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► **To cite this version:**

Moshe Marcus, Laurent Veron. Capacitary estimates of solutions of semilinear parabolic equations. 2006. hal-00282315v2

HAL Id: hal-00282315

<https://hal.science/hal-00282315v2>

Preprint submitted on 24 Sep 2010 (v2), last revised 17 Jun 2012 (v5)

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Capacitary estimates of solutions of semilinear parabolic equations

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Abstract

We prove that any positive solution of $\partial_t u - \Delta u + u^q = 0$ ($q > 1$) in $\mathbb{R}^N \times (0, \infty)$ with initial trace $(F, 0)$, where F is a closed subset of \mathbb{R}^N can be represented, up to two universal multiplicative constants, by a series involving the Bessel capacity $C_{2/q, q'}$. As a consequence we prove that there exists a unique positive solution of the equation with such an initial trace. We also characterize the blow-up set of $u(x, t)$ when $t \downarrow 0$, by using the "density" of F expressed in terms of the $C_{2/q, q'}$ -capacity.

2000 Mathematics Subject Classification. 35K05; 35K55; 31C15; 31B10; 31C40.

Key words. Heat equation; singularities; Borel measures; Bessel capacities; Wiener type test; initial trace.

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1 Introduction

Let $T \in (0, \infty]$ and $Q_T = \mathbb{R}^N \times (0, T]$ ($N \geq 1$). If $q > 1$ and $u \in C^2(Q_T)$ is nonnegative and verifies

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T, \quad (1.1)$$

it has been proven by Marcus and Véron [25] that there exists a unique outer-regular positive Borel measure ν in \mathbb{R}^N such that

$$\lim_{t \rightarrow 0} u(\cdot, t) = \nu, \quad (1.2)$$

in the sense of Borel measures; the set of such measures is denoted by $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$. To each of its element ν is associated a unique couple $(\mathcal{S}_\nu, \mu_\nu)$ (we write $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$) where \mathcal{S}_ν , *the singular part* of ν , is a closed subset of \mathbb{R}^N and μ_ν , *the regular part* is a nonnegative Radon measure on $\mathcal{R}_\nu = \mathbb{R}^N \setminus \mathcal{S}_\nu$. In this setting, relation (1.2) has the following meaning :

$$\begin{aligned} (i) \quad & \lim_{t \rightarrow 0} \int_{\mathcal{R}_\nu} u(\cdot, t) \zeta dx = \int_{\mathcal{R}_\nu} \zeta d\mu_\nu, & \forall \zeta \in C_0(\mathcal{R}_\nu), \\ (ii) \quad & \lim_{t \rightarrow 0} \int_{\mathcal{O}} u(\cdot, t) dx = \infty, & \forall \mathcal{O} \subset \mathbb{R}^N \text{ open, } \mathcal{O} \cap \mathcal{S}_\nu \neq \emptyset. \end{aligned} \quad (1.3)$$

The measure ν is by definition *the initial trace* of u and denoted by $Tr_{\mathbb{R}^N}(u)$. It is wellknown that equation (1.1) admits a critical exponent

$$1 < q < q_c = 1 + \frac{N}{2}.$$

This is due to the fact, proven by Brezis and Friedman [6], that if $q \geq q_c$, isolated singularities of solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ are removable. Conversely, if $1 < q < q_c$, it is proven by the same authors that for any $k > 0$, equation (1.1) admits a unique solution $u_{k\delta_0}$ with initial data $k\delta_0$. This existence and uniqueness results extends in a simple way if the initial data $k\delta_0$ is replaced by any Radon measure μ in \mathbb{R}^N (see [5]). Furthermore, if $k \rightarrow \infty$, $u_{k\delta_0}$ increases and converges to a positive, radial and self-similar solution u_∞ of (1.1). Writing it under the form $u_\infty(x, t) = t^{-\frac{1}{q-1}} f(|x|/\sqrt{t})$, f is a positive solution of

$$\begin{cases} \Delta f + \frac{1}{2}y \cdot Df + \frac{1}{q-1}f - f^q = 0 & \text{in } \mathbb{R}^N \\ \lim_{|y| \rightarrow \infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \quad (1.4)$$

The existence, uniqueness and the expression of the asymptotics of f has been studied thoroughly by Brezis, Peletier and Terman in [7]. Later on, Marcus and Véron proved in [25] that in the same range of exponents, for any $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty, \\ Tr_{\mathbb{R}^N}(u) = \nu, \end{cases} \quad (1.5)$$

admits a unique positive solution. This result means that the initial trace establishes a one to one correspondence between the set of positive solutions of (1.1) and $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$. A key step for proving the uniqueness is the following inequalities

$$t^{-\frac{1}{q-1}} f(|x-a|/\sqrt{t}) \leq u(x, t) \leq ((q-1)t)^{-\frac{1}{q-1}} \quad \forall (x, t) \in Q_\infty, \quad (1.6)$$

valid for any $a \in \mathcal{S}_\nu$. As a consequence of Brezis and Friedman's result if $q \geq q_c$, i.e. in the *supercritical range*, Problem (1.5) may admit no solution at all. If $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$, the necessary and sufficient conditions for the existence of a maximal solution $u = \bar{u}_\nu$ to Problem (1.5) are obtained in [25] and expressed in terms of the the Bessel capacity $C_{2/q, q'}$, (with $q' = q/(q-1)$). Furthermore, uniqueness does not hold in general as it was pointed out by Le Gall [21]. In the particular case where $\mathcal{S}_\nu = \emptyset$ and ν is simply the Radon measure μ_ν , the necessary and sufficient condition for solvability is that μ_ν does not charge Borel subsets with $C_{2/q, q'}$ -capacity zero. This result was already proven by Baras and Pierre [4] in the particular case of bounded measures and extended by Marcus and Véron [25] to the general case. We shall denote by $\mathfrak{M}_+^q(\mathbb{R}^N)$ the positive cone of the space $\mathfrak{M}^q(\mathbb{R}^N)$ of Radon measures which does not charge Borel subsets with zero $C_{2/q, q'}$ -capacity. Notice that $W^{-2/q, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ is a subset of $\mathfrak{M}_+^q(\mathbb{R}^N)$; here $\mathfrak{M}_+^b(\mathbb{R}^N)$ is the cone of positive bounded Radon measures in \mathbb{R}^N . For such measures, uniqueness always holds and we denote $\bar{u}_{\mu_\nu} = u_{\mu_\nu}$.

In view of the already known facts concerning the parabolic equation, it is useful to recall the much more advanced results previously obtained for the stationary equation

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \quad (1.7)$$

in a smooth bounded domain Ω of \mathbb{R}^N . This equation has been intensively studied since 1993, both by probabilists (Le Gall, Dynkin, Kuznetsov) and by analysts (Marcus, Véron). The existence of a *boundary trace* for positive solutions, in the class of outer-regular positive Borel measures on $\partial\Omega$, is proven by Le Gall [20], [21] in the case $q = N = 2$, by probabilistic methods, and by Marcus and Véron in [23], [24] in the general case $q > 1$, $N > 1$. The existence of a critical exponent $q_e = (N+1)/(N-1)$ is due to Gmira and Véron [12] who shew that, if $q \geq q_e$ boundary isolated singularities of solutions of (1.7) are removable, which is not the case if $1 < q < q_e$. In this *subcritical case* Le Gall and Marcus and Véron proved that the boundary trace establishes a one to one correspondence between positive solutions of (1.7) in Ω and outer regular positive Borel measures on $\partial\Omega$, which is not the case in the *supercritical case* $q \geq q_e$. In [10] Dynkin and Kuznetsov introduced the notion of σ -moderate solution which means that u is a positive solution of (1.7) such that there exists an increasing sequence of positive Radon measures on $\partial\Omega$ $\{\mu_n\}$ belonging to $W^{-2/q, q'}(\partial\Omega)$ such that the corresponding solutions $v = v_{\mu_n}$ of

$$\begin{cases} -\Delta v + v^q = 0 & \text{in } \Omega \\ v = \mu_n & \text{in } \partial\Omega \end{cases} \quad (1.8)$$

converges to u locally uniformly in Ω . This class of solutions plays a fundamental role since Dynkin and Kuznetsov proved that a σ -moderate solution of (1.7) is uniquely determined by its *fine trace*, a new notion of trace introduced in order to avoid the non-uniqueness phenomena. Later on, it is proved by Mselati (if $q = 2$) [32] and then by Dynkin (if $q_e \leq q \leq 2$) [8], that all the positive solutions of (1.7) are σ -moderate. The key-stone element in their proof (partially probabilistic) is the fact that the maximal solution \bar{u}_K of (1.7) with a boundary trace vanishing outside a compact subset $K \subset \partial\Omega$ is indeed σ -moderate. This deep result was obtained by a combination of probabilistic and analytic methods by Mselati [32] in the case $q = 2$ and by purely analytic tools by Marcus and Véron [28], [29] in the case $q \geq q_e$. Defining \underline{u}_K as the

largest σ -moderate solution of (1.7) with a boundary trace concentrated on K , the crucial step in Marcus-Véron's proof (non probabilistic) is the bilateral estimate satisfied by \bar{u}_K and \underline{u}_K

$$C^{-1}\rho(x)W_K(x) \leq \underline{u}_K(x) \leq \bar{u}_K(x) \leq C\rho(x)W_K(x). \quad (1.9)$$

In this expression $C = C(\Omega, q)$, $\rho(x) = \text{dist}(x, \partial\Omega)$ and $W_F(x)$ is the *capacitary potential* of K defined by

$$W_K(x) = \sum_{-\infty}^{\infty} 2^{-\frac{m(q+1)}{q-1}} C_{2/q, q'}(2^m K_m(x)), \quad (1.10)$$

where $K_m(x) = K \cap \{z : 2^{-m-1} \leq |z - x| \leq 2^{-m}\}$, the Bessel capacity being relative to \mathbb{R}^{N-1} . Note that, using a technique introduced in [24], inequality $\bar{u}_K \leq C^2 \underline{u}_K$ implies $\underline{u}_K = \bar{u}_K$.

Extending Dynkin's ideas to the parabolic case, we introduce the following notion

Definition 1.1 *A positive solution u of (1.1) is called σ -moderate if there exists an increasing sequence $\{\mu_n\} \subset W^{-2/q, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ such that the corresponding solution $u := u_{\mu_n}$ of*

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty \\ u(x, 0) = \mu_n & \text{in } \mathbb{R}^N, \end{cases} \quad (1.11)$$

converges to u locally uniformly in Q_∞ .

If F is a closed subset of \mathbb{R}^N , we denote by \bar{u}_F the maximal solution of (1.1) with an initial trace vanishing on F^c , and by \underline{u}_F the maximal σ -moderate solution of (1.1) with an initial trace vanishing on F^c . Thus \underline{u}_F is defined by

$$\underline{u}_F = \sup\{u_\mu : \mu \in \mathfrak{M}_+^q(\mathbb{R}^N), \mu(F^c) = 0\}, \quad (1.12)$$

where $\mathfrak{M}_+^q(\mathbb{R}^N) := W^{-2/q, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$. One of the main goal of this article is to prove that \bar{u}_F is σ -moderate and more precisely,

Theorem 1.2 *For any $q > 1$ and any closed subset F of \mathbb{R}^N , $\bar{u}_F = \underline{u}_F$.*

We define below a set function which will play a fundamental role in the sequel.

Definition 1.3 *Let F be a closed subset of \mathbb{R}^N . The $C_{2/q, q'}$ -capacitary potential W_F of F is defined by*

$$W_F(x, t) = t^{-\frac{1}{q-1}} \sum_{n=0}^{\infty} (n+1)^{\frac{N}{2} - \frac{1}{q-1}} e^{-\frac{n}{4}} C_{2/q, q'} \left(\frac{F_n}{\sqrt{(n+1)t}} \right) \quad \forall (x, t) \in Q_\infty, \quad (1.13)$$

where $F_n = F_n(x, t) := \{y \in F : \sqrt{nt} \leq |x - y| \leq \sqrt{(n+1)t}\}$.

One of the tool for proving Theorem 1.2 is the following bilateral estimate which is only meaningful in the supercritical case, otherwise it reduces to (1.6);

Theorem 1.4 For any $q \geq q_c$ there exist two positive constants $C_1 \geq C_2 > 0$, depending only on N and q such that for any closed subset F of \mathbb{R}^N , there holds

$$C_2 W_F(x, t) \leq \underline{u}_F(x, t) \leq \bar{u}_F(x, t) \leq C_1 W_F(x, t) \quad \forall (x, t) \in Q_\infty. \quad (1.14)$$

It is important to notice that the capacity potential is equivariant with respect to the same scaling transformation which let (1.1) invariant in the sense that, for any $\ell > 0$,

$$\ell^{\frac{1}{q-1}} W_F(\sqrt{\ell}x, \ell t) = W_{F/\sqrt{\ell}}(x, t) \quad \forall (x, t) \in Q_\infty. \quad (1.15)$$

This *quasi representation*, up to uniformly upper and lower bounded functions, is also interesting in the sense that it indicates precisely what are the blow-up point of $\bar{u}_F = \underline{u}_F := u_F$. Introducing an integral expression comparable to W_F we show, in particular, the following results

$$\lim_{\tau \rightarrow 0} C_{2/q, q'} \left(\frac{F}{\tau} \cap B_1(x) \right) = \gamma \in [0, \infty) \implies \lim_{t \rightarrow 0} t^{-\frac{1}{q-1}} u_F(x, t) = C\gamma \quad (1.16)$$

for some $C_\gamma = C(N, q, \gamma) > 0$, and

$$\limsup_{\tau \rightarrow 0} \tau^{\frac{2}{q-1}} C_{2/q, q'} \left(\frac{F}{\tau} \cap B_1(x) \right) < \infty \implies \limsup_{t \rightarrow 0} u_F(x, t) < \infty. \quad (1.17)$$

Our paper is organized as follows. In Section 2 we obtain estimates from above on \bar{u}_F . In Section 3 we give estimates from below on \underline{u}_F . In Section 4 we prove the main theorems and expose various consequences. In Appendix we derive a series of sharp integral inequalities.

Acknowledgements The authors are grateful to the European RTN Contract N° HPRN-CT-2002-00274 for the support provided in the realization of this work.

2 Estimates from above

Some notations. Let Ω be a domain in \mathbb{R}^N with a compact C^2 boundary and $T > 0$. Set $B_r(a)$ the open ball of radius $r > 0$ and center a (and $B_r(0) := B_r$) and

$$Q_T^\Omega := \Omega \times (0, T), \quad \partial_t Q_T^\Omega = \partial\Omega \times (0, T), \quad Q_T := Q_T^{\mathbb{R}^N}, \quad Q_\infty := Q_\infty^{\mathbb{R}^N}.$$

Let $\mathbb{H}^\Omega[\cdot]$ (resp. $\mathbb{H}[\cdot]$) denote the heat potential in Ω with zero lateral boundary data (resp. the heat potential in \mathbb{R}^N) with corresponding kernel

$$(x, y, t) \mapsto H^\Omega(x, y, t) \quad (\text{resp. } (x, y, t) \mapsto H(x, y, t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}).$$

We denote by $q_c := 1 + \frac{N}{2}$, the parabolic critical exponent.

Theorem 2.1 Let $q \geq q_c$. Then there exists a positive constant $C_1 = C_1(N, q)$ such that for any closed subset F of \mathbb{R}^N and any $u \in C^2(Q_\infty) \cap C(\overline{Q_\infty} \setminus F)$ satisfying

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty \\ \lim_{t \rightarrow 0} u(x, t) = 0 & \text{locally uniformly in } F^c, \end{cases} \quad (2.1)$$

there holds

$$u(x, t) \leq C_1 W_F(x, t) \quad \forall (x, t) \in Q_\infty, \quad (2.2)$$

where W_F is the $(2/q, q')$ -capacitary potential of F defined by (1.13).

First we shall consider the case where $F = K$ is compact and

$$K \subset B_r \subset \overline{B}_r, \quad (2.3)$$

and then we shall extend to the general case by a covering argument.

2.1 Global L^q -estimates

Let $\rho > 0$, we assume (2.3) holds and we put

$$\mathcal{T}_{r,\rho}(K) = \{\eta \in C_0^\infty(B_{r+\rho}), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\}. \quad (2.4)$$

If $\eta \in \mathcal{T}_{r,\rho}(K)$, we set $\eta^* = 1 - \eta$, $\zeta = \mathbb{H}[\eta^*]^{2q'}$ and

$$R[\eta] = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta] + \Delta \mathbb{H}[\eta]|. \quad (2.5)$$

We fix $T > 0$ and shall consider the equation on Q_T . Throughout this paper C will denote a generic positive constant, depending only on N, q and sometimes T , the value of which may vary from one occurrence to another. We shall also use sometimes the notation $A \approx B$ for meaning that there exists a constant $C > 0$ independent of the data such that $C^{-1}A \leq B \leq CA$.

Except in Lemma 2.12 the only assumption on q is $q > 1$. In the sequel we shall obtain pointwise estimate on the solution expressed in terms of the $L^{q'}$ -norm of $R[\eta]$ for $\eta \in \mathcal{T}_{r,\rho}(K)$. Although these estimates could have been immediately turned into capacitary estimates as in [29], the advantage of keeping them comes from the possibility of performing operations such as dilations or summations on them. The next lemma points out the connection between $R[\eta]$ and the $C_{2/q,q'}$ -capacity of K .

Lemma 2.2 *There exists $C = C(N, q) > 0$ such that*

$$C^{-1} \|\eta\|_{W^{2/q,q'}}^{q'} \leq \iint_{Q_\infty} (R[\eta])^{q'} dx dt := \|R[\eta]\|_{L^{q'}}^{q'} \leq C \|\eta\|_{W^{2/q,q'}}^{q'} \quad \forall \eta \in \mathcal{T}_{r,\rho}(K). \quad (2.6)$$

Therefore

$$\inf \left\{ \|R[\eta]\|_{L^{q'}}^{q'} : \eta \in \mathcal{T}_{r,\rho}(K) \right\} \approx C_{2/q,q'}^{B_{r+\rho}}(K). \quad (2.7)$$

Proof. There holds $\partial_t \mathbb{H}[\eta] = \Delta \mathbb{H}[\eta]$, and

$$\iint_{Q_\infty} |\partial_t \mathbb{H}[\eta]|^{q'} dx dt = \int_0^\infty \left\| t^{1-1/q} \partial_t \mathbb{H}[\eta] \right\|_{L^{q'}(\mathbb{R}^N)}^{q'} \frac{dt}{t} \approx \|\eta\|_{[W^{2,q'}, L^{q'}]_{1/q,q'}}^{q'} \quad (2.8)$$

where $[W^{2,q'}, L^{q'}]_{1/q, q'}$ indicates the real interpolation functor of degree $1/q$ between $W^{2,q'}(\mathbb{R}^N)$ and $L^{q'}(\mathbb{R}^N)$ [35]. Similarly, and using the Gagliardo-Nirenberg inequality,

$$\iint_{Q_\infty} |\nabla(\mathbb{H}[\eta])|^{2q'} dx dt \leq C \|\eta\|_{W^{2/q, q'}}^{q'} \|\eta\|_{L^\infty}^{q'} = C \|\eta\|_{W^{2/q, q'}}^{q'}. \quad (2.9)$$

Inequality (2.6) follows from (2.8) and (2.9), and (2.7) from the definition of the Bessel capacity relative to $B_{r+\rho}$. \square

Lemma 2.3 *There exists $C = C(N, q) > 0$ such that for any $T > 0$,*

$$\iint_{Q_\infty} u^q \zeta dx dt + \int_{\mathbb{R}^N} (u\zeta)(x, T) dx \leq C \|R[\eta]\|_{L^{q'}}^q \quad \forall \eta \in \mathcal{T}_{r, \rho}(K). \quad (2.10)$$

Proof. We recall that there always hold

$$0 \leq u(x, t) \leq \left(\frac{1}{t(q-1)} \right)^{\frac{1}{q-1}} \quad \forall (x, t) \in Q_\infty, \quad (2.11)$$

and (see [6] e.g.)

$$0 \leq u(x, t) \leq \left(\frac{C}{t + (|x| - r)^2} \right)^{\frac{1}{q-1}} \quad \forall (x, t) \in Q_\infty \setminus B_r. \quad (2.12)$$

Since η^* vanishes in an open neighborhood \mathcal{N}_1 , for any open subset \mathcal{N}_2 such that $K \subset \mathcal{N}_2 \subset \overline{\mathcal{N}_2} \subset \mathcal{N}_1$ there exist $c_{\mathcal{N}_2} > 0$ and $C_{\mathcal{N}_2} > 0$ such that

$$\mathbb{H}[\eta^*](x, t) \leq C_{\mathcal{N}_2} \exp(-c_{\mathcal{N}_2} t), \quad \forall (x, t) \in Q_T^{\mathcal{N}_2}.$$

Therefore

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} (u\zeta)(x, t) dx = 0.$$

Thus ζ is an admissible test function and one has

$$\iint_{Q_T} u^q \zeta dx dt + \int_{\mathbb{R}^N} (u\zeta)(x, T) dx = \iint_{Q_T} u(\partial_t \zeta + \Delta \zeta) dx dt. \quad (2.13)$$

Notice that the three terms on the left-hand side are nonnegative. Put $\mathbb{H}_{\eta^*} = \mathbb{H}[\eta^*]$, then

$$\begin{aligned} \partial_t \zeta + \Delta \zeta &= 2q' \mathbb{H}_{\eta^*}^{2q'-1} (\partial_t \mathbb{H}_{\eta^*} + \Delta \mathbb{H}_{\eta^*}) + 2q'(2q' - 1) \mathbb{H}_{\eta^*}^{2q'-2} |\nabla \mathbb{H}_{\eta^*}|^2, \\ &= 2q' \mathbb{H}_{\eta^*}^{2q'-1} (\partial_t \mathbb{H}_\eta + \Delta \mathbb{H}_\eta) + 2q'(2q' - 1) \mathbb{H}_{\eta^*}^{2q'-2} |\nabla \mathbb{H}_\eta|^2, \end{aligned}$$

because $\mathbb{H}_{\eta^*} = 1 - \mathbb{H}_\eta$, hence

$$u(\partial_t \zeta + \Delta \zeta) = u \mathbb{H}_{\eta^*}^{2q'/q} \left[2q'(2q' - 1) \mathbb{H}_{\eta^*}^{2q'-2-2q'/q} |\nabla \mathbb{H}_\eta|^2 - 2q' \mathbb{H}_{\eta^*}^{2q'-1-2q'/q} (\Delta \mathbb{H}_\eta + \partial_t \mathbb{H}_\eta) \right].$$

Since $2q' - 2 - 2q'/q = 0$ and $0 \leq \mathbb{H}_{\eta^*} \leq 1$,

$$\left| \iint_{Q_T} u(\partial_t \zeta + \Delta \zeta) dx dt \right| \leq C(q) \left(\iint_{Q_T} u^q \zeta dx dt \right)^{1/q} \left(\iint_{Q_T} R^{q'}(\eta) dx dt \right)^{1/q'},$$

where

$$R(\eta) = |\nabla \mathbb{H}_\eta|^2 + |\Delta \mathbb{H}_\eta + \partial_t \mathbb{H}_\eta|.$$

Using Lemma 2.2 one obtains (2.10). \square

Proposition 2.4 *Let $r > 0$, $\rho > 0$, $T \geq (r + \rho)^2$*

$$\mathcal{E}_{r+\rho} := \{(x, t) : |x|^2 + t \leq (r + \rho)^2\}$$

and $Q_{r+\rho, T} = Q_T \setminus \mathcal{E}_{r+\rho}$. *There exists $C = C(N, q, T) > 0$ such that*

$$\iint_{Q_{r+\rho, T}} u^q dx dt + \int_{\mathbb{R}^N} u(x, T) dx \leq C \|R[\eta]\|_{L^{q'}}^{q'} \quad \forall \eta \in \mathcal{T}_{r, \rho}(K). \quad (2.14)$$

Proof. Because $K \subset B_r$ and $\eta^* \equiv 1$ outside $B_{r+\rho}$ and takes value between 0 and 1,

$$\begin{aligned} \mathbb{H}[\eta^*](x, t) \geq \mathbb{H}[1 - \chi_{B_{r+\rho}}](x, t) &= \left(\frac{1}{4\pi t} \right)^{\frac{N}{2}} \int_{|y| \geq r+\rho} e^{-\frac{|x-y|^2}{4t}} dy, \\ &= 1 - \left(\frac{1}{4\pi t} \right)^{\frac{N}{2}} \int_{|y| \leq r+\rho} e^{-\frac{|x-y|^2}{4t}} dy. \end{aligned}$$

For $(x, t) \in \mathcal{E}_{r+\rho}$, put $x = (r + \rho)\xi$, $y = (r + \rho)v$ and $t = (r + \rho)^2\tau$. Then $(\xi, \tau) \in \mathcal{E}_1$ and

$$\left(\frac{1}{4\pi t} \right)^{\frac{N}{2}} \int_{|y| \leq r+\rho} e^{-\frac{|x-y|^2}{4t}} dy = \left(\frac{1}{4\pi \tau} \right)^{\frac{N}{2}} \int_{|v| \leq 1} e^{-\frac{|\xi-v|^2}{4\tau}} dv.$$

We claim that

$$\max \left\{ \left(\frac{1}{4\pi \tau} \right)^{\frac{N}{2}} \int_{|v| \leq 1} e^{-\frac{|\xi-v|^2}{4\tau}} dv : (\xi, \tau) \in \mathcal{E}_1 \right\} = \ell, \quad (2.15)$$

and $\ell = \ell(N) \in (0, 1]$. We recall that

$$\left(\frac{1}{4\pi \tau} \right)^{\frac{N}{2}} \int_{|v| \leq 1} e^{-\frac{|\xi-v|^2}{4\tau}} dv < 1 \quad \forall \tau > 0. \quad (2.16)$$

If the maximum is achieved for some $(\bar{\xi}, \bar{\tau}) \in \mathcal{E}_1$, it is smaller than 1 and

$$\mathbb{H}[\eta^*](x, t) \geq \mathbb{H}[1 - \chi_{B_{r+\rho}}](x, t) \geq 1 - \ell > 0, \quad \forall (x, t) \in \mathcal{E}_{r+\rho}. \quad (2.17)$$

Let us assume that the maximum is achieved following a sequence $\{(\xi_n, \tau_n)\}$ with $\tau_n \rightarrow 0$ and $|\xi_n| \downarrow 1$. We can assume that $\xi_n \rightarrow \bar{\xi}$ with $|\bar{\xi}| = 1$, then

$$\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}} \int_{|v| \leq 1} e^{-\frac{|\xi_n - v|^2}{4\tau_n}} dv = \left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}} \int_{B_1(\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv.$$

But $B_1(\xi_n) \cap B_1(-\xi_n) = \emptyset$,

$$\int_{B_1(\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv + \int_{B_1(-\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv < \int_{\mathbb{R}^N} e^{-\frac{|v|^2}{4\tau_n}} dv$$

and

$$\int_{B_1(\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv = \int_{B_1(-\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv.$$

This implies

$$\lim_{n \rightarrow \infty} \left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}} \int_{B_1(\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv \leq 1/2.$$

If the maximum were achieved with a sequence $\{(\xi_n, \tau_n)\}$ with $|\tau_n| \rightarrow \infty$, it would also imply (2.17), since the integral term in (2.16) is always bounded. Therefore (2.16) holds. Put $C = (1 - \ell)^{-1}$, then

$$\iint_{Q_{r,T}} u^q dx dt + \int_{\mathbb{R}^N} u(\cdot, T) dx \leq C \|R[\eta]\|_{L^{q'}}^{q'}, \quad (2.18)$$

and (2.14) follows. \square

2.2 Pointwise estimates

We give first a rough pointwise estimate.

Lemma 2.5 *There exists a constant $C = C(N, q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$,*

$$u(x, (r + 2\rho)^2) \leq \frac{C \|R[\eta]\|_{L^{q'}}^{q'}}{(\rho(r + \rho))^{\frac{N}{2}}}, \quad \forall x \in \mathbb{R}^N. \quad (2.19)$$

Proof. We observe first that

$$\int_s^T \int_{\mathbb{R}^N} u^q dx dt + \int_{\mathbb{R}^N} u(x, T) dx = \int_{\mathbb{R}^N} u(x, s) dx \quad \forall T > s > 0. \quad (2.20)$$

By the maximum principle u is dominated by the maximal solution v which has the indicatrix function I_{B_r} for initial trace. The function v is the limit, as $k \rightarrow \infty$, of the solutions v_k with initial data $k\chi_{B_r}$. Since $v_k \leq k\mathbb{H}[\chi_{B_r}]$, it follows Hence

$$\int_{\mathbb{R}^N} u(\cdot, s) dx \leq C \|R[\eta]\|_{L^{q'}}^{q'} \quad \forall T > s \geq (r + \rho)^2, \quad (2.21)$$

by Lemma 2.3. Using the fact that

$$u(x, \tau + s) \leq \mathbb{H}[u(\cdot, s)](x, \tau) \leq \left(\frac{1}{4\pi\tau} \right)^{\frac{N}{2}} \int_{\mathbb{R}^N} u(\cdot, s) dx,$$

we obtain (2.19) with $s = (r + \rho)^2$ and $\tau = (r + 2\rho)^2 - (r + \rho)^2 \approx \rho(r + \rho)$. \square

The above estimate does not take into account the fact that $u(x, 0) = 0$ if $|x| \geq r$. It is mainly interesting if $|x| \leq r$. In order to derive a sharper estimate which uses the localization of the singularity and not only the $L^{q'}$ -norm of $R[\eta]$. For such a goal, we need some lateral boundary estimates.

Lemma 2.6 *Let $\gamma \geq r + 2\rho$ and $c > 0$ and either $N = 1$ or 2 and $0 \leq t \leq c\gamma^2$ for some $c > 0$, or $N \geq 3$ and $t > 0$. Then, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$\int_0^t \int_{\partial B_\gamma} u dS d\tau \leq C_5 \gamma \|R[\eta]\|_{L^{q'}}^{q'}. \quad (2.22)$$

where $C > 0$ depends on N, q and c if $N = 1, 2$ or depends only on N and q if $N \geq 3$.

Proof. Let us assume that $N = 1$ or 2 . Put $G^\gamma := B_\gamma^c \times (-\infty, 0)$ and $\partial_\ell G^\gamma = \partial B_\gamma \times (-\infty, 0)$. Set

$$h_\gamma(x) = 1 - \frac{\gamma}{|x|},$$

and let ψ_γ be the solution of

$$\begin{aligned} \partial_\tau \psi_\gamma + \Delta \psi_\gamma &= 0 && \text{in } G^\gamma, \\ \psi_\gamma &= 0 && \text{on } \partial_\ell G^\gamma, \\ \psi_\gamma(\cdot, 0) &= h_\gamma && \text{in } B_\gamma^c. \end{aligned} \quad (2.23)$$

Thus the function

$$\tilde{\psi}(x, \tau) = \psi_\gamma(\gamma x, \gamma^2 \tau)$$

satisfies

$$\begin{aligned} \partial_t \tilde{\psi} + \Delta \tilde{\psi} &= 0 && \text{in } G^1 \\ \tilde{\psi} &= 0 && \text{on } \partial_\ell G^1 \\ \tilde{\psi}(\cdot, 0) &= \tilde{h} && \text{in } B_1^c, \end{aligned} \quad (2.24)$$

and $\tilde{h}(x) = 1 - |x|^{-1}$. By the maximum principle $0 \leq \tilde{\psi} \leq 1$, and by Hopf Lemma

$$-\frac{\partial \tilde{\psi}}{\partial \mathbf{n}} \Big|_{\partial B_1 \times [-c, 0]} \geq \theta > 0, \quad (2.25)$$

where $\theta = \theta(N, c)$. Then $0 \leq \psi_\gamma \leq 1$ and

$$-\frac{\partial \psi_\gamma}{\partial \mathbf{n}} \Big|_{\partial B_\gamma \times [-\gamma^2, 0]} \geq \theta/\gamma. \quad (2.26)$$

Multiplying (1.1) by $\psi_\gamma(x, \tau - t) = \psi_\gamma^*(x, \tau)$ and integrating on $B_\gamma^c \times (0, t)$ yields to

$$\int_0^t \int_{B_\gamma^c} u^a \psi_r^* dx d\tau + \int_{B_\gamma^c} (uh_\gamma)(x, t) dx - \int_0^t \int_{\partial B_\gamma} \frac{\partial u}{\partial \mathbf{n}} \psi_\gamma^* dS d\tau = - \int_0^t \int_{\partial B_\gamma} \frac{\partial \psi_\gamma^*}{\partial \mathbf{n}} u d\sigma d\tau. \quad (2.27)$$

Since ψ_γ^* is bounded from above by 1, (2.22) follows from (2.26) and Proposition 2.4 (notice that $B_\gamma^c \times (0, t) \subset \mathcal{E}_\gamma^c$), first by taking $t = T = \gamma^2 \geq (r + 2\rho)^2$, and then for any $t \leq \gamma^2$.

If $N \geq 3$, we proceed as above except that we take

$$h_\gamma(x) = 1 - \left(\frac{\gamma}{|x|} \right)^{N-2}.$$

Then $\psi_\gamma(x, t) = h_\gamma(x)$ and $\theta = N - 2$ is independent of the length of the time interval. This leads to the conclusion. \square

Lemma 2.7 *I- Let $M, a > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that*

$$0 \leq \eta(x) \leq M e^{-a|x|^2} \quad \text{a.e. in } \mathbb{R}^N. \quad (2.28)$$

Then, for any $t > 0$,

$$0 \leq \mathbb{H}[\eta](x, t) \leq \frac{M}{(4at + 1)^{\frac{N}{2}}} e^{-\frac{a|x|^2}{4at+1}} \quad \forall x \in \mathbb{R}^N. \quad (2.29)$$

II- Let $M, a, b > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that

$$0 \leq \eta(x) \leq M e^{-a(|x|-b)_+^2} \quad \text{a.e. in } \mathbb{R}^N. \quad (2.30)$$

Then, for any $t > 0$,

$$0 \leq \mathbb{H}[\eta](x, t) \leq \frac{M e^{-\frac{a(|x|-b)_+^2}{4at+1}}}{(4at + 1)^{\frac{N}{2}}} \quad \forall x \in \mathbb{R}^N, \forall t > 0. \quad (2.31)$$

Proof. For the first statement, put $a = \frac{1}{4}s$. Then

$$0 \leq \eta(x) \leq M(4\pi s)^{\frac{N}{2}} \frac{1}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4s}} = C(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, s).$$

By the order property of the heat kernel,

$$0 \leq \mathbb{H}[\eta](x, t) \leq M(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, t + s) = M \left(\frac{s}{t + s} \right)^{\frac{N}{2}} e^{-\frac{|x|^2}{4(t+s)}},$$

and (2.29) follows by replacing s by $\frac{1}{4}a$.

For the second statement, let $\tilde{a} < a$ and $R = \max\{e^{-a(r-b)_+^2 + \tilde{a}r^2} : r \geq 0\}$. A direct computation gives $R = e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}$, and (2.31) implies

$$0 \leq \eta(x) \leq M e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}} e^{-\tilde{a}|x|^2}.$$

Applying the statement I, we derive

$$0 \leq \mathbb{H}[\eta](x, t) \leq \frac{C e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}}{(4\tilde{a}t + 1)^{\frac{N}{2}}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \quad \forall x \in \mathbb{R}^N, \forall t > 0. \quad (2.32)$$

Since for any $x \in \mathbb{R}^N$ and $t > 0$,

$$(4\tilde{a}t + 1)^{-\frac{N}{2}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \leq e^{-\frac{a\tilde{a}b^2}{a-\tilde{a}}} (4at + 1)^{-\frac{N}{2}} e^{-\frac{a(|x|-b)^2}{4at+1}},$$

(2.31) follows from (2.32). \square

Lemma 2.8 *There exists a constant $C = C(N, q) > 0$ such that, for any $\eta \in \mathcal{T}_{r, \rho}(K)$, there holds*

$$u(x, (r + 2\rho)^2) \leq C \max \left\{ \frac{r + \rho}{(|x| - r - 2\rho)^{N+1}}, \frac{|x| - r - 2\rho}{(r + \rho)^{N+1}} \right\} e^{-\frac{(|x| - (r + 2\rho))^2}{4(r + 2\rho)^2}} \|R[\eta]\|_{L^{q'}}^q, \quad (2.33)$$

for any $x \in \mathbb{R}^N \setminus B_{r+3\rho}$.

Proof. It is classical that the Dirichlet heat kernel $H^{B_1^c}$ in the complement of B_1 satisfies, for some $C = C(N) > 0$,

$$H^{B_1^c}(x', y', t', s') \leq C_7 (t' - s')^{-(N+2)/2} (|x'| - 1) e^{-\frac{|x' - y'|^2}{4(t' - s')}}}, \quad (2.34)$$

for $t' > s'$. By performing the change of variable $x' \mapsto (r + 2\rho)x'$, $t' \mapsto (r + 2\rho)^2 t'$, for any $x \in \mathbb{R}^N \setminus B_{r+2\rho}$ and $0 \leq t \leq T$, one obtains

$$u(x, t) \leq C (|x| - r - 2\rho) \int_0^t \int_{\partial B_{r+2\rho}} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{1+\frac{N}{2}}} u(y, s) d\sigma(y) ds. \quad (2.35)$$

The right-hand side term in (2.35) is smaller than

$$\max \left\{ \frac{C (|x| - r - 2\rho)}{(t-s)^{1+\frac{N}{2}}} e^{-\frac{(|x|-r-2\rho)^2}{4(t-s)}} : s \in (0, t) \right\} \int_0^t \int_{\partial B_{r+2\rho}} u(y, s) d\sigma(y) ds.$$

We fix $t = (r + 2\rho)^2$ and $|x| \geq r + 3\rho$. Since

$$\begin{aligned} & \max \left\{ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : s \in (0, (r + 2\rho)^2) \right\} \\ &= (|x| - r - 2\rho)^{-2-N} \max \left\{ \frac{e^{-\frac{1}{4\sigma}}}{\sigma^{1+\frac{N}{2}}} : 0 < \sigma < \left(\frac{r + 2\rho}{|x| - r - 2\rho} \right)^2 \right\}, \end{aligned}$$

a direct computation gives

$$\begin{aligned} & \max \left\{ \frac{e^{-\frac{1}{4}\sigma}}{\sigma^{1+\frac{N}{2}}} : 0 < \sigma < \left(\frac{r+2\rho}{|x|-r-2\rho} \right)^2 \right\} \\ &= \begin{cases} (2N+4)^{1+\frac{N}{2}} e^{-(N+2)/2} & \text{if } r+3\rho \leq |x| \leq (r+2\rho)(1+\sqrt{4+2N}), \\ \left(\frac{|x|-r-2\rho}{r+2\rho} \right)^{2+N} e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2} & \text{if } |x| \geq (r+2\rho)(1+\sqrt{4+2N}). \end{cases} \end{aligned}$$

Thus there exists a constant $C(N) > 0$ such that

$$\max \left\{ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : s \in (0, (r+2\rho)^2) \right\} \leq C(N)\rho^{-2-N} e^{-\left(\frac{|x|-(r+2\rho)}{2r+4\rho}\right)^2}. \quad (2.36)$$

Combining this estimate with (2.22) with $\gamma = r+2\rho$ and (2.35), one derives (2.33). \square

Lemma 2.9 *Under the assumptions of Lemma 2.8, there exists a constant $C = C(N, q) > 0$ such that*

$$0 \leq u(x, (r+2\rho)^2) \leq C \max \left\{ \frac{(r+\rho)^3}{\rho(|x|-r-2\rho)^{N+1}}, \frac{1}{(r+\rho)^{N-1}\rho} \right\} e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2} \|R[\eta]\|_{L^{q'}}^{q'}, \quad (2.37)$$

for every $x \in \mathbb{R}^N \setminus B_{r+3\rho}$.

Proof. This is a direct consequence of the inequality

$$(|x|-r-2\rho)e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2} \leq \frac{C(r+\rho)^2}{\rho} e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2}, \quad \forall x \in B_{r+2\rho}^c, \quad (2.38)$$

and Lemma 2.8. \square

Lemma 2.10 *There exists a constant $C = C(N, q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, the following estimate holds*

$$u(x, t) \leq \frac{C\tilde{M}e^{-\frac{(|x|-r-3\rho)_+^2}{4t}}}{t^{\frac{N}{2}}} \|R[\eta]\|_{L^{q'}}^{q'}, \quad \forall x \in \mathbb{R}^N, \forall t \geq (r+2\rho)^2, \quad (2.39)$$

where

$$\tilde{M} = \tilde{M}(x, r, \rho) = \begin{cases} \left(1 + \frac{r}{\rho}\right)^{\frac{N}{2}} & \text{if } |x| < r+3\rho \\ \frac{(r+\rho)^{N+3}}{\rho(|x|-r-2\rho)^{N+2}} & \text{if } r+3\rho \leq |x| \leq C_N(r+2\rho) \\ 1 + \frac{r}{\rho} & \text{if } |x| \geq C_N(r+2\rho) \end{cases} \quad (2.40)$$

with $C_N = 1 + \sqrt{4+2N}$.

Proof. It follows by the maximum principle

$$u(x, t) \leq \mathbb{H}[u(\cdot, (r + 2\rho)^2)](x, t - (r + 2\rho)^2).$$

for $t \geq (r + 2\rho)^2$ and $x \in \mathbb{R}^N$. By Lemma 2.5 and Lemma 2.9

$$u(x, (r + 2\rho)^2) \leq C_{10} \tilde{M} e^{-\frac{(|x| - r - 3\rho)^2}{4(r + 2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'},$$

where

$$\tilde{M} = \begin{cases} ((r + \rho)\rho)^{-\frac{N}{2}} & \text{if } |x| < r + 3\rho \\ \frac{(r + \rho)^3}{\rho} (|x| - r - 2\rho)^{N+2} & \text{if } r + 3\rho \leq |x| \leq C_N(r + 2\rho) \\ \frac{1}{(r + \rho)^{N-1}\rho} & \text{if } |x| \geq C_N(r + 2\rho) \end{cases}$$

Applying Lemma 2.7 with $a = (2r + 4\rho)^{-2}$, $b = r + 3\rho$ and t replaced by $t - (r + 2\rho)^2$ implies

$$u(x, t) \leq C \frac{(r + 2\rho)^N \tilde{M}}{t^{\frac{N}{2}}} e^{-\frac{(|x| - r - 3\rho)^2}{4t}} \|R[\eta]\|_{L^{q'}}^{q'}, \quad (2.41)$$

for all $x \in B_{r+3\rho}^c$ and $t \geq (r + 2\rho)^2$, which is (2.39). \square

The next estimate gives a precise upper bound for u when t is not bounded from below.

Lemma 2.11 *Assume that $0 < t \leq (r + 2\rho)^2$ for some $c > 0$, then there exists a constant $C = C(N, q) > 0$ such that the following estimate holds*

$$u(x, t) \leq C(r + \rho) \max \left\{ \frac{1}{(|x| - r - 2\rho)^{N+1}}, \frac{1}{\rho t^{\frac{N}{2}}} \right\} e^{-\frac{(|x| - r - 3\rho)^2}{4t}} \|R[\eta]\|_{L^{q'}}^{q'}, \quad (2.42)$$

for any $(x, t) \in \mathbb{R}^N \setminus B_{r+3\rho} \times (0, (r + 2\rho)^2]$.

Proof. By using (2.22) the following estimate is a straightforward variant of (2.33) for any $\gamma \geq r + 2\rho$,

$$u(x, t) \leq C_8 (|x| - r - 2\rho)(r + 2\rho) \max \left\{ \frac{e^{-\frac{(|x| - r - 2\rho)^2}{4s}}}{s^{1 + \frac{N}{2}}} : 0 < s \leq t \right\} \|R[\eta]\|_{L^{q'}}^{q'}. \quad (2.43)$$

Clearly

$$\begin{aligned} & \max \left\{ \frac{e^{-\frac{(|x| - r - 2\rho)^2}{4s}}}{s^{1 + \frac{N}{2}}} : 0 < s \leq t \right\} \\ &= \begin{cases} (2N + 4)^{1 + \frac{N}{2}} (|x| - r - 2\rho)^{-N-2} e^{-\frac{N+2}{2}} & \text{if } 0 < |x| \leq r + 2\rho + \sqrt{2t(N + 2)} \\ \frac{e^{-\frac{(|x| - r - 2\rho)^2}{4t}}}{t^{1 + \frac{N}{2}}} & \text{if } |x| > r + 2\rho + \sqrt{2t(N + 2)}. \end{cases} \end{aligned}$$

By elementary analysis, if $x \in B_{r+3\rho}^c$,

$$(|x| - r - 2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq e^{-\frac{(|x|-r-3\rho)^2}{4t}} \begin{cases} \rho e^{-\frac{\rho^2}{4t}} & \text{if } 2t < \rho^2 \\ \frac{2t}{\rho} e^{-1+\frac{\rho^2}{4t}} & \text{if } \rho^2 \leq 2t \leq 2(r+2\rho)^2. \end{cases}$$

However, since

$$\frac{\rho}{t} e^{-\frac{\rho^2}{4t}} \leq \frac{4}{\rho},$$

we derive

$$(|x| - r - 2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq \frac{Ct}{\rho} e^{-\frac{(|x|-r-3\rho)^2}{4t}},$$

from which inequality (2.42) follows. \square

Lemma 2.12 *Assume $q \geq q_c$. Let $r > 0$, $\rho > 0$ and K be a compact subset of $B_{r+\rho}$. If $\eta \in \mathcal{T}_{r,\rho}(K)$, denote by η_r the function defined by $\eta_r(x) = \eta(rx)$ and*

$$R_r[\eta_r](x, t) = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta] + \Delta \mathbb{H}[\eta]|(rx, r^2t) \quad \forall (x, t) \in Q_\infty.$$

Then

$$\|R[\eta]\|_{L^{q'}}^{q'} = r^{N-\frac{2}{q-1}} \|R_r[\eta_r]\|_{L^{q'}}^{q'}. \quad (2.44)$$

Furthermore

$$C_{2/q,q'}^{B_{r+\rho}}(K) = r^{N-\frac{2}{q-1}} C_{2/q,q'}^{B_{1+\frac{\rho}{r}}}(K/r), \quad (2.45)$$

and

$$r^{N-\frac{2}{q-1}} C_{2/q,q'}(K/r) \leq C_{2/q,q'}^{B_{r+\rho}}(K) \leq Cr^{N-\frac{2}{q-1}} \left(1 + \frac{r}{\rho}\right)^{\frac{2}{q-1}} C_{2/q,q'}(K/r). \quad (2.46)$$

Proof. Estimate (2.44) follows from the change of variable $(rx, r^2t) = (y, s)$. Thus it implies the scaling property (2.47), since there is a one to one correspondence between $\mathcal{T}_{r,\rho}(K)$ and $\mathcal{T}_{1,\frac{\rho}{r}}(K/r)$. In order to prove (2.44) set $K' = K/r \subset B_1$, thus

$$C_{2/q,q'}^{B_{1+\frac{\rho}{r}}}(K') = \inf \left\{ \|\zeta\|_{W^{2/q,q'}}^{q'} : \zeta \in \mathcal{T}_{1,\frac{\rho}{r}}(K') \right\}.$$

Let $\phi \in C^2(\mathbb{R}^N)$ be a radial cut-off function such that $0 \leq \rho \leq 1$, $\rho = 1$ on B_1 , $\rho = 0$ on $\mathbb{R}^N \setminus B_{1+\frac{\rho}{r}}$, $|\nabla \phi| \leq Cr\rho^{-1}\chi_{B_{1+\frac{\rho}{r}} \setminus B_1}$ and $|D^2 \phi| \leq Cr^2\rho^{-2}\chi_{B_{1+\frac{\rho}{r}} \setminus B_1}$, where C is independent of r and ρ . Let $\zeta \in C_0^2(\mathbb{R}^N)$. Then

$$\nabla(\zeta\phi) = \zeta\nabla\phi + \phi\nabla\zeta, \quad D^2(\zeta\phi) = \zeta D^2\phi + \phi D^2\zeta + 2\nabla\phi \otimes \nabla\zeta.$$

Thus $\|\zeta\phi\|_{L^{q'}(B_{1+\frac{\rho}{r}})} \leq \|\zeta\|_{L^{q'}(\mathbb{R}^N)}$,

$$\int_{B_{1+\frac{\rho}{r}}} |\nabla(\zeta\phi)|^{q'} dx \leq C \left(1 + \frac{r}{\rho}\right)^{q'} \|\zeta\|_{W^{1,q'}}^{q'},$$

and

$$\int_{B_{r+\rho}} |D^2(\zeta\phi)|^{q'} dx \leq C \left(1 + \frac{r^2}{\rho^2}\right)^{q'} \|\zeta\|_{W^{2,q'}}^{q'}.$$

Finally

$$\|\zeta\phi\|_{W^{2/q,q'}} \leq C \left(1 + \frac{r^2}{\rho^2}\right) \|\zeta\|_{W^{2/q,q'}}.$$

Denote by \mathcal{T} the linear mapping $\zeta \mapsto \zeta\phi$. Because

$$W^{2/q,q'} = [W^{2,q'}, L^{q'}]_{1/q,q'},$$

(here we use the Lions-Petree real interpolation notations and results from [22]), it follows

$$\|\mathcal{T}\|_{\mathcal{L}(W_0^{2/q,q'}(\mathbb{R}^N), W_0^{2/q,q'}(B_{1+\frac{\rho}{r}}))} \leq C(q) \left(1 + \frac{r^2}{\rho^2}\right)^{1/q}.$$

Therefore

$$C_{2/q,q'}^{B_{1+\frac{\rho}{r}}}(K') \leq C \left(1 + \frac{r^2}{\rho^2}\right)^{\frac{1}{q-1}} C_{2/q,q'}(K').$$

Thus we get the left-hand side of (2.46). The right-hand side is a straightforward consequence of (2.47). \square

Remark. In the subcritical case $1 < q < q_c$, estimate (2.46) becomes

$$C_{2/q,q'}^{B_{r+\rho}}(K) \leq C \max\{r^N, \rho^N\} \left(1 + \rho^{-\frac{2}{q-1}}\right) C_{2/q,q'}(K/r). \quad (2.47)$$

By using Lemma 2.11, it is easy to derive from this estimate that any positive solution u of (2.1), the initial trace of which vanishes outside 0, satisfies

$$u(x, t) \leq Ct^{-\frac{1}{q-1}} \min \left\{ 1, \left(\frac{|x|}{\sqrt{t}} \right)^{\frac{2}{q-1}-N} e^{-\frac{|x|^2}{4t}} \right\} \quad \forall (x, t) \in Q_\infty. \quad (2.48)$$

This upper estimate corresponds to the one obtained in [7]. If $F = \overline{B}_r$ the upper estimate is less esthetic. However, it is proved in [25] by a barrier method that, if the initial trace of positive solution u of (2.1), vanishes outside F , and if $1 < q < 3$, there holds

$$u(x, t) \leq t^{-\frac{1}{q-1}} f_1(|x| - r) / \sqrt{t} \quad \forall (x, t) \in Q_\infty, |x| \geq r, \quad (2.49)$$

where $f = f_1$ is the unique positive (and radial) solution of

$$\begin{cases} f'' + \frac{y}{2}f' + \frac{1}{q-1}f - f^q = 0 & \text{in } (0, \infty) \\ f'(0) = 0, \lim_{y \rightarrow \infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \quad (2.50)$$

Notice that the existence of f_1 follows from [7] since q is the critical exponent in 1 dim. Furthermore f_1 has the following asymptotic expansion

$$f_1(y) = Cy^{(3-q)/(q-1)} e^{-y^2/4t} (1 + o(1)) \quad \text{as } y \rightarrow \infty.$$

2.3 The upper Wiener test

Definition 2.13 We define on $\mathbb{R}^N \times \mathbb{R}$ the two *parabolic distances* δ_2 and δ_∞ by

$$\delta_2[(x, t), (y, s)] := \sqrt{|x - y|^2 + |t - s|}, \quad (2.51)$$

and

$$\delta_\infty[(x, t), (y, s)] := \max\{|x - y|, \sqrt{|t - s|}\}. \quad (2.52)$$

If $K \subset \mathbb{R}^N$ and $i = 2, \infty$,

$$\delta_i[(x, t), K] = \inf\{\delta_i[(x, t), (y, 0)] : y \in K\} = \begin{cases} \max\{\text{dist}(x, K), \sqrt{|t|}\} & \text{if } i = \infty, \\ \sqrt{\text{dist}^2(x, K) + |t|} & \text{if } i = 2. \end{cases}$$

For $\beta > 0$ and $i = 2, \infty$, we denote by $\mathcal{B}_\beta^i(m)$ the parabolic ball of center $m = (x, t)$ and radius β in the parabolic distance δ_i .

Let K be *any* compact subset of \mathbb{R}^N and \bar{u}_K the maximal solution of (1.1) which blows up on K . The function \bar{u}_K is obtained as the decreasing limit of the \bar{u}_{K_ϵ} ($\epsilon > 0$) when $\epsilon \rightarrow 0$, where

$$K_\epsilon = \{x \in \mathbb{R}^N : \text{dist}(x, K) \leq \epsilon\}$$

and $\bar{u}_{K_\epsilon} = \lim_{k \rightarrow \infty} u_{k, K_\epsilon} = \bar{u}_K$, where u_k is the solution of the classical problem,

$$\begin{cases} \partial_t u_k - \Delta u_k + u_k^q = 0 & \text{in } Q_T, \\ u_k = 0 & \text{on } \partial_\ell Q_T, \\ u_k(\cdot, 0) = k\chi_{K_\epsilon} & \text{in } \mathbb{R}^N. \end{cases} \quad (2.53)$$

If $(x, t) = m \in \mathbb{R}^N \times (0, T]$, we set $d_K = \text{dist}(x, K)$, $D_K = \max\{|x - y| : y \in K\}$ and $\lambda = \sqrt{d_K^2 + t} = \delta_2[m, K]$. We define a slicing of K , by setting $d_n = d_n(K, t) := \sqrt{nt}$ ($n \in \mathbb{N}$),

$$T_n = \bar{B}_{d_{n+1}}(x) \setminus B_{d_n}(x), \quad \forall n \in \mathbb{N},$$

thus $T_0 = B_{\sqrt{t}}(x)$, and

$$K_n(x) = K \cap T_n(x) \quad \text{for } n \in \mathbb{N} \quad \text{and} \quad \mathcal{Q}_n(x) = K \cap B_{d_{n+1}}(x).$$

When there is no ambiguity, we shall skip the x variable in the above sets. The main result of this section is the following discrete upper Wiener-type estimate.

Theorem 2.14 *Assume $q \geq q_c$. Then there exists $C = C(N, q, T) > 0$ such that*

$$\bar{u}_K(x, t) \leq \frac{C}{t^{\frac{N}{2}}} \sum_{n=0}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q, q'} \left(\frac{K_n}{d_{n+1}} \right) \quad \forall (x, t) \in Q_T, \quad (2.54)$$

where a_t is the largest integer j such that $K_j \neq \emptyset$.

With no loss of generality, we can first assume that $x = 0$. Furthermore, in considering the scaling transformation $u_\ell(y, t) = \ell^{\frac{1}{q-1}} u(\sqrt{\ell}y, \ell t)$, with $\ell > 0$, we can assume $t = 1$. Thus the new compact singular set of the initial trace becomes $K/\sqrt{\ell}$, that we shall still denote K . We shall also set $a_K = a_{K,1}$. Since for each $n \in \mathbb{N}$,

$$\frac{1}{2\sqrt{n+1}} \leq d_{n+1} - d_n \leq \frac{1}{\sqrt{n+1}},$$

it is possible to exhibit a collection Θ_n of points $a_{n,j}$ with center on the sphere $\Sigma_n = \{y \in \mathbb{R}^N : |y| = (d_{n+1} + d_n)/2\}$, such that

$$T_n \subset \bigcup_{a_{n,j} \in \Theta_n} B_{1/\sqrt{n+1}}(a_{n,j}), \quad |a_{n,j} - a_{n,k}| \geq 1/2\sqrt{n+1} \quad \text{and} \quad \#\Theta_n \leq Cn^{N-1},$$

for some constant $C = C(N)$. If $K_{n,j} = K_n \cap B_{1/\sqrt{n+1}}(a_{n,j})$, there holds

$$K = \bigcup_{0 \leq n \leq a_K} \bigcup_{a_{n,j} \in \Theta_n} K_{n,j}.$$

The first intermediate step is based on the *quasi-additivity* property of capacities [2].

Lemma 2.15 *Let $q \geq q_c$. There exists a constant $C = C(N, q)$ such that*

$$\sum_{a_{n,j} \in \Theta_n} C_{2/q, q'}(K_{n,j}) \leq Cn^{\frac{N}{2} - \frac{1}{q-1}} C_{2/q, q'} \left(\frac{K_n}{\sqrt{n+1}} \right) \quad \forall n \in \mathbb{N}_*, \quad (2.55)$$

where $B_{n,j} = B_{2/\sqrt{n+1}}(a_{n,j})$ and $C_{2/q, q'}$ stands for the capacity taken with respect to \mathbb{R}^N .

Proof. The following result is proved in [2, Th 3]: if the spheres $B_{\rho_j}(b_j)$ are disjoint in \mathbb{R}^N and G is an analytic subset of $\bigcup B_{\rho_j}(b_j)$ where the ρ_j are positive and smaller than some $\rho^* > 0$, there holds

$$C_{2/q, q'}(G) \leq \sum_j C_{2/q, q'}(G \cap B_{\rho_j}(b_j)) \leq A C_{2/q, q'}(G), \quad (2.56)$$

where $\theta = 1 - 2/N(q-1)$, for some A depending on N , q and ρ^* . This property is called quasi-additivity. We define for $n \in \mathbb{N}_*$,

$$\tilde{T}_n = \sqrt{n+1} T_n, \quad \tilde{K}_n = \sqrt{n+1} K_n \quad \text{and} \quad \tilde{Q}_n = \sqrt{n+1} Q_n.$$

Since $K_{n,j} \subset B_{1/\sqrt{n+1}}(a_{n,j})$ and the $C_{2/q,q'}$ capacities are taken with respect to the balls $B_{2/\sqrt{n+1}}(a_{n,j}) = B_{n,j}$. By Lemma 2.12 with $r = \rho = 1/\sqrt{n+1}$

$$C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \leq C(n+1)^{\frac{1}{q-1} - \frac{N}{2}} C_{2/q,q'}(\tilde{K}_{n,j}), \quad (2.57)$$

where $\tilde{K}_{n,j} = \sqrt{n+1}K_{n,j}$ and $\tilde{B}_{n,j} = \sqrt{n+1}B_{n,j}$. For a fixed $n > 0$ and each repartition Λ of points $\tilde{a}_{n,j} = \sqrt{n+1}a_{n,j}$ such that the balls $B_{2\theta}(\tilde{a}_{n,j})$ are disjoint, the quasi-additivity property holds in the following sense: if we set

$$K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} K_{n,j}, \quad \tilde{K}_{n,\Lambda} = \sqrt{n+1}K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} \tilde{K}_{n,j} \quad \text{and} \quad \tilde{K}_n = \sqrt{n+1}K_n,$$

then

$$\sum_{a_{n,j} \in \Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \leq AC_{2/q,q'}(\tilde{K}_{n,\Lambda}). \quad (2.58)$$

The maximal cardinal of any such repartition Λ is of the order of Cn^{N-1} for some positive constant $C = C(N)$, therefore, the number of repartitions needed for a full covering of the set \tilde{T}_n is of finite order depending upon the dimension. Because \tilde{K}_n is the union of the $\tilde{K}_{n,\Lambda}$,

$$\sum_{a_{n,j} \in \Theta_n} C_{2/q,q'}(\tilde{K}_{n,j}) = \sum_{\Lambda} \sum_{a_{n,j} \in \Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \leq CC_{2/q,q'}(\tilde{K}_n). \quad (2.59)$$

Since, by Lemma 2.12,

$$C_{2/q,q'}(\tilde{K}_n) \leq C_{2/q,q'}^{B_{2(n+1)}}(\tilde{K}_n) = (n+1)^{N - \frac{1}{q-1}} C_{2/q,q'}^{B_2} \left(\frac{K_n}{\sqrt{n+1}} \right) \leq C(n+1)^{N - \frac{1}{q-1}} C_{2/q,q'} \left(\frac{K_n}{\sqrt{n+1}} \right),$$

we obtain (2.55) by combining this last inequality with (2.57) and (2.59). \square

Proof of Theorem 2.14. Step 1. We first notice that

$$\bar{u}_K \leq \sum_{0 \leq n \leq a_K} \sum_{a_{n,j} \in \Theta_n} \bar{u}_{K_{n,j}}. \quad (2.60)$$

Actually, since $K = \bigcup_n \bigcup_{a_{n,j}} K_{n,j}$, for any $0 < \epsilon' < \epsilon$, there holds $\overline{K_{\epsilon'}} \subset \bigcup_n \bigcup_{a_{n,j}} K_{n,j\epsilon}$. Because a finite sum of positive solutions of (1.1) is a super solution,

$$\bar{u}_{K_{\epsilon'}} \leq \sum_{0 \leq n \leq a_K} \sum_{a_{n,j} \in \Theta_n} \bar{u}_{K_{n,j\epsilon}}. \quad (2.61)$$

Letting successively ϵ' and ϵ go to 0 implies (2.60).

Step 2. Let $n \in \mathbb{N}$. Since $K_{n,j} \subset B_{1/\sqrt{n+1}}(a_{n,j})$ and $|x - a_{n,j}| = (d_n + d_{n+1})/2 = (\sqrt{n+1} + \sqrt{n})/2$, we can apply the previous lemmas with $r = 1/\sqrt{n+1}$ and $\rho = r$. For $n \geq n_N$ there

holds $t = 1 \geq (r + 2\rho)^2 = 9/(n + 1)$ and $|x - a_{n,j}| = (\sqrt{n+1} - \sqrt{n})/2 \geq (2 + C_N)(3/\sqrt{n+1})$ (notice that $n_N \geq 8$). Thus

$$u_{K_{n,j}}(0, 1) \leq C e^{(\sqrt{n}-3/\sqrt{n+1})^2/4} C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \leq C e^{3/2} e^{-\frac{n}{4}} C_{2/q,q'}^{B_{n,j}}(K_{n,j}). \quad (2.62)$$

Using Lemma 2.15 we obtain, with $d_n = d_n(1) = \sqrt{n+1}$

$$\sum_{n=n_N}^{a_K} \sum_{a_{n,j} \in \Theta_n} u_{K_{n,j}}(0, 1) \leq C \sum_{n=n_N}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \quad (2.63)$$

Finally, we apply Lemma 2.5 if $1 \leq n < n_N$ and get

$$\begin{aligned} \sum_1^{n_N-1} \sum_{a_{n,j} \in \Theta_n} u_{K_{n,j}}(0, 1) &\leq C \sum_1^{n_N-1} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right) \\ &\leq C' \sum_1^{n_N-1} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \end{aligned} \quad (2.64)$$

For $n = 0$, we proceed similarly, in splitting K_1 in a finite number of $K_{1,i}$, depending only on the dimension, such that $\text{diam } K_{1,i} < 1/3$. Combining (2.63) and (2.64), we derive

$$\bar{u}_K(0, 1) \leq C \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \quad (2.65)$$

In order to derive the same result for any $t > 0$, we notice that

$$\bar{u}_K(y, t) = t^{-\frac{1}{q-1}} \bar{u}_{K\sqrt{t}}(y\sqrt{t}, 1).$$

Going back to the definition of $d_n = d_n(K, t) = \sqrt{nt} = d_n(K\sqrt{t}, 1)$, we derive from (2.65) and the fact that $a_{K,t} = a_{K\sqrt{t},1}$

$$\bar{u}_K(0, t) \leq C t^{-\frac{1}{q-1}} \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right), \quad (2.66)$$

with $d_n = d_n(t) = \sqrt{t(n+1)}$. This is (2.54) with $x = 0$, and a space translation leads to the final result. \square

Proof of Theorem 2.1. Let $m > 0$ and $F_m = F \cap \bar{B}_m$. We denote by $U_{B_m^c}$ the maximal solution of (1.1) in Q_∞ the initial trace of which vanishes on B_m . Such a solution is actually the unique solution of (2.1) which satisfies

$$\lim_{t \rightarrow 0} u(x, t) = \infty$$

uniformly on $B_{m'}^c$, for any $m' > m$: this can be checked by noticing that

$$U_{B_m^c} \ell(y, t) = \ell^{\frac{1}{q-1}} U_{B_m^c}(\sqrt{\ell}y, \ell t) = U_{B_{m/\sqrt{\ell}}^c}(y, t).$$

Furthermore

$$\lim_{m \rightarrow \infty} U_{B_m^c}(y, t) = \lim_{m \rightarrow \infty} m^{-\frac{2}{q-1}} U_{B_1^c}(y/m, t/m^2) = 0$$

uniformly on any compact subset of $\overline{Q_\infty}$. Since $\bar{u}_{F_m} + U_{B_m^c}$ is a super-solution, it is larger than \bar{u}_F and therefore $\bar{u}_{F_m} \uparrow \bar{u}_F$. Because $W_{F_m}(x, t) \leq W_F(x, t)$ and $\bar{u}_{F_m} \leq C_1 W_{F_m}(x, t)$, the result follows. \square

Theorem 2.1 admits the following integral expression.

Theorem 2.16 *Assume $q \geq q_c$. Then there exists a positive constant $C_1^* = C^*(N, q, T)$ such that, for any closed subset F of \mathbb{R}^N , there holds*

$$\bar{u}_F(x, t) \leq \frac{C_1^*}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q, q'} \left(\frac{1}{s} F \cap B_1(x) \right) s ds, \quad (2.67)$$

where $a_t = \min\{n : F \subset B_{\sqrt{(n+1)t}}(x)\}$.

Proof. We first use

$$C_{2/q, q'} \left(\frac{F_n}{d_{n+1}} \right) \leq C_{2/q, q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right),$$

and we denote

$$\Phi(s) = C_{2/q, q'} \left(\frac{F}{s} \cap B_1 \right) \quad \forall s > 0. \quad (2.68)$$

Step 1. The following inequality holds (see [1] and [29])

$$c_1 \Phi(\alpha s) \leq \Phi(s) \leq c_2 \Phi(\beta s) \quad \forall s > 0, \quad \forall 1/2 \leq \alpha \leq 1 \leq \beta \leq 2, \quad (2.69)$$

for some positive constants c_1, c_2 depending on N and q . If $\beta \in [1, 2]$,

$$\Phi(\beta s) = C_{2/q, q'} \left(\frac{1}{\beta} \left(\frac{F}{s} \cap B_\beta \right) \right) \approx C_{2/q, q'} \left(\frac{F}{s} \cap B_\beta \right) \geq c_1 \Phi(s).$$

If $\alpha \in [1/2, 1]$,

$$\Phi(\alpha s) = C_{2/q, q'} \left(\frac{1}{\alpha} \left(\frac{F}{s} \cap B_\alpha \right) \right) \approx C_{2/q, q'} \left(\frac{F}{s} \cap B_\alpha \right) \leq c_2 \Phi(s).$$

Step 2. By (2.69)

$$C_{2/q, q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) \leq c_2 C_{2/q, q'} \left(\frac{F}{s} \cap B_1 \right) \quad \forall s \in [d_{n+1}, d_{n+2}],$$

and $n \leq a_t$. Then

$$\begin{aligned} c_2 \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q, q'} \left(\frac{F}{s} \cap B_1 \right) s ds \\ \geq C_{2/q, q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} s ds. \end{aligned}$$

Using the fact that $N - \frac{2}{q-1} \geq 0$, we get,

$$\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} ds \geq e^{-\frac{d_{n+2}^2}{4t}} d_{n+1}^{N-\frac{2}{q-1}+1} (d_{n+2} - d_{n+1}) \quad (2.70)$$

$$\geq \frac{t}{4e^2} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{d_{n+2}^2}{4t}}. \quad (2.71)$$

Thus

$$\bar{u}_F(x, t) \leq \frac{C}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'} \left(\frac{1}{s} F \cap B_1 \right) s ds, \quad (2.72)$$

which ends the proof. \square

3 Estimate from below

If $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N)$, we denote by $u_\mu = u_{\mu,0}$ the solution of

$$\begin{cases} \partial_t u_\mu - \Delta u_\mu + u_\mu^q = 0 & \text{in } Q_T, \\ u_\mu(\cdot, 0) = \mu & \text{in } \mathbb{R}^N. \end{cases} \quad (3.1)$$

The maximal σ -moderate solution of (1.1) which has an initial trace vanishing outside a closed set F is defined by

$$\underline{u}_F = \sup \left\{ u_\mu : \mu \in \mathfrak{M}_+^q(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N), \mu(F^c) = 0 \right\}. \quad (3.2)$$

The main result of this section is the next one

Theorem 3.1 *Assume $q \geq q_c$. There exists a constant $C_2 = C_2(N, q, T) > 0$ such that, for any closed subset $F \subset \mathbb{R}^N$, there holds*

$$\underline{u}_F(x, t) \geq C_2 W_F(x, t) \quad \forall (x, t) \in Q_T. \quad (3.3)$$

We first assume that F is compact, and we shall denote it by K . The first observation is that if $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N)$, $u_\mu \in L^q(Q_T)$ (see lemma below) and $0 \leq u_\mu \leq \mathbb{H}[\mu] := \mathbb{H}_\mu$. Therefore

$$u_\mu \geq \mathbb{H}_\mu - \mathbb{G}[\mathbb{H}_\mu^q], \quad (3.4)$$

where \mathbb{G} is the Green heat potential in Q_T defined by

$$\mathbb{G}[f](t) = \int_0^t \mathbb{H}[f(s)](t-s) ds = \int_0^t \int_{\mathbb{R}^N} H(\cdot, y, t-s) f(y, s) dy ds.$$

Since the details of the proof are very technical, we shall present its main line. The key idea is to construct, for any $(x, t) \in Q_T$, a measure $\mu = \mu(x, t) \in \mathfrak{M}_+^q(\mathbb{R}^N)$ such that there holds

$$\mathbb{H}_\mu(x, t) \geq C W_K(x, t) \quad \forall (x, t) \in Q_T, \quad (3.5)$$

and

$$\mathbb{G}(\mathbb{H}_\mu)^q \leq C \mathbb{H}_\mu \quad \text{in } Q_T, \quad (3.6)$$

with constants C depends only on N , q , and T , then to replace μ by $\mu_\epsilon = \epsilon\mu$ with $\epsilon = (2C)^{-\frac{1}{q-1}}$ in order to derive

$$u_{\mu_\epsilon} \geq 2^{-1} \mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \quad (3.7)$$

From this follows

$$\underline{u}_K \geq 2^{-1} \mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \quad (3.8)$$

and the proof of Theorem 3.1 with $C_2 = 2^{-1}C$.

We recall the following regularity result which actually can be used for defining the norm in negative Besov spaces [35]

Lemma 3.2 *There exists a constant $c > 0$ such that*

$$c^{-1} \|\mu\|_{W^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}_\mu\|_{L^q(Q_T)} \leq c \|\mu\|_{W^{-2/q,q}(\mathbb{R}^N)} \quad (3.9)$$

for any $\mu \in W^{-2/q,q}(\mathbb{R}^N)$.

3.1 Estimate from below of the solution of the heat equation

The purely spatial slicing used is the trace on $\mathbb{R}^N \times \{0\}$ of an *extended slicing* in Q_T which is constructed as follows: if K is a compact subset of \mathbb{R}^N , $m = (x, t)$, we define d_K , λ , d_n and a_t as in Section 2.3. Let $\alpha \in (0, 1)$ to be fixed later on, we define \mathcal{T}_n for $n \in \mathbb{Z}$ by

$$\mathcal{T}_n = \begin{cases} \mathcal{B}_{\sqrt{t(n+1)}}^2(m) \setminus \mathcal{B}_{\sqrt{tn}}^2(m) & \text{if } n \geq 1, \\ \mathcal{B}_{\alpha^{-n}\sqrt{t}}^2(m) \setminus \mathcal{B}_{\alpha^{1-n}\sqrt{t}}^2(m) & \text{if } n \leq 0, \end{cases}$$

and put

$$\mathcal{T}_n^* = \mathcal{T}_n \cap \{s : 0 \leq s \leq t\}, \quad \text{for } n \in \mathbb{Z}.$$

We recall that for $n \in \mathbb{N}_*$,

$$\mathcal{Q}_n = K \cap \mathcal{B}_{\sqrt{t(n+1)}}^2(m) = K \cap B_{d_n}(x)$$

and

$$K_n = K \cap \mathcal{T}_{n+1} = K \cap (B_{d_{n+1}}(x) \setminus B_{d_n}(x)).$$

Let $\nu_n \in \mathfrak{M}_+(\mathbb{R}^N) \cap W^{-2/q,q}(\mathbb{R}^N)$ be the q -capacitary measure of the set K_n/d_{n+1} (see [1, Sec. 2.2]). Such a measure has support in K_n/d_{n+1} and

$$\nu_n(K_n/d_{n+1}) = C_{2/q,q'}(K_n/d_{n+1}) \quad \text{and} \quad \|\nu_n\|_{W^{-2/q,q'}(\mathbb{R}^N)} = (C_{2/q,q'}(K_n/d_{n+1}))^{1/q}. \quad (3.10)$$

We define μ_n as follows

$$\mu_n(A) = d_{n+1}^{N-\frac{2}{q-1}} \nu_n(A/d_{n+1}) \quad \forall A \subset K_n, A \text{ Borel}, \quad (3.11)$$

and set

$$\mu_{t,K} = \sum_{n=0}^{a_t} \mu_n,$$

and

$$\mathbb{H}_{\mu_{t,K}} = \sum_{n=0}^{a_t} \mathbb{H}_{\mu_n}. \quad (3.12)$$

Proposition 3.3 *Let $q \geq q_c$, then there holds*

$$\mathbb{H}_{\mu_{t,K}}(x, t) \geq \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{n=0}^{a_t} e^{-\frac{n+1}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right), \quad (3.13)$$

in $\mathbb{R}^N \times (0, T)$.

Proof. Since

$$\mathbb{H}_{\mu_n}(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n, \quad (3.14)$$

and

$$y \in K_n \implies |x - y| \leq d_{n+1},$$

(3.13) follows because of (3.11) and (3.12). \square

3.2 Estimate from above of the nonlinear term

We write (3.4) under the form

$$\begin{aligned} u_\mu(x, t) &\geq \sum_{n \in \mathbb{Z}} \mathbb{H}_{\mu_n}(x, t) - \int_0^t \int_{\mathbb{R}^N} H(x, y, t-s) \left[\sum_{n \in A_K} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds \\ &= I_1 - I_2. \end{aligned} \quad (3.15)$$

since $\mu_n = 0$ if $n \notin A_K = \mathbb{N} \cap [1, a_t]$, and

$$\begin{aligned} I_2 &\leq \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \int_{\mathbb{R}^N} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n \in A_K} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds \\ &\leq \frac{1}{(4\pi)^{\frac{N}{2}}} (J_\ell + J'_\ell), \end{aligned} \quad (3.16)$$

for some $\ell \in \mathbb{N}^*$ to be fixed later on, where

$$J_\ell = \sum_{p \in \mathbb{Z}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n < p+\ell} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds,$$

and

$$J'_\ell = \sum_{p \in \mathbb{Z}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n \geq p+\ell} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds.$$

The next estimate will be used several times in the sequel.

Lemma 3.4 *Let $0 < a < b$ and $t > 0$, then,*

$$\max \left\{ \sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}} : 0 \leq \sigma \leq t, at \leq \rho^2 + \sigma \leq bt \right\} = e^{\frac{1}{4}} \begin{cases} t^{-\frac{N}{2}} e^{-\frac{a}{4}} & \text{if } \frac{a}{2N} > 1, \\ \left(\frac{2N}{at} \right)^{\frac{N}{2}} e^{-\frac{N}{2}} & \text{if } \frac{a}{2N} \leq 1. \end{cases}$$

Proof. Set

$$\mathcal{J}(\rho, \sigma) = \sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}}$$

and

$$\mathcal{K}_{a,b,t} = \{(\rho, \sigma) \in [0, \infty) \times (0, t] : at \leq \rho^2 + \sigma \leq bt\}.$$

We first notice that, for fixed σ , the maximum of $\mathcal{J}(\cdot, \sigma)$ is achieved for ρ minimal. If $\sigma \in [at, bt]$ the minimal value of ρ is 0, while if $\sigma \in (0, at)$, the minimum of ρ is $\sqrt{at - \sigma}$.

- Assume first $a \geq 1$, then $\mathcal{J}(\sqrt{at - \sigma}, \sigma) = e^{\frac{1}{4}} \sigma^{-\frac{N}{4}} e^{-\frac{at}{4\sigma}}$, thus, if $1 \leq a/2N$ the minimal value of $\mathcal{J}(\sqrt{at - \sigma}, \sigma)$ is $e^{\frac{1-2N}{4}} \left(\frac{2N}{at} \right)^{\frac{N}{2}}$, while, if $a/2N < 1 \leq a$, the minimum is $e^{\frac{1}{4}} t^{-\frac{N}{2}} e^{-\frac{a}{4}}$.

- Assume now $a \leq 1$. Then

$$\begin{aligned} \max\{\mathcal{J}(\rho, \sigma) : (\rho, \sigma) \in \mathcal{K}_{a,b,t}\} &= \max \left\{ \max_{\sigma \in (at, t]} \mathcal{J}(0, \sigma), \max_{\sigma \in (0, at]} \mathcal{J}(\sqrt{at - \sigma}, \sigma) \right\} \\ &= \max \left\{ (at)^{-\frac{N}{2}}, e^{\frac{1-2N}{4}} \left(\frac{2N}{at} \right)^{\frac{N}{2}} \right\} \\ &= e^{\frac{1-2N}{4}} \left(\frac{2N}{at} \right)^{\frac{N}{2}}. \end{aligned}$$

Combining these two estimates, we derive the result. \square

Remark. The following variant of Lemma 3.4 will be useful in the sequel: *For any $\theta \geq 1/2N$ there holds*

$$\max\{\mathcal{J}(\rho, \sigma) : (\rho, \sigma) \in \mathcal{K}(a, b, t)\} \leq e^{\frac{1}{4}} \left(\frac{2N\theta}{t} \right)^{\frac{N}{2}} e^{-\frac{a}{4}} \quad \text{if } \theta a \geq 1. \quad (3.17)$$

Lemma 3.5 *There exists a positive constant $C = C(N, \ell, q)$ such that*

$$J_\ell \leq C t^{-\frac{N}{2}} \sum_{n=1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-(1+(n-\ell)_+)/4} C_{2/q, q'} \left(\frac{K_n}{d_{n+1}} \right). \quad (3.18)$$

Proof. The set of p for the summation in J_ℓ is reduced to $\mathbb{Z} \cap [-\ell + 2, \infty)$ and we write

$$J_\ell = J_{1,\ell} + J_{2,\ell}$$

where

$$J_{1,\ell} = \sum_{p=2-\ell}^0 \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n < p+\ell} \mathbb{H}_{\mu_n}(y, s) \right]^q$$

and

$$J_{2,\ell} = \sum_{p=1}^{\infty} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n < p+\ell} \mathbb{H}_{\mu_n}(y, s) \right]^q.$$

If $p = 2 - \ell, \dots, 0$,

$$(y, s) \in \mathcal{T}_p^* \implies t\alpha^{2-2p} \leq |x-y|^2 + t-s \leq t\alpha^{-2p},$$

and, if $p \geq 1$

$$(y, s) \in \mathcal{T}_p^* \implies pt \leq |x-y|^2 + t-s \leq (p+1)t.$$

By Lemma 3.4 and (3.17), there exists $C = C(N, \ell, \alpha) > 0$ such that

$$\max \left\{ (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y, s) \in \mathcal{T}_p^* \right\} \leq Ct^{-\frac{N}{2}} e^{-\alpha^{2-2p}/4}, \quad (3.19)$$

if $p = 2 - \ell, \dots, 0$, and

$$\max \left\{ (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y, s) \in \mathcal{T}_p^* \right\} \leq Ct^{-\frac{N}{2}} e^{-p/4}, \quad (3.20)$$

if $p \geq 1$. When $p = 2 - \ell, \dots, 0$

$$\left[\sum_1^{p+\ell-1} \mathbb{H}_{\mu_n}(y, s) \right]^q \leq C \sum_1^{p+\ell-1} \mathbb{H}_{\mu_n}^q(y, s). \quad (3.21)$$

for some $C = C(\ell, q) > 0$, thus

$$\begin{aligned} J_{1,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=2-\ell}^0 e^{-\frac{\alpha^{2-2p}}{4}} \sum_{n=1}^{p+\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q \sum_{p=n-\ell+1}^0 e^{-\frac{\alpha^{2-2p}}{4}} \\ &\leq Ct^{-\frac{N}{2}} e^{-\frac{\alpha^{2\ell-2}}{4}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q. \end{aligned} \quad (3.22)$$

If the set of p 's is not upper bounded, we introduce $\delta > 0$ to be made precise later on. Then

$$\left[\sum_1^{p+\ell-1} \mathbb{H}_{\mu_n}(y, s) \right]^q \leq \left[\sum_1^{p+\ell-1} e^{\delta q' \frac{n}{4}} \right]^{q/q'} \sum_1^{p+\ell-1} e^{-\frac{\delta q n}{4}} \mathbb{H}_{\mu_n}^q(y, s), \quad (3.23)$$

with $q' = q/(q-1)$. If, by convention $\mu_n = 0$ whenever $n > at$, we obtain, for some $C > 0$ which depends also on δ ,

$$\begin{aligned}
J_{2,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \sum_{n=1}^{p+\ell-1} e^{-\frac{\delta qn}{4}} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q \\
&\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q e^{-\frac{\delta qn}{4}} \sum_{p=(n-\ell+1)\vee 1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \\
&\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{1+(n-\ell)_+}{4}} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q.
\end{aligned} \tag{3.24}$$

Notice that we choose δ such that $\delta\ell q < 1$. Combining (3.22) and (3.24), we derive (3.18) from Lemma 3.2, (3.10) and (3.11). \square

The set of indices p for which the μ_n terms are not zero in J'_ℓ is $\mathbb{Z} \cap (-\infty, at - \ell]$. We write

$$J'_\ell = J'_{1,\ell} + J'_{2,\ell},$$

where

$$J'_{1,\ell} = \sum_{p=-\infty}^0 \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=1\vee p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds,$$

and

$$J'_{2,\ell} = \sum_{p=1}^{at-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds.$$

Lemma 3.6 *There exists a constant $C = C(N, q, \ell) > 0$ such that*

$$J'_{1,\ell} \leq Ct^{1-\frac{Nq}{2}} \sum_{n=0}^{at} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} d_{n+1}^{Nq-2q'} C_{2/q, q'}^q \left(\frac{K_n}{d_{n+1}} \right), \tag{3.25}$$

where $\beta_0 = (q-1)/4$ and $h = 2q(q+1)/(q-1)^2$.

Proof. Since

$$(y, s) \in \mathcal{T}_p^*, \text{ and } (z, 0) \in K_n \implies |y-z| \geq (\sqrt{n} - \alpha^{-p})\sqrt{t}, \tag{3.26}$$

there holds

$$\mathbb{H}_{\mu_n}(y, s) \leq (4\pi s)^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2 t}{4s}} \mu_n(K_n) \leq Ct^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n(K_n),$$

by Lemma 3.4. Let $\epsilon_n > 0$ such that

$$A_\epsilon = \sum_{n=1}^{\infty} \epsilon_n^{q'} < \infty,$$

then

$$\begin{aligned}
J'_{1,\ell} &\leq CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{p=-\infty}^0 \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \sum_{n=1 \vee (p+\ell)}^{\infty} \epsilon_n^{-q} e^{-q\frac{(\sqrt{n}-\alpha-p)^2}{4}} \mu_n^q(K_n) ds dy \\
&\leq CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) \sum_{p=0 \wedge (n-\ell)}^{-\infty} e^{-q\frac{(\sqrt{n}-\alpha-p)^2}{4}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds dy \\
&\leq CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-q\frac{(\sqrt{n}-1)^2}{4}} \iint_{\cup_{p \leq 0} \mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds dy \\
&\leq CA_\epsilon^{q/q'} t^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-q\frac{(\sqrt{n}-1)^2}{4}}.
\end{aligned} \tag{3.27}$$

Set $h = 2q(q+1)/(q-1)^2$ and $Q = (1+q)/2$, then $q(\sqrt{n}-1)^2 \geq Q(n-h)_+$ for any $n \geq 1$. If we choose $\epsilon_n = e^{-\frac{(q-1)(n-h)_+}{16q}}$, there holds $\epsilon_n^{-q} e^{-q\frac{(\sqrt{n}-1)^2}{4}} \leq e^{\frac{(q+3)(n-h)_+}{16}}$. Finally

$$J'_{1,\ell} \leq Ct^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} e^{\frac{(1+\epsilon_0)(n-h)_+}{4}} \mu_n^q(K_n),$$

with $\beta_0 = (q-1)/4$, which yields to (3.25) by the choice of the μ_n . \square

In order to make easier the obtention of the estimate of the term $J'_{2,\ell}$, we first give the proof in dimension 1.

Lemma 3.7 *Assume $N = 1$ and ℓ is an integer larger than 1. There exists a positive constant $C = C(q, \ell) > 0$ such that*

$$J'_{2,\ell} \leq Ct^{-1/2} \sum_{n=\ell}^{at} e^{-\frac{n}{4}} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \tag{3.28}$$

Proof. If $(y, s) \in \mathcal{T}_p^*$ and $z \in K_n$ ($p \geq 1, n \geq p = \ell$), there holds $|x-y| \geq \sqrt{t}\sqrt{p}$ and $|y-z| \geq \sqrt{t}(\sqrt{n}-\sqrt{p+1})$. Therefore

$$J'_{2,\ell} \leq C\sqrt{t} \sum_{p=1}^{at-\ell} \frac{1}{\sqrt{p}} \int_0^t e^{-\frac{pt}{4(t-s)}} \left(\sum_{n=p+\ell}^{at} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q.$$

If $\epsilon \in (0, q)$ is some positive parameter which will be made more precise later on, there holds

$$\begin{aligned}
&\left(\sum_{n=p+\ell}^{at} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q \\
&\leq \left(\sum_{n=p+\ell}^{at} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \right)^{q/q'} \sum_{n=p+\ell}^{at} s^{-\frac{q}{2}} e^{-(q-\epsilon) \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n^q(K_n),
\end{aligned}$$

by Hölder's inequality. By comparison between series and integrals and using Gauss integral

$$\begin{aligned}
\sum_{n=p+\ell}^{at} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} &\leq \int_{p+\ell}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\
&= 2 \int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} (x + \sqrt{p+1}) dx \\
&\leq \frac{4s}{\epsilon q' t} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{4s}} + 2\sqrt{p+1} \int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} dx \\
&\leq C \sqrt{\frac{(p+1)s}{t}} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{2s}} \\
&\leq C \sqrt{\frac{(p+1)s}{t}}.
\end{aligned}$$

If we set $q_\epsilon = q - \epsilon$, then

$$J'_{2,\ell} \leq C \epsilon^{-q'/q} t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{\infty} \mu_n^q(K_n) \sum_{p=1}^{n-\ell} p^{\frac{q-2}{2}} \int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds.$$

where $C = C(\epsilon, q) > 0$. Since

$$\begin{aligned}
\int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \\
= \int_0^1 (1-s)^{-1/2} s^{-1/2} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}} ds,
\end{aligned}$$

we can apply Lemma A.1 with $a = 1/2$, $b = 1/2$, $A = \sqrt{p}$ and $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$. In this range of indices $B \geq \sqrt{q_\epsilon}(\sqrt{p+\ell} - \sqrt{p+1}) \geq \sqrt{q_\epsilon}(\ell - 1)\sqrt{p}$, thus $\kappa = \sqrt{q_\epsilon}(\ell - 1)$ and

$$\sqrt{\frac{A}{A+B}} \sqrt{\frac{B}{A+B}} \leq p^{\frac{1}{4}} n^{-1/2} (\sqrt{n} - \sqrt{p})^{1/2}.$$

Therefore

$$\int_0^t (t-s)^{-1/2} s^{-\frac{q}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \leq \frac{C p^{\frac{1}{4}} (\sqrt{n} - \sqrt{p})^{1/2}}{\sqrt{n}} e^{-\frac{(\sqrt{p} + \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1}))^2}{4}}, \quad (3.29)$$

which implies

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{at} \frac{\mu_n^q(K_n)}{\sqrt{n}} \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n} - \sqrt{p})^{1/2} e^{-\frac{(\sqrt{p} + \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1}))^2}{4}}, \quad (3.30)$$

where C depends of ϵ , q and ℓ . By Lemma A.2

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{at} n^{\frac{q-3}{2}} e^{-\frac{n}{4}} \mu_n^q(K_n) \quad (3.31)$$

Because $\mu_n(K_n) = d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right)$ (remember $N = 1$) and $\text{diam} \frac{K_n}{d_{n+1}} \leq n^{-1}$, there holds

$$\mu_n^q(K_n) \leq C \left(\frac{\sqrt{t}}{\sqrt{n}} \right)^{q-3} \mu_n(K_n) = C \left(\frac{\sqrt{t}}{\sqrt{n}} \right)^{q-3} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}(K_n/d_{n+1}) \quad (3.32)$$

and inequality (3.28) follows. \square

Next we give the general proof. For this task we shall use again the quasi-additivity with separated partitions.

Lemma 3.8 *Assume $N \geq 2$ and ℓ is an integer larger than 1. There exist a positive constant $C_1 = C_1(q, N, \ell) > 0$ such that f*

$$J'_{2,\ell} \leq C_1 t^{-\frac{N}{2}} \sum_{n=\ell}^{at} e^{-\frac{n}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \quad (3.33)$$

Proof. As in the proof of Theorem 2.14, we know that there exists a finite number J , depending only on the dimension N , of separated sub-partitions $\{\#\Theta_{t,n}^h\}_{h=1}^J$ of the rescaled sets $\tilde{T}_n = \sqrt{\frac{n+1}{t}} T_n$ by the N -dim balls $B_2(\tilde{a}_{n,j})$ where $\tilde{a}_{n,j} = \sqrt{\frac{n+1}{t}} a_{n,j}$, $|a_{n,j}| = \frac{d_{n+1} + d_n}{2}$ and $|a_{n,j} - a_{n,k}| \geq \sqrt{\frac{4t}{n+1}}$. Furthermore $\#\Theta_{t,n}^h \leq Cn^{N-1}$. We denote $K_{n,j} = K_n \cap B_{\sqrt{\frac{t}{n+1}}}(a_{n,j})$.

We write $\mu_n = \sum_{h=1}^J \mu_n^h$, and accordingly $J'_{2,\ell} = \sum_{h=1}^J J'_{2,\ell}^h$, where $\mu_n^h = \sum_{j \in \Theta_{t,n}^h} \mu_{n,j}$, and $\mu_{n,j}$ are the capacity measures of $K_{n,j}$ relative to $B_{n,j} = B_{6t/5\sqrt{n}}(a_{n,j})$, which means

$$\nu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \quad \text{and} \quad \|\nu_{n,j}\|_{W^{-2/q,q'}(B_{n,j})} = \left(C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \right)^{1/q}. \quad (3.34)$$

Thus

$$J'_{2,\ell} = \sum_{p=1}^{at-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{h=1}^J \sum_{j \in \Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y, s) \right]^q dy ds.$$

We denote

$$J'_{2,\ell}^h = \sum_{p=1}^{at-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{j \in \Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y, s) \right]^q dy ds,$$

and clearly

$$J'_{2,\ell} \leq C \sum_{h=1}^J J'_{2,\ell}^h, \quad (3.35)$$

where C depends only on N and q . For integers n and p such that $n \geq \ell + 1$, we set

$$\lambda_{n,j,y} = \inf\{|y-z| : z \in B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})\} = |y - a_{n,j}| - \frac{\sqrt{t}}{\sqrt{n+1}}.$$

Therefore

$$\begin{aligned} \sum_{n=p+\ell}^{a_t} \int_{K_n} e^{-\frac{|y-z|^2}{4s}} d\mu_n^h(z) &= \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} \int_{K_{n,j}} e^{-\frac{|y-z|^2}{4s}} d\mu_{n,j}(z) \\ &\leq \left(\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \right)^{1/q'} \left(\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-q \lambda_{n,j,y}^2 \frac{1-\epsilon}{4s}} \mu_{n,j}^q(K_{n,j}) \right)^{1/q} \end{aligned}$$

where $\epsilon > 0$ will be made precise later on.

Step 1 We claim that

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \leq C \sqrt{\frac{ps}{t}} \quad (3.36)$$

where C depends on ϵ , q and N . If y is fixed in T_p , we denote by z_y the point of T_n which solves $|y - z_y| = \text{dist}(y, T_n)$. Thus

$$\sqrt{t}(\sqrt{n} - \sqrt{p+1}) \leq |y - z_y| \leq t(\sqrt{n} - \sqrt{p}).$$

Let $Y = y\sqrt{t(p+1)}/|y|$. On the axis $\overrightarrow{0Y}$ we set $\mathbf{e} = Y/|Y|$, consider the points $b_k = (k\sqrt{t}/\sqrt{n})\mathbf{e}$ where $-n \leq k \leq n$ and denote by $G_{n,k}$ the spherical shell obtain by intersecting the spherical shell T_n with the domain $H_{n,k}$ which is the set of points in \mathbb{R}^N limited by the hyperplanes orthogonal to $\overrightarrow{0Y}$ going through $((k+1)\sqrt{t}/\sqrt{n})\mathbf{e}$ and $((k-1)\sqrt{t}/\sqrt{n})\mathbf{e}$. The number of points $a_{n,j} \in G_{n,k}$ is smaller than $C(n+1-|k|)^{N-2}$, where C depends only on N , and we denote by $\Lambda_{n,k}$ the set of $j \in \Theta_{t,n}$ such that $a_{n,j} \in G_{n,k}$. Furthermore, if $a_{n,j} \in G_{n,k}$ elementary geometric considerations (Pythagore's theorem) imply that $\lambda_{n,j,y}^2$ is greater than $t(n+p+1-2k\sqrt{p+1}/\sqrt{n})$. Therefore

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \leq C \sum_{n=p+\ell}^{a_t} \sum_{k=-n}^n (n+1-|k|)^{N-2} e^{-\frac{\epsilon q' (n+p+1-2k\sqrt{p+1})t}{4s\sqrt{n}}} \quad (3.37)$$

Case $N = 2$. Summing a geometric series and using the inequality $\frac{e^u}{e^u-1} \leq 1 + u^{-1}$ for $u > 0$, we obtain

$$\begin{aligned} \sum_{k=-n}^n e^{\frac{\epsilon q' (k\sqrt{p+1})t}{2s\sqrt{n}}} &\leq e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \frac{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}}{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}-1}} \\ &\leq e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \left(1 + \frac{2s\sqrt{n}}{\epsilon q' t\sqrt{p+1}} \right). \end{aligned} \quad (3.38)$$

Thus, by comparison between series and integrals,

$$\begin{aligned}
\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} &\leq C \sum_{n=p+\ell}^{a_t} \left(1 + \frac{s\sqrt{n}}{t\sqrt{p}}\right) e^{-\frac{\epsilon q' (\sqrt{n}-\sqrt{p+1})^2}{4s}} \\
&\leq C \int_{p+1}^{\infty} e^{-\frac{\epsilon q' (\sqrt{x}-\sqrt{p+1})^2}{4s}} dx \\
&\quad + \frac{Cs}{t\sqrt{p}} \int_{p+1}^{\infty} \sqrt{x} e^{-\frac{\epsilon q' (\sqrt{x}-\sqrt{p+1})^2}{4s}} dx.
\end{aligned} \tag{3.39}$$

Next

$$\begin{aligned}
\int_{p+1}^{\infty} e^{-\frac{\epsilon q' (\sqrt{x}-\sqrt{p+1})^2}{4s}} dx &= 2 \int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' (y-\sqrt{p+1})^2}{4s}} y dy \\
&= 2 \int_0^{\infty} e^{-\frac{\epsilon q' y^2}{4s}} y dy + 2\sqrt{p+1} \int_0^{\infty} e^{-\frac{\epsilon q' y^2}{4s}} dy \\
&= \frac{2s}{t} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} z dz + 2\sqrt{\frac{(p+1)s}{t}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz,
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
\int_{p+1}^{\infty} \sqrt{x} e^{-\frac{\epsilon q' (\sqrt{x}-\sqrt{p+1})^2}{4s}} dx &= 2 \int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' (y-\sqrt{p+1})^2}{4s}} y^2 dy \\
&= 2 \int_0^{\infty} e^{-\frac{\epsilon q' y^2}{4s}} (y + \sqrt{p+1})^2 dy \\
&\leq 4 \int_0^{\infty} e^{-\frac{\epsilon q' y^2}{4s}} y^2 dy + 4(p+1) \int_0^{\infty} e^{-\frac{\epsilon q' y^2}{4s}} dy \\
&\leq 4 \left(\frac{s}{t}\right)^{3/2} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} z^2 dz + 4(p+1) \sqrt{\frac{s}{t}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz
\end{aligned} \tag{3.41}$$

Jointly with (3.39), these inequalities imply

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} \leq C \sqrt{\frac{ps}{t}}. \tag{3.42}$$

Case $N > 2$. Because the value of the right-hand side of (3.37) is an increasing value of N , it is sufficient to prove (3.36) when N is even, say $(N-2)/2 = d \in \mathbb{N}_*$. There holds

$$\sum_{k=-n}^n (n+1-|k|)^d e^{\frac{\epsilon q' (k\sqrt{p+1})t}{2s\sqrt{n}}} \leq 2 \sum_{k=0}^n (n+1-k)^d e^{\frac{\epsilon q' (k\sqrt{p+1})t}{2s\sqrt{n}}}. \tag{3.43}$$

We set

$$\alpha = \epsilon q' \frac{t\sqrt{p+1}}{2s\sqrt{n}} \quad \text{and} \quad I_d = \sum_{k=0}^n (n+1-k)^d e^{k\alpha}.$$

Since

$$e^{k\alpha} = \frac{e^{(k+1)\alpha} - e^{k\alpha}}{e^\alpha - 1},$$

we use Abel's transform to obtain

$$\begin{aligned} I_d &= \frac{1}{e^\alpha - 1} \left(e^{(n+1)\alpha} - (n+1)^d + \sum_{k=1}^n ((n+2-k)^d - (n+1-k)^d) e^{k\alpha} \right) \\ &\leq \frac{1}{e^\alpha - 1} \left((1-d)e^{(n+1)\alpha} - (n+1)^d + de^\alpha \sum_{k=1}^n ((n+1-k)^{d-1}) e^{k\alpha} \right). \end{aligned}$$

Therefore the following induction holds

$$I_d \leq \frac{de^\alpha}{e^\alpha - 1} I_{d-1}. \quad (3.44)$$

In (3.38), we have already used the fact that

$$\frac{de^\alpha}{e^\alpha - 1} \leq C \left(1 + \frac{s\sqrt{n}}{t\sqrt{p}} \right),$$

and

$$I_d \leq C \left(1 + \left(\frac{s\sqrt{n}}{t\sqrt{p}} \right)^{d+1} \right) I_0.$$

Thus (3.39) is replaced by

$$\begin{aligned} \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} &\leq C \sum_{n=p+\ell}^{a_t} \left(1 + \left(\frac{s\sqrt{n}}{t\sqrt{p}} \right)^{d+1} \right) e^{-\frac{\epsilon q' (\sqrt{n} - \sqrt{p+1})^2 t}{4s}} \\ &\leq C \int_{p+1}^{\infty} e^{-\frac{\epsilon q' (\sqrt{x} - \sqrt{p+1})^2 t}{4s}} dx \\ &\quad + \left(\frac{Cs}{t\sqrt{p}} \right)^{d+1} \int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q' (\sqrt{x} - \sqrt{p+1})^2 t}{4s}} dx. \end{aligned} \quad (3.45)$$

The first integral on the right-hand side has already been estimated in (3.40), for the second integral, there holds

$$\begin{aligned} \int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q' (\sqrt{x} - \sqrt{p+1})^2 t}{4s}} dx &= \int_0^{\infty} (y + \sqrt{p+1})^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\leq C \int_0^{\infty} y^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dy + Cp^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\leq C \left(\frac{s}{t} \right)^{2+\frac{d}{2}} \int_0^{\infty} z^{(d+1)/2} e^{-\frac{\epsilon q' z^2}{4}} dz \\ &\quad + C \left(\frac{s}{t} \right)^{3/2} p^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz. \end{aligned} \quad (3.46)$$

Combining (3.40), (3.45) and (3.46), we derive (3.36).

Step 2. Since $\mathcal{T}_p^* \subset \Gamma_p \times [0, t]$ where $\Gamma_p = B_{d_{p+1}}(x) \setminus B_{d_{p-1}}(x)$, $(y, s) \in \mathcal{T}_p^*$ implies that $|x - y|^2 \geq (p-1)t$, thus $J'_{2,\ell}$ satisfies

$$\begin{aligned}
J'_{2,\ell} &\leq Ct^{\frac{1-q}{2}} \sum_{p=1}^{\infty} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} \\
&\quad \times \sum_{n=p+\ell}^{at} \sum_{j \in \Theta_{t,n}^h} e^{-\frac{q\lambda_{n,j,y}^2(1-\epsilon)}{4s}} \mu_{n,j}^q(K_{n,j}) ds dy \\
&\leq Ct^{\frac{1-q}{2}} \sum_{n=\ell+1}^{at} \sum_{j \in \Theta_{t,n}^h} \mu_{n,j}^q(K_{n,j}) \\
&\quad \times \sum_{p=1}^{n-\ell} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-|x-y|^2/4(t-s)} e^{-\frac{q\lambda_{n,j,y}^2(1-\epsilon)}{4s}} ds dy
\end{aligned} \tag{3.47}$$

and the constant C depends on N, q and ϵ . Next we set $q_\epsilon = (1-\epsilon)q$. Writting

$$|y - a_{n,j}|^2 = |x - y|^2 + |x - a_{n,j}|^2 - 2\langle y - x, a_{n,j} - x \rangle \geq pt + |x - a_{n,j}|^2 - 2\langle y - x, a_{n,j} - x \rangle,$$

we get

$$\int_{\Gamma_p} e^{-\frac{q_\epsilon |y - a_{n,j}|^2}{4s}} dy = e^{-\frac{q_\epsilon |x - a_{n,j}|^2}{4s}} \int_{\sqrt{tp}}^{\sqrt{t(p+1)}} e^{-\frac{q_\epsilon r^2}{4s}} \int_{|x-y|=r} e^{2q_\epsilon \langle y-x, a_{n,j}-x \rangle / 4s} dS_r(y) dr.$$

For estimating the value of the spherical integral, we can assume that $a_{n,j} - x = (0, \dots, 0, |a_{n,j} - x|)$, $y = (y_1, \dots, y_N)$ and, using spherical coordinates with center at x , that the unit sphere has the representation $S^{N-1} = \{(\sin \phi, \sigma, \cos \phi) \in \mathbb{R}^{N-1} \times \mathbb{R} : \sigma \in S^{N-2}, \phi \in [0, \pi]\}$. With this representation, $dS_r = r^{N-1} \sin^{N-2} \phi d\phi d\sigma$ and $\langle y - x, a_{n,j} - x \rangle = |a_{n,j} - x| |y - x| \cos \phi$. Therefore

$$\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x \rangle}{4s}} dS_r(y) = r^{N-1} |S^{N-2}| \int_0^\pi e^{2q_\epsilon \frac{|a_{n,j}-x| r \cos \phi}{4s}} \sin^{N-2} \phi d\phi.$$

By Lemma A.3

$$\begin{aligned}
\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x \rangle}{4s}} dS_r(y) &\leq C \frac{r^{N-1} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}}{\left(1 + \frac{r|a_{n,j}-x|}{s}\right)^{\frac{N-1}{2}}} \\
&\leq Cs^{\frac{N-1}{2}} \left(\frac{r}{|a_{n,j}-x|}\right)^{\frac{N-1}{2}} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}.
\end{aligned} \tag{3.48}$$

Therefore

$$\int_{\Gamma_p} e^{-q_\epsilon \frac{|y - a_{n,j}|^2}{4s}} dy \leq Ct^{\frac{N-1}{4}} p^{\frac{N-3}{4}} s^{\frac{N-1}{2}} \frac{e^{-q_\epsilon \frac{(|a_{n,j}-x| - \sqrt{t(p+1)})^2}{4s}}}{|a_{n,j} - x|^{\frac{N-1}{2}}}, \tag{3.49}$$

and, since $|a_{n,j} - x| \geq \sqrt{tn}$,

$$\begin{aligned}
& \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon \frac{\lambda_{n,j,y}^2}{4s}} dy ds \\
& \leq C \frac{\sqrt{tp}^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^t (t-s)^{-\frac{N}{2}} s^{-\frac{(q-1)(N-1)+1}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{tn}-\sqrt{t(p+1)})^2}{4s}} ds \\
& \leq C \frac{t^{\frac{1-q(N-1)}{2}} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^1 (1-s)^{-\frac{N}{2}} s^{-\frac{(q-1)(N-1)+1}{2}} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}}.
\end{aligned} \tag{3.50}$$

We apply Lemma A.1, with $A = \sqrt{p}$, $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$, $b = \frac{(q-1)(N-1)+1}{2}$, $a = \frac{N}{2}$ and $\kappa = \sqrt{q_\epsilon}(\ell - 1)/8$ as in the case $N = 1$, and noticing that, for these specific values,

$$\begin{aligned}
A^{1-a} B^{1-b} (A+B)^{a+b-2} &= p^{\frac{2-N}{4}} (\sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1}))^{\frac{1-(q-1)(N-1)}{2}} \\
&\quad \times (\sqrt{p} + \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1}))^{\frac{(q-1)(N-1)+N-3}{2}} \\
&\leq C \left(\frac{n}{p}\right)^{\frac{N}{4}-1/2} \left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}},
\end{aligned}$$

where C depends on N , q and κ . Therefore

$$\begin{aligned}
& \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon |y-z|^2/4s} dy ds \\
& \leq C \frac{t^{(1-q(N-1))/2} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \left(\frac{n}{p}\right)^{\frac{N}{4}-1/2} \left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \\
& \leq C t^{\frac{1-q(N-1)}{2}} p^{-\frac{1}{4}} n^{\frac{(q-1)(N-1)-2}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned} \tag{3.51}$$

We derive from (3.47), (3.51),

$$\begin{aligned}
J'_{2,\ell} h &\leq C t^{1-\frac{Nq}{2}} \\
&\times \sum_{n=\ell+1}^{a_t} \sum_{j \in \Theta_{t,n}^h} n^{\frac{(q-1)(N-1)-2}{4}} \mu_{n,j}^q(K_{n,j}) \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned} \tag{3.52}$$

By Lemma A.2 with $\alpha = \frac{2q-3}{4}$, $\beta = \frac{1-(q-1)(N-1)}{2}$, $\delta = \frac{1}{4}$ and $\gamma = q_\epsilon$, we obtain

$$\sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \leq C n^{\frac{N(q-1)+q-3}{4}} e^{-\frac{n}{4}}, \tag{3.53}$$

thus

$$J'_{2,\ell} h \leq C t^{1-\frac{Nq}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{N(q-1)}{2}-1} e^{-\frac{n}{4}} \sum_{j \in \Theta_{t,n}^h} \mu_{n,j}^q(K_{n,j}). \tag{3.54}$$

Because

$$\mu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}),$$

we use the rescaling procedure as in the proof of Lemma 2.15, except that the scale factor is $\sqrt{(n+1)t}$ instead of $\sqrt{n+1}$ so that the sets \tilde{T}_n , \tilde{K}_n , \tilde{Q}_n and \tilde{K}_n remains unchanged Using again the quasi-additivity and the fact that $J'_{2,\ell} = \sum_{h=1}^J J'_{2,\ell}{}^h$, we deduce

$$J_{2,\ell} \leq C' t^{-\frac{N}{2}} \sum_{n=\ell+1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right), \quad (3.55)$$

which implies (3.33). \square

The proof of Theorem 3.1 follows from the previous estimates on J_1 and J_2 . Furthermore the following integral expression holds

Theorem 3.9 *Assume $q \geq q_c$. Then there exists a positive constants C_2^* , depending on N, q and T , such that for any closed set F , there holds*

$$\underline{u}_F(x, t) \geq \frac{C_2^*}{t^{1+\frac{N}{2}}} \int_0^{\sqrt{tat}} e^{-\frac{s}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'} \left(\frac{F}{s} \cap B_1(x) \right) s ds, \quad (3.56)$$

where a_t is the smallest integer j such that $F \subset B_{\sqrt{jt}}(x)$.

Proof. We shall distinguish according $q = q_c$, or $q > q_c$, and for simplicity we shall denote $B_r = B_r(x)$ for the various values of r .

Case 1: $q = q_c \iff N - \frac{2}{q-1} = 0$. Because $F_n = F \cap (B_{d_{n+1}} \setminus B_{d_n})$ there holds

$$C_{2/q,q'} \left(\frac{F_n}{d_{n+1}} \right) \geq C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) - C_{2/q,q'} \left(\frac{F \cap B_{d_n}}{d_{n+1}} \right),$$

Furthermore, since $d_{n+1} \geq d_n$,

$$C_{2/q,q'} \left(\frac{F \cap B_{d_n}}{d_{n+1}} \right) = C_{2/q,q'} \left(\frac{d_n}{d_{n+1}} \frac{F \cap B_{d_n}}{d_n} \right) \leq C_{2/q,q'} \left(\frac{F}{d_n} \cap B_1 \right),$$

thus

$$C_{2/q,q'} \left(\frac{F_n}{d_{n+1}} \right) \geq C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) - C_{2/q,q'} \left(\frac{F}{d_n} \cap B_1 \right),$$

it follows

$$\begin{aligned} \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{F_n}{d_{n+1}} \right) &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) - \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{F}{d_n} \cap B_1 \right) \\ &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) - e^{-\frac{1}{4}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) \\ &\geq (1 - e^{-\frac{1}{4}}) \sum_{n=1}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) - e^{-\frac{1}{4}} C_{2/q,q'} \left(\frac{F}{\sqrt{t}} \cap B_1 \right). \end{aligned}$$

Since, by (2.69),

$$C_{2/q,q'} \left(\frac{F}{s'} \cap B_1 \right) \geq C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) \geq C_{2/q,q'} \left(\frac{F}{s} \cap B_1 \right),$$

for any $s' \in [d_{n+1}, d_{n+2}]$ and $s \in [d_n, d_{n+1}]$, there holds

$$\begin{aligned} te^{-\frac{n}{4}} C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) &\geq C_{2/q,q'} \left(\frac{F}{d_{n+1}} \cap B_1 \right) \int_{d_n}^{d_{n+1}} e^{-s^2/4t} s \, ds \\ &\geq \int_{d_n}^{d_{n+1}} e^{-s^2/4t} C_{2/q,q'} \left(\frac{F}{s} \cap B_1 \right) s \, ds. \end{aligned}$$

This implies

$$W_F(x, t) \geq (1 - e^{-\frac{1}{4}}) t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} e^{-s^2/4t} C_{2/q,q'} \left(\frac{F}{s} \cap B_1 \right) s \, ds.$$

Case 2: $q > q_c \iff N - \frac{2}{q-1} > 0$. In that case it is known [1] that

$$C_{2/q,q'} \left(\frac{F_n}{d_{n+1}} \right) \approx d_{n+1}^{\frac{2}{q-1}-N} C_{2/q,q'} (F_n)$$

thus

$$W_F(x, t) \approx t^{-1-\frac{N}{2}} \sum_{n=0}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} (F_n).$$

Since

$$C_{2/q,q'} (F_n) \geq C_{2/q,q'} (F \cap B_{d_{n+1}}) - C_{2/q,q'} (F \cap B_{d_n}),$$

and again

$$\begin{aligned} t^{-\frac{N}{2}} \sum_{n=0}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} (F_n) &\geq (1 - e^{-\frac{1}{4}}) t^{-\frac{N}{2}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'} (F \cap B_{d_{n+1}}) \\ &\geq (1 - e^{-\frac{1}{4}}) t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} C_{2/q,q'} (F \cap B_s) s \, ds. \end{aligned}$$

Because $C_{2/q,q'} (F \cap B_s) \approx s^{N-\frac{2}{q-1}} C_{2/q,q'} (s^{-1}F \cap B_1)$, (3.56) follows. \square

4 Applications

The first result of this section is the following

Theorem 4.1 *Assume $N \geq 1$ and $q > 1$. Then $\bar{u}_K = \underline{u}_K$.*

Proof. If $1 < q < q_c$, the result is already proved in [25]. The proof in the super-critical case is an adaptation that we shall recall, for the sake of completeness. By Theorem 2.16 and Theorem 3.9 there exists a positive constant C , depending on N , q and T such that

$$\bar{u}_F(x, t) \leq \underline{u}_F(x, t) \quad \forall (x, t) \in Q_T.$$

By convexity $\tilde{u} = \underline{u}_F - \frac{1}{2C}(\bar{u}_F - \underline{u}_F)$ is a super-solution, which is smaller than \underline{u}_F if we assume that $\bar{u}_F \neq \underline{u}_F$. If we set $\theta := 1/2 + 1/(2C)$, then $u_\theta = \theta \bar{u}_F$ is a subsolution. Therefore there exists a solution u_1 of (1.1) in Q_∞ such that $u_\theta \leq u_1 \leq \tilde{u} < \underline{u}_F$. If $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N)$ satisfies $\mu(F^c) = 0$, then $u_{\theta\mu}$ is the smallest solution of (1.1) which is above the subsolution θu_μ . Thus $u_{\theta\mu} \leq u_1 < \underline{u}_F$ and finally $\underline{u}_F \leq u_1 < \underline{u}_F$, a contradiction. \square

If we combine Theorem 2.16 and Theorem 3.9 we derive the following integral approximation of the capacitary potential

Proposition 4.2 *Assume $q \geq q_c$. Then there exist two positive constants C_1^\dagger, C_2^\dagger , depending only on N, q and T such that*

$$\begin{aligned} C_2^\dagger t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'} \left(\frac{F}{s} \cap B_1(x) \right) s ds &\leq W_F(x, t) \\ &\leq C_1^\dagger t^{-(1+\frac{N}{2})} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'} \left(\frac{F}{s} \cap B_1(x) \right) s ds \end{aligned} \quad (4.57)$$

for any $(x, t) \in Q_T$.

Definition 4.3 *If F is a closed subset of \mathbb{R}^N , we define the $(2/q, q')$ -integral capacitary potential \mathcal{W}_F by*

$$\mathcal{W}_F(x, t) = t^{-1-\frac{N}{2}} \int_0^{D_F(x)} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q, q'} \left(\frac{F}{s} \cap B_1(x) \right) s ds \quad \forall (x, t) \in Q_\infty, \quad (4.58)$$

where $D_F(x) = \max\{|x - y| : y \in F\}$.

An easy computation shows that

$$\begin{aligned} 0 \leq \mathcal{W}_F(x, t) - t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'} \left(\frac{F}{s} \cap B_1(x) \right) s ds \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-D_F^2(x)/4t}, \end{aligned} \quad (4.59)$$

and

$$\begin{aligned} 0 \leq t^{-(1+\frac{N}{2})} \int_0^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'} \left(\frac{F}{s} \cap B_1(x) \right) s ds - \mathcal{W}_F(x, t) \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-\frac{D_F^2(x)}{4t}}, \end{aligned} \quad (4.60)$$

for some $C = C(N, q) > 0$. Furthermore

$$\mathcal{W}_F(x, t) = t^{-\frac{1}{q-1}} \int_0^{D_F(x)/\sqrt{t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4}} C_{2/q, q'} \left(\frac{F}{s\sqrt{t}} \cap B_1(x) \right) s \, ds. \quad (4.61)$$

The following result gives a sufficient condition in order \bar{u}_F has not a strong blow-up at some point x .

Proposition 4.4 *Assume $q \geq q_c$ and F is a closed subset of \mathbb{R}^N . If there exists $\gamma \in [0, \infty)$ such that*

$$\lim_{\tau \rightarrow 0} C_{2/q, q'} \left(\frac{F}{\tau} \cap B_1(x) \right) = \gamma, \quad (4.62)$$

then

$$\lim_{t \rightarrow 0} t^{\frac{1}{q-1}} \bar{u}_F(x, t) = C\gamma, \quad (4.63)$$

for some $C = C(N, q) > 0$.

Proof. Clearly, condition (4.62) implies

$$\lim_{t \rightarrow 0} C_{2/q, q'} \left(\frac{F}{\sqrt{t}s} \cap B_1(x) \right) = \gamma$$

for any $s > 0$. Then (4.63) follows by Lebesgue's theorem. Notice also that the set of γ is bounded from above by a constant depending on N and q . \square

In the next result we give a condition in order the solution remains bounded at some point x . The proof is similar to the previous one.

Proposition 4.5 *Assume $q \geq q_c$ and F is a closed subset of \mathbb{R}^N . If*

$$\limsup_{\tau \rightarrow 0} \tau^{-\frac{2}{q-1}} C_{2/q, q'} \left(\frac{F}{\tau} \cap B_1(x) \right) < \infty, \quad (4.64)$$

then $\bar{u}_F(x, t)$ remains bounded when $t \rightarrow 0$.

A Appendix

The next estimate is crucial in the study of semilinear parabolic equations.

Lemma A.1 *Let a and b be two real numbers, $a > 0$ and $\kappa > 0$. Then there exists a constant $C = C(a, b, \kappa) > 0$ such that for any $A > 0$, $B > \kappa/A$ there holds*

$$\int_0^1 (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x} dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \quad (A.1)$$

Proof. We first notice that

$$\max\{e^{-A^2/4(1-x)}e^{-B^2/4x} : 0 \leq x \leq 1\} = e^{-(A+B)^2/4}, \quad (\text{A.2})$$

and it is achieved for $x_0 = B/(A+B)$. Set $\Phi(x) = (1-x)^{-a}x^{-b}e^{-A^2/4(1-x)}e^{-B^2/4x}$, thus

$$\int_0^1 \Phi(x)dx = \int_0^{x_0} \Phi(x)dx + \int_{x_0}^1 \Phi(x)dx = I_{a,b} + J_{a,b}.$$

Put

$$u = \frac{A^2}{4(1-x)} + \frac{B^2}{4x}, \quad (\text{A.3})$$

then

$$4ux^2 - (4u + B^2 - A^2)x + B^2 = 0. \quad (\text{A.4})$$

If $0 < x < x_0$ this equation admits the solution

$$x = x(u) = \frac{1}{8u} \left(4u + B^2 - A^2 - \sqrt{16u^2 - 8u(A^2 + B^2) + (A^2 - B^2)^2} \right)$$

$$\int_0^{x_0} (1-x)^{-a}x^{-b}e^{-A^2/4(1-x)-B^2/4x}dx = - \int_{(A+B)^2/4}^{\infty} (1-x(u))^{-a}x(u)^{-b}e^{-u}x'(u)du$$

Putting $x' = x'(u)$ and differentiating (A.4),

$$4x^2 + 8uxx' - (4u + B^2 - A^2)x' - 4x = 0 \implies -x' = \frac{4x(1-x)}{4u + B^2 - A^2 - 8ux}.$$

Thus

$$\int_0^{x_0} \Phi(x)dx = 4 \int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1}x(u)^{-b+1}e^{-u}du}{4u + B^2 - A^2 - 8ux(u)}. \quad (\text{A.5})$$

Using the explicit value of the root $x(u)$, we finally get

$$\int_0^{x_0} \Phi(x)dx = 4 \int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1}x(u)^{-b+1}e^{-u}du}{\sqrt{16u^2 - 8u(A^2 + B^2) + (A^2 - B^2)^2}}, \quad (\text{A.6})$$

and the factorization below holds

$$16u^2 - 8u(A^2 + B^2) + (A^2 - B^2)^2 = 16(u - (A+B)^2/4)(u - (A-B)^2/4).$$

We set $u = v + (A+B)^2/4$ and obtain

$$x(u) = \frac{v + (AB + B^2)/2 - \sqrt{v(v+AB)}}{2(v + (A+B)^2/4)},$$

and

$$1 - x(u) = \frac{v + (A^2 + AB)/2 + \sqrt{v(v+AB)}}{2(v + (A+B)^2/4)}.$$

We introduce the relation \approx linking two positive quantities depending on A and B . It means that the two sided-inequalities up to multiplicative constants independent of A and B . Therefore

$$\int_0^{x_0} \Phi(x) dx = 2^{a-b-4} e^{-(A+B)^2/4} \int_0^\infty \tilde{\Phi}(v) dv \quad \text{where}$$

$$\tilde{\Phi}(v) = \frac{\left(v + (AB + B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b} \left(v + (A^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-a}}{\left(v + (A+B)^2/4\right)^{2-a-b} \sqrt{v(v+AB)}} e^{-v} dv. \quad (\text{A.7})$$

Case 1: $a \geq 1, b \geq 1$. First

$$\frac{\left(v + (A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+AB)}} \leq \frac{\left(v + (A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \approx \frac{\left(v + (A+B)^2\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \quad (\text{A.8})$$

since $a+b-2 \geq 0$ and $AB \geq \kappa$. Next

$$\left(v + (A^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-a} \approx \left(v + A(A+B)\right)^{1-a}. \quad (\text{A.9})$$

Furthermore

$$v + (AB + B^2)/2 - \sqrt{v(v+AB)} = B^2 \frac{v + (A+B)^2/4}{v + B(A+B)/2 + \sqrt{v(v+AB)}} \approx B^2 \frac{v + (A+B)^2}{v + B(A+B)}. \quad (\text{A.10})$$

Then

$$\left(v + (AB + B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b} \approx B^{2-2b} \left(\frac{v + B(A+B)}{v + (A+B)^2}\right)^{b-1} \quad (\text{A.11})$$

It follows

$$\begin{aligned} \tilde{\Phi}(v) &\leq C B^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)}\right)^{a-1} \frac{\left(v + B(A+B)\right)^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq C B^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)}\right)^{a-1} \frac{v^{b-1} + (B^2 + AB)^{b-1}}{\sqrt{v(v+\kappa)}} \end{aligned} \quad (\text{A.12})$$

where C depends on a, b and κ . The function $v \mapsto (v + (A+B)^2)/(v + A(A+B))$ is decreasing on $(0, \infty)$. If we set

$$C_1 = \int_0^\infty \frac{v^{b-1} e^{-v} dv}{\sqrt{v(v+\kappa)}} \quad \text{and} \quad C_2 = \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}}$$

then

$$C_1 \leq K (B^2 + AB)^{b-1} C_2$$

with $K = C_1 \kappa^{1-b} / C_2$. Therefore

$$\int_0^{x_0} \Phi(x) dx \leq C e^{-(A+B)^2/4} B^{1-b} A^{1-a} (A+B)^{a+b-2}. \quad (\text{A.13})$$

The estimate of $J_{a,b}$ is obtained by exchanging (A, a) with (B, b) and replacing x by $1 - x$. *Mutadis mutandis*, this yields directly to the same expression as in A.13 and finally

$$\int_0^1 \Phi(x) dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \quad (\text{A.14})$$

Case 2: $a \geq 1, b < 1$. Estimates (A.7), (A.8), (A.9), (A.10) and (A.11) are valid. Because $v \mapsto (v + B(A+B))^{b-1}$ is decreasing, (A.12) has to be replaced by

$$\tilde{\Phi}(v) \leq C B^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)} \right)^{a-1} \frac{(AB + B^2)^{b-1}}{\sqrt{v(v+\kappa)}}. \quad (\text{A.15})$$

This implies (A.13) directly. The estimate of $J_{a,b}$ is performed by the change of variable $x \mapsto 1 - x$. If $x_1 = 1 - x_0$, there holds

$$J_{a,b} = \int_0^{x_1} x^{-a} (1-x)^{-b} e^{-A^2/4x} e^{-B^2/4(1-x)} dx = \int_0^{x_1} \Psi(x) dx.$$

Then

$$\int_0^{x_1} \Psi(x) dx = 2^{b-a-4} e^{-(A+B)^2/4} \int_0^{x_1} \tilde{\Psi}(v) dv \quad \text{where}$$

$$\tilde{\Psi}(v) = \frac{\left(v + (AB + A^2)/2 - \sqrt{v(v+AB)} \right)^{1-a} \left(v + (B^2 + AB)/2 + \sqrt{v(v+AB)} \right)^{1-b}}{(v + (A+B)^2/4)^{2-a-b} \sqrt{v(v+AB)}} e^{-v} dv. \quad (\text{A.16})$$

Equivalence (A.8) is unchanged; (A.9) is replaced by

$$\left(v + (B^2 + AB)/2 + \sqrt{v(v+AB)} \right)^{1-b} \approx (v + B(A+B))^{1-b}, \quad (\text{A.17})$$

(A.10) by

$$v + (AB + A^2)/2 - \sqrt{v(v+AB)} \approx A^2 \frac{v + (A+B)^2}{v + A(A+B)}, \quad (\text{A.18})$$

and (A.11) by

$$\left(v + (AB + A^2)/2 - \sqrt{v(v+AB)} \right)^{1-a} \approx A^{2-2a} \left(\frac{v + A(A+B)}{v + (A+B)^2} \right)^{a-1}. \quad (\text{A.19})$$

Because $a > 1$, (A.12) turns into

$$\begin{aligned} \tilde{\Psi}(v) &\leq C A^{2-2b} (v + (A+B)^2)^{b-1} \frac{(v + A^2 + AB)^{a-1} (v + B^2 + AB)^{1-b}}{\sqrt{v(v+\kappa)}} \\ &\leq C e^{-(A+B)^2/4} A^{2-2b} (A+B)^{2b-2} \\ &\quad \times \frac{v^{a-b} + (A^2 + AB)^{a-1} v^{1-b} + (B^2 + AB)^{1-b} v^{a-1} + A^{a-1} B^{1-b} (A+B)^{a-b}}{\sqrt{v(v+\kappa)}}. \end{aligned} \quad (\text{A.20})$$

Because $AB \geq \kappa$, there exists a positive constant C , depending on κ , such that

$$\int_0^\infty \frac{v^{a-b} + (A^2 + AB)^{a-1}v^{1-b} + (B^2 + AB)^{1-b}v^{a-1}}{\sqrt{v(v+\kappa)}} e^{-v} dv \leq CA^{a-1}B^{1-b}(A+B)^{a-b} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}}. \quad (\text{A.21})$$

Combining (A.20) and (A.21) yields to

$$\int_0^{x_1} \Psi(x) dx \leq Ce^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \quad (\text{A.22})$$

This, again, implies that (A.1) holds.

Case 3: $\max\{a, b\} < 1$. Inequalities (A.7)-(A.11) hold, but (A.12) has to be replaced by

$$\begin{aligned} \tilde{\Phi}(v) &\leq CB^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)} \right)^{a-1} \frac{(v + B^2 + AB)^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq CB^{1-b}(A+B)^{2a+b-3} \frac{v^{1-a} + (A^2 + AB)^{1-a}}{\sqrt{v(v+\kappa)}} \end{aligned} \quad (\text{A.23})$$

Noticing that

$$\int_0^\infty \frac{v^{1-a} e^{-v} dv}{\sqrt{v(v+\kappa)}} \leq C (A^2 + AB)^{1-a} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}},$$

it follows that (A.13) holds. Finally (A.14) holds by exchanging (A, a) and (B, b) . \square

Lemma A.2 . *Let $\alpha, \beta, \gamma, \delta$ be real numbers and ℓ an integer. We assume $\gamma > 1, \delta > 0$ and $\ell \geq 2$. Then there exists a positive constant C such that, for any integer $n > \ell$*

$$\sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \leq C n^{\alpha-\beta/2} e^{-\delta n}. \quad (\text{A.24})$$

Proof. The function $x \mapsto (\sqrt{x} + \sqrt{\gamma}(\sqrt{n} - \sqrt{x+1}))^2$ is decreasing on $[(\gamma-1)^{-1}, \infty)$. Furthermore there exists $C > 0$ depending on ℓ, α and β such that $p^\alpha (\sqrt{n} - \sqrt{p})^\beta \leq C x^\alpha (\sqrt{n} - \sqrt{x+1})^\beta$ for $x \in [p, p+1]$ If we denote by p_0 the smallest integer larger than $(\gamma-1)^{-1}$, we derive

$$\begin{aligned} S &= \sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2/4} = \sum_{p=1}^{p_0-1} + \sum_{p_0}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \\ &\leq \sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \\ &\quad + C \int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n} - \sqrt{x})^\beta e^{-\delta(\sqrt{x} + \sqrt{\gamma}(\sqrt{n} - \sqrt{x+1}))^2} dx, \end{aligned}$$

(notice that $\sqrt{n} - \sqrt{x} \approx \sqrt{n} - \sqrt{x+1}$ for $x \leq n - \ell$). Clearly

$$\sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \leq C_0 n^\alpha (\sqrt{n} - \sqrt{n-\ell})^\beta e^{-\delta n} \quad (\text{A.25})$$

for some C_0 independent of n . We set $y = y(x) = \sqrt{x+1} - \sqrt{x}/\sqrt{\gamma}$. Obviously

$$y'(x) = \frac{1}{2} \left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{\gamma}\sqrt{x}} \right) \quad \forall x \geq p_0,$$

and there exists $\epsilon = \epsilon(\delta, \gamma) > 0$ such that $\sqrt{2}\sqrt{x} \geq y(x) \geq \epsilon\sqrt{x}$ and $y'(x) \geq \epsilon/\sqrt{x}$. Furthermore

$$\begin{aligned} \sqrt{x} &= \frac{\sqrt{\gamma} \left(y + \sqrt{\gamma y^2 + 1 - \gamma} \right)}{\gamma - 1}, \\ \sqrt{n} - \sqrt{x} &= \frac{\sqrt{n}(\gamma - 1) - \sqrt{\gamma}y - \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}}{\gamma - 1} \\ &= \frac{n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}(\gamma - 1) - \sqrt{\gamma}y + \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}} \\ &\approx \frac{n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}} \end{aligned}$$

since $y(x) \leq \sqrt{n}$. Furthermore

$$\begin{aligned} n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2 &= \gamma(\sqrt{n+1} + \sqrt{n}/\sqrt{\gamma} + y)(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y) \\ &\approx \sqrt{n}(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y), \end{aligned}$$

because y ranges between $\sqrt{n+2-\ell} - \sqrt{n+1-\ell}\sqrt{\gamma} \approx \sqrt{n}$ and $\sqrt{p_0+1} - \sqrt{p_0}\sqrt{\gamma}$. Thus

$$(\sqrt{n} - \sqrt{x})^\beta \approx (\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y)^\beta.$$

This implies

$$\begin{aligned} &\int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n} - \sqrt{x})^\beta e^{-\delta(\sqrt{x} + \sqrt{\gamma}(\sqrt{n} - \sqrt{x+1}))^2} dx \\ &\leq C \int_{y(p_0)}^{y(n+1-\ell)} y^{2\alpha+1} (\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y)^\beta e^{-\gamma\delta(\sqrt{n}-y)^2} dy \\ &\leq C n^{\alpha+\beta/2+1} \int_{1-y(n+1-\ell)/\sqrt{n}}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz. \end{aligned} \quad (\text{A.26})$$

Moreover

$$\begin{aligned} 1 - \frac{y(p_0)}{\sqrt{n}} &= 1 - \frac{1}{\sqrt{n}} \left(\sqrt{p_0+1} - \frac{\sqrt{p_0}}{\sqrt{\gamma}} \right), \\ 1 - \frac{y(n-\ell+1)}{\sqrt{n}} &= 1 - \frac{\sqrt{n-\ell+2}}{\sqrt{n}} + \frac{\sqrt{n-\ell+1}}{\sqrt{n}\sqrt{\gamma}} \\ &= \frac{1}{\sqrt{\gamma}} \left(1 + \frac{\sqrt{\gamma}(\ell-2) - \ell + 1}{2n} + \frac{\sqrt{\gamma}(\ell-2)^2 - (\ell-1)^2}{8n^2} \right) + O(n^{-3}). \end{aligned} \quad (\text{A.27})$$

Let θ fixed such that $1 - \frac{y(n - \ell + 1)}{\sqrt{n}} < \theta < 1 - \frac{y(p_0)}{\sqrt{n}}$ for any $n > p_0$. Then

$$\begin{aligned} \int_{\theta}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^{\beta} e^{-\gamma\delta n z^2} dz &\leq C_{\theta} \int_{\theta}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} e^{-\gamma\delta n z^2} dz \\ &\leq C_{\theta} e^{-\gamma\delta n \theta^2} \int_{\theta}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} dz \\ &\leq C e^{-\gamma\delta n \theta^2} \max\{1, n^{-\alpha-1/2}\}. \end{aligned}$$

Because $\gamma\theta^2 > 1$ we derive

$$\int_{\theta}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^{\beta} e^{-\gamma\delta n z^2} dz \leq C n^{-\beta} e^{-\delta n}, \quad (\text{A.28})$$

for some constant $C > 0$. On the other hand

$$\begin{aligned} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^{\beta} e^{-\gamma\delta n z^2} dz \\ \leq C'_{\theta} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^{\beta} e^{-\gamma\delta n z^2} dz. \end{aligned}$$

The minimum of $z \mapsto (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^{\beta}$ is achieved at $1 - y(n+1-\ell)$ with value

$$\frac{\sqrt{\gamma}(\ell+1) + 1 - \ell}{2n\sqrt{\gamma}} + O(n^{-2}),$$

and the maximum of the exponential term is achieved at the same point with value

$$e^{-n\delta + ((\ell-2)\sqrt{\gamma} + 1 - \ell)/2} (1 + o(1)) = C_{\gamma} e^{-n\delta} (1 + o(1)).$$

We denote

$$z_{\gamma,n} = 1 + 1/\sqrt{\gamma} - \sqrt{1+1/n} \quad \text{and} \quad I_{\beta} = \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^{\beta} e^{-\gamma\delta n z^2} dz.$$

Since $1 - y(n+1-\ell) \geq 1/\sqrt{2\gamma}$ for n large enough,

$$\begin{aligned} I_{\beta} &\leq \sqrt{2\gamma} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^{\beta} z e^{-\gamma\delta n z^2} dz \\ &\leq \frac{-\sqrt{2\gamma}}{2n\gamma\delta} \left[(z - z_{\gamma,n})^{\beta} e^{-\gamma\delta n z^2} \right]_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} + \frac{\beta\sqrt{2\gamma}}{2n\gamma\delta} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^{\beta-1} z e^{-\gamma\delta n z^2} dz \end{aligned}$$

But $1 - y(n+1-\ell)/\sqrt{n} - z_{\gamma,n} = (\ell-1)(1 - 1/\sqrt{\gamma})/2n$, therefore

$$I_{\beta} \leq C_1 n^{-\beta-1} e^{-\delta n} + \beta C'_1 n^{-1} I_{\beta-1}. \quad (\text{A.29})$$

If $\beta \leq 0$, we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n},$$

which inequality, combined with (A.26) and (A.28), yields to (A.24). If $\beta > 0$, we iterate and get

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C'_1 n^{-1} (C_1 n^{-\beta} e^{-\delta n} + (\beta - 1) C'_1 n^{-1} I_{\beta-2})$$

If $\beta - 1 \leq 0$ we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C_1 C'_1 n^{-1-\beta} e^{-\delta n} = C_2 n^{-\beta-1} e^{-\delta n},$$

which again yields to (A.24). If $\beta - 1 > 0$, we continue up we find a positive integer k such that $\beta - k \leq 0$, which again yields to

$$I_\beta \leq C_k n^{-\beta-1} e^{-\delta n}$$

and to (A.24). □

The next estimate is fundamental in deriving the N -dimensional estimate.

Lemma A.3 *For any integer $N \geq 2$ there exists a constant $c_N > 0$ such that*

$$\int_0^\pi e^{m \cos \theta} \sin^{N-2} \theta d\theta \leq c_N \frac{e^m}{(1+m)^{(N-1)/2}} \quad \forall m > 0. \quad (\text{A.30})$$

Proof. Put $\mathcal{I}_N(m) = \int_0^\pi e^{m \cos \theta} \sin^{N-2} \theta d\theta$. Then $\mathcal{I}'_2(m) = \int_0^\pi e^{m \cos \theta} \cos \theta d\theta$ and

$$\begin{aligned} \mathcal{I}''_2(m) &= \int_0^\pi e^{m \cos \theta} \cos^2 \theta d\theta = \mathcal{I}_2(m) - \int_0^\pi e^{m \cos \theta} \sin^2 \theta d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m} \int_0^\pi e^{m \cos \theta} \cos \theta d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m} \mathcal{I}'_2(m). \end{aligned}$$

Thus \mathcal{I}_2 satisfies a Bessel equation of order 0. Since $\mathcal{I}_2(0) = \pi$ and $\mathcal{I}'_2(0) = 0$, $\pi^{-1} \mathcal{I}_2$ is the modified Bessel function of index 0 (usually denoted by I_0) the asymptotic behaviour of which is well known, thus (A.30) holds. If $N = 3$

$$\mathcal{I}_3(m) = \int_0^\pi e^{m \cos \theta} \sin \theta d\theta = \left[\frac{-e^{m \cos \theta}}{m} \right]_0^\pi = \frac{2 \sinh m}{m}.$$

For $N > 3$ arbitrary

$$\mathcal{I}_N(m) = \int_0^\pi \frac{-1}{m} \frac{d}{d\theta} (e^{m \cos \theta}) \sin^{N-3} \theta d\theta = \frac{N-3}{m} \int_0^\pi e^{m \cos \theta} \cos \theta \sin^{N-4} \theta d\theta. \quad (\text{A.31})$$

Therefore,

$$\mathcal{I}_4(m) = \frac{1}{m} \int_0^\pi e^{m \cos \theta} \cos \theta d\theta = \mathcal{I}'_2(m),$$

and, again (A.30) holds since $I'_0(m)$ has the same behaviour as $I_0(m)$ at infinity. For $N \geq 5$

$$\mathcal{I}_N(m) = \frac{3-N}{m^2} \left[e^{m \cos \theta} \cos \theta \sin^{N-5} \theta \right]_0^\pi + \frac{N-3}{m^2} \int_0^\pi e^{m \cos \theta} \frac{d}{d\theta} (\cos \theta \sin^{N-5} \theta) d\theta.$$

Differentiating $\cos \theta \sin^{N-5} \theta$ and using (A.31), we obtain

$$\mathcal{I}_5(m) = \frac{4 \sinh m}{m^2} - \frac{4 \sinh m}{m^3},$$

while

$$\mathcal{I}_N(m) = \frac{(N-3)(N-5)}{m^2} (\mathcal{I}_{N-4}(m) - \mathcal{I}_{N-2}(m)), \quad (\text{A.32})$$

for $N \geq 6$. Since the estimate (A.30) for $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ and \mathcal{I}_5 has already been obtained, a straightforward induction yields to the general result. \square

Remark. Although it does not have any importance for our use, it must be noticed that \mathcal{I}_N can be expressed either with hyperbolic functions if N is odd, or with Bessel functions if N is even.

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