{a-b}+(A^2+AB)^{a-1}v^{1-b}+(B^2+AB)^{1-b}v^{a-1}+A^{a-1}B^{1-b}(A+B)^{a-b}}{\sqrt{v(v+\kappa)}}.\end{aligned} \tag{A.20}$$

Because $AB \geq \kappa$, there exists a positive constant C, depending on κ, such that

$$\begin{aligned}\int_0^\infty \frac{v^{a-b}+(A^2+AB)^{a-1}v^{1-b}+(B^2+AB)^{1-b}v^{a-1}}{\sqrt{v(v+\kappa)}}e^{-v}dv&\\ \leq CA^{a-1}B^{1-b}(A+B)^{a-b}\int_0^\infty\frac{e^{-v}dv}{\sqrt{v(v+\kappa)}}.&\end{aligned} \tag{A.21}$$

Combining $(A.20)$ and $(A.21)$ yields to

$$\int_0^{x_1}\Psi(x)dx \leq Ce^{-(A+B)^2/4}A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{A.22}$$

This, again, implies that $(A.1)$ holds.

Case 3: $\max\{a,b\} < 1$. Inequalities $(A.7)$-$(A.11)$ hold, but $(A.12)$ has to be replaced by

$$\begin{aligned}\tilde{\Phi}(v) &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{\left(v+B^2+AB\right)^{b-1}}{\sqrt{v(v+\kappa)}}\\ &\leq CB^{1-b}(A+B)^{2a+b-3}\frac{v^{1-a}+\left(A^2+AB\right)^{1-a}}{\sqrt{v(v+\kappa)}}\end{aligned} \tag{A.23}$$

Noticing that

$$\int_0^\infty\frac{v^{1-a}e^{-v}dv}{\sqrt{v(v+\kappa)}} \leq C\left(A^2+AB\right)^{1-a}\int_0^\infty\frac{e^{-v}dv}{\sqrt{v(v+\kappa)}},$$

it follows that $(A.13)$ holds. Finally $(A.14)$ holds by exchanging (A,a) and (B,b). □

Lemma A.2 . *Let α, β, γ, δ be real numbers and ℓ an integer. We assume $\gamma > 1$, $\delta > 0$ and $\ell \geq 2$. Then there exists a positive constant C such that, for any integer $n > \ell$*

$$\sum_{p=1}^{n-\ell}p^\alpha(\sqrt{n}-\sqrt{p})^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \leq Cn^{\alpha-\beta/2}e^{-\delta n}. \tag{A.24}$$

Proof. The function $x \mapsto (\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2$ is decreasing on $[(\gamma-1)^{-1},\infty)$. Furthermore there exists $C > 0$ depending on ℓ, α and β such that $p^\alpha(\sqrt{n}-\sqrt{p})^\beta \leq Cx^\alpha(\sqrt{n}-\sqrt{x+1})^\beta$

for $x \in [p, p+1]$ If we denote by p_0 the smallest integer larger than $(\gamma-1)^{-1}$, we derive

$$\begin{aligned} S &= \sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n}-\sqrt{p}\,)^\beta e^{-(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2/4} = \sum_{p=1}^{p_0-1} + \sum_{p_0}^{n-\ell} p^\alpha (\sqrt{n}-\sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \\ &\le \sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n}-\sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \\ &\qquad + C\int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n}-\sqrt{x}\,)^\beta e^{-\delta(\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2} dx, \end{aligned}$$

(notice that $\sqrt{n}-\sqrt{x} \approx \sqrt{n}-\sqrt{x+1}$ for $x \le n-\ell$). Clearly

$$\sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n}-\sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \le C_0 n^\alpha (\sqrt{n}-\sqrt{n-\ell}\,)^\beta e^{-\delta n} \tag{A.25}$$

for some C_0 independent of n. We set $y = y(x) = \sqrt{x+1} - \sqrt{x}/\sqrt{\gamma}$. Obviously

$$y'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{\gamma}\sqrt{x}}\right) \qquad \forall x \ge p_0,$$

and their exists $\epsilon = \epsilon(\delta,\gamma) > 0$ such that $\sqrt{2}\sqrt{x} \ge y(x) \ge \epsilon\sqrt{x}$ and $y'(x) \ge \epsilon/\sqrt{x}$. Furthermore

$$\sqrt{x} = \frac{\sqrt{\gamma}\left(y + \sqrt{\gamma y^2 + 1 - \gamma}\right)}{\gamma - 1},$$

$$\begin{aligned} \sqrt{n}-\sqrt{x} &= \frac{\sqrt{n}(\gamma-1) - \sqrt{\gamma}y - \sqrt{\gamma}\sqrt{\gamma y^2+1-\gamma}}{\gamma-1} \\ &= \frac{n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2}{\sqrt{n}(\gamma-1)-\sqrt{\gamma}y+\sqrt{\gamma}\sqrt{\gamma y^2+1-\gamma}} \\ &\approx \frac{n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2}{\sqrt{n}} \end{aligned}$$

since $y(x) \le \sqrt{n}$. Furthermore

$$\begin{aligned} n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2 &= \gamma(\sqrt{n+1}+\sqrt{n}/\sqrt{\gamma}+y)(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y) \\ &\approx \sqrt{n}(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y), \end{aligned}$$

because y ranges between $\sqrt{n+2-\ell}-\sqrt{n+1-\ell}\sqrt{\gamma} \approx \sqrt{n}$ and $\sqrt{p_0+1}-\sqrt{p_0}\sqrt{\gamma}$. Thus

$$(\sqrt{n}-\sqrt{x}\,)^\beta \approx \left(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y\right)^\beta.$$

This implies

$$\begin{aligned}\int_{p_0}^{n+1-\ell} & x^{\alpha}(\sqrt{n}-\sqrt{x}\,)^{\beta}e^{-\delta(\sqrt{x}+\gamma(\sqrt{n}-\sqrt{x+1}))^2}dx \\ &\le C\int_{y(p_0)}^{y(n+1-\ell)} y^{2\alpha+1}\left(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y\right)^{\beta}e^{-\gamma\delta(\sqrt{n}-y)^2}dy \\ &\le Cn^{\alpha+\beta/2+1}\int_{1-y(n+1-\ell)/\sqrt{n}}^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^{\beta}e^{-\gamma\delta nz^2}dz.\end{aligned} \tag{A.26}$$

Moreover

$$\begin{aligned}1-\frac{y(p_0)}{\sqrt{n}} &= 1-\frac{1}{\sqrt{n}}\left(\sqrt{p_0+1}-\frac{\sqrt{p_0}}{\sqrt{\gamma}}\right), \\ 1-\frac{y(n-\ell+1)}{\sqrt{n}} &= 1-\frac{\sqrt{n-\ell+2}}{\sqrt{n}}+\frac{\sqrt{n-\ell+1}}{\sqrt{n\gamma}} \\ &= \frac{1}{\sqrt{\gamma}}\left(1+\frac{\sqrt{\gamma}\,(\ell-2)-\ell+1}{2n}+\frac{\sqrt{\gamma}\,(\ell-2)^2-(\ell-1)^2}{8n^2}\right)+O(n^{-3}).\end{aligned} \tag{A.27}$$

Let θ fixed such that $1-\dfrac{y(n-\ell+1)}{\sqrt{n}}<\theta<1-\dfrac{y(p_0)}{\sqrt{n}}$ for any $n>p_0$. Then

$$\begin{aligned}\int_{\theta}^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^{\beta}e^{-\gamma\delta nz^2}dz &\le C_{\theta}\int_{\theta}^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}e^{-\gamma\delta nz^2}dz \\ &\le C_{\theta}\,e^{-\gamma\delta n\theta^2}\int_{\theta}^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}dz \\ &\le C\,e^{-\gamma\delta n\theta^2}\max\{1,n^{-\alpha-1/2}\}.\end{aligned}$$

Because $\gamma\theta^2>1$ we derive

$$\int_{\theta}^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^{\beta}e^{-\gamma\delta nz^2}dz\le Cn^{-\beta}e^{-\delta n}, \tag{A.28}$$

for some constant $C>0$. On the other hand

$$\begin{aligned}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}&(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^{\beta}e^{-\gamma\delta nz^2}dz \\ &\le C'_{\theta}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^{\beta}e^{-\gamma\delta nz^2}dz.\end{aligned}$$

The minimum of $z\mapsto(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^{\beta}$ is achieved at $1-y(n+1-\ell)$ with value

$$\frac{\sqrt{\gamma}(\ell+1)+1-\ell}{2n\sqrt{\gamma}}+O(n^{-2}),$$

and the maximum of the exponential term is achieved at the same point with value

$$e^{-n\delta+((\ell-2)\sqrt{\gamma}+1-\ell)/2}(1+\circ(1))=C_{\gamma}e^{-n\delta}(1+\circ(1)).$$

We denote

$$z_{\gamma,n} = 1 + 1/\sqrt{\gamma} - \sqrt{1 + 1/n} \quad \text{and} \quad I_\beta = \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^\beta e^{-\gamma\delta n z^2} dz.$$

Since $1 - y(n + 1 - \ell) \geq 1/\sqrt{2\gamma}$ for n large enough,

$$\begin{aligned} I_\beta &\leq \sqrt{2\gamma} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^\beta z e^{-\gamma\delta n z^2} dz \\ &\leq \frac{-\sqrt{2\gamma}}{2n\gamma\delta} \left[(z - z_{\gamma,n})^\beta e^{-\gamma\delta n z^2} \right]_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} + \frac{\beta\sqrt{2\gamma}}{2n\gamma\delta} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^{\beta-1} z e^{-\gamma\delta n z^2} dz \end{aligned}$$

But $1 - y(n + 1 - \ell)/\sqrt{n} - z_{\gamma,n} = (\ell - 1)(1 - 1/\sqrt{\gamma})/2n$, therefore

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + \beta C_1' n^{-1} I_{\beta-1}. \tag{A.29}$$

If $\beta \leq 0$, we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n},$$

which inequality, combined with $(A.26)$ and $(A.28)$, yields to $(A.24)$. If $\beta > 0$, we iterate and get

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C_1' n^{-1} (C_1 n^{-\beta} e^{-\delta n} + (\beta - 1) C_1' n^{-1} I_{\beta-2})$$

If $\beta - 1 \leq 0$ we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C_1 C_1' n^{-1-\beta} e^{-\delta n} = C_2 n^{-\beta-1} e^{-\delta n},$$

which again yields to $(A.24)$. If $\beta - 1 > 0$, we continue up we find a positive integer k such that $\beta - k \leq 0$, which again yields to

$$I_\beta \leq C_k n^{-\beta-1} e^{-\delta n}$$

and to $(A.24)$. □

The next estimate is fundamental in deriving the N-dimensional estimate.

Lemma A.3 *For any integer $N \geq 2$ there exists a constant $c_N > 0$ such that*

$$\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta \leq c_N \frac{e^m}{(1+m)^{(N-1)/2}} \qquad \forall m > 0. \tag{A.30}$$

Proof. Put $\mathcal{I}_N(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta$. Then $\mathcal{I}_2'(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \cos\theta \, d\theta$ and

$$\begin{aligned} \mathcal{I}_2''(m) &= \int_0^\pi e^{m\cos\theta} \cos^2\theta \, d\theta = \mathcal{I}_2(m) - \int_0^\pi e^{m\cos\theta} \sin^2\theta \, d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m} \int_0^\pi e^{m\cos\theta} \cos\theta \, d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m} \mathcal{I}_2'(m). \end{aligned}$$

Thus $\mathcal{I}_2$ satisfies a Bessel equation of order 0. Since $\mathcal{I}_2(0) = \pi$ and $\mathcal{I}_2'(0) = 0$, $\pi^{-1}\mathcal{I}_2$ is the modified Bessel function of index 0 (usually denoted by I_0) the asymptotic behaviour of which is well known, thus $(A.30)$ holds. If $N = 3$

$$\mathcal{I}_3(m) = \int_0^\pi e^{m\cos\theta}\sin\theta\,d\theta = \left[\frac{-e^{m\cos\theta}}{m}\right]_0^\pi = \frac{2\sinh m}{m}.$$

For $N > 3$ arbitrary

$$\mathcal{I}_N(m) = \int_0^\pi \frac{-1}{m}\frac{d}{d\theta}(e^{m\cos\theta})\sin^{N-3}\theta\,d\theta = \frac{N-3}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\sin^{N-4}\theta\,d\theta. \tag{A.31}$$

Therefore,

$$\mathcal{I}_4(m) = \frac{1}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\,d\theta = \mathcal{I}_2'(m),$$

and, again $(A.30)$ holds since $I_0'(m)$ has the same behaviour as $I_0(m)$ at infinity. For $N \geq 5$

$$\mathcal{I}_N(m) = \frac{3-N}{m^2}\left[e^{m\cos\theta}\cos\theta\sin^{N-5}\theta\right]_0^\pi + \frac{N-3}{m^2}\int_0^\pi e^{m\cos\theta}\frac{d}{d\theta}\left(\cos\theta\sin^{N-5}\theta\right)d\theta.$$

Differentiating $\cos\theta\sin^{N-5}\theta$ and using $(A.31)$, we obtain

$$\mathcal{I}_5(m) = \frac{4\sinh m}{m^2} - \frac{4\sinh m}{m^3},$$

while

$$\mathcal{I}_N(m) = \frac{(N-3)(N-5)}{m^2}\left(\mathcal{I}_{N-4}(m) - \mathcal{I}_{N-2}(m)\right), \tag{A.32}$$

for $N \geq 6$. Since the estimate $(A.30)$ for $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ has already been obtained, a straigthforward induction yields to the general result. □

Remark. Although it does not has any importance for our use, it must be noticed that $\mathcal{I}_N$ can be expressed either with hyperbolic functions if N is odd, or with Bessel functions if N is even.

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