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Abstract

We prove that any positive solution of $\partial_t u - \Delta u + u^q = 0$ $(q > 1)$ in $\mathbb{R}^N \times (0, \infty)$ with initial trace $(F, 0)$, where F is a closed subset of $\mathbb{R}^N$ can be estimated from above and below and up to two universal multiplicative constants, by a series involving the Bessel capacity $C_{2/q,q'}$. As a consequence we prove that there exists a unique positive solution of the equation with such an initial trace. We also characterize the blow-up set of $u(x,t)$ when $t \downarrow 0$, by using the "density" of F expressed in terms of the $C_{2/q,q'}$-capacity.

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Contents

1 Introduction

Let $T \in (0,\infty]$ and $Q_T = \mathbb{R}^N \times (0,T]$ ($N \geq 1$). If $q > 1$ and $u \in C^2(Q_T)$ is nonnegative and verifies

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T, \tag{1.1}$$

it has been proven by Marcus and Véron [26] that there exists a unique outer-regular positive Borel measure ν in $\mathbb{R}^N$ such that

$$\lim_{t\to 0} u(.,t) = \nu, \tag{1.2}$$

in the sense of Borel measures; the set of such measures is denoted by $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$. To each of its element ν is associated a unique couple $(\mathcal{S}_\nu, \mu_\nu)$ (we write $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$) where $\mathcal{S}_\nu$, *the singular part* of ν, is a closed subset of $\mathbb{R}^N$ and μ_ν, *the regular part* is a nonnegative Radon measure on $\mathcal{R}_\nu = \mathbb{R}^N \setminus \mathcal{S}_\nu$. In this setting, relation (1.2) has the following meaning :

$$\begin{array}{lll} (i) & \lim_{t\to 0} \int_{\mathcal{R}_\nu} u(.,t)\zeta dx = \displaystyle\int_{\mathcal{R}_\nu} \zeta d\mu_\nu, & \forall \zeta \in C_0(\mathcal{R}_\nu), \\ (ii) & \lim_{t\to 0} \displaystyle\int_{\mathcal{O}} u(.,t)dx = \infty, & \forall \mathcal{O} \subset \mathbb{R}^N \text{ open}, \ \mathcal{O} \cap \mathcal{S}_\nu \neq \emptyset. \end{array} \tag{1.3}$$

The measure ν is by definition *the initial trace* of u and denoted by $Tr_{\mathbb{R}^N}(u)$. It is wellknown that equation (1.1) admits a critical exponent

$$1 < q < q_c = 1 + \frac{N}{2}.$$

This is due to the fact, proven by Brezis and Friedman [6], that if $q \geq q_c$, isolated singularities of solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ are removable. Conversely, if $1 < q < q_c$, it is proven by the same authors that for any $k > 0$, equation (1.1) admits a unique solution $u_{k\delta_0}$ with initial data $k\delta_0$. This existence and uniqueness results extends in a simple way if the initial data $k\delta_0$ is replaced by any Radon measure μ in $\mathbb{R}^N$ (see [5]). Furthermore, if $k \to \infty$, $u_{k\delta_0}$ increases and converges to a positive, radial and self-similar solution u_∞ of (1.1) known as the *very singular solution* of (1.1). Writing it under the form $u_\infty(x,t) = t^{-\frac{1}{q-1}} f(|x|/\sqrt{t})$, f is a positive solution of

$$\begin{cases} \Delta f + \frac{1}{2} y.Df + \frac{1}{q-1} f - f^q = 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|y|\to\infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \tag{1.4}$$

The existence, uniqueness and the expression of the asymptotics of f has been studied thoroughly by Brezis, Peletier and Terman in [7]. Later on, Marcus and Véron proved in [26] that in the same range of exponents, for any $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_\infty, \\ Tr_{\mathbb{R}^N}(u) = \nu, \end{cases} \tag{1.5}$$

admits a unique positive solution. This result means that the initial trace establishes a one to one correspondence between the set of positive solutions of (1.1) and $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$. A key step for proving the uniqueness is the following inequalities

$$t^{-\frac{1}{q-1}} f(|x-a|/\sqrt{t}) \leq u(x,t) \leq ((q-1)t)^{-\frac{1}{q-1}} \qquad \forall (x,t) \in Q_\infty, \tag{1.6}$$

valid for any $a \in \mathcal{S}_\nu$. As a consequence of Brezis and Friedman's removability result, if $q \geq q_c$, i.e. in the *supercritical range*, the key estimate (1.6) cannot hold; actually Problem (1.5) may admit no solution at all. If $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$, the necessary and sufficient conditions for the existence of a maximal solution $u = \overline{u}_\nu$ to Problem (1.5) are obtained in [26] and expressed in terms of the the Bessel capacity $C_{2/q,q'}$, (with $q' = q/(q-1)$). Furthermore, uniqueness does not hold in general as it was pointed out by Le Gall [21]. In the particular case where $\mathcal{S}_\nu = \emptyset$ and ν is simply the Radon measure μ_ν, the necessary and sufficient condition for solvability is that μ_ν does not charge Borel subsets with $C_{2/q,q'}$-capacity zero. This result was already proven by Baras and Pierre [4] in the particular case of bounded measures and extended by Marcus and Véron [26] to the general case. We denote by $\mathfrak{M}_+^q(\mathbb{R}^N)$ the positive cone of the space $\mathfrak{M}^q(\mathbb{R}^N)$ of Radon measures which do not charge Borel subsets with zero $C_{2/q,q'}$-capacity. Notice that $W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ is a subset of $\mathfrak{M}_+^q(\mathbb{R}^N)$ where $\mathfrak{M}_+^b(\mathbb{R}^N)$ is the cone of positive bounded Radon mesures in $\mathbb{R}^N$. For such measures, uniqueness always holds and we denote $\overline{u}_{\mu_\nu} = u_{\mu_\nu}$.

In view of the already known results concerning the parabolic equation, it is useful to recall the main advanced results previously obtained for the stationary equation

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{1.7}$$

in a smooth bounded domain Ω of $\mathbb{R}^N$. This equation has been intensively studied since 1993, both by probabilists (Le Gall, Dynkin, Kuznetsov) and by analysts (Marcus, Véron). The existence of a *boundary trace* for positive solutions, in the class of outer-regular positive Borel measures on $\partial\Omega$, is proven by Le Gall [20], [21] in the case $q = N = 2$, by probabilistic methods, and by Marcus and Véron in [24], [25] in the general case $q > 1$, $N > 1$. The existence of a critical exponent $q_e = (N+1)/(N-1)$ is due to Gmira and Véron [12] who shew that, if $q \geq q_e$ boundary isolated singularities of solutions of (1.7) are removable, which is not the case if $1 < q < q_e$. In this *subcritical case* Le Gall and Marcus and Véron proved that the boundary trace establishes a one to one correspondence between positive solutions of (1.7) in Ω and outer regular positive Borel measures on $\partial\Omega$. This fundamental result does not hold in the *supercritical case* $q \geq q_e$. In [10] Dynkin and Kuznetsov introduced the notion of σ-moderate solution which means that u is a positive solution of (1.7) such that there exists an increasing sequence of positive Radon measures on $\partial\Omega$ $\{\mu_n\}$ belonging to $W^{-2/q,q'}(\partial\Omega)$ such that the corresponding solutions $v = v_{\mu_n}$ of

$$\begin{cases} -\Delta v + v^q = 0 & \text{in } \Omega \\ \qquad\quad v = \mu_n & \text{in } \partial\Omega \end{cases} \tag{1.8}$$

converges to u locally uniformly in Ω. This class of solutions plays a fundamental role since Dynkin and Kuznetsov proved that a σ-moderate solution of (1.7) is uniquely determined by its *fine trace*, a new notion of trace introduced in order to avoid the non-uniqueness phenomena. Later on, it is proved by Mselati (if $q = 2$) [33], then by Dynkin (if $q_e \leq q \leq 2$) [8] and finally by Marcus with no restriction on q [23], that all the positive solutions of (1.7) are σ-moderate. One of the key-stones element in their proof (partially probabilistic) is the fact that the maximal solution $\overline{u}_K$ of (1.7) with a boundary trace vanishing outside a compact subset $K \subset \partial\Omega$ is indeed σ-moderate. This deep result was obtained by a combination of probabilistic and analytic methods by Mselati [33] in the case $q = 2$ and by purely analytic tools by Marcus and Véron [29], [30] in the case $q \geq q_e$. Defining $\underline{u}_K$ as the largest σ-moderate solution of

(1.7) with a boundary trace concentrated on K, the crucial step in Marcus-Véron's proof (non probabilistic) is the bilateral estimate satisfied by $\overline{u}_K$ and $\underline{u}_K$

$$C^{-1}\rho(x)W_K(x) \le \underline{u}_K(x) \le \overline{u}_K(x) \le C\rho(x)W_K(x). \tag{1.9}$$

In this expression $C = C(\Omega, q)$, $\rho(x) = \text{dist}\,(x, \partial\Omega)$ and $W_F(x)$ is the *elliptic capacitary potential* of K defined by

$$W_K(x) = \sum_{-\infty}^{\infty} 2^{-\frac{m(q+1)}{q-1}} C_{2/q,q'}(2^m K_m(x)), \tag{1.10}$$

where $K_m(x) = K \cap \{z : 2^{-m-1} \le |z-x| \le 2^{-m}\}$, the Bessel capacity being relative to $\mathbb{R}^{N-1}$. Note that, using a technique introduced in [25], inequality $\overline{u}_K \le C^2 \underline{u}_K$ implies $\underline{u}_K = \overline{u}_K$.

The aim of this article is to initiate the fine study of the complete initial trace problem for positive solutions of (1.1) in the supercritical case $q \ge q_c$ and to give in particular the parabolic counterparts of the results of [33], [29] and [30]. The main step is to obtain a kind of parabolic analogous of the nonlinear potential estimate (1.10)

Extending Dynkin's ideas to the parabolic case, we introduce the following notion

Definition 1.1 *A positive solution u of (1.1) is called σ-moderate if their exists an increasing sequence $\{\mu_n\} \subset W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ such that the corresponding solution $u := u_{\mu_n}$ of*

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_\infty \\ \qquad\qquad u(x,0) = \mu_n \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.11}$$

converges to u locally uniformly in Q_∞.

If F is a closed subset of $\mathbb{R}^N$, we denote by $\overline{u}_F$ the maximal solution of (1.1) with an initial trace vanishing on F^c, and by $\underline{u}_F$ the maximal σ-moderate solution of (1.1) with an initial trace vanishing on F^c. Thus $\underline{u}_F$ is defined by

$$\underline{u}_F = \sup\{u_\mu : \mu \in W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N), \mu(F^c) = 0\}, \tag{1.12}$$

(and clearly $W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ can be replaced by $\mathfrak{M}^q_+(\mathbb{R}^N)$). One of the main goal of this article is to prove that $\overline{u}_F$ is σ-moderate and more precisely,

Theorem 1.2 *For any $q > 1$ and any closed subset F of $\mathbb{R}^N$, $\overline{u}_F = \underline{u}_F$.*

We define below a set function which will play a fundamental role in the sequel.

Definition 1.3 *Let F be a closed subset of $\mathbb{R}^N$. The $C_{2/q,q'}$-parabolic capacitary potential W_F of F is defined by*

$$W_F(x,t) = t^{-\frac{1}{q-1}} \sum_{n=0}^{\infty} (n+1)^{\frac{N}{2}-\frac{1}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F_n}{\sqrt{(n+1)t}}\right) \qquad \forall (x,t) \in Q_\infty, \tag{1.13}$$

where $F_n = F_n(x,t) := \{y \in F : \sqrt{nt} \le |x-y| \le \sqrt{(n+1)t}\}$.

One of the tool for proving Theorem 1.2 is the following bilateral estimate which is only meaningful in the supercritical case, otherwhile it reduces to (1.6);

Theorem 1.4 *For any $q \geq q_c$ there exist two positive constants $C_1 \geq C_2 > 0$, depending only on N and q such that for any closed subset F of $\mathbb{R}^N$, there holds*

$$C_2 W_F(x,t) \leq \underline{u}_F(x,t) \leq \overline{u}_F(x,t) \leq C_1 W_F(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.14}$$

Such a parabolic estimate is much more difficult to obtain than the corresponding elliptic one, although some of the techniques we introduce here could simplify the derivation of (1.10). It is important to notice that the parabolic capacitary potential is equivariant with respect to the same scaling transformation which let (1.1) invariant in the sense that, for any $\ell > 0$,

$$\ell^{\frac{1}{q-1}} W_F(\sqrt{\ell}x, \ell t) = W_{F/\sqrt{\ell}}(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.15}$$

This *quasi representation*, up to uniformly upper and lower bounded functions, is also interesting in the sense that it indicates precisely *how to characterize the blow-up points* of $\overline{u}_F = \underline{u}_F := u_F$. Introducing an integral expression comparable to W_F, we show in particular the following results

$$\lim_{\tau \to 0} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) = \gamma \in [0,\infty) \Longrightarrow \lim_{t \to 0} t^{-\frac{1}{q-1}} u_F(x,t) = C\gamma \tag{1.16}$$

for some $C_\gamma = C(N,q,\gamma) > 0$, and

$$\limsup_{\tau \to 0} \tau^{\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) < \infty \Longrightarrow \limsup_{t \to 0} u_F(x,t) < \infty. \tag{1.17}$$

Our paper is organized as follows. In Section 2 we obtain estimates from above on $\overline{u}_F$. In Section 3 we give estimates from below on $\underline{u}_F$. In Section 4 we prove the main theorems and expose various consequences. In Appendix we derive a series of sharp integral inequalities.

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2 Estimates from above

Some notations. Let Ω be a domain in $\mathbb{R}^N$ with a compact C^2 boundary and $T > 0$. Set $B_r(a)$ the open ball of radius $r > 0$ and center a (and $B_r(0) := B_r$) and

$$Q_T^\Omega := \Omega \times (0,T), \quad \partial_\ell Q_T^\Omega = \partial\Omega \times (0,T), \quad Q_T := Q_T^{\mathbb{R}^N}, \quad Q_\infty := Q_\infty^{\mathbb{R}^N}.$$

Let $\mathbb{H}^\Omega[.]$ (resp. $\mathbb{H}[.]$) denote the heat potential in Ω with zero lateral boundary data (resp. the heat potential in $\mathbb{R}^N$) with corresponding kernel

$$(x,y,t) \mapsto H^\Omega(x,y,t) \quad (\text{resp.}(x,y,t) \mapsto H(x,y,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}).$$

We denote by $q_c := 1 + \frac{N}{2}$, the Brezis-Friedman critical exponent.

Theorem 2.1 *Let $q \geq q_c$. Then there exists a positive constant $C_1 = C_1(N,q)$ such that for any closed subset F of $\mathbb{R}^N$ and any $u \in C^2(Q_\infty) \cap C(\overline{Q_\infty} \setminus F)$ satisfying*

$$\begin{aligned} \partial_t u - \Delta u + u^q = 0 \quad & \text{in } Q_\infty \\ \lim_{t\to 0} u(x,t) = 0 \quad & \text{locally uniformly in } F^c, \end{aligned} \tag{2.1}$$

there holds

$$u(x,t) \leq C_1 W_F(x,t) \qquad \forall (x,t) \in Q_\infty, \tag{2.2}$$

where W_F is the $(2/q, q')$-parabolic capacitary potential of F defined by (1.13).

First we consider the case where $F = K$ is compact and

$$K \subset B_r \subset \overline{B}_r, \tag{2.3}$$

and then we extend to the general case by a covering argument.

2.1 Global L^q-estimates

Let $\rho > 0$, we assume (2.3) holds and we put

$$\mathcal{T}_{r,\rho}(K) = \{\eta \in C_0^\infty(B_{r+\rho}), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\}. \tag{2.4}$$

If $\eta \in \mathcal{T}_{r,\rho}(K)$, we set $\eta^* = 1 - \eta$, $\zeta = \mathbb{H}[\eta^*]^{2q'}$ and

$$R[\eta] = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta] + \Delta \mathbb{H}[\eta]|\,. \tag{2.5}$$

We fix $T > 0$ and consider the equation on Q_T. Throughout this paper C will denote a generic positive constant, depending only on N, q and sometimes T, the value of which may vary from one ocurrence to another. We also use sometimes the notation $A \approx B$ for meaning that there exists a constant $C > 0$ independent of the data such that $C^{-1}A \leq B \leq CA$.

Except in Lemma 2.12 the only assumption on q is $q > 1$. In the sequel we will obtain pointwise estimates on the solution expressed in terms of the $L^{q'}$-norm of $R[\eta]$ for $\eta \in \mathcal{T}_{r,\rho}(K)$. Although these estimates could have been immediately turned into capacitary estimates as in [30], the advantage of keeping them comes from the possibility of performing operations such as dilations or summations on them. The next key lemma points out the connection between $R[\eta]$ and the the $C_{2/q,q'}$- capacity of K.

Lemma 2.2 *There exists $C = C(N,q) > 0$ such that*

$$C^{-1}\|\eta\|^{q'}_{W^{2/q,q'}} \leq \iint_{Q_\infty} (R[\eta])^{q'}\,dx\,dt := \|R[\eta]\|^{q'}_{L^{q'}} \leq C\|\eta\|^{q'}_{W^{2/q,q'}} \qquad \forall \eta \in \mathcal{T}_{r,\rho}(K). \tag{2.6}$$

Therefore

$$\inf\left\{\|R[\eta]\|^{q'}_{L^{q'}} : \eta \in \mathcal{T}_{r,\rho}(K)\right\} \approx C^{B_{r+\rho}}_{2/q,q'}(K). \tag{2.7}$$

Proof. There holds $\partial_t \mathbb{H}[\eta] = \Delta \mathbb{H}[\eta]$, and

$$\iint_{Q_\infty} |\partial_t \mathbb{H}[\eta]|^{q'} \, dx \, dt = \int_0^\infty \left\| t^{1-1/q} \partial_t \mathbb{H}[\eta] \right\|^{q'}_{L^{q'}(\mathbb{R}^N)} \frac{dt}{t} \approx \|\eta\|^{q'}_{\left[W^{2,q'}, L^{q'}\right]_{1/q,q'}} \tag{2.8}$$

where $\left[W^{2,q'}, L^{q'}\right]_{1/q,q'}$ indicates the real interpolation functor of degree $1/q$ between $W^{2,q'}(\mathbb{R}^N)$ and $L^{q'}(\mathbb{R}^N)$. See [36]. Similarly, and using the Gagliardo-Nirenberg inequality,

$$\iint_{Q_\infty} |\nabla(\mathbb{H}[\eta])|^{2q'} \, dx \, dt \le C \|\eta\|^{q'}_{W^{2/q,q'}} \|\eta\|^{q'}_{L^\infty} = C \|\eta\|^{q'}_{W^{2/q,q'}}. \tag{2.9}$$

Inequality (2.6) follows from (2.8) and (2.9), and (2.7) from the definition of the Bessel capacity relative to $B_{r+\rho}$. □

Lemma 2.3 *Assume u is a positive solution of (2.1) in Q_∞. Then there exists $C = C(N,q) > 0$ such that for any $T > 0$,*

$$\iint_{Q_T} u^q \zeta dx \, dt + \int_{\mathbb{R}^N} (u\zeta)(x,T) dx \le C \|R[\eta]\|^{q'}_{L^{q'}} \qquad \forall \eta \in \mathcal{T}_{r,\rho}(K). \tag{2.10}$$

Proof. We recall that there always hold

$$0 \le u(x,t) \le \left(\frac{1}{t(q-1)} \right)^{\frac{1}{q-1}} \qquad \forall (x,t) \in Q_\infty, \tag{2.11}$$

and

$$0 \le u(x,t) \le \left(\frac{C}{t + (|x| - r)^2} \right)^{\frac{1}{q-1}} \qquad \forall (x,t) \in Q_\infty \setminus B_r, \tag{2.12}$$

See [6]. Since η^* vanishes in an open neighborhood $\mathcal{N}_1$, for any open subset $\mathcal{N}_2$ such that $K \subset \mathcal{N}_2 \subset \overline{\mathcal{N}}_2 \subset \mathcal{N}_1$ there exist $c_{\mathcal{N}_2} > 0$ and $C_{\mathcal{N}_2} > 0$ such that

$$\mathbb{H}[\eta^*](x,t) \le C_{\mathcal{N}_2} e^{-\frac{c_{\mathcal{N}_2}}{t}}, \qquad \forall (x,t) \in Q_T^{\mathcal{N}_2}.$$

Therefore

$$\lim_{t \to 0} \int_{\mathbb{R}^N} (u\zeta)(x,t) dx = 0.$$

Thus ζ is an admissible test function and one has

$$\iint_{Q_T} u^q \zeta dx \, dt + \int_{\mathbb{R}^N} (u\zeta)(x,T) dx = \iint_{Q_T} u(\partial_t \zeta + \Delta \zeta) dx \, dt. \tag{2.13}$$

Notice that the two terms on the left-hand side are nonnegative. Put $\mathbb{H}_{\eta^*} = \mathbb{H}[\eta^*]$, then

$$\begin{aligned} \partial_t \zeta + \Delta \zeta &= 2q' \mathbb{H}_{\eta^*}^{2q'-1} \left(\partial_t \mathbb{H}_{\eta^*} + \Delta \mathbb{H}_{\eta^*} \right) + 2q'(2q'-1) \mathbb{H}_{\eta^*}^{2q'-2} |\nabla \mathbb{H}_{\eta^*}|^2, \\ &= 2q' \mathbb{H}_{\eta^*}^{2q'-1} \left(\partial_t \mathbb{H}_{\eta} + \Delta \mathbb{H}_{\eta} \right) + 2q'(2q'-1) \mathbb{H}_{\eta}^{2q'-2} |\nabla \mathbb{H}_{\eta}|^2, \end{aligned}$$

because $\mathbb{H}_{\eta^*} = 1 - \mathbb{H}_\eta$, hence

$$u(\partial_t\zeta + \Delta\zeta) = u\mathbb{H}_{\eta^*}^{2q'/q}\left[2q'(2q'-1)\mathbb{H}_{\eta^*}^{2q'-2-2q'/q}|\nabla\mathbb{H}_\eta|^2 - 2q'\mathbb{H}_{\eta^*}^{2q'-1-2q'/q}(\Delta\mathbb{H}_\eta + \partial_t\mathbb{H}_\eta)\right].$$

Finally, since $2q' - 2 - 2q'/q = 0$ and $0 \le \mathbb{H}_{\eta^*} \le 1$, there holds

$$\left|\iint_{Q_T} u(\partial_t\zeta + \Delta\zeta)dx\,dt\right| \le C(q)\left(\iint_{Q_T} u^q\zeta dx\,dt\right)^{1/q}\left(\iint_{Q_T} R^{q'}(\eta)dx\,dt\right)^{1/q'},$$

where

$$R(\eta) = |\nabla\mathbb{H}_\eta|^2 + |\Delta\mathbb{H}_\eta + \partial_t\mathbb{H}_\eta|.$$

Using Lemma 2.2 one obtains (2.10). □

Proposition 2.4 *Under the assumptions of Lemma 2.3, let $r > 0$, $\rho > 0$, $T \ge (r+\rho)^2$*

$$\mathcal{E}_{r+\rho} := \{(x,t) : |x|^2 + t \le (r+\rho)^2\}$$

and $Q_{r+\rho,T} = Q_T \setminus \mathcal{E}_{r+\rho}$. There exists $C = C(N,q,T) > 0$ such that

$$\iint_{Q_{r+\rho,T}} u^q dx\,dt + \int_{\mathbb{R}^N} u(x,T)dx \le C\|R[\eta]\|_{L^{q'}}^{q'} \qquad \forall\eta \in \mathcal{T}_{r,\rho}(K). \tag{2.14}$$

Proof. Because $K \subset B_r$ and $\eta^* \equiv 1$ outside $B_{r+\rho}$ and takes value between 0 and 1,

$$\begin{aligned}\mathbb{H}[\eta^*](x,t) \ge \mathbb{H}[1-\chi_{B_{r+\rho}}](x,t) &= \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\ge r+\rho} e^{-\frac{|x-y|^2}{4t}}dy,\\ &= 1 - \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\le r+\rho} e^{-\frac{|x-y|^2}{4t}}dy.\end{aligned}$$

For $(x,t) \in Q_{r+\rho,T}$, put $x = (r+\rho)\xi$, $y = (r+\rho)\upsilon$ and $t = (r+\rho)^2\tau$. Then $(\xi,\tau) \in Q_{1,\frac{T}{(r+\rho)^2}}$ and

$$\left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\le r+\rho} e^{-\frac{|x-y|^2}{4t}}dy = \left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\le 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon.$$

We claim that

$$\max\left\{\left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\le 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon : (\xi,\tau) \in Q_{1,\frac{T}{(r+\rho)^2}}\right\} = \ell, \tag{2.15}$$

and $\ell = \ell(N, \frac{T}{(r+\rho)^2}) \in (0,1]$ is actually independent of $\frac{T}{(r+\rho)^2}$when large enough. We recall that

$$\left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\le 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon < 1 \qquad \forall\tau > 0. \tag{2.16}$$

If the maximum is achieved for some $(\bar\xi,\bar\tau)\in Q_{1,\frac{T}{(r+\rho)^2}}$, it is smaller that 1 and

$$\mathbb{H}[\eta^*](x,t)\geq \mathbb{H}[1-\chi_{B_{r+\rho}}](x,t)\geq 1-\ell>0,\qquad \forall (x,t)\in Q_{r+\rho,T}. \tag{2.17}$$

Let us assume that the maximum is achieved following a sequence $\{(\xi_n,\tau_n)\}$ with $\tau_n\to 0$ and $|\xi_n|\to\alpha\geq 1$, and clearly $\alpha=1$. We can assume that $\xi_n\to\bar\xi$ with $\left|\bar\xi\right|=1$, then

$$\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}}\int_{|\upsilon|\leq 1}e^{-\frac{|\xi_n-\upsilon|^2}{4\tau_n}}d\upsilon=\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}}\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon.$$

But $B_1(\xi_n)\cap B_1(-\xi_n)=\emptyset$,

$$\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon+\int_{B_1(-\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon<\int_{\mathbb{R}^N}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon$$

and

$$\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon=\int_{B_1(-\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon.$$

This implies

$$\lim_{n\to\infty}\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}}\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon\leq 1/2.$$

If the maximum were achieved with a sequence $\{(\xi_n,\tau_n)\}$ with $\tau_n+|\xi_n|\to\infty$, it would also imply (2.17), since the integral term in (2.16) is always bounded. Therefore (2.16) holds. Put $C=(1-\ell)^{-1}$, then

$$\iint_{Q_{r,T}}u^q dx\,dt+\int_{\mathbb{R}^N}u(.,T)dx\leq C\left\|R[\eta]\right\|_{L^{q'}}^{q'}, \tag{2.18}$$

and (2.14) follows. □

2.2 Pointwise estimates

In all this subsection u is a positive solution of (2.1) in Q_∞ and the assumptions of Lemma 2.3 hold. We give first a rough pointwise estimate.

Lemma 2.5 *There exists a constant $C=C(N,q)>0$ such that, for any $\eta\in\mathcal{T}_{r,\rho}(K)$,*

$$u(x,(r+2\rho)^2)\leq\frac{C\left\|R[\eta]\right\|_{L^{q'}}^{q'}}{(\rho(r+\rho))^{\frac{N}{2}}},\qquad\forall x\in\mathbb{R}^N. \tag{2.19}$$

Proof. We observe first that

$$\int_s^T\int_{\mathbb{R}^N}u^q dx\,dt+\int_{\mathbb{R}^N}u(x,T)dx=\int_{\mathbb{R}^N}u(x,s)dx\qquad\forall T>s>0, \tag{2.20}$$

and

$$\int_{\mathbb{R}^N} u(.,s)dx \le C \left\|R[\eta]\right\|_{L^{q'}}^{q'} \qquad \forall T > s \ge (r+\rho)^2, \tag{2.21}$$

from Proposition 2.4. By the maximum principle u is dominated by the maximal solution v of (1.1) having the indicatrix function I_{B_r} for initial trace. The function v is the limit, as $k \to \infty$, of the solutions v_k with initial data $k\chi_{B_r}$. Since $v_k \le k\mathbb{H}[\chi_{B_r}]$, it follows by Lemma 2.3. Using the fact that

$$u(x,\tau+s) \le \mathbb{H}[u(.,s)](x,\tau) \le \left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}} \int_{\mathbb{R}^N} u(.,s)dx,$$

we obtain (2.19) with $s = (r+\rho)^2$ and $\tau = (r+2\rho)^2 - (r+\rho)^2 \approx \rho(r+\rho)$. □

The above estimate does not take into account the fact that $u(x,0) = 0$ if $|x| \ge r$. It is mainly interesting if $|x| \le r$. In order to derive a sharper estimate which uses the localization of the singularity and not only the $L^{q'}$-norm of $R[\eta]$. For such a goal, we need some lateral boundary estimates.

Lemma 2.6 *Let $\gamma \ge r + 2\rho$ and $c > 0$ and either $N = 1$ or 2 and $0 \le t \le c\gamma^2$ for some $c > 0$, or $N \ge 3$ and $t > 0$. Then, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$\int_0^t \int_{\partial B_\gamma} u dS d\tau \le C_5 \gamma \left\|R[\eta]\right\|_{L^{q'}}^{q'}. \tag{2.22}$$

where $C > 0$ depends on N, q and c if $N = 1$, 2 or depends only on N and q if $N \ge 3$.

Proof. First we assume $N = 1$ or 2. Put $G^\gamma := B_\gamma^c \times (-\infty, 0)$ and $\partial_\ell G^\gamma = \partial B_\gamma \times (-\infty, 0)$. We set

$$h_\gamma(x) = 1 - \frac{\gamma}{|x|},$$

and let ψ_γ be the solution of

$$\begin{aligned} \partial_\tau \psi_\gamma + \Delta \psi_\gamma &= 0 \qquad \text{in } G^\gamma, \\ \psi_\gamma &= 0 \qquad \text{on } \partial_\ell G^\gamma, \\ \psi_\gamma(.,0) &= h_\gamma \qquad \text{in } B_\gamma^c. \end{aligned} \tag{2.23}$$

Thus the function

$$\tilde{\psi}(x,\tau) = \psi_\gamma(\gamma x, \gamma^2 \tau)$$

satisfies

$$\begin{aligned} \partial_t \tilde{\psi} + \Delta \tilde{\psi} &= 0 \qquad \text{in } G^1 \\ \tilde{\psi} &= 0 \qquad \text{on } \partial_\ell G^1 \\ \tilde{\psi}(.,0) &= \tilde{h} \qquad \text{in } B_1^c, \end{aligned} \tag{2.24}$$

and $\tilde{h}(x) = 1 - |x|^{-1}$. By the maximum principle $0 \le \tilde{\psi} \le 1$, and by Hopf Lemma

$$-\frac{\partial \tilde{\psi}}{\partial \mathbf{n}}\Big|_{\partial B_1 \times [-c,0]} \ge \theta > 0, \tag{2.25}$$

where $\theta = \theta(N, c)$. Then $0 \le \psi_\gamma \le 1$ and

$$-\frac{\partial \psi_\gamma}{\partial \mathbf{n}}\Big|_{\partial B_\gamma \times [-\gamma^2, 0]} \ge \theta/\gamma. \tag{2.26}$$

Multiplying (1.1) by $\psi_\gamma(x, \tau - t) = \psi^*_\gamma(x, \tau)$ and integrating on $B^c_\gamma \times (0, t)$ yields to

$$\int_0^t \int_{B^c_\gamma} u^q \psi^*_r dx d\tau + \int_{B^c_\gamma} (uh_\gamma)(x, t) dx - \int_0^t \int_{\partial B_\gamma} \frac{\partial u}{\partial \mathbf{n}} \psi^*_\gamma dS d\tau = -\int_0^t \int_{\partial B_\gamma} \frac{\partial \psi^*_\gamma}{\partial \mathbf{n}} u d\sigma d\tau. \tag{2.27}$$

Since ψ^*_γ is bounded from above by 1, estimate (2.22) follows from (2.26) and Proposition 2.4 (notice that $B^c_\gamma \times (0, t) \subset \mathcal{E}^c_\gamma$), first by taking $t = T = \gamma^2 \ge (r + 2\rho)^2$, and then for any $t \le \gamma^2$.

If $N \ge 3$, we proceed as above except that we take

$$h_\gamma(x) = 1 - \left(\frac{\gamma}{|x|}\right)^{N-2}.$$

Then $\psi_\gamma(x, t) = h_\gamma(x)$ and $\theta = N - 2$ is independent of the length of the time interval. This leads to the conclusion. □

Lemma 2.7 *I- Let M, $a > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that*

$$0 \le \eta(x) \le M e^{-a|x|^2} \qquad a.e.\ in\ \mathbb{R}^N. \tag{2.28}$$

Then, for any $t > 0$,

$$0 \le \mathbb{H}[\eta](x, t) \le \frac{M}{(4at + 1)^{\frac{N}{2}}} e^{-\frac{a|x|^2}{4at+1}} \qquad \forall x \in \mathbb{R}^N. \tag{2.29}$$

II- Let M, a, $b > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that

$$0 \le \eta(x) \le M e^{-a(|x|-b)^2_+} \qquad a.e.\ in\ \mathbb{R}^N. \tag{2.30}$$

Then, for any $t > 0$,

$$0 \le \mathbb{H}[\eta](x, t) \le \frac{M e^{-\frac{a(|x|-b)^2_+}{4at+1}}}{(4at + 1)^{\frac{N}{2}}} \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0. \tag{2.31}$$

Proof. For the first statement, put $a = \frac{1}{4}s$. Then

$$0 \le \eta(x) \le M(4\pi s)^{\frac{N}{2}} \frac{1}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4s}} = C(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, s).$$

By the order property of the heat kernel,

$$0 \le \mathbb{H}[\eta](x, t) \le M(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, t + s) = M \left(\frac{s}{t + s}\right)^{\frac{N}{2}} e^{-\frac{|x|^2}{4(t+s)}},$$

and (2.29) follows by replacing s by $\frac{1}{4}a$.

For the second statement, let $\tilde{a} < a$ and $R = \max\{e^{-a(r-b)_+^2 + \tilde{a}r^2} : r \geq 0\}$. A direct computation gives $R = e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}$, and (2.31) implies

$$0 \leq \eta(x) \leq Me^{\frac{a\tilde{a}b^2}{a-\tilde{a}}} e^{-\tilde{a}|x|^2}.$$

Applying the statement I, we derive

$$0 \leq \mathbb{H}[\eta](x,t) \leq \frac{Ce^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}}{(4\tilde{a}t+1)^{\frac{N}{2}}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0. \tag{2.32}$$

Since for any $x \in \mathbb{R}^N$ and $t > 0$,

$$(4\tilde{a}t+1)^{-\frac{N}{2}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \leq e^{-\frac{a\tilde{a}b^2}{a-\tilde{a}}} (4at+1)^{-\frac{N}{2}} e^{-\frac{a(|x|-b)^2}{4at+1}},$$

(2.31) follows from (2.32). □

Lemma 2.8 *There exists a constant $C = C(N,q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$u(x,(r+2\rho)^2) \leq C \max\left\{ \frac{r+\rho}{(|x|-r-2\rho)^{N+1}}, \frac{|x|-r-2\rho}{(r+\rho)^{N+1}} \right\} e^{-\frac{(|x|-(r+2\rho))^2}{4(r+2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.33}$$

for any $x \in \mathbb{R}^N \setminus B_{r+3\rho}$.

Proof. It is classical that the Dirichlet heat kernel $H^{B_1^c}$ in the complement of B_1 satisfies, for some $C = C(N) > 0$,

$$H^{B_1^c}(x',y',t',s') \leq C_7 (t'-s')^{-(N+2)/2} (|x'|-1) e^{-\frac{|x'-y'|^2}{4(t'-s')}}, \tag{2.34}$$

for $t' > s'$. By performing the change of variable $x' \mapsto (r+2\rho)x'$, $t' \mapsto (r+2\rho)^2 t'$, for any $x \in \mathbb{R}^N \setminus B_{r+2\rho}$ and $0 \leq t \leq T$, one obtains

$$u(x,t) \leq C(|x|-r-2\rho) \int_0^t \int_{\partial B_{r+2\rho}} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{1+\frac{N}{2}}} u(y,s) d\sigma(y) ds. \tag{2.35}$$

The right-hand side term in (2.35) is smaller than

$$\max\left\{ \frac{C(|x|-r-2\rho)}{(t-s)^{1+\frac{N}{2}}} e^{-\frac{(|x|-r-2\rho)^2}{4(t-s)}} : s \in (0,t) \right\} \int_0^t \int_{\partial B_{r+2\rho}} u(y,s) d\sigma(y) ds.$$

We fix $t = (r+2\rho)^2$ and $|x| \geq r+3\rho$. Since

$$\max\left\{ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : s \in \left(0,(r+2\rho)^2\right) \right\}$$
$$= (|x|-r-2\rho)^{-2-N} \max\left\{ \frac{e^{-\frac{1}{4\sigma}}}{\sigma^{1+\frac{N}{2}}} : 0 < \sigma < \left(\frac{r+2\rho}{|x|-r-2\rho} \right)^2 \right\},$$

a direct computation gives

$$\max\left\{\frac{e^{-\frac{1}{4}\sigma}}{\sigma^{1+\frac{N}{2}}}:0<\sigma<\left(\frac{r+2\rho}{|x|-r-2\rho}\right)^2\right\}$$

$$=\begin{cases}(2N+4)^{1+\frac{N}{2}}e^{-(N+2)/2} & \text{if } r+3\rho\le|x|\le(r+2\rho)(1+\sqrt{4+2N}),\\ \left(\dfrac{|x|-r-2\rho}{r+2\rho}\right)^{2+N}e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2} & \text{if } |x|\ge(r+2\rho)(1+\sqrt{4+2N}).\end{cases}$$

Thus there exists a constant $C(N)>0$ such that

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}}:s\in\left(0,(r+2\rho)^2\right)\right\}\le C(N)\rho^{-2-N}e^{-\left(\frac{|x|-(r+2\rho)}{2r+4\rho}\right)^2}. \tag{2.36}$$

Combining this estimate with (2.22) with $\gamma=r+2\rho$ and (2.35), one derives (2.33). □

Lemma 2.9 *There exists a constant $C=C(N,q)>0$ such that*

$$0\le u(x,(r+2\rho)^2)\le C\max\left\{\frac{(r+\rho)^3}{\rho(|x|-r-2\rho)^{N+1}},\frac{1}{(r+\rho)^{N-1}\rho}\right\}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2}\|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.37}$$

for every $x\in\mathbb{R}^N\setminus B_{r+3\rho}$.

Proof. This is a direct consequence of the inequality

$$(|x|-r-2\rho)e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2}\le\frac{C(r+\rho)^2}{\rho}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2},\qquad\forall x\in B_{r+2\rho}^c, \tag{2.38}$$

and Lemma 2.8. □

Lemma 2.10 *There exists a constant $C=C(N,q)>0$ such that, for any $\eta\in\mathcal{T}_{r,\rho}(K)$, the following estimate holds*

$$u(x,t)\le\frac{C\tilde{M}e^{-\frac{(|x|-r-3\rho)_+^2}{4t}}}{t^{\frac{N}{2}}}\|R[\eta]\|_{L^{q'}}^{q'},\qquad\forall x\in\mathbb{R}^N,\ \forall t\ge(r+2\rho)^2, \tag{2.39}$$

where

$$\tilde{M}=\tilde{M}(x,r,\rho)=\begin{cases}\left(1+\frac{r}{\rho}\right)^{\frac{N}{2}} & \text{if } |x|<r+3\rho\\ \frac{(r+\rho)^{N+3}}{\rho(|x|-r-2\rho)^{N+2}} & \text{if } r+3\rho\le|x|\le c_N^*(r+2\rho)\\ 1+\frac{r}{\rho} & \text{if } |x|\ge c_N^*(r+2\rho)\end{cases} \tag{2.40}$$

with $c_N^*=1+\sqrt{4+2N}$.

Proof. It follows by the maximum principle

$$u(x,t) \le \mathbb{H}[u(.,(r+2\rho)^2)](x,t-(r+2\rho)^2).$$

for $t \ge (r+2\rho)^2$ and $x \in \mathbb{R}^N$. By Lemma 2.5 and Lemma 2.9

$$u(x,(r+2\rho)^2) \le C_{10}\tilde{M} e^{-\frac{(|x|-r-3\rho)^2}{4(r+2\rho)^2}} \left\|R[\eta]\right\|_{L^{q'}}^{q'},$$

where

$$\tilde{M} = \begin{cases} ((r+\rho)\rho)^{-\frac{N}{2}} & \text{if } |x| < r+3\rho \\ \frac{(r+\rho)^3}{\rho}\left(|x|-r-2\rho)\right)^{N+2} & \text{if } r+3\rho \le |x| \le c_N^*(r+2\rho) \\ \frac{1}{(r+\rho)^{N-1}\rho} & \text{if } |x| \ge c_N^*(r+2\rho) \end{cases}$$

Applying Lemma 2.7 with $a=(2r+4\rho)^{-2}$, $b=r+3\rho$ and t replaced by $t-(r+2\rho)^2$ implies

$$u(x,t) \le C\frac{(r+2\rho)^N \tilde{M}}{t^{\frac{N}{2}}} e^{-\frac{(|x|-r-3\rho)^2}{4t}} \left\|R[\eta]\right\|_{L^{q'}}^{q'}, \tag{2.41}$$

for all $x \in B_{r+3\rho}^c$ and $t \ge (r+2\rho)^2$, which is (2.39). □

The next estimate gives a precise upper bound for u when t is not bounded from below.

Lemma 2.11 *Assume that $0 < t \le (r+2\rho)^2$, then there exists a constant $C = C(N,q) > 0$ such that the following estimate holds*

$$u(x,t) \le C(r+\rho)\max\left\{\frac{1}{(|x|-r-2\rho)^{N+1}}, \frac{1}{\rho t^{\frac{N}{2}}}\right\} e^{-\frac{(|x|-r-3\rho)^2}{4t}} \left\|R[\eta]\right\|_{L^{q'}}^{q'}, \tag{2.42}$$

for any $(x,t) \in \mathbb{R}^N \setminus B_{r+3\rho} \times (0,(r+2\rho)^2]$.

Proof. Thanks to (2.22) the following estimate is a straightforward variant of (2.33) for any $|x| \ge r+2\rho$,

$$u(x,t) \le C_8(|x|-r-2\rho)(r+2\rho)\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \le t\right\} \left\|R[\eta]\right\|_{L^{q'}}^{q'}. \tag{2.43}$$

Clearly

$$\begin{aligned} &\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \le t\right\} \\ &= \begin{cases} (2N+4)^{1+\frac{N}{2}}(|x|-r-2\rho)^{-N-2}e^{-\frac{N+2}{2}} & \text{if } 0 < |x| \le r+2\rho+\sqrt{2t(N+2)} \\ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4t}}}{t^{1+\frac{N}{2}}} & \text{if } |x| > r+2\rho+\sqrt{2t(N+2)}. \end{cases} \end{aligned}$$

By elementary analysis, if $x \in B^c_{r+3\rho}$,

$$(|x|-r-2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq e^{-\frac{(|x|-r-3\rho)^2}{4t}} \begin{cases} \rho e^{-\frac{\rho^2}{4t}} & \text{if } 2t < \rho^2 \\ \dfrac{2t}{\rho} e^{-1+\frac{\rho^2}{4t}} & \text{if } \rho^2 \leq 2t \leq 2(r+2\rho)^2. \end{cases}$$

However, since

$$\frac{\rho}{t} e^{-\frac{\rho^2}{4t}} \leq \frac{4}{\rho},$$

we derive

$$(|x|-r-2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq \frac{Ct}{\rho} e^{-\frac{(|x|-r-3\rho)^2}{4t}},$$

and (2.42) follows. □

Lemma 2.12 *Assume $q \geq q_c$. Let $r > 0$, $\rho > 0$ and K be a compact subset of $B_{r+\rho}$. If $\eta \in \mathcal{T}_{r,\rho}(K)$, denote by η_r the function defined by $\eta_r(x) = \eta(rx)$ and*

$$R_r[\eta_r](x,t) = \left(|\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta] + \Delta \mathbb{H}[\eta]|\right)(rx, r^2 t) \qquad \forall (x,t) \in Q_\infty.$$

Then

$$\|R[\eta]\|^{q'}_{L^{q'}} = r^{N-\frac{2}{q-1}} \|R_r[\eta_r]\|^{q'}_{L^{q'}}. \tag{2.44}$$

Furthermore

$$C^{B_{r+\rho}}_{2/q,q'}(K) = r^{N-\frac{2}{q-1}} C^{B_{1+\frac{\rho}{r}}}_{2/q,q'}(K/r), \tag{2.45}$$

and

$$r^{N-\frac{2}{q-1}} C_{2/q,q'}(K/r) \leq C^{B_{r+\rho}}_{2/q,q'}(K) \leq C r^{N-\frac{2}{q-1}} \left(1+\frac{r}{\rho}\right)^{\frac{2}{q-1}} C_{2/q,q'}(K/r). \tag{2.46}$$

Proof. Estimate (2.44) follows from the change of variable $(rx, r^2 t) = (y,s)$. Thus it implies the scaling property (2.47), since there is a one to one correspondence between $\mathcal{T}_{r,\rho}(K)$ and $\mathcal{T}_{1,\frac{\rho}{r}}(K/r)$. In order to prove (2.44) set $K' = K/r \subset B_1$, thus

$$C^{B_{1+\frac{\rho}{r}}}_{2/q,q'}(K') = \inf\left\{\|\zeta\|^{q'}_{W^{2/q,q'}} : \zeta \in \mathcal{T}_{1,\frac{\rho}{r}}(K')\right\}.$$

Let $\phi \in C^2(\mathbb{R}^N)$ be a radial cut-off function such that $0 \leq \rho \leq 1$, $\rho = 1$ on B_1, $\rho = 0$ on $\mathbb{R}^N \setminus B_{1+\frac{\rho}{r}}$, $|\nabla \phi| \leq Cr\rho^{-1}\chi_{B_{1+\frac{\rho}{r}} \setminus B_1}$ and $\left|D^2\phi\right| \leq Cr^2\rho^{-2}\chi_{B_{1+\frac{\rho}{r}} \setminus B_1}$, where C is independent of r and ρ. Let $\zeta \in C^2_0(\mathbb{R}^N)$. Then

$$\nabla(\zeta\phi) = \zeta\nabla\phi + \phi\nabla\zeta\,,\ D^2(\zeta\phi) = \zeta D^2\phi + \phi D^2\zeta + 2\nabla\phi \boxtimes \nabla\zeta.$$

Thus $\|\zeta\phi\|_{L^{q'}(B_{1+\frac{\rho}{r}})} \leq \|\zeta\|_{L^{q'}(\mathbb{R}^N)}$,

$$\int_{B_{1+\frac{\rho}{r}}} |\nabla(\zeta\phi)|^{q'}\, dx \leq C\left(1+\frac{r}{\rho}\right)^{q'} \|\zeta\|^{q'}_{W^{1,q'}},$$

and

$$\int_{B_{r+\rho}} \left|D^2(\zeta\phi)\right|^{q'} dx \leq C\left(1+\frac{r^2}{\rho^2}\right)^{q'} \|\zeta\|^{q'}_{W^{2,q'}}.$$

Finally

$$\|\zeta\phi\|_{W^{2/q,q'}} \leq C\left(1+\frac{r^2}{\rho^2}\right)\|\zeta\|_{W^{2/q,q'}}.$$

Denote by $\mathcal{T}$ the linear mapping $\zeta \mapsto \zeta\phi$. Because

$$W^{2/q,q'} = \left[W^{2,q'}, L^{q'}\right]_{1/q,q'},$$

(here we use the Lions-Petree real interpolation notations and some classical results from [22]), it follows

$$\|\mathcal{T}\|_{\mathcal{L}(W_0^{2/q,q'}(\mathbb{R}^N), W_0^{2/q,q'}(B_{1+\frac{\rho}{r}}))} \leq C(q)\left(1+\frac{r^2}{\rho^2}\right)^{1/q}.$$

Therefore

$$C^{B_{1+\frac{\rho}{r}}}_{2/q,q'}(K') \leq C\left(1+\frac{r^2}{\rho^2}\right)^{\frac{1}{q-1}} C_{2/q,q'}(K').$$

Thus we get the left-hand side of (2.46). The right-hand side is a straightforward consequence of (2.47). □

Remark. In the subcritical case $1 < q < q_c$, estimate (2.46) becomes

$$C^{B_{r+\rho}}_{2/q,q'}(K) \leq C\max\left\{r^N, \rho^N\right\}\left(1+\rho^{-\frac{2}{q-1}}\right) C_{2/q,q'}(K/r). \tag{2.47}$$

By using Lemma 2.11, it is easy to derive from this estimate that any positive solution u of (2.1), the initial trace of which vanishes outside 0, satisfies

$$u(x,t) \leq Ct^{-\frac{1}{q-1}} \min\left\{1, \left(\frac{|x|}{\sqrt{t}}\right)^{\frac{2}{q-1}-N} e^{-\frac{|x|^2}{4t}}\right\} \qquad \forall (x,t) \in Q_\infty. \tag{2.48}$$

This upper estimate corresponds to the one obtained in [7]. If $F = \overline{B}_r$ the upper estimate is less esthetic. However, it is proved in [26] by a barrier method that, if the initial trace of positive solution u of (2.1), vanishes outside F, and if $1 < q < 3$, there holds

$$u(x,t) \leq t^{-\frac{1}{q-1}} f_1((|x|-r)/\sqrt{t}) \qquad \forall (x,t) \in Q_\infty,\ |x| \geq r, \tag{2.49}$$

where $f = f_1$ is the unique positive (and radial) solution of

$$\begin{cases} f'' + \dfrac{y}{2}f' + \dfrac{1}{q-1}f - f^q = 0 \quad \text{in } (0,\infty) \\ f'(0) = 0\,,\ \lim_{y\to\infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \tag{2.50}$$

Notice that the existence of f_1 follows from [7] since q belongs to the subcritical range on exponents in dimension one. Furthermore f_1 has the following asymptotic expansion

$$f_1(y) = Cy^{(3-q)/(q-1)}e^{-y^2/4t}(1+\circ(1))) \quad \text{as } y \to \infty.$$

2.3 The upper Wiener test

Definition 2.13 We define on $\mathbb{R}^N \times \mathbb{R}$ the two *parabolic distances* δ_2 and δ_∞ by

$$\delta_2[(x,t),(y,s)] := \sqrt{|x-y|^2 + |t-s|}, \tag{2.51}$$

and

$$\delta_\infty[(x,t),(y,s)] := \max\{|x-y|, \sqrt{|t-s|}\}. \tag{2.52}$$

If $K \subset \mathbb{R}^N$ and $i = 2, \infty$,

$$\delta_i[(x,t),K] = \inf\{\delta_i[(x,t),(y,0)] : y \in K\} = \begin{cases} \max\left\{\text{dist}\,(x,K), \sqrt{|t|}\right\} & \text{if } i = \infty, \\ \sqrt{\text{dist}^{\,2}(x,K) + |t|} & \text{if } i = 2. \end{cases}$$

For $\beta > 0$ and $i = 2, \infty$, we denote by $\mathcal{B}^i_\beta(m)$ the parabolic ball of center $m = (x,t)$ and radius β in the parabolic distance δ_i.

Let K be *any* compact subset of $\mathbb{R}^N$ and $\overline{u}_K$ the maximal solution of (1.1) which blows up on K. The function $\overline{u}_K$ is constructed in [26] as being the decreasing limit of the $\overline{u}_{K_\epsilon}$ ($\epsilon > 0$) when $\epsilon \to 0$, where

$$K_\epsilon = \{x \in \mathbb{R}^N : \text{dist}\,(x,K) \le \epsilon\}$$

and $\overline{u}_{K_\epsilon} = \lim_{k\to\infty} u_{k,K_\epsilon} = \overline{u}_K$, where u_k is the solution of the classical problem,

$$\begin{cases} \partial_t u_k - \Delta u_k + u_k^q = 0 & \text{in } Q_T, \\ u_k = 0 & \text{on } \partial_\ell Q_T, \\ u_k(.,0) = k\chi_{K_\epsilon} & \text{in } \mathbb{R}^N. \end{cases} \tag{2.53}$$

If $(x,t) = m \in \mathbb{R}^N \times (0,T]$, we set $d_K = \text{dist}\,(x,K)$, $D_K = \max\{|x-y| : y \in K\}$ and $\lambda = \sqrt{d_K^2 + t} = \delta_2[m,K]$. We define a slicing of K, by setting $d_n = d_n(K,t) := \sqrt{nt}$ ($n \in \mathbb{N}$),

$$T_n = \overline{B}_{d_{n+1}}(x) \setminus B_{d_n}(x), \qquad \forall n \in \mathbb{N},$$

thus $T_0 = B_{\sqrt{t}}(x)$, and

$$K_n(x) = K \cap T_n(x) \ \text{ for } n \in \mathbb{N} \text{ and } \mathcal{Q}_n(x) = K \cap B_{d_{n+1}}(x).$$

When there is no ambiguity, we will skip the x variable in the above sets. The main result of this section is the following discrete upper Wiener-type estimate.

Theorem 2.14 *Assume $q \geq q_c$. Then there exists $C = C(N, q, T) > 0$ such that*

$$\overline{u}_K(x,t) \leq \frac{C}{t^{\frac{N}{2}}} \sum_{n=0}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \qquad \forall (x,t) \in Q_T, \tag{2.54}$$

where a_t is the largest integer j such that $K_j \neq \emptyset$.

With no loss of generality, we can first assume that $x = 0$. Furthermore, in considering the scaling transformation $u_\ell(y,t) = \ell^{\frac{1}{q-1}} u(\sqrt{\ell}y, \ell t)$, with $\ell > 0$, we can assume $t = 1$. Thus the new compact singular set of the initial trace becomes $K/\sqrt{\ell}$, that we still denote K. We also set $a_{_K} = a_{_{K,1}}$ Since for each $n \in \mathbb{N}$,

$$\frac{1}{2\sqrt{n+1}} \leq d_{n+1} - d_n \leq \frac{1}{\sqrt{n+1}},$$

it is possible to exhibit a collection Θ_n of points $a_{n,j}$ with center on the sphere $\Sigma_n = \{y \in \mathbb{R}^N : |y| = (d_{n+1} + d_n)/2\}$, such that

$$T_n \subset \bigcup_{a_{n,j} \in \Theta_n} B_{1/\sqrt{n+1}}(a_{n,j}), \quad |a_{n,j} - a_{n,k}| \geq 1/2\sqrt{n+1} \text{ and } \#\Theta_n \leq C n^{N-1},$$

for some constant $C = C(N)$. If $K_{n,j} = K_n \cap B_{1/\sqrt{n+1}}(a_{n,j})$, there holds

$$K = \bigcup_{0 \leq n \leq a_{_K}} \bigcup_{a_{n,j} \in \Theta_n} K_{n,j}.$$

The first intermediate step is based on the *quasi-additivity* property of capacities developed in [2].

Lemma 2.15 *Let $q \geq q_c$. There exists a constant $C = C(N, q)$ such that*

$$\sum_{a_{n,j} \in \Theta_n} C_{2/q,q'}(K_{n,j}) \leq C n^{\frac{N}{2} - \frac{1}{q-1}} C_{2/q,q'}\left(\frac{K_n}{\sqrt{n+1}}\right) \qquad \forall n \in \mathbb{N}_*, \tag{2.55}$$

where $B_{n,j} = B_{2/\sqrt{n+1}}(a_{n,j})$ and $C_{2/q,q'}$ stands for the capacity taken with respect to $\mathbb{R}^N$.

Proof. The following result is proved in [2, Th 3]: if the spheres $B_{\rho_j^\theta}(b_j)$ are disjoint in $\mathbb{R}^N$ and G is an analytic subset of $\bigcup B_{\rho_j}(b_j)$ where the ρ_j are positive and smaller than some $\rho^* > 0$, there holds

$$C_{2/q,q'}(G) \leq \sum_j C_{2/q,q'}(G \cap B_{\rho_j}(b_j)) \leq A C_{2/q,q'}(G), \tag{2.56}$$

where $\theta = 1 - 2/N(q-1)$, for some A depending on N, q and ρ^*. This property is called *quasi-additivity*. We define for $n \in \mathbb{N}_*$,

$$\tilde{T}_n = \sqrt{n+1}\, T_n, \quad \tilde{K}_n = \sqrt{n+1}\, K_n \quad \text{and} \quad \tilde{\mathcal{Q}}_n = \sqrt{n+1}\, \mathcal{Q}_n.$$

Since $K_{n,j} \subset B_{1/\sqrt{n+1}}(a_{n,j})$ and the $C_{2/q,q'}$ capacities are taken with respect to the balls $B_{2/\sqrt{n+1}}(a_{n,j}) = B_{n,j}$. By Lemma 2.12 with $r = \rho = 1/\sqrt{n+1}$

$$C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \leq C(n+1)^{\frac{1}{q-1}-\frac{N}{2}} C_{2/q,q'}(\tilde{K}_{n,j}), \tag{2.57}$$

where $\tilde{K}_{n,j} = \sqrt{n+1}\,K_{n,j}$ and $\tilde{B}_{n,j} = \sqrt{n+1}\,B_{n,j}$. For a fixed $n > 0$ and each repartition Λ of points $\tilde{a}_{n,j} = \sqrt{n+1}\,a_{n,j}$ such that the balls $B_{2^\theta}(\tilde{a}_{n,j})$ are disjoint, the quasi-additivity property holds in the following sense: if we set

$$K_{n,\Lambda} = \bigcup_{a_{n,j}\in\Lambda} K_{n,j}\ , \quad \tilde{K}_{n,\Lambda} = \sqrt{n+1}\,K_{n,\Lambda} = \bigcup_{a_{n,j}\in\Lambda} \tilde{K}_{n,j} \quad \text{and} \quad \tilde{K}_n = \sqrt{n+1}\,K_n,$$

then

$$\sum_{a_{n,j}\in\Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \leq A C_{2/q,q'}(\tilde{K}_{n,\Lambda}). \tag{2.58}$$

The maximal cardinal of any such repartition Λ is of the order of Cn^{N-1} for some positive constant $C = C(N)$, therefore, the number of repartitions needed for a full covering of the set $\tilde{T}_n$ is of finite order depending upon the dimension. Because $\tilde{K}_n$ is the union of the $\tilde{K}_{n,\Lambda}$,

$$\sum_{a_{n,j}\in\Theta_n} C_{2/q,q'}(\tilde{K}_{n,j}) = \sum_{\Lambda} \sum_{a_{n,j}\in\Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \leq C\, C_{2/q,q'}(\tilde{K}_n). \tag{2.59}$$

Since, by Lemma 2.12,

$$C_{2/q,q'}(\tilde{K}_n) \leq C^{B_{2(n+1)}}_{2/q,q'}(\tilde{K}_n) = (n+1)^{N-\frac{1}{q-1}} C^{B_2}_{2/q,q'}\left(\frac{K_n}{\sqrt{n+1}}\right) \leq C(n+1)^{N-\frac{1}{q-1}} C_{2/q,q'}\left(\frac{K_n}{\sqrt{n+1}}\right),$$

we obtain (2.55) by combining this last inequality with (2.57) and (2.59). □

Proof of Theorem 2.14. Step 1. We first notice that

$$\overline{u}_K \leq \sum_{0\leq n\leq a_K} \sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j}}. \tag{2.60}$$

Actually, since $K = \bigcup_n \bigcup_{a_{n,j}} K_{n,j}$, for any $0 < \epsilon' < \epsilon$, there holds $\overline{K_{\epsilon'}} \subset \bigcup_n \bigcup_{a_{n,j}} K_{n,j\,\epsilon}$. Because a finite sum of positive solutions of (1.1) is a super solution,

$$\overline{u}_{K_{\epsilon'}} \leq \sum_{0\leq n\leq a_K} \sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j\,\epsilon}}. \tag{2.61}$$

Letting successively ϵ' and ϵ go to 0 implies (2.60).

Step 2. Let $n \in \mathbb{N}$. Since $K_{n,j} \subset B_{1/\sqrt{n+1}}(a_{n,j})$ and $|x - a_{n,j}| = (d_n + d_{n+1})/2 = (\sqrt{n+1} + \sqrt{n})/2$, we can apply the previous lemmas with $r = 1/\sqrt{n+1}$ and $\rho = r$. For $n \geq n_N$ there

holds $t = 1 \geq (r + 2\rho)^2 = 9/(n + 1)$ and $|x - a_{n,j}| = (\sqrt{n+1} - \sqrt{n})/2 \geq (2 + C_N)(3/\sqrt{n+1})$ (notice that $n_N \geq 8$). Thus

$$u_{K_{n,j}}(0,1) \leq Ce^{(\sqrt{n}-3/\sqrt{n+1})^2/4} C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \leq Ce^{3/2} e^{-\frac{n}{4}} C_{2/q,q'}^{B_{n,j}}(K_{n,j}). \tag{2.62}$$

Using Lemma 2.15 we obtain, with $d_n = d_n(1) = \sqrt{n+1}$

$$\sum_{n=n_N}^{a_K} \sum_{a_{n,j} \in \Theta_n} u_{K_{n,j}}(0,1) \leq C \sum_{n=n_N}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \tag{2.63}$$

Finally, we apply Lemma 2.5 if $1 \leq n < n_N$ and get

$$\begin{aligned} \sum_{1}^{n_N-1} \sum_{a_{n,j} \in \Theta_n} u_{K_{n,j}}(0,1) &\leq C \sum_{1}^{n_N-1} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right) \\ &\leq C' \sum_{1}^{n_N-1} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \end{aligned} \tag{2.64}$$

For $n = 0$, we proceed similarly, in splitting K_1 in a finite number of $K_{1,i}$, depending only on the dimension, such that $\operatorname{diam} K_{1,i} < 1/3$. Combining (2.63) and (2.64), we derive

$$\overline{u}_K(0,1) \leq C \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \tag{2.65}$$

In order to derive the same result for any $t > 0$, we notice that

$$\overline{u}_K(y,t) = t^{-\frac{1}{q-1}} \overline{u}_{K\sqrt{t}}(y\sqrt{t}, 1).$$

Going back to the definition of $d_n = d_n(K,t) = \sqrt{nt} = d_n(K\sqrt{t}, 1)$, we derive from (2.65) and the fact that $a_{K,t} = a_{K\sqrt{t},1}$

$$\overline{u}_K(0,t) \leq Ct^{-\frac{1}{q-1}} \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right), \tag{2.66}$$

with $d_n = d_n(t) = \sqrt{t(n+1)}$. This is (2.54) with $x = 0$, and a space translation leads to the final result. □

Proof of Theorem 2.1. Let $m > 0$ and $F_m = F \cap \overline{B}_m$. We denote by $U_{B_m^c}$ the maximal solution of (1.1) in Q_∞ the initial trace of which vanishes on B_m. Such a solution is actually the unique solution of (2.1) which satisfies

$$\lim_{t \to 0} u(x,t) = \infty$$

uniformly on $B_{m'}^c$, for any $m' > m$: this can be easily proved by noticing that

$$U_{B_m^c}{}_\ell(y,t) = \ell^{\frac{1}{q-1}} U_{B_m^c}(\sqrt{\ell}y, \ell t) = U_{B^c_{m/\sqrt{\ell}}}(y,t).$$

Furthermore

$$\lim_{m\to\infty} U_{B_m^c}(y,t) = \lim_{m\to\infty} m^{-\frac{2}{q-1}} U_{B_1^c}(y/m, t/m^2) = 0$$

uniformly on any compact subset of $\overline{Q}_\infty$. Since $\overline{u}_{F_m} + U_{B_m^c}$ is a super-solution, it is larger that $\overline{u}_F$ and therefore $\overline{u}_{F_m} \uparrow \overline{u}_F$. Because $W_{F_m}(x,t) \le W_F(x,t)$ and $\overline{u}_{F_m} \le C_1 W_{F_m}(x,t)$, the result follows. □

Theorem 2.1 admits the following integral expression.

Theorem 2.16 *Assume $q \ge q_c$. Then there exists a positive constant $C_1^* = C^*(N,q,T)$ such that, for any closed subset F of $\mathbb{R}^N$, there holds*

$$\overline{u}_F(x,t) \le \frac{C_1^*}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{1}{s}F \cap B_1(x)\right) s\, ds, \tag{2.67}$$

where $a_t = \min\{n : F \subset B_{\sqrt{n+1)t}}(x)\}$.

Proof. We first use

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \le C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right),$$

and we denote

$$\Phi(s) = C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \qquad \forall s > 0. \tag{2.68}$$

Step 1. The following inequality holds

$$c_1\Phi(\alpha s) \le \Phi(s) \le c_2\Phi(\beta s) \qquad \forall s > 0, \quad \forall 1/2 \le \alpha \le 1 \le \beta \le 2, \tag{2.69}$$

for some positive constants c_1, c_2 depending on N and q. See [1] and [30]. If $\beta \in [1,2]$,

$$\Phi(\beta s) = C_{2/q,q'}\left(\frac{1}{\beta}\left(\frac{F}{s} \cap B_\beta\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\beta\right) \ge c_1\Phi(s).$$

If $\alpha \in [1/2,1]$,

$$\Phi(\alpha s) = C_{2/q,q'}\left(\frac{1}{\alpha}\left(\frac{F}{s} \cap B_\alpha\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\alpha\right) \le c_2\Phi(s).$$

Step 2. By (2.69)

$$C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \le c_2 C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \qquad \forall\, s \in [d_{n+1}, d_{n+2}],$$

and $n \le a_t$. Then

$$c_2 \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\, ds$$

$$\ge C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} s\, ds.$$

Using the fact that $N - \frac{2}{q-1} \geq 0$, we get,

$$\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} s\, ds \geq e^{-\frac{n+2}{4}} d_{n+1}^{N-\frac{2}{q-1}+1} (d_{n+2} - d_{n+1}) \tag{2.70}$$

$$\geq \frac{t}{4e^2} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}}. \tag{2.71}$$

Thus

$$\overline{u}_F(x,t) \leq \frac{C}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'} \left(\frac{1}{s} F \cap B_1 \right) s\, ds, \tag{2.72}$$

which ends the proof. □

3 Estimate from below

If $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N)$, we denote by $u_\mu = u_{\mu,0}$ the solution of

$$\begin{cases} \partial_t u_\mu - \Delta u_\mu + u_\mu^q = 0 & \text{in } Q_T, \\ u_\mu(.,0) = \mu & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

The maximal σ-moderate solution of (1.1) which has an initial trace vanishing outside a closed set F is defined by

$$\underline{u}_F = \sup \left\{ u_\mu : \mu \in \mathfrak{M}_+^q(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N),\ \mu(F^c) = 0 \right\}. \tag{3.2}$$

The main result of this section is the next one

Theorem 3.1 *Assume $q \geq q_c$. There exists a constant $C_2 = C_2(N,q,T) > 0$ such that, for any closed subset $F \subset \mathbb{R}^N$, there holds*

$$\underline{u}_F(x,t) \geq C_2 W_F(x,t) \qquad \forall (x,t) \in Q_T. \tag{3.3}$$

We first assume that F is compact, and we denote it by K. The first observation is that if $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N)$, $u_\mu \in L^q(Q_T)$ (see lemma below) and $0 \leq u_\mu \leq \mathbb{H}[\mu] := \mathbb{H}_\mu$. Therefore

$$u_\mu \geq \mathbb{H}_\mu - \mathbb{G}\left[\mathbb{H}_\mu^q\right], \tag{3.4}$$

where $\mathbb{G}$ is the Green heat potential in Q_T defined by

$$\mathbb{G}[f](t) = \int_0^t \mathbb{H}[f(s)](t-s)ds = \int_0^t \int_{\mathbb{R}^N} H(.,y,t-s) f(y,s) dy ds.$$

Since the details of the proof are very technical, we present below its main streams. The key idea is to construct, for any $(x,t) \in Q_T$, a measure $\mu = \mu(x,t) \in \mathfrak{M}_+^q(\mathbb{R}^N)$ such that there holds

$$\mathbb{H}_\mu(x,t) \geq C W_K(x,t) \qquad \forall (x,t) \in Q_T, \tag{3.5}$$

and

$$\mathbb{G}\,(\mathbb{H}_\mu)^q \le C\,\mathbb{H}_\mu \quad \text{in } Q_T, \tag{3.6}$$

with constants C depends only on N, q, and T, then to replace μ by $\mu_\epsilon = \epsilon\mu$ with $\epsilon = (2C)^{-\frac{1}{q-1}}$ in order to derive

$$u_{\mu_\epsilon} \ge 2^{-1}\mathbb{H}_{\mu_\epsilon} \ge 2^{-1}CW_K. \tag{3.7}$$

From this follows

$$\underline{u}_K \ge 2^{-1}\mathbb{H}_{\mu_\epsilon} \ge 2^{-1}CW_K. \tag{3.8}$$

and the proof of Theorem 3.1 with $C_2 = 2^{-1}C$.

We recall the following regularity result which actually can be used for defining the norm in negative Besov spaces [36]

Lemma 3.2 *There exists a constant $c > 0$ such that*

$$c^{-1}\|\mu\|_{W^{-2/q,q}(\mathbb{R}^N)} \le \|\mathbb{H}_\mu\|_{L^q(Q_T)} \le c\|\mu\|_{W^{-2/q,q}(\mathbb{R}^N)} \tag{3.9}$$

for any $\mu \in W^{-2/q,q}(\mathbb{R}^N)$.

3.1 Estimate from below of the solution of the heat equation

The purely spatial slicing used is the trace on $\mathbb{R}^N \times \{0\}$ of an *extended slicing* in Q_T which is constructed as follows: if K is a compact subset of $\mathbb{R}^N$, $m = (x,t)$, we define d_K, λ, d_n and a_t as in Section 2.3. Let $\alpha \in (0,1)$ to be fixed later on, we define $\mathcal{T}_n$ for $n \in \mathbb{Z}$ by

$$\mathcal{T}_n = \begin{cases} \mathcal{B}^2_{\sqrt{t(n+1)}}(m) \setminus \mathcal{B}^2_{\sqrt{tn}}(m) & \text{if } n \ge 1, \\ \mathcal{B}^2_{\alpha^{-n}\sqrt{t}}(m) \setminus \mathcal{B}^2_{\alpha^{1-n}\sqrt{t}}(m) & \text{if } n \le 0, \end{cases}$$

and put

$$\mathcal{T}^*_n = \mathcal{T}_n \cap \{s : 0 \le s \le t\}, \quad \text{for } n \in \mathbb{Z}.$$

We recall that for $n \in \mathbb{N}_*$,

$$\mathcal{Q}_n = K \cap \mathcal{B}^2_{\sqrt{t(n+1)}}(m) = K \cap B_{d_n}(x)$$

and

$$K_n = K \cap \mathcal{T}_{n+1} = K \cap \left(B_{d_{n+1}}(x) \setminus B_{d_n}(x)\right).$$

Let $\nu_n \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap W^{-2/q,q}(\mathbb{R}^N)$ be the *q-capacitary measure* of the set K_n/d_{n+1}. See [1, Sec. 2.2]. Such a measure has support in K_n/d_{n+1} and

$$\nu_n(K_n/d_{n+1}) = C_{2/q,q'}(K_n/d_{n+1}) \text{ and } \|\nu_n\|_{W^{-2/q,q'}(\mathbb{R}^N)} = \left(C_{2/q,q'}(K_n/d_{n+1})\right)^{1/q}. \tag{3.10}$$

We define μ_n as follows

$$\mu_n(A) = d_{n+1}^{N-\frac{2}{q-1}}\nu_n(A/d_{n+1}) \qquad \forall A \subset K_n,\ A \text{ Borel}, \tag{3.11}$$

and set

$$\mu_{t,K} = \sum_{n=0}^{a_t} \mu_n,$$

and

$$\mathbb{H}_{\mu_{t,K}} = \sum_{n=0}^{a_t} \mathbb{H}_{\mu_n}. \tag{3.12}$$

Proposition 3.3 *Let $q \geq q_c$, then there holds*

$$\mathbb{H}_{\mu_{t,K}}(x,t) \geq \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{n=0}^{a_t} e^{-\frac{n+1}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}}\right), \tag{3.13}$$

in $\mathbb{R}^N \times (0,T)$.

Proof. Since

$$\mathbb{H}_{\mu_n}(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n, \tag{3.14}$$

and

$$y \in K_n \Longrightarrow |x-y| \leq d_{n+1},$$

(3.13) follows because of (3.11) and (3.12). □

3.2 Estimate from above of the nonlinear term

We write (3.4) under the form

$$\begin{aligned} u_\mu(x,t) &\geq \sum_{n\in\mathbb{Z}} \mathbb{H}_{\mu_n}(x,t) - \int_0^t \int_{\mathbb{R}^N} H(x,y,t-s) \left[\sum_{n\in A_K} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\ &= I_1 - I_2. \end{aligned} \tag{3.15}$$

since $\mu_n = 0$ if $n \notin A_K = \mathbb{N} \cap [1, a_t]$, and

$$\begin{aligned} I_2 &= \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \int_{\mathbb{R}^N} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n\in A_K} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\ &= \frac{1}{(4\pi)^{\frac{N}{2}}} (J_\ell + J'_\ell), \end{aligned} \tag{3.16}$$

for some $\ell \in \mathbb{N}^*$ to be fixed later on, where

$$J_\ell = \sum_{p\in\mathbb{Z}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n<p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds,$$

and

$$J'_\ell = \sum_{p\in\mathbb{Z}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n\geq p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds.$$

The next estimate will be used several times in the sequel.

Lemma 3.4 *Let $0 < a < b$ and $t > 0$, then,*

$$\max\left\{\sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}} : 0 \leq \sigma \leq t,\ at \leq \rho^2 + \sigma \leq bt\right\} = e^{\frac{1}{4}} \begin{cases} t^{-\frac{N}{2}} e^{-\frac{a}{4}} & \text{if } \dfrac{a}{2N} > 1, \\ \left(\dfrac{2N}{at}\right)^{\frac{N}{2}} e^{-\frac{N}{2}} & \text{if } \dfrac{a}{2N} \leq 1. \end{cases}$$

Proof. Set

$$\mathcal{J}(\rho,\sigma) = \sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}}$$

and

$$\mathcal{K}_{a,b,t} = \left\{(\rho,\sigma) \in [0,\infty)\times(0,t] :\ at \leq \rho^2 + \sigma \leq bt\right\}.$$

We first notice that, for fixed σ, the maximum of $\mathcal{J}(.,\sigma)$ is achieved for ρ minimal. If $\sigma \in [at, bt]$ the minimal value of ρ is 0, while if $\sigma \in (0, at)$, the minimum of ρ is $\sqrt{at-s}$.

- Assume first $a \geq 1$, then $\mathcal{J}(\sqrt{at-\sigma},\sigma) = e^{\frac{1}{4}}\sigma^{-\frac{N}{4}} e^{-\frac{at}{4\sigma}}$. Thus if $1 \leq a/2N$, the minimal value of $\mathcal{J}(\sqrt{at-\sigma},\sigma)$ is $e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}$, while if $a/2N < 1 \leq a$, the minimum is $e^{\frac{1}{4}} t^{-\frac{N}{2}} e^{-\frac{a}{4}}$.
- Assume now $a \leq 1$. Then

$$\begin{aligned} \max\{\mathcal{J}(\rho,\sigma) :\ (\rho,\sigma) \in \mathcal{K}_{a,b,t}\} &= \max\left\{\max_{\sigma\in(at,t]} \mathcal{J}(0,\sigma),\ \max_{\sigma\in(0,at]} \mathcal{J}(\sqrt{at-\sigma},\sigma)\right\} \\ &= \max\left\{(at)^{-\frac{N}{2}},\ e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}\right\} \\ &= e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}. \end{aligned}$$

Combining these two estimates, we derive the result. □

Remark. The following variant of Lemma 3.4 will be useful in the sequel: *For any $\theta \geq 1/2N$ there holds*

$$\max\{\mathcal{J}(\rho,\sigma) :\ (\rho,\sigma) \in \mathcal{K}(a,b,t)\} \leq e^{\frac{1}{4}}\left(\frac{2N\theta}{t}\right)^{\frac{N}{2}} e^{-\frac{a}{4}} \quad \text{if } \theta a \geq 1. \tag{3.17}$$

Lemma 3.5 *There exists a positive constant $C = C(N,\ell,q)$ such that*

$$J_\ell \leq C t^{-\frac{N}{2}} \sum_{n=1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-(1+(n-\ell)_+)/4}\, C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.18}$$

Proof. The set of the p's for the summation in J_ℓ is reduced to $\mathbb{Z}\cap[-\ell+2,\infty)$, thus we write

$$J_\ell = J_{1,\ell} + J_{2,\ell}$$

where

$$J_{1,\ell} = \sum_{p=2-\ell}^{0} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n<p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q$$

and

$$J_{2,\ell} = \sum_{p=1}^{\infty} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n<p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q .$$

If $p = 2-\ell,\dots,0$,

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow t\alpha^{2-2p} \le |x-y|^2 + t - s \le t\alpha^{-2p},$$

and, if $p\ge 1$

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow pt \le |x-y|^2 + t - s \le (p+1)t.$$

By Lemma 3.4 and (3.17), there exists $C = C(N,\ell,\alpha) > 0$ such that

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y,s)\in\mathcal{T}_p^*\right\} \le Ct^{-\frac{N}{2}} e^{-\alpha^{2-2p}/4}, \tag{3.19}$$

if $p = 2-\ell,\dots,0$, and

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y,s)\in\mathcal{T}_p^*\right\} \le Ct^{-\frac{N}{2}} e^{-p/4}, \tag{3.20}$$

if $p\ge 1$. When $p = 2-\ell,\dots,0$

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \le C\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}^q(y,s), \tag{3.21}$$

for some $C = C(\ell,q) > 0$, thus

$$\begin{aligned} J_{1,\ell} &\le Ct^{-\frac{N}{2}} \sum_{p=2-\ell}^{0} e^{-\frac{\alpha^{2-2p}}{4}} \sum_{n=1}^{p+\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q \\ &\le Ct^{-\frac{N}{2}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q \sum_{p=n-\ell+1}^{0} e^{-\frac{\alpha^{2-2p}}{4}} \\ &\le Ct^{-\frac{N}{2}} e^{-\frac{\alpha^{2\ell-2}}{4}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q . \end{aligned} \tag{3.22}$$

If the set of p's is not upper bounded, we introduce some parameter $\delta > 0$ to be made precise later on. Then

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \le \left[\sum_{1}^{p+\ell-1} e^{\delta q'\frac{n}{4}}\right]^{q/q'} \sum_{1}^{p+\ell-1} e^{-\frac{\delta q n}{4}} \mathbb{H}_{\mu_n}^q(y,s), \tag{3.23}$$

with $q' = q/(q-1)$. If, by convention $\mu_n = 0$ whenever $n > a_t$, we obtain, for some $C > 0$ which depends also on δ,

$$\begin{aligned} J_{2,\ell} &\le Ct^{-\frac{N}{2}} \sum_{p=1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \sum_{n=1}^{p+\ell-1} e^{-\frac{\delta q n}{4}} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \\ &\le Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} e^{-\frac{\delta q n}{4}} \sum_{p=(n-\ell+1)\vee 1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \\ &\le Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{1+(n-\ell)_+}{4}} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)}. \end{aligned} \tag{3.24}$$

Notice that we choose δ such that $\delta \ell q < 1$. Combining (3.22) and (3.24), we derive (3.18) from Lemma 3.2, (3.10) and (3.11). □

The set of indices p for which the μ_n terms are not zero in J'_ℓ is $\mathbb{Z} \cap (-\infty, a_t - \ell]$. We write

$$J'_\ell = J'_{1,\ell} + J'_{2,\ell},$$

where

$$J'_{1,\ell} = \sum_{p=-\infty}^{0} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=1\vee p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s) \right]^q dyds,$$

and

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s) \right]^q dyds.$$

Lemma 3.6 *There exists a constant $C = C(N, q, \ell) > 0$ such that*

$$J'_{1,\ell} \le Ct^{1-\frac{Nq}{2}} \sum_{n=0}^{a_t} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} d_{n+1}^{Nq-2q'} C^q_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right), \tag{3.25}$$

where $\beta_0 = (q-1)/4$ and $h = 2q(q+1)/(q-1)^2$.

Proof. Since

$$(y,s) \in \mathcal{T}_p^*, \text{ and } (z,0) \in K_n \Longrightarrow |y-z| \ge (\sqrt{n} - \alpha^{-p})\sqrt{t}, \tag{3.26}$$

there holds

$$\mathbb{H}_{\mu_n}(y,s) \le (4\pi s)^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2 t}{4s}} \mu_n(K_n) \le Ct^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n(K_n),$$

by Lemma 3.4. Let $\{\epsilon_n\}$ be a sequence of positive numbers such that

$$A_\epsilon = \sum_{n=1}^{\infty} \epsilon_n^{q'} < \infty,$$

then

$$
\begin{aligned}
J'_{1,\ell} &\le CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{p=-\infty}^{0} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \sum_{n=1\vee(p+\ell)}^{\infty} \epsilon_n^{-q} e^{-q\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n^q(K_n) ds\, dy \\
&\le CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) \sum_{-\infty}^{p=0\wedge(n-\ell)} e^{-\frac{q(\sqrt{n}-\alpha^{-p})^2}{4}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds\, dy \\
&\le CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-\frac{q(\sqrt{n}-1)^2}{4}} \iint_{\{\cup_{p\le 0}\mathcal{T}_p^*\}} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds\, dy \\
&\le CA_\epsilon^{q/q'} t^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-\frac{q(\sqrt{n}-1)^2}{4}}.
\end{aligned}
\tag{3.27}
$$

Set $h = 2q(q+1)/(q-1)^2$ and $Q = (1+q)/2$, then $q(\sqrt{n}-1)^2 \ge Q(n-h)_+$ for any $n \ge 1$. If we choose $\epsilon_n = e^{-\frac{(q-1)(n-h)_+}{16q}}$, there holds $\epsilon_n^{-q} e^{-\frac{q(\sqrt{n}-1)^2}{4}} \le e^{-\frac{(q+3)(n-h)_+}{16}}$. Finally

$$
J'_{1,\ell} \le Ct^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} \mu_n^q(K_n),
$$

with $\beta_0 = (q-1)/4$, which yields to (3.25) by the choice of the μ_n. □

In order to make easier the obtention of the estimate of the term $J'_{2,\ell}$, we first give the proof in dimension 1.

Lemma 3.7 *Assume $N = 1$ and ℓ is an integer larger than 1. There exists a positive constant $C = C(q,\ell) > 0$ such that*

$$
J'_{2,\ell} \le Ct^{-1/2} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}}\right). \tag{3.28}
$$

Proof. If $(y,s) \in \mathcal{T}_p^*$ and $z \in K_n$ ($p \ge 1$, $n \ge p = \ell$) , there holds $|x-y| \ge \sqrt{t}\sqrt{p}$ and $|y-z| \ge \sqrt{t}(\sqrt{n} - \sqrt{p+1})$. Therefore

$$
J'_{2,\ell} \le C\sqrt{t} \sum_{p=1}^{a_t-\ell} \frac{1}{\sqrt{p}} \int_0^t e^{-\frac{pt}{4(t-s)}} \left(\sum_{n=p+\ell}^{a_t} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q.
$$

If $\epsilon \in (0,q)$ is some positive parameter which will be made more precise later on, there holds

$$
\begin{aligned}
&\left(\sum_{n=p+\ell}^{a_t} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q \\
&\qquad \le \left(\sum_{n=p+\ell}^{a_t} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \right)^{q/q'} \sum_{n=p+\ell}^{a_t} s^{-\frac{q}{2}} e^{-(q-\epsilon)\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n^q(K_n),
\end{aligned}
$$

by Hölder's inequality. By comparison between series and integrals and using Gauss integral

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} &\leq \int_{p+\ell}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\ &= 2\int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} (x+\sqrt{p+1}) dx \\ &\leq \frac{4s}{\epsilon q' t} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{4s}} + 2\sqrt{p+1}\int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} dx \\ &\leq C\sqrt{\frac{(p+1)s}{t}} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{2s}} \\ &\leq C\sqrt{\frac{(p+1)s}{t}}.\end{aligned}$$

If we set $q_\epsilon = q - \epsilon$, then

$$J'_{2,\ell} \leq C\epsilon^{-q'/q} t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{\infty} \mu_n^q(K_n) \sum_{p=1}^{n-\ell} p^{\frac{q-2}{2}} \int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds.$$

where $C = C(\epsilon, q) > 0$. Since

$$\begin{aligned}&\int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \\ &\qquad = \int_0^1 (1-s)^{-1/2} s^{-1/2} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}} ds,\end{aligned}$$

we can apply Lemma A.1 with $a = 1/2$, $b = 1/2$, $A = \sqrt{p}$ and $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$. In this range of indices $B \geq \sqrt{q_\epsilon}(\sqrt{p+\ell} - \sqrt{p+1}) \geq \frac{\sqrt{q_\epsilon}(\ell-1)}{\sqrt{p}}$, thus $\kappa = \sqrt{q_\epsilon}(\ell-1)$ and

$$\sqrt{\frac{A}{A+B}}\sqrt{\frac{B}{A+B}} \leq p^{\frac{1}{4}} n^{-1/2} (\sqrt{n} - \sqrt{p})^{1/2}.$$

Therefore

$$\int_0^t (t-s)^{-1/2} s^{-\frac{q}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \leq \frac{C p^{\frac{1}{4}} (\sqrt{n}-\sqrt{p})^{1/2}}{\sqrt{n}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \quad (3.29)$$

which implies

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} \frac{\mu_n^q(K_n)}{\sqrt{n}} \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{1/2} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \quad (3.30)$$

where C depends of ϵ, q and ℓ. By Lemma A.2

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{q-3}{2}} e^{-\frac{n}{4}} \mu_n^q(K_n) \quad (3.31)$$

Because $\mu_n(K_n) = d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right)$ (remember $N=1$) and diam $\frac{K_n}{d_{n+1}} \le n^{-1}$, there holds

$$\mu_n^q(K_n) \le C\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{q-3} \mu_n(K_n) = C\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{q-3} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}(K_n/d_{n+1}) \tag{3.32}$$

and inequality (3.28) follows. □

Next we give the general proof, unfortunately much more technical. For this task we will use again the notion of quasi-additivity with separated partitions and all the results presented in the Appendix.

Lemma 3.8 *Assume $N \ge 2$ and ℓ is an integer larger than 1. There exists a positive constant $C_1 = C_1(q, N, \ell) > 0$ such that*

$$J'_{2,\ell} \le C_1 t^{-\frac{N}{2}} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.33}$$

Proof. As in the proof of Theorem 2.14, we know that there exists a finite number J, depending only on the dimension N, of separated sub-partitions $\{\#\Theta_{t,n}^h\}_{h=1}^J$ of the rescaled sets $\tilde{T}_n = \sqrt{\frac{n+1}{t}}\, T_n$ by the N-dim balls $B_2(\tilde{a}_{n,j})$ where $\tilde{a}_{n,j} = \sqrt{\frac{n+1}{t}}\, a_{n,j}$, $|a_{n,j}| = \dfrac{d_{n+1}+d_n}{2}$ and $|a_{n,j} - a_{n,k}| \ge \sqrt{\frac{4t}{n+1}}$. Furthermore $\#\Theta_{t,n}^h \le Cn^{N-1}$. We denote $K_{n,j} = K_n \cap B_{\sqrt{\frac{t}{n+1}}}(a_{n,j})$. We write $\mu_n = \sum_{h=1}^J \mu_n^h$, and accordingly $J'_{2,\ell} = \sum_{h=1}^J J'^{\,h}_{2,\ell}$, where $\mu_n^h = \sum_{j\in\Theta_{t,n}^h} \mu_{n,j}$, and $\mu_{n,j}$ are the capacitary measures of $K_{n,j}$ relative to $B_{n,j} = B_{6t/5\sqrt{n}}(a_n, j)$, which means

$$\nu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \quad \text{and} \quad \|\nu_{n,j}\|_{W^{-2/q,q'}(B_{n,j})} = \left(C_{2/q,q'}^{B_{n,j}}(K_{n,j})\right)^{1/q}. \tag{3.34}$$

Thus

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{h=1}^{J} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s)\right]^q dyds.$$

We denote

$$J'^{\,h}_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s)\right]^q dyds,$$

and clearly

$$J'_{2,\ell} \le C \sum_{h=1}^{J} J'^{\,h}_{2,\ell}, \tag{3.35}$$

where C depends only on N and q. For integers n and p such that $n \geq \ell + 1$, we set

$$\lambda_{n,j,y} = \inf\{|y - z| : z \in B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})\} = |y - a_{n,j}| - \frac{\sqrt{t}}{\sqrt{n+1}}.$$

Therefore

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} \int_{K_n} e^{-\frac{|y-z|^2}{4s}} d\mu_n^h(z) &= \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} \int_{K_{n,j}} e^{-\frac{|y-z|^2}{4s}} d\mu_{n,j}(z) \\ &\leq \left(\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \right)^{1/q'} \left(\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-q\lambda_{n,j,y}^2 \frac{1-\epsilon}{4s}} \mu_{n,j}^q(K_{n,j}) \right)^{1/q}\end{aligned}$$

where $\epsilon > 0$ will be made precise later on.

Step 1. We claim that

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \leq C \sqrt{\frac{ps}{t}} \tag{3.36}$$

where C depends on ϵ, q and N. If y is fixed in T_p, we denote by z_y the point of T_n which solves $|y - z_y| = \operatorname{dist}(y, T_n)$. Thus

$$\sqrt{t}(\sqrt{n} - \sqrt{p+1}) \leq |y - z_y| \leq t(\sqrt{n} - \sqrt{p}).$$

Let $Y = \frac{\sqrt{t(p+1)}}{|y|} y$. On the axis $\overrightarrow{0Y}$ we set $\mathbf{e} = \frac{Y}{|Y|}$, consider the points $b_k = \frac{k\sqrt{t}}{\sqrt{n}}\mathbf{e}$ where $-n \leq k \leq n$ and denote by $G_{n,k}$ the spherical shell obtain by intersecting the spherical shell T_n with the domain $H_{n,k}$ which is the set of points in $\mathbb{R}^N$ limited by the hyperplanes orthogonal to $\overrightarrow{0Y}$ going through $\frac{(k+1)\sqrt{t}}{\sqrt{n}}\mathbf{e}$ and $\frac{(k-1)\sqrt{t}}{\sqrt{n}}\mathbf{e}$. The number of points $a_{n,j} \in G_{n,k}$ is smaller than $C(n+1-|k|)^{N-2}$, where C depends only on N, and we denote by $\Lambda_{n,k}$ the set of $j \in \Theta_{t,n}$ such that $a_{n,j} \in G_{n,k}$. Furthermore, if $a_{n,j} \in G_{n,k}$ elementary geometric considerations (Pythagore's theorem) imply that $\lambda_{n,j,y}^2$ is greater than $t(n + p + 1 - 2k\frac{\sqrt{p+1}}{\sqrt{n}})$. Therefore

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \leq C \sum_{n=p+\ell}^{a_t} \sum_{k=-n}^{n} (n+1-|k|)^{N-2} e^{-\frac{\epsilon q'(n+p+1-2k\sqrt{p+1}/)t}{4s\sqrt{n}}}. \tag{3.37}$$

Case $N = 2$: Summing a geometric series and using the inequality $\frac{e^u}{e^u - 1} \leq 1 + u^{-1}$ for $u > 0$, we obtain

$$\begin{aligned}\sum_{k=-n}^{n} e^{\frac{\epsilon q' (k\sqrt{p+1})t}{2s\sqrt{n}}} &\leq e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \frac{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}}{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}} - 1} \\ &\leq e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \left(1 + \frac{2s\sqrt{n}}{\epsilon q' t\sqrt{p+1}}\right).\end{aligned} \tag{3.38}$$

Thus, by comparison between series and integrals,

$$\begin{aligned}\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q'\lambda_{n,j,y}^2}{4s}} &\le C\sum_{n=p+\ell}^{a_t}\left(1+\frac{s\sqrt{n}}{t\sqrt{p}}\right)e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2}{4s}}\\ &\le C\int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx \\ &\qquad + \frac{Cs}{t\sqrt{p}}\int_{p+1}^{\infty}\sqrt{x}e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx.\end{aligned} \tag{3.39}$$

Next

$$\begin{aligned}\int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}}ydy\\ &= 2\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}ydy + 2\sqrt{p+1}\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}dy\\ &= \frac{2s}{t}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}}zdz + 2\sqrt{\frac{(p+1)s}{t}}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}}dz,\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}\int_{p+1}^{\infty}\sqrt{x}e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}}y^2dy\\ &= 2\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}(y+\sqrt{p+1})^2dy\\ &\le 4\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}y^2dy + 4(p+1)\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}dy\\ &\le 4\left(\frac{s}{t}\right)^{3/2}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}}z^2dz + 4(p+1)\sqrt{\frac{s}{t}}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}}dz.\end{aligned} \tag{3.41}$$

Jointly with (3.39), these inequalities imply

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q'\lambda_{n,j,y}^2}{4s}} \le C\sqrt{\frac{ps}{t}}. \tag{3.42}$$

Case $N>2$: Because the value of the right-hand side of (3.37) is an increasing value of N, it is sufficient to prove (3.36) when N is even, say $\frac{N-2)}{2} = d \in \mathbb{N}_*$. There holds

$$\sum_{k=-n}^{n}(n+1-|k|)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}} \le 2\sum_{k=0}^{n}(n+1-k)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}}. \tag{3.43}$$

We set

$$\alpha = \epsilon q'\frac{t\sqrt{p+1}}{2s\sqrt{n}} \quad \text{and} \quad I_d = \sum_{k=0}^{n}(n+1-k)^d e^{k\alpha}.$$

Since

$$e^{k\alpha} = \frac{e^{(k+1)\alpha}-e^{k\alpha}}{e^{\alpha}-1},$$

we use Abel's transform to obtain

$$\begin{aligned} I_d &= \frac{1}{e^\alpha - 1}\left(e^{(n+1)\alpha} - (n+1)^d + \sum_{k=1}^{n} \left((n+2-k)^d - (n+1-k)^d \right) e^{k\alpha} \right) \\ &\leq \frac{1}{e^\alpha - 1}\left((1-d)e^{(n+1)\alpha} - (n+1)^d + de^\alpha \sum_{k=1}^{n} \left((n+1-k)^{d-1} \right) e^{k\alpha} \right). \end{aligned}$$

Therefore the following induction holds

$$I_d \leq \frac{de^\alpha}{e^\alpha - 1} I_{d-1}. \tag{3.44}$$

In (3.38), we have already used the fact that

$$\frac{de^\alpha}{e^\alpha - 1} \leq C\left(1 + \frac{s\sqrt{n}}{t\sqrt{p}}\right),$$

and

$$I_d \leq C\left(1 + \left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right) I_0.$$

Thus (3.39) is replaced by

$$\begin{aligned} \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} &\leq C \sum_{n=p+\ell}^{a_t} \left(1 + \left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right) e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \\ &\leq C\int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\ &\qquad + \left(\frac{Cs}{t\sqrt{p}}\right)^{d+1} \int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx. \end{aligned} \tag{3.45}$$

The first integral on the right-hand side has already been estimated in (3.40), for the second integral, there holds

$$\begin{aligned} \int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= \int_0^{\infty} (y + \sqrt{p+1}\,)^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dx \\ &\leq C\int_0^{\infty} y^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dy + Cp^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\leq C\left(\frac{s}{t}\right)^{2+\frac{d}{2}} \int_0^{\infty} z^{(d+1)/2} e^{-\frac{\epsilon q' z^2}{4}} dz \\ &\qquad + C\left(\frac{s}{t}\right)^{3/2} p^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz. \end{aligned} \tag{3.46}$$

Combining (3.40), (3.45) and (3.46), we derive (3.36).

Step 2. Since $\mathcal{T}_p^* \subset \Gamma_p \times [0,t]$ where $\Gamma_p = B_{d_{p+1}}(x) \setminus B_{d_{p-1}}(x)$, $(y,s) \in \mathcal{T}_p^*$ implies that $|x-y|^2 \geq (p-1)t$, thus $J'^{\,h}_{2,\ell}$ satisfies

$$\begin{aligned} J'^{\,h}_{2,\ell} &\leq Ct^{\frac{1-q}{2}} \sum_{p=1}^{\infty} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} \\ &\qquad \times \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta^h_{t,n}} e^{-\frac{q\lambda^2_{n,j,y}(1-\epsilon)}{4s}} \mu^q_{n,j}(K_{n,j}) ds dy \\ &\leq Ct^{\frac{1-q}{2}} \sum_{n=\ell+1}^{a_t} \sum_{j \in \Theta^h_{t,n}} \mu^q_{n,j}(K_{n,j}) \\ &\quad \times \sum_{p=1}^{n-\ell} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-|x-y|^2/4(t-s)} e^{-\frac{q\lambda^2_{n,j,y}(1-\epsilon)}{4s}} ds dy \end{aligned} \tag{3.47}$$

and the constant C depends on N, q and ϵ. Next we set $q_\epsilon = (1-\epsilon)q$. Writting

$$|y - a_{n,j}|^2 = |x-y|^2 + |x - a_{n,j}|^2 - 2\langle y - x, a_{n,j} - x\rangle \geq pt + |x - a_{n,j}|^2 - 2\langle y - x, a_{n,j} - x\rangle,$$

we get

$$\int_{\Gamma_p} e^{-\frac{q_\epsilon \left|y - a_{n,j}\right|^2}{4s}} dy = e^{-\frac{q_\epsilon \left|x - a_{n,j}\right|^2}{4s}} \int_{\sqrt{tp}}^{\sqrt{t(p+1)}} e^{-\frac{q_\epsilon r^2}{4s}} \int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) dr.$$

For estimating the value of the spherical integral, we can assume that $a_{n,j} - x = (0, \ldots, 0, |a_{n,j} - x|)$, $y = (y_1, \ldots, y_N)$ and, using spherical coordinates with center at x, that the unit sphere has the representation $S^{N-1} = \{(\sin\phi.\sigma, \cos\phi) \in \mathbb{R}^{N-1} \times \mathbb{R} : \sigma \in S^{N-2},\ \phi \in [0,\pi]\}$. With this representation, $dS_r = r^{N-1} \sin^{N-2}\phi \, d\phi \, d\sigma$ and $\langle y - x, a_{n,j} - x\rangle = |a_{n,j} - x|\,|y - x| \cos\phi$. Therefore

$$\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) = r^{N-1} \left|S^{N-2}\right| \int_0^\pi e^{2q_\epsilon \frac{\left|a_{n,j}-x\right| r\cos\phi}{4s}} \sin^{N-2}\phi \, d\phi.$$

By Lemma A.3

$$\begin{aligned} \int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) &\leq C \frac{r^{N-1} e^{2q_\epsilon \frac{r\left|a_{n,j}-x\right|}{4s}}}{\left(1 + \frac{r|a_{n,j}-x|}{s}\right)^{\frac{N-1}{2}}} \\ &\leq C s^{\frac{N-1}{2}} \left(\frac{r}{|a_{n,j}-x|}\right)^{\frac{N-1}{2}} e^{2q_\epsilon \frac{r\left|a_{n,j}-x\right|}{4s}}. \end{aligned} \tag{3.48}$$

Therefore

$$\int_{\Gamma_p} e^{-q_\epsilon \frac{\left|y - a_{n,j}\right|^2}{4s}} dy \leq Ct^{\frac{N-1}{4}} p^{\frac{N-3}{4}} \frac{s^{\frac{N-1}{2}} e^{-q_\epsilon \frac{(\left|a_{n,j}-x\right| - \sqrt{t(p+1)})^2}{4s}}}{|a_{n,j}-x|^{\frac{N-1}{2}}}, \tag{3.49}$$

and, since $|a_{n,j}-x|\geq\sqrt{tn}$,

$$\begin{aligned}\int_0^t\int_{\Gamma_p}(t-s)^{-\frac{N}{2}}s^{-(q(N-1)+1)/2}e^{-\frac{|x-y|^2}{4(t-s)}}e^{-q_\epsilon\frac{\lambda_{n,j,y}^2}{4s}}dy\,ds\\ \leq C\frac{\sqrt{t}p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}}\int_0^t(t-s)^{-\frac{N}{2}}s^{-\frac{(q-1)(N-1)+1}{2}}e^{-\frac{pt}{4(t-s)}}e^{-q_\epsilon\frac{(\sqrt{tn}-\sqrt{t(p+1)})^2}{4s}}ds\\ \leq C\frac{t^{\frac{1-q(N-1)}{2}}p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}}\int_0^1(1-s)^{-\frac{N}{2}}s^{-\frac{(q-1)(N-1)+1}{2}}e^{-\frac{p}{4(1-s)}}e^{-q_\epsilon\frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}}.\end{aligned}\tag{3.50}$$

We apply Lemma A.1, with $A=\sqrt{p}$, $B=\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1})$, $b=\frac{(q-1)(N-1)+1}{2}$, $a=\frac{N}{2}$ and $\kappa=\frac{\sqrt{q_\epsilon}(\ell-1)}{8}$ as in the case $N=1$, and noticing that, for these specific values,

$$\begin{aligned}A^{1-a}B^{1-b}(A+B)^{a+b-2}&=p^{\frac{2-N}{4}}(\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{\frac{1-(q-1)(N-1)}{2}}\\ &\quad\times(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{\frac{(q-1)(N-1)+N-3}{2}}\\ &\leq C\left(\frac{n}{p}\right)^{\frac{N}{4}-1/2}\left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}},\end{aligned}$$

where C depends on N, q and κ. Therefore

$$\begin{aligned}&\int_0^t\int_{\Gamma_p}(t-s)^{-\frac{N}{2}}s^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}e^{-q_\epsilon\frac{|y-z|^2}{4s}}dy\,ds\\ &\leq C\frac{t^{(1-q(N-1))/2}p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}}\left(\frac{n}{p}\right)^{\frac{N}{4}-1/2}\left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}}e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}\\ &\leq Ct^{\frac{1-q(N-1)}{2}}p^{-\frac{1}{4}}n^{\frac{(q-1)(N-1)-2}{4}}(\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}}e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.\end{aligned}\tag{3.51}$$

We derive from (3.47), (3.51),

$$\begin{aligned}&J'^{\,h}_{2,\ell}\leq Ct^{1-\frac{Nq}{2}}\\ &\times\sum_{n=\ell+1}^{a_t}\sum_{j\in\Theta^h_{t,n}}n^{\frac{(q-1)(N-1)-2}{4}}\mu_{n,j}^q(K_{n,j})\sum_{p=1}^{n-\ell}p^{\frac{2q-3}{4}}(\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}}e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.\end{aligned}\tag{3.52}$$

By Lemma A.2 with $\alpha=\dfrac{2q-3}{4}$, $\beta=\frac{1-(q-1)(N-1)}{2}$, $\delta=\frac{1}{4}$ and $\gamma=q_\epsilon$, we obtain

$$\sum_{p=1}^{n-\ell}p^{\frac{2q-3}{4}}(\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}}e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}\leq Cn^{\frac{N(q-1)+q-3}{4}}e^{-\frac{n}{4}},\tag{3.53}$$

thus

$$J'^{\,h}_{2,\ell}\leq Ct^{1-\frac{Nq}{2}}\sum_{n=\ell+1}^{a_t}n^{\frac{N(q-1)}{2}-1}e^{-\frac{n}{4}}\sum_{j\in\Theta^h_{t,n}}\mu_{n,j}^q(K_{n,j}).\tag{3.54}$$

Because

$$\mu_{n,j}(K_{n,j}) = C^{B_{n,j}}_{2/q,q'}(K_{n,j}),$$

we use the rescaling procedure as in the proof of Lemma 2.15, except that the scale factor is $\sqrt{(n+1)t}$ instead of $\sqrt{n+1}$ so that the sets $\tilde{T}_n$, $\tilde{K}_n$, $\tilde{\mathcal{Q}}_n$ and $\tilde{K}_n$ remains unchanged Using again the quasi-additivity and the fact that $J'_{2,\ell} = \sum_{h=1}^{J} J'^{\,h}_{2,\ell}$, we deduce

$$J_{2,\ell} \leq C' t^{-\frac{N}{2}} \sum_{n=\ell+1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\tfrac{K_n}{d_{n+1}}\right), \tag{3.55}$$

which implies (3.33). □

The proof of Theorem 3.1 follows from the previous estimates on J_1 and J_2. Furthermore the following integral expression holds

Theorem 3.9 *Assume $q \geq q_c$. Then there exists a positive constants C_2^* , depending on N,q and T, such that for any closed set F, there holds*

$$\underline{u}_F(x,t) \geq \frac{C_2^*}{t^{1+\frac{N}{2}}} \int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds, \tag{3.56}$$

where a_t is the smallest integer j such that $F \subset B_{\sqrt{jt}}(x)$.

Proof. We distinguish according $q = q_c$, or $q > q_c$, and for simplicity we denote $B_r = B_r(x)$ for the various values of r.

Case 1: $q = q_c \Longleftrightarrow N - \frac{2}{q-1} = 0$. Because $F_n = F \cap (B_{d_{n+1}} \setminus B_{d_n})$ there holds

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - C_{2/q,q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right),$$

Furthermore, since $d_{n+1} \geq d_n$,

$$C_{2/q,q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right) = C_{2/q,q'}\left(\frac{d_n}{d_{n+1}} \frac{F \cap B_{d_n}}{d_n}\right) \leq C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right),$$

thus

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right),$$

it follows

$$\begin{aligned}
\sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right) \\
&\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-\frac{1}{4}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \\
&\geq (1 - e^{-\frac{1}{4}}) \sum_{n=1}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-\frac{1}{4}} C_{2/q,q'}\left(\frac{F}{\sqrt{t}} \cap B_1\right).
\end{aligned}$$

Since, by (2.69),

$$C_{2/q,q'}\left(\frac{F}{s'}\cap B_1\right)\geq C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right)\geq C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right),$$

for any $s'\in[d_{n+1},d_{n+2}]$ and $s\in[d_n,d_{n+1}]$, there holds

$$\begin{aligned}te^{-\frac{n}{4}}C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right)&\geq C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right)\int_{d_n}^{d_{n+1}}e^{-s^2/4t}s\,ds\\&\geq\int_{d_n}^{d_{n+1}}e^{-s^2/4t}C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right)s\,ds.\end{aligned}$$

This implies

$$W_F(x,t)\geq(1-e^{-\frac{1}{4}})t^{-(1+\frac{N}{2})}\int_0^{\sqrt{ta_t}}e^{-s^2/4t}C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right)s\,ds.$$

Case 2: $q>q_c\Longleftrightarrow N-\frac{2}{q-1}>0$. In that case it is known that

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right)\approx d_{n+1}^{\frac{2}{q-1}-N}C_{2/q,q'}\left(F_n\right),$$

see [1]. Thus

$$W_F(x,t)\approx t^{-1-\frac{N}{2}}\sum_{n=0}^{a_t}e^{-\frac{n}{4}}C_{2/q,q'}\left(F_n\right).$$

Since there holds

$$C_{2/q,q'}\left(F_n\right)\geq C_{2/q,q'}\left(F\cap B_{d_{n+1}}\right)-C_{2/q,q'}\left(F\cap B_{d_n}\right),$$

we derive

$$\begin{aligned}t^{-\frac{N}{2}}\sum_{n=0}^{a_t}e^{-\frac{n}{4}}C_{2/q,q'}\left(F_n\right)&\geq(1-e^{-\frac{1}{4}})t^{-\frac{N}{2}}\sum_{n=0}^{a_t-1}e^{-\frac{n}{4}}C_{2/q,q'}\left(F\cap B_{d_{n+1}}\right)\\&\geq(1-e^{-\frac{1}{4}})t^{-(1+\frac{N}{2})}\int_0^{\sqrt{ta_t}}e^{-\frac{s^2}{4t}}C_{2/q,q'}\left(F\cap B_s\right)s\,ds.\end{aligned}$$

Because $C_{2/q,q'}\left(F\cap B_s\right)\approx s^{N-\frac{2}{q-1}}C_{2/q,q'}\left(s^{-1}F\cap B_1\right)$, (3.56) follows. □

4 Applications

The first result of this section is the following

Theorem 4.1 *Assume $N\geq 1$ and $q>1$. Then $\overline{u}_K=\underline{u}_K$.*

Proof. If $1 < q < q_c$, the result is already proved in [26]. The proof in the super-critical case is an adaptation that we recall, for the sake of completeness. By Theorem 2.16 and Theorem 3.9 there exists a positive constant C, depending on N, q and T such that

$$\overline{u}_F(x,t) \le C\underline{u}_F(x,t) \qquad \forall (x,t) \in Q_T.$$

By convexity $\tilde{u} = \underline{u}_F - \dfrac{1}{2C}(\overline{u}_F - \underline{u}_F)$ is a super-solution, which is smaller than $\underline{u}_F$ if we assume that $\overline{u}_F \neq \underline{u}_F$. If we set $\theta := 1/2 + 1/(2C)$, then $u_\theta = \theta\overline{u}_F$ is a subsolution. Therefore there exists a solution u_1 of (1.1) in Q_∞ such that $u_\theta \le u_1 \le \tilde{u} < \underline{u}_F$. If $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N)$ satisfies $\mu(F^c) = 0$, then $u_{\theta\mu}$ is the smallest solution of (1.1) which is above the subsolution θu_μ. Thus $u_{\theta\mu} \le u_1 < \underline{u}_F$ and finally $\underline{u}_F \le u_1 < \underline{u}_F$, a contradiction. □

If we combine Theorem 2.16 and Theorem 3.9 we derive the following integral approximation of the parabolic capacitary potential

Proposition 4.2 *Assume $q \ge q_c$. Then there exist two positive constants $C_1^\dagger$, $C_2^\dagger$, depending only on N, q and T such that*

$$\begin{aligned} C_2^\dagger t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \le W_F(x,t) \\ \le C_1^\dagger t^{-(1+\frac{N}{2})} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \end{aligned} \tag{4.57}$$

for any $(x,t) \in Q_T$.

Definition 4.3 *If F is a closed subset of $\mathbb{R}^N$, we define the $(2/q,q')$-integral parabolic capacitary potential $\mathcal{W}_F$ by*

$$\mathcal{W}_F(x,t) = t^{-1-\frac{N}{2}} \int_0^{D_F(x)} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \qquad \forall (x,t) \in Q_\infty, \tag{4.58}$$

where $D_F(x) = \max\{|x-y| : y \in F\}$.

An easy computation shows that

$$\begin{aligned} 0 \le \mathcal{W}_F(x,t) - t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \\ \le C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-D_F^2(x)/4t}, \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} 0 \le t^{-(1+\frac{N}{2})} \int_0^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds - \mathcal{W}_F(x,t) \\ \le C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-\frac{D_F^2(x)}{4t}}, \end{aligned} \tag{4.60}$$

for some $C = C(N,q) > 0$. Furthermore

$$\mathcal{W}_F(x,t) = t^{-\frac{1}{q-1}} \int_0^{D_F(x)/\sqrt{t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4}} C_{2/q,q'} \left(\frac{F}{s\sqrt{t}} \cap B_1(x) \right) s\, ds. \tag{4.61}$$

The next result gives a sufficient condition in order $\overline{u}_F$ has not a strong blow-up at some point x.

Proposition 4.4 *Assume $q \geq q_c$ and F is a closed subset of $\mathbb{R}^N$. If there exists $\gamma \in [0,\infty)$ such that*

$$\lim_{\tau \to 0} C_{2/q,q'} \left(\frac{F}{\tau} \cap B_1(x) \right) = \gamma, \tag{4.62}$$

then

$$\lim_{t \to 0} t^{\frac{1}{q-1}} \overline{u}_F(x,t) = C\gamma, \tag{4.63}$$

for some $C = C(N,q) > 0$.

Proof. Clearly, condition (4.62) implies

$$\lim_{t \to 0} C_{2/q,q'} \left(\frac{F}{\sqrt{t}s} \cap B_1(x) \right) = \gamma$$

for any $s > 0$. Then (4.63) follows by Lebesgue's theorem. Notice also that the set of γ is bounded from above by a constant depending on N and q. □

In the next result we give a condition in order the solution remains bounded at some point x. The proof is similar to the previous one.

Proposition 4.5 *Assume $q \geq q_c$ and F is a closed subset of $\mathbb{R}^N$. If*

$$\limsup_{\tau \to 0} \tau^{-\frac{2}{q-1}} C_{2/q,q'} \left(\frac{F}{\tau} \cap B_1(x) \right) < \infty, \tag{4.64}$$

then $\overline{u}_F(x,t)$ remains bounded when $t \to 0$.

A Appendix

The next highly non-trivial computational estimates, are crucial in the estimate of the term $J'_{2,\ell}$ in Lemma 3.7 and Lemma 3.8.

Lemma A.1 *Let a and b be two real numbers, $a > 0$ and $\kappa > 0$. Then there exists a constant $C = C(a,b,\kappa) > 0$ such that for any $A > 0$, $B > \kappa/A$ there holds*

$$\int_0^1 (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x} dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \tag{A.1}$$

Proof. We first notice that

$$\max\{e^{-A^2/4(1-x)}e^{-\frac{B^2}{4x}} : 0 \leq x \leq 1\} = e^{-\frac{(A+B)^2}{4}}, \tag{A.2}$$

and this maximum is achieved for $x_0 = \frac{B}{A+B}$. Set $\Phi(x) = (1-x)^{-a}x^{-b}e^{-\frac{A^2}{4(1-x)}}e^{-\frac{B^2}{4x}}$, thus

$$\int_0^1 \Phi(x)dx = \int_0^{x_0} \Phi(x)dx + \int_{x_0}^1 \Phi(x)dx = I_{a,b} + J_{a,b}.$$

Put

$$u = \frac{A^2}{4(1-x)} + \frac{B^2}{4x}, \tag{A.3}$$

then

$$4ux^2 - (4u + B^2 - A^2)x + B^2 = 0. \tag{A.4}$$

If $0 < x < x_0$ this equation admits the solution

$$x = x(u) = \frac{1}{8u}\left(4u + B^2 - A^2 - \sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}\right)$$

$$\int_0^{x_0} (1-x)^{-a}x^{-b}e^{-\frac{A^2}{4(1-x)}-\frac{B^2}{4x}}dx = -\int_{(A+B)^2/4}^{\infty} (1-x(u))^{-a}x(u)^{-b}e^{-u}x'(u)du$$

Putting $x' = x'(u)$ and differentiating $(A.4)$,

$$4x^2 + 8uxx' - (4u + B^2 - A^2)x' - 4x = 0 \Longrightarrow -x' = \frac{4x(1-x)}{4u + B^2 - A^2 - 8ux}.$$

Thus

$$\int_0^{x_0} \Phi(x)dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1}x(u)^{-b+1}e^{-u}du}{4u + B^2 - A^2 - 8ux(u)}. \tag{A.5}$$

Using the explicit value of the root $x(u)$, we finally get

$$\int_0^{x_0} \Phi(x)dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1}x(u)^{-b+1}e^{-u}du}{\sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}}, \tag{A.6}$$

and the factorization below holds

$$16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2 = 16(u - (A+B)^2/4)(u - (A-B)^2/4).$$

We set $u = \upsilon + \frac{(A+B)^2}{4}$ and obtain

$$x(u) = \frac{v + (AB+B^2)/2 - \sqrt{v(v+AB)}}{2\left(v + (A+B)^2/4\right)},$$

and

$$1 - x(u) = \frac{v + (A^2+AB)/2 + \sqrt{v(v+AB)}}{2\left(v + (A+B)^2/4\right)}.$$

We introduce the relation $\approx$ linking two positive quantities depending on A and B. It means that the two sided-inequalities up to multiplicative constants independent of A and B. Therefore

$$\int_0^{x_0}\Phi(x)dx = 2^{a-b-4}e^{-(A+B)^2/4}\int_0^{\infty}\tilde{\Phi}(v)dv \quad \text{where}$$
$$\tilde{\Phi}(v) = \frac{\left(v+(AB+B^2)/2-\sqrt{v(v+AB)}\right)^{1-b}\left(v+(A^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-a}}{(v+(A+B)^2/4)^{2-a-b}\sqrt{v(v+AB)}}e^{-v}dv. \tag{A.7}$$

Case 1: $a \geq 1$, $b \geq 1$. First

$$\frac{\left(v+(A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+AB)}} \leq \frac{\left(v+(A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \approx \frac{\left(v+(A+B)^2\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \tag{A.8}$$

since $a+b-2 \geq 0$ and $AB \geq \kappa$. Next

$$\left(v+(A^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-a} \approx (v+A(A+B))^{1-a}. \tag{A.9}$$

Furthermore

$$\begin{aligned} v+(AB+B^2)/2-\sqrt{v(v+AB)} &= B^2\frac{v+(A+B)^2/4}{v+B(A+B)/2+\sqrt{v(v+AB)}} \\ &\approx B^2\frac{v+(A+B)^2}{v+B(A+B)}. \end{aligned} \tag{A.10}$$

Then

$$\left(v+(AB+B^2)/2-\sqrt{v(v+AB)}\right)^{1-b} \approx B^{2-2b}\left(\frac{v+B(A+B)}{v+(A+B)^2}\right)^{b-1} \tag{A.11}$$

It follows

$$\begin{aligned} \tilde{\Phi}(v) &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{(v+B(A+B))^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{v^{b-1}+(B^2+AB)^{b-1}}{\sqrt{v(v+\kappa)}} \end{aligned} \tag{A.12}$$

where C depends on a, b and κ. The function $v \mapsto \frac{v+(A+B)^2}{v+A(A+B)}$ is decreasing on $(0,\infty)$. If we set

$$C_1 = \int_0^{\infty}\frac{v^{b-1}e^{-v}dv}{\sqrt{v(v+\kappa)}} \quad \text{and} \quad C_2 = \int_0^{\infty}\frac{e^{-v}dv}{\sqrt{v(v+\kappa)}},$$

then

$$C_1 \leq K(B^2+AB)^{b-1}C_2$$

with $K = C_1\kappa^{1-b}/C_2$. Therefore

$$\int_0^{x_0}\Phi(x)dx \leq Ce^{-(A+B)^2/4}B^{1-b}A^{1-a}(A+B)^{a+b-2}. \tag{A.13}$$

The estimate of $J_{a,b}$ is obtained by exchanging (A,a) with (B,b) and replacing x by $1-x$. *Mutadis mutandis*, this yields directely to the same expression as in $(A.13)$ and finally

$$\int_0^1 \Phi(x)dx \leq Ce^{-\frac{(A+B)^2}{4}} A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{A.14}$$

Case 2: $a \geq 1$, $b < 1$. Estimates $(A.7)$, $(A.8)$, $(A.9)$, $(A.10)$ and $(A.11)$ are valid. Because $v \mapsto (v + B(A+B))^{b-1}$ is decreasing, $(A.12)$ has to be replaced by

$$\tilde{\Phi}(v) \leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1} \frac{\left(AB+B^2\right)^{b-1}}{\sqrt{v(v+\kappa)}}. \tag{A.15}$$

This implies $(A.13)$ directly. The estimate of $J_{a,b}$ is performed by the change of variable $x \mapsto 1-x$. If $x_1 = 1 - x_0$, there holds

$$J_{a,b} = \int_0^{x_1} x^{-a}(1-x)^{-b}e^{-\frac{A^2}{4x}}e^{-\frac{B^2}{4(1-x)}}dx = \int_0^{x_1}\Psi(x)dx.$$

Then

$$\int_0^{x_1}\Psi(x)dx = 2^{b-a-4}e^{-\frac{(A+B)^2}{4}}\int_0^{x_1}\tilde{\Psi}(v)dv \quad \text{where}$$
$$\tilde{\Psi}(v) = \frac{\left(v+(AB+A^2)/2-\sqrt{v(v+AB)}\right)^{1-a}\left(v+(B^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-b}}{\left(v+(A+B)^2/4\right)^{2-a-b}\sqrt{v(v+AB)}}e^{-v}dv. \tag{A.16}$$

Equivalence $(A.8)$ is unchanged; $(A.9)$ is replaced by

$$\left(v+(B^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-b} \approx (v+B(A+B))^{1-b}, \tag{A.17}$$

$(A.10)$ by

$$v+(AB+A^2)/2-\sqrt{v(v+AB)} \approx A^2\frac{v+(A+B)^2}{v+A(A+B)}, \tag{A.18}$$

and $(A.11)$ by

$$\left(v+(AB+A^2)/2-\sqrt{v(v+AB)}\right)^{1-a} \approx A^{2-2a}\left(\frac{v+A(A+B)}{v+(A+B)^2}\right)^{a-1}. \tag{A.19}$$

Because $a > 1$, $(A.12)$ turns into

$$\begin{aligned}\tilde{\Psi}(v) &\leq CA^{2-2b}(v+(A+B)^2)^{b-1}\frac{(v+A^2+AB)^{a-1}(v+B^2+AB)^{1-b}}{\sqrt{v(v+\kappa)}} \\ &\leq Ce^{-(A+B)^2/4}A^{2-2b}(A+B)^{2b-2} \\ &\qquad \times \frac{v^{a-b}+(A^2+AB)^{a-1}v^{1-b}+(B^2+AB)^{1-b}v^{a-1}+A^{a-1}B^{1-b}(A+B)^{a-b}}{\sqrt{v(v+\kappa)}}.\end{aligned} \tag{A.20}$$

Because $AB \geq \kappa$, there exists a positive constant C, depending on κ, such that

$$\int_0^\infty \frac{v^{a-b} + (A^2+AB)^{a-1} v^{1-b} + (B^2+AB)^{1-b} v^{a-1}}{\sqrt{v(v+\kappa)}} e^{-v} dv \\ \leq C A^{a-1} B^{1-b} (A+B)^{a-b} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}}. \tag{A.21}$$

Combining $(A.20)$ and $(A.21)$ yields to

$$\int_0^{x_1} \Psi(x) dx \leq C e^{-\frac{(A+B)^2}{4}} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \tag{A.22}$$

This, again, implies that $(A.1)$ holds.

Case 3: $\max\{a,b\} < 1$. Inequalities $(A.7)$-$(A.11)$ hold, but $(A.12)$ has to be replaced by

$$\begin{aligned} \tilde{\Phi}(v) &\leq C B^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)} \right)^{a-1} \frac{\left(v + B^2 + AB\right)^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq C B^{1-b} (A+B)^{2a+b-3} \frac{v^{1-a} + \left(A^2+AB\right)^{1-a}}{\sqrt{v(v+\kappa)}} \end{aligned} \tag{A.23}$$

Noticing that

$$\int_0^\infty \frac{v^{1-a} e^{-v} dv}{\sqrt{v(v+\kappa)}} \leq C \left(A^2+AB\right)^{1-a} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}},$$

it follows that $(A.13)$ holds. Finally $(A.14)$ holds by exchanging (A,a) and (B,b). □

Lemma A.2 . *Let α, β, γ, δ be real numbers and ℓ an integer. We assume $\gamma > 1$, $\delta > 0$ and $\ell \geq 2$. Then there exists a positive constant C such that, for any integer $n > \ell$*

$$\sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \leq C n^{\alpha-\beta/2} e^{-\delta n}. \tag{A.24}$$

Proof. The function $x \mapsto (\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2$ is decreasing on $[(\gamma-1)^{-1}, \infty)$. Furthermore there exists $C > 0$ depending on ℓ, α and β such that $p^\alpha(\sqrt{n} - \sqrt{p}\,)^\beta \leq C x^\alpha (\sqrt{n} - \sqrt{x+1}\,)^\beta$ for $x \in [p, p+1]$ If we denote by p_0 the smallest integer larger than $(\gamma-1)^{-1}$, we derive

$$\begin{aligned} S &= \sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\frac{(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2}{4}} = \sum_{p=1}^{p_0-1} + \sum_{p_0}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \\ &\leq \sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \\ &\qquad + C \int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n} - \sqrt{x}\,)^\beta e^{-\delta(\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2} dx, \end{aligned}$$

(notice that $\sqrt{n}-\sqrt{x}\approx\sqrt{n}-\sqrt{x+1}$ for $x\le n-\ell$). Clearly

$$\sum_{p=1}^{p_0-1}p^\alpha(\sqrt{n}-\sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2}\le C_0 n^\alpha(\sqrt{n}-\sqrt{n-\ell}\,)^\beta e^{-\delta n} \tag{A.25}$$

for some C_0 independent of n. We set $y=y(x)=\sqrt{x+1}-\sqrt{x}/\sqrt{\gamma}$. Obviously

$$y'(x)=\frac{1}{2}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{\gamma}\sqrt{x}}\right)\qquad\forall x\ge p_0,$$

and their exists $\epsilon=\epsilon(\delta,\gamma)>0$ such that $\sqrt{2}\sqrt{x}\ge y(x)\ge\epsilon\sqrt{x}$ and $y'(x)\ge\epsilon/\sqrt{x}$. Furthermore

$$\sqrt{x}=\frac{\sqrt{\gamma}\left(y+\sqrt{\gamma y^2+1-\gamma}\right)}{\gamma-1},$$

$$\begin{aligned}\sqrt{n}-\sqrt{x}&=\frac{\sqrt{n}(\gamma-1)-\sqrt{\gamma}y-\sqrt{\gamma}\sqrt{\gamma y^2+1-\gamma}}{\gamma-1}\\&=\frac{n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2}{\sqrt{n}(\gamma-1)-\sqrt{\gamma}y+\sqrt{\gamma}\sqrt{\gamma y^2+1-\gamma}}\\&\approx\frac{n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2}{\sqrt{n}}\end{aligned}$$

since $y(x)\le\sqrt{n}$. Furthermore

$$\begin{aligned}n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2&=\gamma(\sqrt{n+1}+\sqrt{n}/\sqrt{\gamma}+y)(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y)\\&\approx\sqrt{n}(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y),\end{aligned}$$

because y ranges between the two values $\sqrt{n+2-\ell}-\sqrt{n+1-\ell}\sqrt{\gamma}$ (which of order of $\sqrt{n}$) and $\sqrt{p_0+1}-\sqrt{p_0}\sqrt{\gamma}$. Thus

$$(\sqrt{n}-\sqrt{x}\,)^\beta\approx\left(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y\right)^\beta.$$

This implies

$$\begin{aligned}&\int_{p_0}^{n+1-\ell}x^\alpha(\sqrt{n}-\sqrt{x}\,)^\beta e^{-\delta(\sqrt{x}+\gamma(\sqrt{n}-\sqrt{x+1}))^2}dx\\&\qquad\le C\int_{y(p_0)}^{y(n+1-\ell)}y^{2\alpha+1}\left(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y\right)^\beta e^{-\gamma\delta(\sqrt{n}-y)^2}dy\\&\qquad\le Cn^{\alpha+\beta/2+1}\int_{1-y(n+1-\ell)/\sqrt{n}}^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta nz^2}dz.\end{aligned} \tag{A.26}$$

Moreover

$$\begin{aligned}
1-\frac{y(p_0)}{\sqrt{n}} &= 1-\frac{1}{\sqrt{n}}\left(\sqrt{p_0+1}-\frac{\sqrt{p_0}}{\sqrt{\gamma}}\right),\\
1-\frac{y(n-\ell+1)}{\sqrt{n}} &= 1-\frac{\sqrt{n-\ell+2}}{\sqrt{n}}+\frac{\sqrt{n-\ell+1}}{\sqrt{n\gamma}}\\
&= \frac{1}{\sqrt{\gamma}}\left(1+\frac{\sqrt{\gamma}\,(\ell-2)-\ell+1}{2n}+\frac{\sqrt{\gamma}\,(\ell-2)^2-(\ell-1)^2}{8n^2}\right)+O(n^{-3}).
\end{aligned} \tag{A.27}$$

Let θ fixed such that $1-\dfrac{y(n-\ell+1)}{\sqrt{n}} < \theta < 1-\dfrac{y(p_0)}{\sqrt{n}}$ for any $n > p_0$. Then

$$\begin{aligned}
\int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz &\le C_\theta \int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1}e^{-\gamma\delta n z^2}dz\\
&\le C_\theta\, e^{-\gamma\delta n\theta^2}\int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1}dz\\
&\le C\, e^{-\gamma\delta n\theta^2}\max\{1, n^{-\alpha-1/2}\}.
\end{aligned}$$

Because $\gamma\theta^2 > 1$ we derive

$$\int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz \le Cn^{-\beta}e^{-\delta n}, \tag{A.28}$$

for some constant $C > 0$. On the other hand

$$\begin{aligned}
&\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz\\
&\qquad\le C'_\theta\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz.
\end{aligned}$$

The minimum of $z\mapsto (z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta$ is achieved when $z = 1-y(n+1-\ell)$ with corresponding value

$$\frac{\sqrt{\gamma}(\ell+1)+1-\ell}{2n\sqrt{\gamma}}+O(n^{-2}),$$

and the maximum of the exponential term is achieved at the same point with value

$$e^{-n\delta+((\ell-2)\sqrt{\gamma}+1-\ell)/2}(1+\circ(1)) = C_\gamma e^{-n\delta}(1+\circ(1)).$$

We denote

$$z_{\gamma,n} = 1+1/\sqrt{\gamma}-\sqrt{1+1/n} \quad\text{and}\quad I_\beta = \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z-z_{\gamma,n})^\beta e^{-\gamma\delta n z^2}dz.$$

Since $1-y(n+1-\ell)\ge 1/\sqrt{2\gamma}$ for n large enough,

$$\begin{aligned}
I_\beta &\le \sqrt{2\gamma}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z-z_{\gamma,n})^\beta z e^{-\gamma\delta n z^2}dz\\
&\le \frac{-\sqrt{2\gamma}}{2n\gamma\delta}\left[(z-z_{\gamma,n})^\beta e^{-\gamma\delta n z^2}\right]_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} + \frac{\beta\sqrt{2\gamma}}{2n\gamma\delta}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z-z_{\gamma,n})^{\beta-1} z e^{-\gamma\delta n z^2}dz.
\end{aligned}$$

But $1 - y(n+1-\ell)/\sqrt{n} - z_{\gamma,n} = (\ell-1)(1-1/\sqrt{\gamma})/2n$, therefore

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n} + \beta C_1' n^{-1} I_{\beta-1}. \tag{A.29}$$

If $\beta \le 0$, we derive

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n},$$

which inequality, combined with $(A.26)$ and $(A.28)$, yields to $(A.24)$. If $\beta > 0$, we iterate and get

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n} + C_1' n^{-1}(C_1 n^{-\beta} e^{-\delta n} + (\beta-1) C_1' n^{-1} I_{\beta-2})$$

If $\beta - 1 \le 0$ we derive

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n} + C_1 C_1' n^{-1-\beta} e^{-\delta n} = C_2 n^{-\beta-1} e^{-\delta n},$$

which again yields to $(A.24)$. If $\beta - 1 > 0$, we continue this process up we find a positive integer k such that $\beta - k \le 0$, which again yields to

$$I_\beta \le C_k n^{-\beta-1} e^{-\delta n}$$

and ultimately to $(A.24)$. □

The next estimate is also fundamental for deriving the N-dimensional estimate of $J_{2,\ell}'$ in Lemma 3.8.

Lemma A.3 *For any integer $N \ge 2$ there exists a constant $c_N > 0$ such that*

$$\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta \le c_N \frac{e^m}{(1+m)^{(N-1)/2}} \qquad \forall m > 0. \tag{A.30}$$

Proof. Put $\mathcal{I}_N(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta$. Then $\mathcal{I}_2'(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \cos\theta \, d\theta$ and

$$\begin{aligned}
\mathcal{I}_2''(m) &= \int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \cos^2\theta \, d\theta = \mathcal{I}_2(m) - \int_0^\pi e^{m\cos\theta} \sin^N\theta \, d\theta \\
&= \mathcal{I}_2(m) - \frac{N-1}{m} \int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \cos\theta \, d\theta \\
&= \mathcal{I}_2(m) - \frac{N-1}{m} \mathcal{I}_2'(m).
\end{aligned}$$

Thus $\mathcal{I}_2$ satisfies a Bessel equation of order 0. Since $\mathcal{I}_2(0) = \pi$ and $\mathcal{I}_2'(0) = 0$, $\pi^{-1}\mathcal{I}_2$ is the modified Bessel function of index 0 (usually denoted by I_0) the asymptotic behaviour of which is well known, thus $(A.30)$ holds. If $N = 3$

$$\mathcal{I}_3(m) = \int_0^\pi e^{m\cos\theta} \sin\theta \, d\theta = \left[\frac{-e^{m\cos\theta}}{m}\right]_0^\pi = \frac{2\sinh m}{m}.$$

For $N > 3$ arbitrary

$$\mathcal{I}_N(m) = \int_0^\pi \frac{-1}{m} \frac{d}{d\theta}(e^{m\cos\theta}) \sin^{N-3}\theta \, d\theta = \frac{N-3}{m} \int_0^\pi e^{m\cos\theta} \cos\theta \sin^{N-4}\theta \, d\theta. \tag{A.31}$$

Therefore,

$$\mathcal{I}_4(m) = \frac{1}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\, d\theta = \mathcal{I}_2'(m),$$

and, again $(A.30)$ holds since $I_0'(m)$ has the same behaviour as $I_0(m)$ at infinity. For $N \geq 5$

$$\mathcal{I}_N(m) = \frac{3-N}{m^2}\left[e^{m\cos\theta}\cos\theta\sin^{N-5}\theta\right]_0^\pi + \frac{N-3}{m^2}\int_0^\pi e^{m\cos\theta}\frac{d}{d\theta}\left(\cos\theta\sin^{N-5}\theta\right)d\theta.$$

Differentiating $\cos\theta\sin^{N-5}\theta$ and using $(A.31)$, we obtain

$$\mathcal{I}_5(m) = \frac{4\sinh m}{m^2} - \frac{4\sinh m}{m^3},$$

while

$$\mathcal{I}_N(m) = \frac{(N-3)(N-5)}{m^2}\left(\mathcal{I}_{N-4}(m) - \mathcal{I}_{N-2}(m)\right), \tag{A.32}$$

for $N \geq 6$. Since the estimate $(A.30)$ for $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ has already been obtained, a straigthforward induction yields to the general result. □

Remark. Although it does not has any importance for our use, it must be noticed that $\mathcal{I}_N$ can be expressed either with hyperbolic functions if N is odd, or with Bessel functions if N is even.

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