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Cédric Bonnafé, Matthew J. Dyer

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# SEMIDIRECT PRODUCT DECOMPOSITION OF COXETER GROUPS

CÉDRIC BONNAFÉ AND MATTHEW J. DYER

ABSTRACT. Let  $(W, S)$  be a Coxeter system, let  $S = I \dot{\cup} J$  be a partition of  $S$  such that no element of  $I$  is conjugate to an element of  $J$ , let  $\tilde{J}$  be the set of  $W_I$ -conjugates of elements of  $J$  and let  $\tilde{W}$  be the subgroup of  $W$  generated by  $\tilde{J}$ . We show that  $W = \tilde{W} \rtimes W_I$  and that  $(\tilde{W}, \tilde{J})$  is a Coxeter system.

**Internal semidirect products.** Let  $(W, S)$  be a Coxeter system and assume that  $S$  is the union of two subsets  $I$  and  $J$  such that no element of  $I$  is conjugate to an element of  $J$ . Let  $W_I$  be the subgroup of  $W$  generated by  $I$ . Let  $\tilde{J}$  be the set of elements of the form  $ws w^{-1}$  where  $w$  is in  $W_I$  and  $s$  is in  $J$ . Let  $\tilde{W}$  be the subgroup of  $W$  generated by  $\tilde{J}$ . The main result of this paper is the following:

**Theorem.** *With the above notation, we have:*

- (a)  $W = \tilde{W} \rtimes W_I$  (semidirect product with  $\tilde{W}$  normal).
- (b)  $(\tilde{W}, \tilde{J})$  is a Coxeter system and  $\tilde{J}$  is the set of canonical Coxeter generators of  $\tilde{W}$ .
- (c) Each element  $w$  of  $W_I$  is the unique element of minimal length in its coset  $\tilde{W}w = w\tilde{W}$ .

NOTATION, REMARK, DEFINITION - Let  $T = \bigcup_{w \in W} wSw^{-1}$  be the set of reflections of  $W$ . If  $w \in W$ , we set  $N(w) = \{t \in T \mid \ell(wt) < \ell(w)\}$  where  $\ell$  is the length function of  $(W, S)$ . If  $W'$  is a subgroup of  $W$  generated by reflections, we set

$$\chi(W') = \{t \in T \mid N(t) \cap W' = \{t\}\}.$$

Then [4, (3.3)]  $(W', \chi(W'))$  is a Coxeter system:  $\chi(W')$  is called the set of *canonical Coxeter generators* of  $W'$ .

COMMENT - This Theorem will be used by the first author [1] for studying the Hecke algebra and the Kazhdan-Lusztig theory with unequal parameters whenever the parameters are zero on the set  $I$ .

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We shall actually state and prove a more precise version (Theorem 2) of the Theorem which explicitly describes the Coxeter matrix of  $(\widetilde{W}, \widetilde{J})$ . This requires some additional notation. If  $s$  and  $t$  are two elements of  $T$ , we denote by  $m_{s,t}$  the order of  $st$ . It is well known that two simple reflections are  $W$ -conjugate iff, regarded as vertices of the Coxeter graph of  $(W, S)$ , there is a path from one to the other such that each edge of the path has either an odd label or no label (i.e. a label of 3, which is omitted by the standard convention). In particular:

$$(1) \quad \text{If } s \in I \text{ and } t \in J, \text{ then } m_{s,t} \text{ is even.}$$

If  $s \in S$ , we set  $s^\perp = \{t \in S \mid st = ts\}$ . If  $A, B$  and  $C$  are three subsets of  $S$  such that  $B \subseteq A$  and  $C \subseteq A$ , we denote by  $X_{BC}^A$  the set of  $x \in W_A$  which have minimal length in  $W_BxW_C$ . For simplicity, we set  $X_{\emptyset C}^A = X_C^A$ . Deodhar's Lemma [5, Lemma 2.1.2] amounts to the statement that if  $w \in X_C^A$  and  $s \in A$  with  $sw \notin W_C^A$  then  $\ell(sw) > \ell(w)$  and  $sw = wr$  for some  $r \in C$ .

If  $\tilde{t} \in \widetilde{J}$  and if  $t, t' \in J$  and  $x, x' \in W_I$  are such that  $\tilde{t} = txt^{-1} = x't'x'^{-1}$ , then

$$(2) \quad t = t'.$$

Indeed, in this case, then  $t' \in W_{I \cup \{t\}} \cap W_J = \langle t \rangle$ . Therefore, if  $\tilde{t} \in \widetilde{J}$ , we can define  $\nu(\tilde{t})$  as the unique element of  $J$  which is conjugate to  $\tilde{t}$  under  $W_I$ .

Now, in order to parametrize  $\widetilde{J}$ , we must determine the centralizer of  $t \in J$  in  $W_I$ .

**Lemma 1.** *Let  $t \in J$ . Then  $C_{W_I}(t) = W_{I \cap t^\perp}$ .*

*Proof.* First it is clear that  $W_{I \cap t^\perp} \subseteq C_{W_I}(t)$ . Conversely, let  $w \in W_I$  be such that  $wt = tw$ . Let  $w = s_1 \cdots s_r$  be a reduced expression of  $w$  (so that  $s_i \in I$ ). Then,  $s_1 \cdots s_r t$  and  $t s_1 \cdots s_r$  are reduced expression of the same element  $wt = tw$  of  $W$ . By Matsumoto's lemma, this means that one can obtain one of these reduced expression by applying only braid relations. But  $t$  occurs only once in both reduced expressions: this means that, in order to make  $t$  pass from the first position to the last position,  $t$  must commute with all the  $s_i$ . So  $w \in W_{I \cap t^\perp}$ .  $\square$

Now, let  $\tilde{s}, \tilde{t} \in \widetilde{J}$  and let  $s = \nu(\tilde{s})$  and  $t = \nu(\tilde{t})$ . Then there exists  $x$  and  $y \in W_I$  such that  $\tilde{s} = xsx^{-1}$  and  $\tilde{t} = yty^{-1}$ . We denote by  $f(\tilde{s}, \tilde{t})$  the unique element of  $X_{I \cap s^\perp, I \cap t^\perp}^I$  such that  $x^{-1}y \in W_{I \cap s^\perp} f(\tilde{s}, \tilde{t}) W_{I \cap t^\perp}$ . It is readily seen that  $f(\tilde{s}, \tilde{t})$  depends only on  $\tilde{s}$  and  $\tilde{t}$  and not on the choice of  $x$  and  $y$ . Note that  $\tilde{s} = \tilde{t}$  if and

only if  $s = t$  and  $f(\tilde{s}, \tilde{t}) = 1$ . We then set:

$$\tilde{m}_{\tilde{s}, \tilde{t}} = \begin{cases} 1 & \text{if } \tilde{s} = \tilde{t}, \\ m_{s,u}/2 & \text{if } s = t \text{ and } f(\tilde{s}, \tilde{t}) = u \in I, \\ \infty & \text{if } s = t \text{ and } \ell(f(\tilde{s}, \tilde{t})) \geq 2, \\ m_{s,t} & \text{if } s \neq t \text{ and } f(\tilde{s}, \tilde{t}) = 1, \\ \infty & \text{if } s \neq t \text{ and } f(\tilde{s}, \tilde{t}) \neq 1. \end{cases}$$

We denote by  $\widetilde{M}$  the matrix  $(\tilde{m}_{\tilde{s}, \tilde{t}})_{\tilde{s}, \tilde{t} \in \widetilde{J}}$ .

Since  $f(\tilde{s}, \tilde{t}) = f(\tilde{t}, \tilde{s})^{-1}$  and since  $m_{su}$  is even if  $s \in J$  and  $u \in I$  by (1), we have, for all  $\tilde{s}, \tilde{t} \in \widetilde{J}$  and  $x \in W_I$ ,

$$(3) \quad \tilde{m}_{\tilde{s}, \tilde{t}} \in \mathbb{Z}_{\geq 1}, \quad \tilde{m}_{\tilde{s}, \tilde{t}} = \tilde{m}_{\tilde{t}, \tilde{s}}, \quad \tilde{m}_{x\tilde{s}x^{-1}, x\tilde{t}x^{-1}} = \tilde{m}_{\tilde{s}, \tilde{t}}, \quad \tilde{m}_{\tilde{s}, \tilde{s}} = 1 \text{ and } \tilde{m}_{\tilde{s}, \tilde{t}} \geq 2 \text{ (if } \tilde{s} \neq \tilde{t}\text{)}.$$

The last inequality follows from the fact that, if  $f(\tilde{s}, \tilde{t}) = u \in I$ , then  $us \neq su$ . Here is a more precise version of the main theorem of this paper:

**Theorem 2.** *With the above notation, we have:*

- (a)  $W = \widetilde{W} \rtimes W_I$  (semidirect product with  $\widetilde{W}$  normal).
- (b)  $(\widetilde{W}, \widetilde{J})$  is a Coxeter system with Coxeter matrix  $\widetilde{M}$  and  $\widetilde{J}$  is the set of canonical Coxeter generators of  $\widetilde{W}$ .
- (c) Each element  $w$  of  $W_I$  is the unique element of minimal length in its coset  $\widetilde{W}w = w\widetilde{W}$ .

*Proof.* We first prove (a). Let

$$\begin{aligned} \varphi: S &\longrightarrow W_I \\ s &\longmapsto \begin{cases} s & \text{if } s \in I, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

It follows easily from (1) that  $(\varphi(s)\varphi(t))^{m_{st}} = 1$  for all  $s, t \in S$ . Therefore, there exists a unique morphism of groups  $W \rightarrow W_I$  extending  $\varphi$ : we still denote it by  $\varphi$ . Since  $\varphi(w) = w$  for all  $w \in W_I$ , it is sufficient to prove that

$$(4) \quad \text{Ker } \varphi = \widetilde{W}.$$

Let us prove (4). First of all, note that  $\widetilde{W} \subseteq \text{Ker } \varphi$ . So it is sufficient to show that  $W = \widetilde{W}W_I$ . For this, it is sufficient to show that if  $w \in W \setminus W_I$ , there is some  $t \in \widetilde{J}$  with  $\ell(tw) < \ell(w)$ . Write  $w = s_1 \cdots s_n$  (reduced) with all  $s_i \in S$ . Since  $w \notin W_I$ , there is some  $j$  with  $s_j \in J$ . Without loss of generality, assume that  $j$  is minimal with this property. Then  $t := s_1 \cdots s_{j-1} s_j s_{j-1} \cdots s_1 \in \widetilde{J}$  and  $\ell(tw) < \ell(w)$  as required. This completes the proof of (a).

Let us now prove (b). For this, we use the standard geometric realization of  $(W, S)$  as a reflection group associated to a based root system. For use later, we introduce

a slightly more general class of geometric realizations with better “functoriality” properties.

Let  $\mathcal{E}$  be a  $\mathbb{R}$ -vector space equipped with a symmetric  $\mathbb{R}$ -bilinear form  $\langle \cdot, \cdot \rangle$ . We say a subset  $\Pi$  of  $\mathcal{E}$  is positively independent if  $\sum_{\alpha \in \Pi} c_\alpha \alpha = 0$  with all  $c_\alpha \geq 0$  implies that all  $c_\alpha = 0$ . For example, any  $\mathbb{R}$ -linearly independent set is positively independent. If  $\alpha \in \mathcal{E}$  is such that  $\langle \alpha, \alpha \rangle = 1$ , we set  $\alpha^\vee = 2\alpha$  and we define

$$s_\alpha : \mathcal{E} \longrightarrow \mathcal{E} \\ v \longmapsto v - \langle v, \alpha^\vee \rangle \alpha.$$

Then  $s_\alpha$  is an orthogonal reflection (with respect to  $\langle \cdot, \cdot \rangle$ ). Let

$$(5) \quad \text{COS} = \{\cos(\pi/m) \mid m \in \mathbb{N}_{\geq 2}\} \cup \mathbb{R}_{\geq 1}.$$

Assume that  $\Pi$  is a subset of  $\mathcal{E}$  with the following properties (i)–(iii):

- (i)  $\Pi$  is positively independent.
- (ii) For all  $\alpha \in \Pi$ ,  $\langle \alpha, \alpha \rangle = 1$ .
- (iii) For all  $\alpha, \beta \in \Pi$  with  $\alpha \neq \beta$ , one has  $\langle \alpha, \beta \rangle \in -\text{COS}$ .

Let  $S := \{s_\alpha \mid \alpha \in \Pi\}$ , let  $W$  be the subgroup of  $O(\mathcal{E}, \langle \cdot, \cdot \rangle)$  generated by  $S$ ,

$$\Phi := \{w(\alpha) \mid w \in W \text{ and } \alpha \in \Pi\}, \quad \Phi_+ = \Phi \cap \left( \sum_{\alpha \in \Pi} \mathbb{R}_{\geq 0} \alpha \right).$$

Then  $(W, S)$  is a Coxeter system, in which the order  $m_{s_\alpha, s_\beta}$  of the product  $s_\alpha s_\beta$  for  $\alpha, \beta \in \Pi$  is given by

$$(6) \quad m_{s_\alpha, s_\beta} = \begin{cases} m, & \text{if } \langle \alpha, \beta \rangle = -\cos \frac{\pi}{m}, \quad m \in \mathbb{N}_{\geq 1} \\ \infty, & \text{if } \langle \alpha, \beta \rangle \leq -1. \end{cases}$$

One has

$$(7) \quad \Phi = \Phi_+ \dot{\cup} -\Phi_+$$

(we use  $\dot{\cup}$  to denote disjoint union throughout this paper).

When the above conditions hold, we say that  $(\Phi, \Pi)$  is a based root system in  $(\mathcal{E}, \langle \cdot, \cdot \rangle)$  with associated Coxeter system  $(W, S)$ . Every Coxeter system is isomorphic to the Coxeter system of some based root system (and even to one with  $\langle \alpha, \beta \rangle = -\cos \frac{\pi}{m_{s_\alpha, s_\beta}}$  for all  $\alpha, \beta \in \Pi$ , and with  $\Pi$  a basis of  $\mathcal{E}$ ; a based root system of this type is called a standard based root system). All the usual results for standard based root systems which we use in this paper, and their proofs, extend mutatis mutandis to the based root systems as defined above, unless additional hypotheses are indicated in our statements here (as in Lemma 6 below, for example).

Let us recall some additional facts needed for the proof of Theorem 2:

**Lemma 3.** For  $w \in W$  and  $\alpha \in \Phi_+$ , one has  $w(\alpha) \in \Phi_+$  iff  $\ell(ws_\alpha) > \ell(w)$ .

**Lemma 4.** Let  $\Delta \subseteq \Phi_+$ , let  $T' = \{s_\alpha \mid \alpha \in \Delta\}$  and let  $W'$  denote the subgroup of  $W$  generated by  $T'$ . Then  $T'$  is the set of canonical Coxeter generators of  $W'$  if and only if  $-\langle \alpha, \beta \rangle \in \text{COS}$  for all  $\alpha, \beta \in \Delta$  such that  $\alpha \neq \beta$ .

*Proof.* See [4, (4.4)] □

**Lemma 5.** Let  $W'$  be a subgroup of  $W$  generated by reflections, let  $T' = \chi(W')$  and let  $X'$  be the set of elements  $x \in W$  such that  $x$  has minimal length in  $xW'$ . Then:

- (a) Every coset in  $W/W'$  contains a unique element of  $X'$ .
- (b) An element  $x \in W$  belongs to  $X'$  if and only if  $\ell(xt) > \ell(x)$  for all  $t \in T'$ .
- (c) If  $x \in X'$ ,  $w \in W'$  and  $t \in W' \cap T$ , then  $\ell(xwt) > \ell(xw)$  iff  $\ell(wt) > \ell(w)$  iff  $\ell'(wt) > \ell'(w)$  where  $\ell'$  is the length function of  $(W', T')$ .

*Proof.* See [4, (3.4)]. □

**Lemma 6** (Brink). Let  $\gamma \in \Phi_+$ . Then one may write  $\gamma = \sum_{\alpha \in \Pi} c_\alpha \alpha$  with  $c_\alpha/2 \in \text{COS}$  for all  $\alpha \in \Pi$ . In particular, if  $c_\alpha \notin \{0, 1\}$ , then  $c_\alpha \geq \sqrt{2}$ . If  $\Pi$  is linearly independent, the  $c_\alpha$  are uniquely determined by the conditions  $\gamma = \sum_{\alpha \in \Pi} c_\alpha \alpha$  and  $c_\alpha \in \mathbb{R}$ .

*Proof.* For the standard reflection representation, for which  $\Pi$  is linearly independent, see [3, Proposition 2.1]. A quick sketch in general is as follows. One checks the statement for dihedral Coxeter systems (for which  $\Pi$  is automatically linearly independent) by direct calculations (see [4, (4.1)]). Then in general, a standard proof (loc cit) of Lemma 3 by reduction to rank two shows that there is some choice of root coefficients  $c_\alpha$  such that all  $c_\alpha$  are expressible as polynomials with non-negative integer coefficients in the (positive) root coefficients for rank two standard parabolic subgroups, and the result follows. □

**Lemma 7.** Let  $\beta \in \Pi$  and  $\alpha \in \Phi_+ \setminus \{\beta\}$ . Then

- (1)  $s_\beta(\alpha) \in \Phi_+$  and  $s_{s_\beta(\alpha)} = s_\beta s_\alpha s_\beta$ .
- (2)  $\ell(s_\beta s_\alpha s_\beta)$  is equal to  $\ell(s_\alpha) + 2$ ,  $\ell(s_\alpha)$  or  $\ell(s_\alpha) - 2$  according as whether  $\langle \alpha, \beta \rangle < 0$ ,  $\langle \alpha, \beta \rangle = 0$  or  $\langle \alpha, \beta \rangle > 0$ . If  $\langle \alpha, \beta \rangle = 0$ , then  $s_\beta s_\alpha s_\beta = s_\alpha$ .

*Proof.* Part (a) is well-known. Part (b) may be verified by direct calculation for dihedral Coxeter systems, and in general, may be reduced to the dihedral case as follows. Let  $W' := \langle s_\alpha, s_\beta \rangle$ ,  $T' = \chi(W')$  and  $l'$  be the length function of  $(W', T')$ . In case  $\langle \alpha, \beta \rangle = 0$ , then by the dihedral case,  $s_\beta s_\alpha = s_\alpha s_\beta$  and so  $\ell(s_\beta s_\alpha s_\beta) = \ell(s_\alpha)$ . In case  $\langle \alpha, \beta \rangle < 0$ , then by the dihedral case, one has  $l'(s_\beta) < l'(s_\beta s_\alpha) < l'(s_\beta s_\alpha s_\beta)$ . Hence by Lemma 5 (c), one has  $\ell(s_\beta) < \ell(s_\beta s_\alpha) < \ell(s_\beta s_\alpha s_\beta)$  and thus  $\ell(s_\beta s_\alpha s_\beta) = \ell(s_\alpha) + 2$  as required. The remaining case  $\langle \alpha, \beta \rangle > 0$  follows from (a) and the second case applied to  $\alpha' := s_\beta(\alpha)$  in place of  $\alpha$ , since  $\langle \alpha', \beta \rangle < 0$ . □

Now we begin the proof of Theorem 2 (b). We assume without loss of generality that  $(W, S)$  is the Coxeter system associated to a based root system  $(\Phi, \Pi)$  such that  $\Pi$  is linearly independent. We keep other notation as above.

Let  $\Pi_I := \{\alpha \in \Pi \mid s_\alpha \in I\}$  and  $\Pi_J := \{\alpha \in \Pi \mid s_\alpha \in J\}$ . By (6), the assumption that no element of  $I$  is conjugate to any element of  $J$  is therefore equivalent to the assertion that if  $\gamma \in \Pi_I$  and  $\delta \in \Pi_J$ , then  $\langle \gamma, \delta \rangle$  is either of the form  $\langle \gamma, \delta \rangle = -\cos \frac{\pi}{2m}$  for some  $m \in \mathbb{N}_{\geq 1}$  or satisfies  $\langle \gamma, \delta \rangle \leq -1$ . In particular,

$$(8) \quad \text{If } \gamma \in \Pi_I \text{ and } \delta \in \Pi_J, \text{ then } \langle \gamma, \delta \rangle \leq -\frac{\sqrt{2}}{2}.$$

Now, let

$$\tilde{\Pi} = \{w(\alpha) \mid w \in W_I \text{ and } \alpha \in \Pi_J\}.$$

Then  $\tilde{\Pi} \subseteq \Phi_+$  by Lemma 3, and  $\tilde{J} = \{s_\alpha \mid \alpha \in \tilde{\Pi}\}$ .

By Lemma 4 and (6), it is sufficient to show that, if  $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Pi}$  are such that  $\tilde{\alpha} \neq \tilde{\beta}$  and if  $\tilde{s} = s_{\tilde{\alpha}}$  and  $\tilde{t} = s_{\tilde{\beta}}$ , then

$$(*) \quad \begin{cases} \langle \tilde{\alpha}, \tilde{\beta} \rangle = -\cos\left(\frac{\pi}{\tilde{m}_{\tilde{s}, \tilde{t}}}\right) & \text{if } \tilde{m}_{\tilde{s}, \tilde{t}} < \infty, \\ \langle \tilde{\alpha}, \tilde{\beta} \rangle \leq -1 & \text{if } \tilde{m}_{\tilde{s}, \tilde{t}} = \infty, \end{cases}$$

For this, let  $s = \nu(\tilde{s})$ ,  $t = \nu(\tilde{t})$  and let  $x, y \in W_I$  be such that  $\tilde{s} = xsx^{-1}$  and  $\tilde{t} = yty^{-1}$ . Let  $\alpha = x^{-1}(\tilde{\alpha})$ ,  $\beta = y^{-1}(\tilde{\beta})$  and  $w = f(\tilde{s}, \tilde{t})$ . Then  $\alpha, \beta \in \Pi_J$ ,  $s = s_\alpha$ ,  $t = s_\beta$  and

$$\langle \tilde{\alpha}, \tilde{\beta} \rangle = \langle \alpha, w(\beta) \rangle.$$

Indeed, if we write  $x^{-1}y = awb$  with  $a \in W_{I \cap s^+}$  and  $b \in W_{I \cap t^+}$ , then

$$\langle \tilde{\alpha}, \tilde{\beta} \rangle = \langle x(\alpha), y(\beta) \rangle = \langle \alpha, awb(\beta) \rangle = \langle a^{-1}(\alpha), wb(\beta) \rangle = \langle \alpha, w(\beta) \rangle.$$

We shall now need the notion of the support of a positive root. If  $\delta \in \Phi_+$ , write  $\delta = \sum_{\gamma \in \Pi} c_\gamma \gamma$  with  $c_\gamma \geq 0$ : the *support*  $\text{supp}(\delta)$  of  $\delta$  is the subset of  $\Pi$  defined by  $\text{supp}(\delta) := \{\gamma \in \Pi \mid c_\gamma \neq 0\}$ . This is well-defined since we have assumed  $\Pi$  is linearly independent. We recall the following facts:

**Lemma 8.** *Let  $\delta \in \Phi_+$  and  $A := \{s_\gamma \mid \gamma \in \text{supp}(\delta)\}$ . Then  $s_\delta \in W_A$ .*

(2) *The full subgraph of the Coxeter graph of  $(W, S)$  with vertex set  $A$  is connected.*

*Proof.* We prove (a)–(b) by induction on  $l(s_\delta)$ . If  $l(s_\delta) = 1$ , then  $\delta \in \Pi$  and (a)–(b) are clear. Otherwise, write  $\delta = \sum_{\alpha \in \Pi} c_\alpha \alpha$  with all  $c_\alpha \geq 0$ . Since  $0 < 1 = \langle \alpha, \delta \rangle = \sum_{\alpha \in \Pi} c_\alpha \langle \alpha, \delta \rangle$  there is some  $\alpha \in \text{supp}(\delta)$  with  $\langle \alpha, \delta \rangle > 0$ . Note  $\alpha \neq \delta$  since  $\delta \notin \Pi$ , so  $\gamma := s_\alpha(\delta) \in \Phi_+$ . By Lemma 7,  $l(s_\gamma) = l(s_\delta) - 2$ . Let  $B := \{s_\beta \mid \beta \in \text{supp}(\delta)\}$ . By induction,  $s_\gamma \in W_B$  and the full subgraph of the Coxeter graph of  $(W, S)$  on vertex set  $B$  is

connected. Since  $\delta = s_\alpha(\gamma) = \gamma + \langle \delta, \alpha \rangle \alpha$ , we have  $\text{supp}(\delta) = \text{sup}(\gamma) \cup \{\alpha\}$  and  $A = B \cup \{s_\alpha\}$ . Since  $0 > -\langle \alpha, \delta \rangle = \langle \alpha, \gamma \rangle$ , an argument like that above shows that there is some  $\beta \in \text{supp}(\gamma)$  with  $\langle \alpha, \beta \rangle < 0$ . Therefore  $s_\alpha$  is joined by an edge in the Coxeter graph of  $(W, S)$  to  $s_\beta \in B$ , completing the inductive proof of (b). Since  $s_\delta = s_\alpha s_\gamma s_\alpha \in W_A$ , the inductive proof of (a) is also finished  $\square$

Now, let  $\Gamma$  be the unique subset of  $\Pi_I$  such that  $\text{supp}(w(\beta)) = \Gamma \cup \{\beta\}$  and set  $I_\Gamma = \{s_\gamma \mid \gamma \in \Gamma\}$ . We write

$$w(\beta) = \beta + \sum_{\gamma \in \Gamma} c_\gamma \gamma,$$

with  $c_\gamma > 0$ . In order to prove (\*), we shall need the following lemmas:

**Lemma 9.** *Let  $\gamma \in \Pi_I$ . Then:*

*If  $\gamma \in \Gamma$ , then  $c_\gamma \geq \sqrt{2}$ .*

**(b)** *If  $s_\gamma$  appears in a reduced expression for  $w$  and  $\langle \beta, \gamma^\vee \rangle \neq 0$ , then  $\gamma \in \Gamma$  and either  $c_\gamma = -\langle \beta, \gamma^\vee \rangle$  or  $c_\gamma \geq 2\sqrt{2}$ .*

*Proof.* We shall argue by induction on  $\ell(w)$ . If  $\ell(w) = 0$ , this is vacuously true. Otherwise, write  $w = xs_\delta$  where  $\delta \in \Pi_I$  and  $\ell(x) < \ell(w)$ . We have  $s_\delta(\beta) = \beta + c\delta$  where  $c := -\langle \beta, \delta^\vee \rangle$ . If  $c = 0$ , then  $w(\beta) = x(\beta)$  and the desired result follows by induction. Otherwise,  $c \geq \sqrt{2}$  and  $w(\beta) = x(\beta) + cx(\delta)$ . Note  $x(\delta) \in \Phi_+$  by Lemma 3 since  $\ell(xs_\delta) > \ell(x)$ . Using the inductive hypothesis (a)–(b) for  $x(\beta)$  and Lemma 6 for  $x(\delta)$ , one gets (a)–(b) for  $w(\beta)$  (for (b), one has to consider the cases  $\gamma = \delta$ ,  $\gamma \neq \delta$  separately, and note that if  $s_\delta$  does not appear in a reduced expression for  $x$ , then the coefficient of  $\delta$  in  $x(\delta)$  is 1).  $\square$

**Lemma 10.** *If  $I_\Gamma \subseteq s^\perp$ , then  $w = 1$ .*

*Proof.* Indeed, if  $I_\Gamma \subseteq s^\perp$ , then Lemma 8(a) implies that we have  $wtw^{-1} \in W_{\{t\} \cup (I \cap s^\perp)}$ . In other words,  $wt \in W_{\{t\} \cup (I \cap s^\perp)} w$ . But  $w$  has minimal length in  $W_{\{t\} \cup (I \cap s^\perp)} w$  by construction, so  $wt$  does not have minimal length in  $W_{\{t\} \cup (I \cap s^\perp)} wt$ . By Deodhar's Lemma, there exists  $u \in \{t\} \cup (I \cap s^\perp)$ , such that  $wt = uw$ . In other words,  $u = wtw^{-1}$  and, since no element of  $I$  is conjugate to  $t$ , we have  $u = t$  and  $wt = tw$ . So  $w \in W_{I \cap t^\perp}$  (see Lemma 1), and thus  $w = 1$  because  $w$  has minimal length in  $wW_{I \cap t^\perp}$ .  $\square$

We shall now prove (\*) by a case-by-case analysis:

• If  $s = t$  and  $w \in I$ , let us write  $w = s_\gamma$  with  $\gamma \in \Pi_I$ . Then  $\alpha = \beta$ ,  $\tilde{m}_{\bar{s}, \bar{t}} = m_{s,w}/2$  and  $w(\beta) = \alpha - \langle \alpha, \gamma^\vee \rangle \gamma$ , so

$$\langle \alpha, w(\beta) \rangle = \langle \alpha, \alpha \rangle - 2\langle \alpha, \gamma \rangle^2 = 1 - 2\cos^2\left(\frac{\pi}{m_{s,w}}\right) = -\cos\left(\frac{2\pi}{m_{s,w}}\right),$$

as required.



- If  $s = t$  and  $\ell(w) \geq 2$ , then  $\tilde{m}_{\tilde{s}, \tilde{t}} = \infty$ . First, note that

$$I_\Gamma \not\subseteq s^\perp$$

(see Lemma 10). Moreover,

$$\langle \alpha, w(\beta) \rangle = \langle \alpha, \beta \rangle + \sum_{\gamma \in \Gamma} c_\gamma \langle \alpha, \gamma \rangle = 1 + \sum_{\substack{\gamma \in \Gamma \\ s_\gamma \notin s^\perp}} c_\gamma \langle \alpha, \gamma \rangle$$

But, if  $\gamma \in \Gamma$  is such that  $s_\gamma \notin s^\perp$ , then  $c_\gamma \geq \sqrt{2}$  by Lemma 9 (a) and  $\langle \alpha, \gamma \rangle = -\cos(\pi/m_{s, s_\gamma}) \leq -\sqrt{2}/2$  by (8) (since  $\alpha \in \Pi_J$  and  $\gamma \in \Pi_I$ ). Therefore,

$$\langle \alpha, w(\beta) \rangle \leq 1 - |I_\Gamma \setminus s^\perp|.$$

So, if  $|I_\Gamma \setminus s^\perp| \geq 2$ , then  $\langle \alpha, w(\beta) \rangle \leq -1$ , as required.

So we may assume that  $I_\Gamma \setminus s^\perp = \{\gamma\}$  with  $\gamma \in \Gamma$ . Note that  $\langle \alpha, w(\beta) \rangle = 1 - c_\gamma \langle \alpha, \gamma \rangle$  and that  $s_\gamma$  appears in a reduced expression of  $w$ . By Lemma 9 (b), two cases may occur:

- If  $c_\gamma \geq 2\sqrt{2}$  then, since  $\langle \alpha, \gamma \rangle \leq -\sqrt{2}/2$  (again by the inequality (8)), we get that  $\langle \alpha, w(\beta) \rangle \leq -1$ , as required.
- If  $c_\gamma = -\langle \beta, \gamma^\vee \rangle$  then

$$\text{supp}(s_\gamma w\beta) = \text{supp}(w\beta) \setminus \{\gamma\} = (\Gamma \setminus \{\gamma\}) \cup \{\beta\}.$$

But no element of  $\{s_\delta \mid \delta \in \Gamma \setminus \{\gamma\}\}$  is connected to  $s_\beta$  in the Coxeter graph of  $(W, S)$ , so by Lemma 8 (b) we get that  $\Gamma = \{\gamma\}$ ,  $\text{supp}(s_\gamma w\beta) = \{\beta\}$  and so  $s_\gamma w\beta = \beta$ . Hence  $s_\gamma w \in W_{I_\Gamma t^\perp}$ . By Deodhar's Lemma, this can only happen if  $w = s_\gamma$ , which contradicts the fact that  $\ell(w) \geq 2$ .

- If  $s \neq t$  and  $w = 1$ , then  $\tilde{m}_{\tilde{s}, \tilde{t}} = m_{s, t}$  and

$$\langle \tilde{\alpha}, \tilde{\beta} \rangle = \langle \alpha, \beta \rangle = -\cos\left(\frac{\pi}{m_{s, t}}\right),$$

as required.

- If  $s \neq t$  and  $w \neq 1$ , then  $\tilde{m}_{\tilde{s}, \tilde{t}} = \infty$ . First, note that

$$I_\Gamma \not\subseteq s^\perp$$

(see Lemma 10). So let  $\gamma \in \Gamma$  be such that  $\langle \alpha, \gamma \rangle \neq 0$ . Then  $c_\gamma \geq \sqrt{2}$  by Lemma 9 and, by (8), we have  $\langle \alpha, \gamma \rangle \leq -\sqrt{2}/2$  (since  $\alpha \in \Pi_J$  and  $\gamma \in \Pi_I$ ). So

$$\langle \alpha, w(\beta) \rangle \leq \langle \alpha, \beta \rangle - 1 + \sum_{\gamma' \neq \gamma} c_{\gamma'} \langle \alpha, \gamma' \rangle \leq -1$$

because  $\langle \alpha, \beta \rangle \leq 0$  and  $\langle \alpha, \gamma' \rangle \leq 0$  for all  $\gamma' \in \Pi_I$ .

The proof of (b) is now complete.

Finally, let us prove (c). If  $t \in \tilde{\mathcal{J}}$  and  $w \in W_I$ , then  $l(tw) > l(w)$  since  $t \notin W_I$ . Since  $\tilde{\mathcal{J}}$  is the set of canonical generators of  $\tilde{W}$ , this implies that  $w \in W_I$  is the

(unique) element of minimal length in its coset  $\widetilde{W}w$  by Lemma 5. The conclusion now follows from Lemma 5 and the fact that  $W = \widetilde{W}W_I$ .  $\square$

**Corollary 11.** *Let  $\tilde{\ell}: \widetilde{W} \rightarrow \mathbb{N}$  denote the length function of  $(\widetilde{W}, \widetilde{J})$ . Let  $w \in \widetilde{W}$  and  $t \in J$ . Then  $\tilde{\ell}(tw) = 1 + \tilde{\ell}(w)$  if and only if  $\ell(tw) = 1 + \ell(w)$ .*

*Proof.* Let  $w \in \widetilde{W}$  and  $t \in \widetilde{W} \cap T$ . Then  $\tilde{\ell}(tw) < \tilde{\ell}(w)$  iff  $\ell(tw) < \ell(w)$ , by Lemma 5 (c). Note that  $J \subseteq \widetilde{J} \cap S \subseteq \widetilde{W} \cap T$ . Assume that  $t \in J$ . Then  $\tilde{\ell}(tw) = \tilde{\ell}(w) \pm 1$ , since  $t \in \widetilde{J}$ . Also,  $\ell(tw) = \ell(w) \pm 1$ , since  $t \in S$ . The desired conclusion follows.  $\square$

**Corollary 12.** *If  $\tilde{t}, \tilde{t}' \in \widetilde{J}$  are  $\widetilde{W}$ -conjugate, then  $\nu(\tilde{t})$  and  $\nu(\tilde{t}')$  are  $W$ -conjugate.*

We conclude this section with a result about the elements of  $\widetilde{J}$ . We set

$$\mathcal{J} = \{(x, t) \mid t \in J \text{ and } x \in X_{I \cap t^\perp}^I\}.$$

Then it follows from (2) and Lemma 1 that the map

$$(9) \quad j: \begin{array}{ccc} \mathcal{J} & \longrightarrow & \widetilde{J} \\ (x, t) & \longmapsto & xtx^{-1} \end{array}$$

is bijective.

**Proposition 13.** *Let  $(x, t) \in \mathcal{J}$ . Then:*

- (a) *For  $w \in W_I$ , one has  $wj(x, t)w^{-1} = j(x', t)$  where  $x'$  is the unique element of  $X_{I \cap t^\perp}^I$  with  $x'W_{I \cap t^\perp} = wxW_{I \cap t^\perp}$ .*
- (b) *The palindromic reduced expressions of  $xtx^{-1}$  in  $(W, S)$  are precisely the expressions  $t_n \cdots t_1 t_0 t_1 \cdots t_n$  such that  $t_n \cdots t_1$  is a reduced expression for  $x$  in  $(W_I, I)$  and  $t_0 = t$ .*

*Proof.* Part (a) is immediate from the definitions. For (b), we first recall the following result:

**Lemma 14.** *If  $r_1 \cdots r_{2m+1}$  is a reduced expression for a reflection  $t \in T$ , then  $r_1 \cdots r_m r_{m+1} r_m \cdots r_1$  is a palindromic reduced expression of  $t$ .*

*Proof.* See [4, (2.7)].  $\square$

Write  $l(xtx^{-1}) = 2m + 1$ . We have  $xtx^{-1} \in W_{I \cup \{t\}}$ , so any reduced expression  $xtx^{-1} = s_1 \cdots s_{2m+1}$  for  $xtx^{-1}$  has all  $s_i \in I \cup \{t\}$ . Note  $s_1 \cdots s_m s_{m+1} s_m \cdots s_1$  is also a reduced expression for  $xtx^{-1}$  by Lemma 14. Thus,  $t \in J$  is  $W$ -conjugate to  $s_{m+1} \in I \cup \{t\}$  and so  $s_{m+1} = t$ . Let  $t_n \cdots t_1$  be a reduced expression for  $x$ , and  $t_0 = t$ . Then  $xtx^{-1} = t_n \cdots t_1 t_0 t_1 \cdots t_n$  and the right hand side contains some reduced expression

$s_1 \cdots s_{2m+1}$  for  $xtx^{-1}$  as a subexpression. By the above, we have  $s_{m+1} = t = t_0$ , which is the only occurrence of  $t$  in  $t_n \cdots t_0 \cdots t_n$ . Hence  $s_1 \cdots s_m s_{m+1} s_m \cdots s_1$  is also a reduced expression for  $xtx^{-1}$  contained as a subexpression of  $t_n \cdots t_0 \cdots t_1$ . Let  $y = s_1 \cdots s_m \in W_I$ . Then  $xtx^{-1} = yty^{-1}$  so  $z := y^{-1}x \in C_{W_I}(t) = W_{I \cap t^\perp}$ . We have  $y = xz^{-1}$  with  $\ell(y) = m = \ell(xz^{-1}) = \ell(x) + \ell(z^{-1}) = n + \ell(z^{-1})$  and  $m \leq n$ , so  $m = n$ . This shows  $t_n \cdots t_0 \cdots t_n$  is a reduced expression for  $xtx^{-1}$ .

Since every reduced expression for  $xtx^{-1}$  has  $t$  as its middle element, it follows that this central  $t$  can never be involved in a braid move between reduced expressions for  $xtx^{-1}$ , and the conclusion of (b) is clear.  $\square$

**REMARK** - Recall that an isomorphism of Coxeter systems  $(W_1, S_1) \rightarrow (W_2, S_2)$  is a group isomorphism  $W_1 \rightarrow W_2$  inducing a bijection  $S_1 \rightarrow S_2$ . In the semidirect product decomposition  $W = \widetilde{W} \rtimes W_I$  of Theorem 1, it is clear that the induced action by conjugation of  $W_I$  on  $\widetilde{W}$  is by automorphisms of the Coxeter system  $(\widetilde{W}, \widetilde{J})$ . Moreover, the set of Coxeter generators  $S$  of  $W$  is the disjoint union of the set  $I$  of Coxeter generators of  $W_I$  and the set  $J$  of  $W_I$ -orbit representatives on  $\widetilde{J}$ .

**External semidirect products.** We now discuss the converse of Theorem 2, giving conditions which imply that an external semidirect product of Coxeter groups is a Coxeter group.

Let  $(W', I)$  and  $(\widetilde{W}, \widetilde{J})$  be Coxeter systems and  $\theta: W' \rightarrow \text{Aut}(\widetilde{W}, \widetilde{J})$  be a group homomorphism, where the right hand side is the group of automorphisms of  $(\widetilde{W}, \widetilde{J})$ . One may regard  $\theta$  as a homomorphism from  $W'$  to the automorphism group of  $\widetilde{W}$ , and form the semidirect product of groups  $W := \widetilde{W} \rtimes W'$ , with  $\widetilde{W}$  normal. We regard  $W'$  and  $\widetilde{W}$  as subgroups of  $W$  in the usual way. Thus, every element  $w$  of  $W$  has a unique expression  $w = \widetilde{w}w'$  with  $w' \in W'$  and  $\widetilde{w} \in \widetilde{W}$ . The product in  $W$  is determined by the equation  $w'\widetilde{w}w'^{-1} = \theta(w')(\widetilde{w})$  for  $w' \in W'$ ,  $\widetilde{w} \in \widetilde{W}$ .

**Theorem 15.** *Fix a set  $J$  of  $W_I$ -orbit representatives on  $\widetilde{J}$ , and set  $S := I \dot{\cup} J$ . For any  $s \in S$ , let  $s^\perp := \{r \in S \mid rs = sr\}$ . Then  $(W, S)$  is a Coxeter system iff the conditions (1) and (2) below hold:*

- (1) for all  $r, s \in J$  and  $u \in W'$  with  $r = usu^{-1}$ , one has  $r = s$  and  $u \in W'_{I \cap r^\perp}$ .
- (2) for all  $r \in J$  and  $s \in \widetilde{J}$  with  $r \neq s$  and  $rs$  of finite order, either (i) or (ii) below holds:
  - (i)  $s = utu^{-1}$  for some  $u \in W'_{I \cap r^\perp}$  and  $t \in J$  with  $t \neq r$  and  $rt$  of finite order
  - (ii)  $s = uvrvu^{-1}$  for some  $u \in W'_{I \cap r^\perp}$  and  $v \in I$  with  $rv$  of finite order greater than 2.

*Proof.* It is easy to see that  $S$  is a set of involutions generating  $W$ . No element of  $I$  is  $W$ -conjugate to an element of  $J$  (since any  $W$ -conjugate of an element of  $J$  is in  $\widetilde{W}$ ). Moreover, a simple computation shows that for  $s \in I$  and  $r \in J$ , the order of  $sr$  in  $W$  is even, equal to twice the order of  $r'r$  in  $\widetilde{W}$  where  $r' = \theta(s)(r) = srs$ .

For  $r, s \in S$ , let  $m_{r,s}$  denote the order of  $rs$ . We have  $m_{r,r} = 1$  and  $m_{r,s} = m_{s,r} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$  for all  $r \neq s$ . Let  $(U, S)$  be a Coxeter system with Coxeter matrix  $m_{r,s}$  i.e.  $U$  is a Coxeter group with  $S$  as its set of Coxeter generators, and the order of  $rs$  in  $U$  is  $m_{r,s}$  for all  $r, s \in S$ .

For any  $K \subseteq S$ , let  $U_K$  denote the standard parabolic subgroup of  $U$  generated by  $K$ . Let  $\widetilde{J}'$  denote the subset of  $U$  consisting of all products  $usu^{-1}$  in  $U$  with  $s \in J$  and  $u \in U_I$ , and let  $\widetilde{U}$  denote the subgroup of  $U$  generated by  $\widetilde{J}'$ . No element of  $I$  is conjugate in  $U$  to an element of  $J$ , since  $m_{r,s}$  is even for all  $r \in I$  and  $s \in J$ . Hence, by Theorem 2, there is a semidirect product decomposition  $U = U_I \ltimes \widetilde{U}$  with  $\widetilde{U}$  normal in  $U$ .

Since  $rs$  has the same order  $m_{r,s}$  in both  $U$  and  $W$ , for any  $r, s \in S$ , there is a group epimorphism  $\pi: U \rightarrow W$  which is the identity on  $S$ . The homomorphism  $\pi$  restricts to an isomorphism of Coxeter systems  $(U_I, I) \rightarrow (W', I)$  (which we henceforward regard as an identification) and  $\pi$  also restricts to an isomorphism of Coxeter systems  $(U_J, J) \rightarrow (\widetilde{W}_J, J)$ . Further,  $\pi$  restricts to a surjective,  $W'$ -equivariant (for the conjugation actions by  $W'$ ) group homomorphism  $\tilde{\pi}: \widetilde{U} \rightarrow \widetilde{W}$  and  $\tilde{\pi}$  restricts further to a surjective map of  $W'$ -sets  $\pi': \widetilde{J}' \rightarrow \widetilde{J}$ .

Now if  $(W, S)$  is a Coxeter system, the validity of the conditions (1) and (2) follows from Theorem 2 (b). (In this case, the map  $\tilde{\pi}$  is of course an isomorphism of Coxeter systems).

Conversely, suppose that (1) and (2) hold. It will suffice to show that  $\tilde{\pi}$  is an isomorphism of Coxeter systems. First, we show that  $\pi'$  is injective. Consider two arbitrary elements  $uru^{-1}$  and  $vsv^{-1}$  of  $\widetilde{J}'$ , with  $u, v \in W'$  and  $r, s \in J$ . Assume  $\pi(uru^{-1}) = \pi(vsv^{-1})$  i.e.  $u\pi(r)u^{-1} = v\pi(s)v^{-1}$ . Then  $\pi(r) = x\pi(s)x^{-1}$  where  $x = u^{-1}v \in W'$ . By (1),  $r = s$  and  $x \in W'_{I \cap r^\perp}$ . By the defining relations for  $(U, S)$ , it follows that  $r = xsx^{-1}$  in  $U$ , so  $uru^{-1} = vsv^{-1}$  in  $U$ . Hence  $\pi'$  is injective, and in fact bijective since we noted above that  $\pi'$  is a surjection.

Now it will suffice to show that for all distinct  $r', s' \in \widetilde{J}'$ ,  $r's'$  has the same order in  $U$  as  $\pi(r')\pi(s')$  has in  $W$ . Using the  $W'$ -equivariance of  $\tilde{\pi}$ , we may assume that  $r' = r \in J$  and  $s' = s \in \widetilde{J}'$ . Also, we may assume that  $\pi(r)\pi(s)$  has finite order  $n > 1$  in  $W$ , without loss of generality. We have by (2) that either  $\pi(s) = u\pi(t)u^{-1}$  for  $u \in W'_{I \cap r^\perp}$ ,  $t \in J$  with  $t \neq r$  and  $\pi(t)\pi(r)$  of finite order, or  $\pi(s) = uv\pi(r)vu^{-1}$  for some  $u \in W'_{I \cap r^\perp}$  and  $v \in I$  with  $v\pi(r)$  of finite order greater than 2. In the first

(resp., second) case,  $\pi(r)\pi(s) = u\pi(r)\pi(t)u^{-1}$  (resp.,  $\pi(r)\pi(s) = u\pi(r)v\pi(r)vu^{-1}$ ) and  $n$  is the order of  $\pi(r)\pi(t)$  (resp., half the order of  $\pi(r)v$ ) in  $W$ . In the first case,  $s = utu^{-1}$ . The relations of  $(U, S)$  imply that  $rs = urtu^{-1}$ , which has the same order as  $rt$  in  $U$ . In the second case,  $s = uvrvu^{-1}$  and the relations of  $(U, S)$  imply that  $rs = urvrvu^{-1}$ , which has order equal to half the order of  $rv$  in  $U$ . The definition of  $U$  implies that the order of  $rt$  (resp.,  $rv$ ) in  $U$  is the same as that of  $\pi(r)\pi(t)$  (resp.,  $\pi(r)v$ ) in  $W$  and so the order of  $rs$  in  $U$  is equal to the order  $n$  of  $\pi(r)\pi(s)$  in  $W$  in either case, completing the proof.  $\square$

REMARK - We leave open the question of whether different choices of the set  $J$  of  $W_I$ -orbit representatives satisfying the conditions in Theorem 15 would give rise to isomorphic Coxeter systems  $(W, I \cup J)$ .

Theorem 18 below is a variant of Theorem 15, providing geometric conditions for the external direct product of two Coxeter systems to be a Coxeter system, when each is attached to a root system in the same ambient real vector space and the Coxeter group attached to the first root system acts as a group of automorphisms of the second based root system. To formulate it naturally (so that it applies in the context of the proof of Theorem 2, for instance), we require the general notion of a based root system as defined in the proof of Theorem 2. The chief technical advantage of this class of root systems is explained by Lemma 16 below, which follows from the definition and previously given facts about this class (especially Lemma 3 and (6)).

**Lemma 16.** *Let  $(\Phi, \Pi)$  be a based root system in  $(\mathcal{E}, \langle, \rangle)$ , with associated Coxeter system  $(W, S)$ . Let  $W'$  be a reflection subgroup of  $(W, S)$  and set  $S' := \chi(W')$ . Let  $\Psi := \{\alpha \in \Phi \mid s_\alpha \in W'\}$  and  $\Delta := \{\alpha \in \Phi_+ \mid s_\alpha \in S'\}$ . Then  $(\Psi, \Delta)$  is a based root system in  $(\mathcal{E}, \langle, \rangle)$  with associated Coxeter system  $(W', S')$ .*

REMARK - Note that even if  $(\Phi, \Pi)$  is a standard based root system and  $S'$  is finite, the elements of  $\Delta$  need not be linearly independent, and for elements  $\alpha, \beta$  of  $\Delta$  such that  $s_\alpha s_\beta$  has infinite order, one may have  $\langle \alpha, \beta \rangle < -1$ . Thus, the lemma fails for the class of standard based root systems in two important respects.

To formulate Theorem 18, we need also the notions of automorphisms, fundamental chamber and Tits cone of a based root system. The latter two are principally of interest when the form  $\langle, \rangle$  on  $\mathcal{E}$  is non-degenerate, but our application won't require this (and non-degeneracy can always be achieved by enlarging the space  $\mathcal{E}$  and extending the form  $\langle, \rangle$ , anyway).

Let  $(\Phi, \Pi)$  be a based root system in  $(\mathcal{E}, \langle, \rangle)$ , with associated Coxeter system  $(W, S)$ . By an automorphism of  $(\Phi, \Pi)$ , we mean an element  $\theta$  of  $O(\mathcal{E}, \langle, \rangle)$  which

restricts to permutations of both  $\Pi$  and  $\Phi$ . For example, in the setting of the proof of Theorem 2,  $W_I$  acts naturally as a group of based root system automorphisms of the based root system attached by Lemma 16 to  $\widetilde{W}$ .

In general, we define the fundamental chamber of  $(W, S)$  on  $\mathcal{E}$  to be the subset  $\mathcal{C} = \mathcal{C}_{(W,S)} := \{ \rho \in \mathcal{E} \mid \langle \alpha, \rho \rangle \geq 0 \text{ for all } \alpha \in \Pi \}$  of  $\mathcal{E}$ , and we call  $\mathcal{X} = \mathcal{X}_{(W,S)} = W\mathcal{C} := \cup_{w \in W} w(\mathcal{C})$  the Tits cone. The most basic properties of  $\mathcal{C}$  and  $\mathcal{X}$  (see [2]) are recalled in the following Lemma.

- Lemma 17.** (a)  $\mathcal{X} = \{ \rho \in \mathcal{E} \mid |\{ \alpha \in \Phi_+ \mid \langle \alpha, \rho \rangle < 0 \}| < \infty \}$ . In particular,  $\mathcal{X}$  is a convex cone in  $\mathcal{E}$ .
- (b) Any  $W$ -orbit on  $X$  contains a unique element of  $\mathcal{C}$ .
- (c) For  $\alpha \in \mathcal{C}$ , the stabilizer  $W_\alpha := \{ w \in W \mid w(\alpha) = \alpha \}$  of  $\alpha$  is the standard parabolic subgroup of  $W$  generated by  $\{ s \in S \mid s(\alpha) = \alpha \}$ .

Now we may state:

**Theorem 18.** Let  $(\Psi, \Delta)$  and  $(\widetilde{\Phi}, \widetilde{\Pi})$  be two based root systems in  $(\mathcal{E}, \langle, \rangle)$  with associated Coxeter systems  $(W', I)$  and  $(\widetilde{W}, \widetilde{J})$  respectively. Let  $\mathcal{C} := \mathcal{C}_{(W', I)}$  and  $\mathcal{X} := \mathcal{X}_{(W', I)}$ . Assume that  $W'(\widetilde{\Pi}) \subseteq \widetilde{\Pi}$ . Then  $W'$  acts as a group of based root system automorphisms of  $(\widetilde{\Phi}, \widetilde{\Pi})$  and also as a group of automorphisms of the Coxeter system  $(\widetilde{W}, \widetilde{J})$ . Let  $W$  be the subgroup of  $O(\mathcal{E}, \langle, \rangle)$  generated by  $W'$  and  $\widetilde{W}$ . Then  $W = \widetilde{W} \rtimes W'$ . Under these assumptions, the following conditions are equivalent:

- (i) There is a based root system  $(\Phi, \Pi)$  with  $\Delta \subseteq \Pi \subseteq \Delta \cup \widetilde{\Pi}$  and  $\widetilde{\Pi} = W'(\Pi \setminus \Delta)$ .
- (ii)  $\Delta \cup \widetilde{\Pi}$  is positively independent and  $\widetilde{\Pi} \subseteq -\mathcal{X}$ .

Assume these conditions hold. Then  $\Pi = \Delta \dot{\cup} (\widetilde{\Pi} \cap -\mathcal{C})$  (so  $(\Phi, \Pi)$  is uniquely determined in (i)),  $\Psi \cup \widetilde{\Phi} \subseteq \Phi$ , and  $\widetilde{\Phi}_+ \subseteq -\mathcal{X}$ . Set  $S := \{ s_\alpha \mid \alpha \in \Pi \}$  and  $J = S \setminus I$ . Then  $(W, S)$  is the Coxeter system associated to the based root system  $(\Phi, \Pi)$ ,  $\widetilde{J} = \{ wsw^{-1} \mid w \in W', s \in J \}$ , and no element of  $I$  is conjugate to any element of  $J$ . The semidirect product decomposition  $W = \widetilde{W} \rtimes W'$  is that attached by Theorem 2 to the subsets  $I$  and  $J$  of  $S$ .

*Proof.* For any  $\theta \in O(\mathcal{E}, \langle, \rangle)$  and  $\alpha \in \mathcal{E}$  with  $\langle \alpha, \alpha \rangle = 1$ , one has  $\langle \theta(\alpha), \theta(\alpha) \rangle = 1$  and  $s_{\theta(\alpha)} = \theta s_\alpha \theta^{-1}$ . Assume further that  $\theta(\Pi') \subseteq \Pi'$ . Then this implies that  $\widetilde{J}$ , and hence  $\widetilde{W}$ , is stable under conjugation by  $\theta$ , and so  $\theta$  acts as an automorphism of  $(\widetilde{W}, \widetilde{J})$ . If  $\alpha \in \widetilde{\Phi}$ , we can write  $\alpha = x(\beta)$  for some  $\beta \in \widetilde{\Pi}$  and  $x \in \widetilde{W}$ . Then  $\theta(\alpha) = \theta x(\beta) = (\theta x \theta^{-1})(\theta(\beta)) \in \widetilde{\Phi}$  since  $\theta x \theta^{-1} \in \widetilde{W}$  and  $\theta(\beta) \in \widetilde{\Pi}$ . Hence  $\theta(\widetilde{\Phi}) \subseteq \widetilde{\Phi}$ . For  $\gamma \in \widetilde{\Phi}_+$ , we may write  $\gamma = \sum_{\alpha \in \widetilde{\Pi}} c_\alpha \alpha$  with all  $c_\alpha \geq 0$ . Then  $\theta(\gamma) = \sum_{\alpha \in \widetilde{\Pi}} c_\alpha \theta(\alpha) \in \widetilde{\Phi}_+$  since all  $\theta(\alpha) \in \widetilde{\Pi}$ , showing that  $\theta(\widetilde{\Phi}_+) \subseteq \widetilde{\Phi}_+$ .

The above all applies with  $\theta \in W'$ , proving that  $W'$  acts as automorphisms of  $(\widetilde{W}, \widetilde{J})$  and  $(\widetilde{\Phi}, \widetilde{\Pi})$ . In particular,  $W'$  normalizes  $\widetilde{W}$ . If  $w \in W'$ , then  $w$  permutes  $\widetilde{\Phi}_+$ . If  $w \in W' \cap \widetilde{W}$ , this implies that  $\tilde{\ell}(w) = 0$  (since  $w$  makes no element of  $\widetilde{\Phi}_+$  negative) so  $w = 1_{W'}$ . From the above, we see that  $W = W'\widetilde{W} = \widetilde{W} \rtimes W'$  as claimed. We also see that  $\Psi \cap \widetilde{\Phi} = \emptyset$ , for if  $\alpha \in \Psi \cap \widetilde{\Phi}$ , then  $s_\alpha \in W' \cap \widetilde{W} = \{1_{W'}\}$  which is a contradiction. From this, one sees further that  $\widetilde{\Phi}$  is stable under the  $W$ -action on  $\mathcal{E}$  and hence that no element of  $\Psi$  is  $W$ -conjugate to any element of  $\widetilde{\Phi}$ .

Now suppose that the assumptions of (i) hold. Since  $\Pi_+$  is positively independent, it follows that  $\Phi_+$  is positively independent, and hence so also is the subset  $\Delta \cup \widetilde{\Pi}$  of  $\Phi_+$ . Let  $\alpha \in \Pi \setminus \Delta \subseteq \widetilde{\Pi}$ . Since  $\alpha \notin \Delta$ , we have  $\langle \alpha, \beta \rangle \in -\text{COS}$  for all  $\beta \in \Delta$ . In particular,  $\langle \alpha, \beta \rangle \leq 0$  so  $\alpha \in -\mathcal{E}$ . Thus,  $\Pi \setminus \Delta \subseteq -\mathcal{E}$ . Thus,

$$\widetilde{\Pi} = W'(\Pi \setminus \Delta) \subseteq W'(-\mathcal{E}) = -\mathcal{X}.$$

Therefore  $\widetilde{\Phi}_+ \subseteq -\mathcal{X}$  also since  $\mathcal{X}$  is a convex cone. Since every  $W'$ -orbit on  $-\mathcal{X}$  contains a unique point of  $-\mathcal{E}$ ,  $\widetilde{\Pi}$  is  $W'$ -stable and  $\widetilde{\Pi} \subseteq W'(\Pi \setminus \Delta)$ , it follows using Lemma 17 (b) that  $\Pi \setminus \Delta = \widetilde{\Pi} \cap -\mathcal{E}$ . Observe also that we have  $\Psi \cup \widetilde{\Phi} \subseteq \Phi$  and so

$$W = \langle s_\alpha \mid \alpha \in \Psi \cup \widetilde{\Phi} \rangle \subseteq \langle s_\alpha \mid \alpha \in \Phi \rangle = \langle s_\alpha \mid \alpha \in \Pi \rangle \subseteq \langle s_\alpha \mid \alpha \in \Delta \cup \widetilde{\Pi} \rangle = W$$

which implies that if (i) holds, then the Coxeter system associated to  $(\Phi, \Pi)$  is  $(W, S)$  where  $S := \{s_\alpha \mid \alpha \in \Pi\}$ .

Now suppose that the assumptions of (ii) hold. Set  $\Pi = \Delta \dot{\cup} (\widetilde{\Pi} \cap -\mathcal{E})$ . Clearly,  $\Delta \subseteq \Pi \subseteq \Delta \cup \widetilde{\Pi}$ . We also have  $\widetilde{\Pi} = W'(\Pi \setminus \Delta)$  since  $\widetilde{\Pi} \subseteq -\mathcal{X}$  and  $\widetilde{\Pi}$  is  $W'$ -stable. Let  $S := \{s_\alpha \mid \alpha \in \Pi\}$  and  $W''$  be the subgroup generated by  $S$ . It is clear  $W''$  contains  $W'$  and  $s_\alpha$  for  $\alpha \in \Pi \setminus \Delta$ , so it also contains  $ws_\alpha w^{-1}$  for such  $\alpha$  and all  $w \in W'$ . That is,  $W''$  contains the group generated by  $s_\beta$  for all  $\beta \in W'(\Pi \setminus \Delta) = \widetilde{\Pi}$ . So  $W'' \supseteq W'\widetilde{W} = W$ . But clearly,  $S \subseteq W$ , so  $W'' = W$ . Let  $\Phi = W\Pi$ .

Since  $\Delta \cup \widetilde{\Pi}$  is positively independent, to show that  $(\Phi, \Pi)$  is a based root system, it will suffice to show that if  $\alpha, \beta \in \Pi$  with  $\alpha \neq \beta$ , then  $c := -\langle \alpha, \beta \rangle \in \text{COS}$ . If both  $\alpha, \beta$  are in  $\Delta$ , or both are in  $\widetilde{\Pi}$ , this follows since  $(\Psi, \Delta)$  and  $(\widetilde{\Phi}, \widetilde{\Pi})$  are based root systems. The remaining case is that, say,  $\alpha \in \Delta$  and  $\beta \in \widetilde{\Pi}$ . We show that in this case,  $c \in \text{COS}' := \{-\cos \pi/2m \mid m \in \mathbb{N}_{\geq 1}\} \cup \mathbb{R}_{\geq 1}$ . We have  $c \geq 0$  since  $\beta \in -\mathcal{E}$ . Also,  $s_\alpha(\beta) = \beta + 2c\alpha \in \widetilde{\Pi}$ . If  $s_\alpha(\beta) = \beta$ , then  $c = 0 \in \text{COS}'$ . Otherwise,  $s_\alpha(\beta) \neq \beta$  are both in  $\widetilde{\Pi}$ , so  $d := -\langle s_\alpha(\beta), \beta \rangle \in \text{COS}$  because  $(\widetilde{\Phi}, \widetilde{\Pi})$  is a based root system. But  $d = -\langle \beta + 2c\alpha, \beta \rangle = -1 + 2c^2$ . So  $c = \sqrt{\frac{d+1}{2}}$  with  $d \in \text{COS}$ . If  $d \geq 1$ , say  $d = \cosh \lambda$  where  $\lambda \in \mathbb{R}$ , then  $c = \cosh \frac{\lambda}{2} \geq 1$  so  $c \in \text{COS}'$ . Otherwise,  $d = \cos \frac{\pi}{m}$  for some  $m \in \mathbb{N}_{\geq 2}$ , so  $c = \cos \frac{\pi}{2m} \in \text{COS}'$ . This shows that (ii) implies

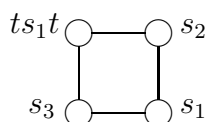
(i). Note that  $J = S \setminus I = \{s_\alpha \mid \alpha \in \Pi \setminus \Delta\}$ . The argument above also shows that no element of  $I$  is  $W$ -conjugate to any element of  $J$ .

Assuming that (i) and (ii) both hold, the remaining assertions of the Theorem follow directly from the consequences of (i)–(ii) proved above.  $\square$

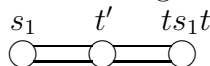
**Examples.** We shall describe in detail some examples of (internal) semidirect product decompositions of Coxeter systems  $(W, S)$ . If  $\Delta$  is a Coxeter graph, we shall denote by  $W(\Delta)$  the associated Coxeter group. In the following table, we have drawn the diagram of  $(W, S, I)$  by marking with *black* nodes the elements of  $I$ . The elements of  $\tilde{J}$  and their reduced expressions have been obtained using the bijection (9) and Proposition 13 (b). The Coxeter graph of  $(\tilde{W}, \tilde{J})$  is obtained from Theorem 2 (b), and the action of the Coxeter generators  $I$  of  $W_I$  by diagram automorphisms of the Coxeter graph of  $(\tilde{W}, \tilde{J})$  may be determined using Proposition 13 (a).

The table contains all possible triples  $(W, S, I)$  where  $W$  is a finite Coxeter group or an affine Weyl group and is irreducible and  $I$  is a proper non-empty subset of  $S$ . (For compactness, we include  $\tilde{A}_1$  as  $I_2(\infty)$ ). In degenerate cases, that is, for small values of  $|S|$ , the diagram for  $(\tilde{W}, \tilde{J})$  given in the table is not correct, but the semi-direct product decomposition is still correct (see the marks (1), (2) and (3) in the table). Here are some detailed explanations:

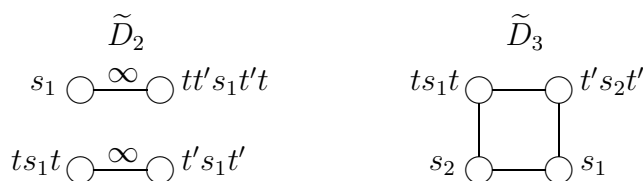
- (1) If  $W$  is of type  $\tilde{B}_3$ , then, since  $D_3 = A_3$ , we have  $\tilde{D}_3 = \tilde{A}_3$ . So the correct Coxeter graph of  $(\tilde{W}, \tilde{J})$  is a square of this form



- (2) If  $W$  is of type  $\tilde{C}_2$ , then, since  $B_2 = C_2$ , we have  $\tilde{B}_2 = \tilde{C}_2$ . So the correct Coxeter graph of  $(\tilde{W}, \tilde{J})$  is of the following form



- (3) For the diagram marked (3) in the table, there are two values of  $n$  for which the graph degenerates: if  $n = 2$ , then  $D_2 = A_1 \times A_1$  (this is a standard convention) and so  $\tilde{D}_2 = \tilde{A}_1 \times \tilde{A}_1$  and, if  $n = 3$ , then  $D_3 = A_3$  so again  $\tilde{D}_3 = \tilde{A}_3$  is a square. We obtain the following diagrams:





We next explain the notation  $t_i$  and  $t'_i$  in the Coxeter graphs marked (a), (b), (c) and (d) in the table.

- (a) Here,  $t_1 = t$  and  $t_{i+1} = s_i t_i s_i$  ( $1 \leq i \leq n-1$ ).
- (b) Here,  $t_1 = t$  and  $t_{i+1} = s_i t_i s_i$  ( $1 \leq i \leq n-1$ ),  $t'_n = s_n t_{n-1} s_n$  and  $t'_i = s_i t'_{i+1} s_i$  ( $1 \leq i \leq n-1$ ).
- (c) Here,  $t_1 = t$  and  $t_{i+1} = s_i t_i s_i$  ( $1 \leq i \leq n-1$ ),  $t'_n = t'$  and  $t'_i = s_i t'_{i+1} s_i$  ( $1 \leq i \leq n-1$ ).
- (d) Here,  $t_1 = t$  and  $t_{i+1} = s_i t_i s_i$  ( $1 \leq i \leq n-1$ ),  $t'_n = t' t_n t'$  and  $t'_i = s_i t'_{i+1} s_i$  ( $1 \leq i \leq n-1$ ).

Finally, it remains to describe the  $W_I$ -action by automorphisms of  $(\widetilde{W}, \widetilde{J})$ . This may be done by describing the automorphism of the Coxeter graph given by the simple reflections  $I$  of  $W_I$ . Each  $s \in I$  acts by conjugation on the vertex set  $\widetilde{J}$  of the Coxeter graph, and in most cases the action is clear by inspection of the graph. It may be specified by giving the induced permutation of the vertex set  $\widetilde{J}$  of the Coxeter graph. For example, in type  $\widetilde{G}_2$  with  $I = \{s_1, s_2\}$ , the action is given by  $s_1 \mapsto (t, s_1 t s_1)$  and  $s_2 \mapsto (s_1 t s_1, s_2 s_1 t s_1 s_2)$  where the image permutations are written in disjoint cycle notation. We will not explicitly list the action in the cases in which it is obvious by inspection.

The four graphs in the table (or amongst the degenerate graphs discussed above) for which the action is not obvious by inspection are again those designated (a), (b), (c) and (d). For these, the actions are as follows:

- (a) Here,  $s_i \mapsto (t_i, t_{i+1})$  for  $1 \leq i \leq n-1$ .
- (b) Here,  $s_i \mapsto (t_i, t_{i+1})(t'_i, t'_{i+1})$  for  $1 \leq i \leq n-1$ , and  $s_n \mapsto (t_{n-1}, t'_n)(t'_{n-1}, t_n)$ .
- (c) Here,  $s_i \mapsto (t_i, t_{i+1})(t'_i, t'_{i+1})$  for  $1 \leq i \leq n-1$ .
- (d) Here,  $s_i \mapsto (t_i, t_{i+1})(t'_i, t'_{i+1})$  for  $1 \leq i \leq n-1$  and  $s_n \mapsto (t_n, t'_n)$ .

The resulting permutation representation of  $W_I$  is in each case (a)–(d) isomorphic in an obvious way to a standard permutation representation of the classical Weyl group  $W_I$  as a group of permutations or signed permutations.

Type	Graph of $(W, S, I)$	Decomposition	Graph of $(\widetilde{W}, \widetilde{J})$
$I_2(2m)$		$(\mathbb{Z}/2\mathbb{Z}) \times W(I_2(m))$	
$F_4$		$\mathfrak{S}_3 \times W(D_4)$	
$B_n$ $(n \geq 2)$		$(\mathbb{Z}/2\mathbb{Z}) \times W(D_n)$ $\mathfrak{S}_n \times (\mathbb{Z}/2\mathbb{Z})^n$	
$\widetilde{G}_2$		$(\mathbb{Z}/2\mathbb{Z}) \times W(\widetilde{A}_2)$ $\mathfrak{S}_3 \times W(\widetilde{A}_2)$	
$\widetilde{F}_4$		$\mathfrak{S}_3 \times W(\widetilde{D}_4)$ $\mathfrak{S}_4 \times W(\widetilde{D}_4)$	
$\widetilde{B}_n$ $(n \geq 3)$		$(\mathbb{Z}/2\mathbb{Z}) \times W(\widetilde{D}_n)$ $W(D_n) \times (W(\widetilde{A}_1))^n$	
$\widetilde{C}_n$ $(n \geq 2)$		$(\mathbb{Z}/2\mathbb{Z}) \times W(\widetilde{B}_n)$ $\mathfrak{S}_n \times (W(\widetilde{A}_1))^n$ $W(B_n) \times (W(\widetilde{A}_1))^n$ $(\mathfrak{S}_2 \times \mathfrak{S}_2) \times W(\widetilde{D}_n)$	

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LABO. DE MATH. DE BESANÇON (CNRS: UMR 6623), UNIVERSITÉ DE FRANCHE-COMTÉ,  
16 ROUTE DE GRAY, 25030 BESANÇON CEDEX, FRANCE  
*E-mail address:* `cedric.bonnafe@univ-fcomte.fr`

DEPARTMENT OF MATHEMATICS, 255 HURLEY BUILDING, UNIVERSITY OF NOTRE DAME,  
NOTRE DAME, INDIANA, 46556-4618, U.S.A.  
*E-mail address:* `dyer.1@nd.edu`