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ALGEBRAS WITH INVOLUTION THAT BECOME HYPERBOLIC OVER THE FUNCTION FIELD OF A CONIC

ANNE QUÉGUINER-MATHIEU AND JEAN-PIERRE TIGNOL

Abstract. We study central simple algebras with involution of the first kind that become hyperbolic over the function field of the conic associated to a given quaternion algebra Q. We classify these algebras in degree 4 and give an example of such a division algebra with orthogonal involution of degree 8 that does not contain $(Q, \overline{})$, even though it contains Q and is totally decomposable into a tensor product of quaternion algebras.

Given two central simple algebras with involution (A, σ) and (B, τ) over a field F, we say that (A, σ) contains (B, τ) if A contains a σ-stable subalgebra isomorphic to B over which the involution induced by σ is conjugate to τ. By the double centralizer theorem [11, (1.5)], this is also equivalent to saying that (A, σ) is isomorphic to a tensor product $(A, \sigma) \simeq (B, \tau) \otimes (C, \gamma)$ for some central simple algebra with involution (C, γ) over F.

Let $(Q, \overline{})$ be a quaternion division algebra over F, endowed with its canonical involution. We denote by F_Q the function field of the associated conic, which is the Severi–Brauer variety of Q. Since $\overline{}$ is of symplectic type, it becomes hyperbolic over any field that splits Q, hence in particular over F_Q. From this, one may easily deduce that *any* (A, σ) *that contains* $(Q, \overline{})$ *becomes hyperbolic over* F_Q. The main theme of this paper is to investigate the reverse implication. In the case where A is split and σ is anisotropic, it is an easy consequence of the Cassels–Pfister subform theorem in the algebraic theory of quadratic forms that the converse holds, see Proposition 2.1. When A is not split, the problem is much more delicate, and comparable to the characterization of quadratic forms that become *isotropic* over F_Q, which was studied by Hoffmann, Lewis, and Van Geel [8], [9], [10]. Using an example from [10] of a 7-dimensional quadratic form over a suitable field F that becomes isotropic over F_Q, we construct in §5 a division algebra A of degree 8 with an orthogonal involution σ such that (A, σ) becomes hyperbolic over F_Q but does not contain $(Q, \overline{})$, even though A contains Q and (A, σ) decomposes into a tensor product of quaternion algebras with involution. This situation does not occur in lower degrees.

Algebras with involution that become split hyperbolic over F_Q are considered in §2. It is shown in Proposition 2.3 that their anisotropic kernel contains $(Q, \overline{})$ (if it is not trivial). This applies in particular to algebras of degree $2m$ with m odd, see Corollary 2.4. The case of algebras of degree 4 is completely elucidated in §3, using Clifford algebras for orthogonal involutions and a relative cohomological invariant of degree 3 due to Knus–Lam–Shapiro–Tignol for symplectic involutions. In the symplectic case, we classify the algebras of degree 4 that become hyperbolic

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over F_Q but do not contain $(Q, \overline{})$, see Theorem 3.6. We show in §4 that our result is equivalent to the Hoffmann–Lewis–Van Geel classification of 5-dimensional quadratic forms that become isotropic over F_Q without containing a Pfister neighbour of the norm form of Q, see Corollary 4.2. Orthogonal involutions on algebras of degree 8 are considered in §§5 and 6, using triality. In §5 we relate tensor products of quaternion algebras to quadratic forms of dimension 8 with trivial discriminant. The algebras with involution that do not decompose into tensor products of quaternion algebras with involution and become hyperbolic over F_Q are determined in §6, see Theorem 6.3. They are isotropic, and their anisotropic kernel contains $(Q, \overline{})$. Finally, we use Laurent power series in §7 to construct algebras with involution of large degree that do not contain $(Q, \overline{})$.

1. Notations and preliminary observations

We work over a base field F of characteristic different from 2, and only consider algebras with involution of the first kind. We refer the reader to [11] and to [12] for background information on central simple algebras with involution and on quadratic forms. However, we depart from the notation in [12] by using $\langle\langle a_1, \dots, a_n \rangle\rangle$ to denote the n-fold Pfister form $\otimes_{i=1}^n \langle 1, -a_i \rangle$.

If $h\colon V \times V \to D$ is a regular hermitian or skew-hermitian form on a finite-dimensional vector space V over a division algebra D, we denote by ad_h the involution on $\mathrm{End}_D(V)$ that is adjoint to h. In the particular case where D is split and h is the polar form of a quadratic form q, we also denote ad_q for ad_h. Following Becher [4], for any n-dimensional quadratic form q over F, we denote by Ad_q the split orthogonal algebra with involution $(M_n(F), \mathrm{ad}_q)$.

Recall that a central simple F-algebra with involution (A, σ) is hyperbolic if and only if it is isomorphic to $(\mathrm{End}_D V, \mathrm{ad}_h)$ for some hyperbolic hermitian or skew-hermitian form h on a vector space V over a division algebra D. In particular, A admits a hyperbolic involution if and only if the index of A divides $\frac{1}{2}\deg(A)$. If so, then A admits up to conjugation a unique orthogonal hyperbolic involution, and a unique symplectic one; see [2] or [11, 6.B].

For any central simple F-algebra with involution (A, σ) and any field extension K/F, we let $(A, \sigma)_K = (A \otimes_F K, \sigma \otimes \mathrm{Id})$. As pointed out in the introduction, we shall be mostly interested in the special case where $K = F_Q$ is the function field of the conic associated to a quaternion algebra Q, i.e. its Severi–Brauer variety. Throughout the paper, we fix the notation

$$Q = (a, b)_F$$

with a, $b \in F^\times$, and we assume Q is not split. We may then identify F_Q with the quadratic extension $F(\sqrt{at^2 + b})$ of $F(t)$, where t is an indeterminate. For any central simple F-algebra A, we let $A(t) = A \otimes_F F(t)$ and $A[t] = A \otimes_F F[t]$.

Proposition 1.1. *Let (A, σ) be a central simple F-algebra with involution. Assume σ is anisotropic. The following conditions are equivalent:*

(a) *$(A, \sigma)_{F_Q}$ is hyperbolic;*

(b) *$A(t)$ contains an element y satisfying*

$$\sigma(y) = -y \qquad \text{and} \qquad y^2 = at^2 + b. \tag{1}$$

If conditions (a) *and* (b) *hold, then $A[t]$ contains an element y_0 such that*

$$\sigma(y_0) y_0 = -(at^2 + b). \tag{2}$$

Moreover, the following conditions are equivalent:

(a') (A,σ) *contains* $(Q,\bar{\ })$;

(b') $A[t]$ *contains an element* y *satisfying* (1).

Proof. The equivalence of (a) and (b) readily follows from the description of anisotropic involutions that become hyperbolic over a quadratic extension in [2, 3.3]. If $y \in A(t)$ satisfies (1), then the version of the Cassels–Pfister theorem for algebras with involution in [19] yields an element $u \in A(t)$ such that $\sigma(u)u = 1$ and $uy \in A[t]$. Then $y_0 = uy$ satisfies (2).

If (a') holds, then A contains two skew-symmetric elements i, j such that $i^2 = a$, $j^2 = b$, and $ji = -ij$. Then $y = it + j \in A[t]$ satisfies (1), so (b') holds. Conversely, suppose (b') holds and let $y \in A[t]$ satisfy (1). We then have $\sigma(y)y = -(at^2 + b)$, hence the degree of y is 1 since σ is anisotropic. Thus, $y = \lambda t + \mu$ for some λ, $\mu \in A$. It follows from (1) that λ and μ are skew-symmetric and satisfy $\lambda^2 = a$, $\mu^2 = b$, and $\mu\lambda = -\lambda\mu$, hence they generate a σ-stable subalgebra of (A,σ) isomorphic to $(Q,\bar{\ })$. □

Proposition 1.2. *Let* (A,σ) *be a central simple* F*-algebra with involution. Assume* A *is division. If* $(A,\sigma)_{F_Q}$ *is hyperbolic, then* A *contains* Q *and* $A[t]$ *contains an element* y_1 *such that* $y_1^2 = at^2 + b$.

Proof. Since $(A,\sigma)_{F_Q}$ is hyperbolic, the algebra A_{F_Q} is not division, hence Merkurjev's index reduction theorem [13, Th. 1] shows that A contains Q, hence also two elements i, j such that $i^2 = a$, $j^2 = b$, and $ji = -ij$. Then $y_1 = it + b$ satisfies $y_1^2 = at^2 + b$. □

Thus, the algebra with involution (A,σ) in Theorem 5.2 below is such that $A(t)$ contains an element y satisfying (1), and $A[t]$ contains elements y_0, y_1 satisfying

$$\sigma(y_0)y_0 = -(at^2 + b), \qquad y_1^2 = at^2 + b,$$

but no element satisfying both equations.

For the rest of this section, we focus on the case where σ is orthogonal. We then have a discriminant $\operatorname{disc}\sigma \in F^\times/F^{\times 2}$ and a Clifford algebra $C(A,\sigma)$, see [11, §§7, 8]. Recall from [11, (8.25)] that the center of $C(A,\sigma)$ is the quadratic étale F-algebra obtained by adjoining a square root of $\operatorname{disc}\sigma$. Therefore, if $\operatorname{disc}\sigma = 1$ the algebra $C(A,\sigma)$ decomposes into a direct product of two components, which are central simple F-algebras,

$$C(A,\sigma) \simeq C_+(A,\sigma) \times C_-(A,\sigma).$$

If $\deg A \equiv 0 \bmod 4$, then the canonical involution $\underline{\sigma}$ on $C(A,\sigma)$ restricts to involutions σ_+ and σ_- on $C_+(A,\sigma)$ and $C_-(A,\sigma)$. Moreover, the tensor product $C_+(A,\sigma) \otimes_F C_-(A,\sigma)$ is Brauer-equivalent to A, see [11, (8.12), (9.14)].

Proposition 1.3. *Let* (A,σ) *be a central simple* F*-algebra with orthogonal involution and* $\deg A \equiv 0 \bmod 4$. *If* $(A,\sigma)_{F_Q}$ *is hyperbolic, then* $\operatorname{disc}\sigma = 1$ *and at least one of* $(C_+(A,\sigma),\sigma_+)$, $(C_-(A,\sigma),\sigma_-)$ *is split and isotropic.*

Proof. Since σ_{F_Q} is hyperbolic, $\operatorname{disc}\sigma$ is a square in F_Q, hence also in F since F is quadratically closed in F_Q. By [11, (8.31)], the Clifford algebra of any hyperbolic involution has a split component; therefore $C_+(A,\sigma)_{F_Q}$ or $C_-(A,\sigma)$ is split. The corresponding involution $\sigma_\pm$ is isotropic over F_Q by the main theorem in [6]. □

Since the Brauer group kernel of the scalar extension map from F to F_Q is $\{0,[Q]\}$, the description of algebras (A,σ) with σ orthogonal and $\deg A \equiv 0 \bmod 4$ such that $(A,\sigma)_{F_Q}$ is hyperbolic falls into two cases:

Case 1: $\bigl\{[C_+(A,\sigma)],[C_-(A,\sigma)]\bigr\} = \bigl\{0,[A]\bigr\}$;
Case 2: $\bigl\{[C_+(A,\sigma)],[C_-(A,\sigma)]\bigr\} = \bigl\{[Q],[A\otimes_F Q]\bigr\}$.

Following Garibaldi's definition in [7], case 1 is when $(A,\sigma)\in I^3$. Note that these two cases are not exclusive: they intersect if and only if A is Brauer-equivalent to Q; then (A,σ) becomes split hyperbolic over F_Q. This case is easily dealt with in §2. If $\deg A = 4$, the first case arises if and only if (A,σ) is hyperbolic, and the second case if and only if (A,σ) contains $(Q,\bar{\ })$, see §3.1. For algebras of degree 8, case 1 is when (A,σ) is totally decomposable, see §5; case 2 is considered in §6.

2. Algebras with involution that become split hyperbolic over F_Q

Since F_Q is the function field of the Severi–Brauer variety of Q, the Brauer group kernel of the scalar extension from F to F_Q is $\{0,[Q]\}$. Therefore, if (A,σ) is a central simple F-algebra with involution that becomes split hyperbolic over F_Q, then either A is split or A is Brauer-equivalent to Q. We consider each case separately.

Proposition 2.1. *Let (A,σ) be a split central simple F-algebra with involution.*

(1) *If σ is symplectic, then it is hyperbolic. In this case, (A,σ) contains $(Q,\bar{\ })$ if and only if $\deg A\equiv 0 \bmod 4$.*
(2) *If σ is orthogonal, then $(A,\sigma)\simeq \mathrm{Ad}_q$ for some quadratic form q. We have $(\mathrm{Ad}_q)_{F_Q}$ hyperbolic if and only if the anisotropic kernel of q is a multiple of the norm form n_Q. Assuming this condition holds, Ad_q contains $(Q,\bar{\ })$ if and only if the Witt index of q is a multiple of 4.*

Proof. (1) Since every alternating form on an F-vector space is hyperbolic, it follows that every split algebra with hyperbolic involution is hyperbolic. If ρ is an orthogonal involution on Q, then $(Q,\bar{\ })\otimes(Q,\rho)$ is a split algebra of degree 4 with symplectic involution. If $\deg A = 4m$, we have

$$(A,\sigma)\simeq (Q,\bar{\ })\otimes(Q,\rho)\otimes \mathrm{Ad}_q$$

for any quadratic form q of dimension m since all the hyperbolic involutions on A are conjugate. Conversely, if A contains Q, then the centralizer of Q in A is Brauer-equivalent to Q, hence of even degree. Therefore, $\deg A\equiv 0 \bmod 4$.

(2) By definition, $(\mathrm{Ad}_q)_{F_Q}$ is hyperbolic if and only if q_{F_Q} is hyperbolic, hence the first statement follows from [12, X(4.11)]. Let $n_Q\otimes q_0$ be the anisotropic kernel of q. If the Witt index of q is $4m$, then denoting by $\mathbb{H}$ the hyperbolic plane over F we have

$$q\simeq n_Q\otimes(q_0\perp m\mathbb{H}).$$

Since $\mathrm{Ad}_{n_Q}\simeq (Q,\bar{\ })\otimes(Q,\bar{\ })$ (see [11, (11.1)]) it follows that

$$\mathrm{Ad}_q\simeq (Q,\bar{\ })\otimes(Q,\bar{\ })\otimes \mathrm{Ad}_{q_0\perp m\mathbb{H}},$$

so Ad_q contains $(Q,\bar{\ })$. Conversely, assume Ad_q contains $(Q,\bar{\ })$ and let (A_1,σ_1) be the centralizer of $(Q,\bar{\ })$ in (A,σ). Then

$$\mathrm{Ad}_q\simeq (Q,\bar{\ })\otimes(A_1,\sigma_1)$$

hence σ_1 is symplectic and A_1 is Brauer-equivalent to Q. Therefore, there is a hermitian Q-form h_1 such that $\sigma_1 \simeq \mathrm{ad}_{h_1}$. The anisotropic kernel of h_1 has a diagonalization $\langle a_1, \dots, a_r \rangle_Q$ with $a_1, \dots, a_r \in F^\times$; letting m be the Witt index of h_1, we have

$$h_1 \simeq \langle a_1, \dots, a_r \rangle_Q \perp m\mathbb{H}_Q.$$

Consider the following quadratic form over F:

$$q_1 = \langle a_1, \dots, a_r \rangle \perp m\mathbb{H}.$$

Then $(A_1, \sigma_1) \simeq (Q, \bar{\ }) \otimes \mathrm{Ad}_{q_1}$, hence $\mathrm{Ad}_q \simeq \mathrm{Ad}_{n_Q \otimes q_1}$, and therefore q is isometric to $n_Q \otimes q_1$ up to a scalar factor. Now,

$$n_Q \otimes q_1 = n_Q \otimes \langle a_1, \dots, a_r \rangle \perp m(n_Q \otimes \mathbb{H}) = n_Q \otimes \langle a_1, \dots, a_r \rangle \perp 4m\mathbb{H},$$

and $n_Q \otimes \langle a_1, \dots, a_r \rangle$ is anisotropic since $\langle a_1, \dots, a_r \rangle_Q$ is anisotropic (see [16, Ch. 10, Th. 1.1]). Therefore, the Witt index of q is $4m$. □

For algebras that are Brauer-equivalent to Q, the result is as follows:

Proposition 2.2. *Let (A, σ) be a central simple F-algebra with involution such that A is Brauer-equivalent to Q.*

(1) *If σ is symplectic, then (A, σ) contains $(Q, \bar{\ })$ and $(A, \sigma)_{F_Q}$ is hyperbolic.*
(2) *If σ is orthogonal and $(A, \sigma)_{F_Q}$ is hyperbolic, then (A, σ) is hyperbolic and contains $(Q, \bar{\ })$.*

Proof. (1) It was already observed in the proof of Proposition 2.1(2) that every central simple algebra with symplectic involution that is Brauer-equivalent to Q contains $(Q, \bar{\ })$. Therefore, every such algebra becomes hyperbolic over F_Q.

(2) The first statement follows from the injectivity of the scalar extension map $W^-(Q, \bar{\ }) \to W(F_Q)$ proved independently in [5] and [14]. If (A, σ) is hyperbolic, then $A = M_r(F) \otimes Q$ for some even integer r, and σ is conjugate to $\tau \otimes \bar{\ }$ for any symplectic (hyperbolic) involution τ on $M_r(F)$. In particular, (A, σ) contains $(Q, \bar{\ })$. □

Focusing on the anisotropic case in the propositions above, we have:

Corollary 2.3. *Let (A, σ) be a central simple F-algebra with involution. Assume σ is anisotropic and $(A, \sigma)_{F_Q}$ is split hyperbolic. Then either*

- *A is split, σ is orthogonal, and there is a quadratic form q such that*

$$(A, \sigma) \simeq \mathrm{Ad}_{n_Q \otimes q} \simeq (Q, \bar{\ }) \otimes (Q, \bar{\ }) \otimes \mathrm{Ad}_q;$$

or

- *A is Brauer-equivalent to Q, σ is symplectic, and there is a quadratic form q such that*

$$(A, \sigma) \simeq (Q, \bar{\ }) \otimes \mathrm{Ad}_q .$$

In both cases, (A, σ) contains $(Q, \bar{\ })$.

Using the results in this section, we may describe the algebras with involution of degree 2 mod 4 that become hyperbolic over F_Q:

Corollary 2.4. *Let (A, σ) be a central simple F-algebra with involution. If $\deg A \equiv 2 \bmod 4$ and $(A, \sigma)_{F_Q}$ is hyperbolic, then A_{F_Q} is split. If moreover σ is anisotropic, then it is symplectic and there is an odd-dimensional quadratic form q over F such that $(A, \sigma) \simeq (Q, \bar{\ }) \otimes \mathrm{Ad}_q$.*

Proof. Since $\deg A \equiv 2 \bmod 4$, we have $A \simeq M_r(H)$ for some odd integer r and some quaternion algebra H over F. If H is division, A does not admit a hyperbolic involution. Therefore, the hypothesis that $(A,\sigma)_{F_Q}$ is hyperbolic implies that H_{F_Q} is split, hence $(A,\sigma)_{F_Q}$ is split hyperbolic. The last statement then readily follows from Corollary 2.3. □

In view of this corollary, we only consider central simple algebras of degree divisible by 4 in the following sections.

3. Algebras of degree 4

Throughout this section, A is a central simple algebra of degree 4 over an arbitrary field F of characteristic different from 2. Involutions on A are classified by cohomological invariants (see [11, Ch. 4]), which we use to give an explicit description of the involutions on A that become hyperbolic over F_Q.

3.1. **The orthogonal case.** This case is easy to handle using Clifford algebras.

Proposition 3.1. *Let σ be an orthogonal involution on A.*

(1) *(A,σ) is hyperbolic if and only if $\operatorname{disc}\sigma = 1$ and one of the components of $C(A,\sigma)$ is split;*
(2) *(A,σ) contains $(Q,\bar{\ })$ if and only if $\operatorname{disc}\sigma = 1$ and one of the components of $C(A,\sigma)$ is isomorphic to Q;*
(3) *A contains Q if and only if $A\otimes_F Q$ is Brauer-equivalent to a quaternion algebra.*

Proof. (1) This readily follows from [2, 2.5].

(2) If (A,σ) contains $(Q,\bar{\ })$, then there is a quaternion F-algebra Q' with canonical involution $\bar{\ }$ such that

$$(A,\sigma)\simeq (Q,\bar{\ })\otimes(Q',\bar{\ }).$$

By [11, (15.12)], this relation holds if and only if $C(A,\sigma)\simeq Q\times Q'$. This proves (2).

(3) If A contains (a copy of) Q, then the centralizer of Q in A is a quaternion F-algebra Q', and we have

$$A\simeq Q\otimes_F Q'.$$

This relation holds if and only if $A\otimes_F Q$ is Brauer-equivalent to Q', hence (3) follows. □

Corollary 3.2. *Let σ be an orthogonal involution on A. Then $(A,\sigma)_{F_Q}$ is hyperbolic if and only if (A,σ) is hyperbolic or contains $(Q,\bar{\ })$.*

Proof. As noticed in the introduction, any (A,σ) containing $(Q,\bar{\ })$ is hyperbolic over F_Q, so we only have to prove the converse. Assume $(A,\sigma)_{F_Q}$ hyperbolic. Since F is quadratically closed in F_Q, the discriminant of σ is trivial, so $C(A,\sigma)$ is a direct product of two quaternion algebras. At least one of these quaternion algebras splits over F_Q by Proposition 3.1(1), hence that component must be either split or isomorphic to Q. It follows that (A,σ) is either hyperbolic or contains $(Q,\bar{\ })$, by Proposition 3.1. □

Note that a central simple algebra of degree 4 with hyperbolic involution does not necessarily contain $(Q,\bar{\ })$, even if it contains Q: if Q' is a quaternion F-algebra such that $Q\otimes_F Q'$ has index 2, then $(A,\sigma)=(M_2(F),\bar{\ })\otimes(Q',\bar{\ })$ is hyperbolic over F and satisfies $C(A,\sigma)\simeq M_2(F)\times Q'$ by [11, (15.12)]. Therefore, Proposition 3.1(2) shows that (A, σ) does not contain $(Q, \overline{})$, even though Proposition 3.1(3) shows that A contains Q. This situation does not occur in the symplectic case, in view of Proposition 3.3 below.

3.2. **The symplectic case.** Symplectic involutions on A are classified up to conjugation by a relative invariant Δ with values in the Galois cohomology group $H^3(F, \mu_2)$, see[1] [11, (16.9)]. We denote by $[A]$ the Brauer class of A, viewed as an element of $H^2(F, \mu_2)$, and for $\lambda \in F^\times$ we denote by (λ) the square class of λ, viewed as an element in $H^1(F, \mu_2)$. Using the invariant Δ, we show:

Proposition 3.3. *Let σ be a symplectic involution on A. If A contains Q and $(A, \sigma)_{F_Q}$ is hyperbolic, then (A, σ) contains $(Q, \overline{})$.*

Proof. Since A contains Q, it decomposes as $A = Q \otimes Q'$ for some quaternion algebra $Q' = (a', b')_F$. Let τ' be an orthogonal involution on Q' of discriminant a'. The involution $\tau = \overline{} \otimes \tau'$ on $A = Q \otimes Q'$ is of symplectic type, and clearly hyperbolic over F_Q since (A, τ) contains $(Q, \overline{})$. Similarly, for every invertible $y \in \mathrm{Sym}(Q', \tau')$, the involution

$$\tau_y = \mathrm{Int}(1 \otimes y) \circ \tau = \overline{} \otimes (\mathrm{Int}(y) \circ \tau)$$

is such that (A, τ_y) contains $(Q, \overline{})$. We prove below that σ is conjugate to τ_y for a suitable y. By [11, (16.18)], the relative discriminant $\Delta_\tau(\tau')$ is given by

$$\Delta_\tau(\tau') = (\mathrm{Nrp}_\tau(1 \otimes y)) \cup [A] \in H^3(F, \mu_2),$$

where Nrp_τ is the Pfaffian norm, as defined in [11, (2.9)]. Since this relative discriminant classifies symplectic involutions on A up to conjugation, it suffices to show:

Lemma 3.4. *There exists an invertible element $y \in \mathrm{Sym}(Q', \tau')$ such that*

$$\Delta_\tau(\sigma) = (\mathrm{Nrp}_\tau(1 \otimes y)) \cup [A].$$

Proof. The involution σ can be written as $\sigma = \mathrm{Int}(x) \circ \tau$ for some $x \in \mathrm{Sym}(A, \sigma)$, hence we already have

$$\Delta_\tau(\sigma) = (\mathrm{Nrp}_\tau(x)) \cup [A], \tag{3}$$

and we want to prove we can substitute for x some element $1 \otimes y$ with $y \in \mathrm{Sym}(Q', \tau')$. Since both σ and τ become hyperbolic over F_Q, the relative discriminant $\Delta_\tau(\sigma)$ is killed by F_Q, hence $(\mathrm{Nrp}_\tau(x)) \cup [Q'] = (\lambda) \cup [Q]$ for some $\lambda \in F^\times$. By the common slot lemma [1, Lemma 1.7], we may even assume

$$(\mathrm{Nrp}_\tau(x)) \cup [Q'] = (\lambda) \cup [Q'] = (\lambda) \cup [Q], \tag{4}$$

from which we deduce

$$(\lambda) \cup ([Q] + [Q']) = 0, \quad \text{and} \quad (\lambda\,\mathrm{Nrp}_\tau(x)) \cup [Q'] = 0. \tag{5}$$

Hence the relative discriminant of σ is

$$\Delta_\tau(\sigma) = (\mathrm{Nrp}_\tau(x)) \cup ([Q] + [Q']) = (\lambda\,\mathrm{Nrp}_\tau(x)) \cup ([Q] + [Q']) = (\lambda\,\mathrm{Nrp}_\tau(x)) \cup [Q].$$

Moreover, the quadratic space $(\mathrm{Sym}(A, \sigma), \mathrm{Nrp}_\tau)$ is an Albert quadratic space for the biquaternion algebra A by [11, (16.8)]. Hence, its Clifford invariant is $e_2(\mathrm{Nrp}_\tau) = [A] = [Q] + [Q']$. We deduce from (5) that the quadratic form $\langle\langle \lambda \rangle\rangle \otimes \mathrm{Nrp}_\tau$ has trivial Arason invariant e_3, and hence is hyperbolic by the Arason–Pfister

[1] The discussion in [11] is in terms of the 3-fold Pfister form j whose Arason invariant is Δ.

Hauptsatz. So there exists an element $x' \in \mathrm{Sym}(A,\sigma)$ such that $\lambda \mathrm{Nrp}_\tau(x) = \mathrm{Nrp}_\tau(x')$, and we have proven

$$\Delta_\tau(\sigma) = (\mathrm{Nrp}_\tau(x')) \cup [Q] \quad \text{and} \quad (\mathrm{Nrp}_\tau(x')) \cup [Q'] = 0 \tag{6}$$

for some $x' \in \mathrm{Sym}(A,\sigma)$.

Let us now pick a pure quaternion $i' \in Q'$ such that $i'^2 = a'$ and $\tau' = \mathrm{Int}(i') \circ \bar{\ }$. Denoting by Q^0 the vector space of pure quaternions in Q, we have

$$\mathrm{Sym}(A,\tau) = (1 \otimes \mathrm{Sym}(Q',\tau')) \oplus (Q^0 \otimes i').$$

Hence x' can be written as $x' = x_0 + 1 \otimes \xi_1 + \xi_2 \otimes i'$ for some $x_0 \in F$, some pure quaternion $\xi_1 \in Q'^0 \cap \mathrm{Sym}(Q',\tau')$ and some $\xi_2 \in Q^0$. The pure quaternion ξ_1 being τ'-symmetric, it anticommutes with i', and changing of quaternionic basis if necessary, we may assume that $\xi_1^2 = b'x_1^2$ for some $x_1 \in F$. Similarly, we may assume that $\xi_2^2 = ax_2^2$ for some $x_2 \in F$. Note that we allow $x_i = 0$ for $i = 0$, 1, and 2 since some term might be zero in the decomposition of x'. We then have

$$\mathrm{Nrp}_\tau(x') = x_0^2 - b'x_1^2 - aa'x_2^2, \tag{7}$$

and the following lemma finishes the proof:

Lemma 3.5. *There exists z_0, z_1, y_0, y_1 and y_2 such that* $\mathrm{Nrp}_\tau(x')(z_0^2 - az_1^2) = y_0^2 - b'y_1^2 + a'b'y_2^2 \neq 0$.

Indeed, if we let $y = y_0 + y_1 j' + y_2 i'j'$, where j' is a pure quaternion in Q' such that $j'^2 = b'$ and $i'j' = -j'i'$, we have $\mathrm{Nrp}_\tau(1 \otimes y) = Nrd_{Q'}(y) = y_0^2 - b'y_1^2 + a'b'y_2^2$ (see [11, (2.11)]). Hence,

$$\begin{aligned}\Delta_\tau(\sigma) &= (\mathrm{Nrp}_\tau(x')) \cup [(a,b)_F] = (\mathrm{Nrp}_\tau(x')(z_0^2 - az_1^2)) \cup [(a,b)_F] \\ &= (\mathrm{Nrd}_{Q'}(y)) \cup [A] = (\mathrm{Nrp}_\tau(1 \otimes y)) \cup [A], \quad \text{with } y \in \mathrm{Sym}(Q',\tau').\end{aligned}$$

Proof of Lemma 3.5. If $\langle 1, -b', a'b' \rangle$ is isotropic, then it is universal, hence represents $\mathrm{Nrp}_\tau(x')$ and we are done. Otherwise, denote $\mu = \mathrm{Nrp}_\tau(x')$; by (7), we have $(\mu) \cup [(aa',b')_F] = 0$, and combining with (6), we get $(\mu) \cup [(a,b')] = 0$. In terms of quadratic forms, this means that the 3-fold Pfister form

$$\langle\langle \mu, a, b' \rangle\rangle = \langle 1, -\mu, -a, \mu b' \rangle \perp \langle \mu a, ab', -\mu ab' \rangle \perp \langle -b' \rangle \tag{8}$$

is hyperbolic. On the other hand, by (7), the quadratic form $\langle 1, -b', -aa', -\mu \rangle$ is isotropic. Multiplying by $(-ab'\mu)$, we get that $-a'b'\mu$ is represented by the form $\langle \mu a, ab', -\mu ab' \rangle$. Hence, in view of (8), the quadratic form $\langle 1, -\mu, -a, \mu b', -a'b'\mu \rangle$, which is a 5-dimensional subform of $\langle\langle \mu, a, b' \rangle\rangle$, is necessarily isotropic. Since the quadratic forms $\langle 1, -a \rangle$ and $\langle 1, -b', a'b' \rangle$ are anisotropic, this completes the proof. □

Using Proposition 3.3, we may characterize the symplectic involutions on A that become hyperbolic over F_Q:

Theorem 3.6. *Let σ be a symplectic involution on A. If $(A,\sigma)_{F_Q}$ is hyperbolic, then either*

(a) *$(A,\sigma) \simeq (Q, \bar{\ }) \otimes (Q',\rho)$ for some quaternion F-algebra with orthogonal involution (Q',ρ), or*

(b) *$(A,\sigma) \simeq \mathrm{Ad}_{\langle\langle \lambda \rangle\rangle} \otimes (Q', \bar{\ })$ for some quaternion F-algebra Q' and some $\lambda \in F^\times$ with the following properties: $Q \otimes_F Q'$ is a division algebra and the norm forms n_Q, $n_{Q'}$ satisfy $\langle\langle \lambda \rangle\rangle \cdot n_Q \simeq \langle\langle \lambda \rangle\rangle \cdot n_{Q'}$.*

Conversely, if (A,σ) is of either type above, then $(A,\sigma)_{F_Q}$ is hyperbolic.

Note that (a) and (b) are mutually exclusive since in the first case $A\otimes_F Q$ has Schur index 1 or 2, whereas it has index 4 in case (b).

Proof. Suppose (A,σ) is not as in case (a) and $(A,\sigma)_{F_Q}$ is hyperbolic. Then Proposition 3.3 shows that A does not contain Q. In particular, A is not division, by Proposition 1.2, so $A\simeq M_2(Q')$ for some quaternion F-algebra Q', and $Q\otimes_F Q'$ is a division algebra by Proposition 3.1(3). As observed in the proof of Proposition 2.1(2), (A,σ) contains $(Q',\overline{})$, hence

$$(A,\sigma)\simeq \mathrm{Ad}_{\langle\langle\lambda\rangle\rangle}\otimes(Q',\overline{})\qquad \text{for some } \lambda\in F^\times.$$

The algebra A carries a hyperbolic symplectic involution τ, and we have by [11, (16.21)]

$$\Delta_\tau(\sigma)=(\lambda)\cup[Q'].$$

Since $(A,\sigma)_{F_Q}$ is hyperbolic, this invariant vanishes over F_Q. Hence, by [12, X(4.11)] and Arason's common slot lemma we may assume as in the proof of Lemma 3.4 that $(\lambda)\cup[Q']=(\lambda)\cup[Q]$. Therefore, $\langle\langle\lambda\rangle\rangle\cdot n_{Q'}\simeq\langle\langle\lambda\rangle\rangle\cdot n_Q$, which shows that (A,σ) is as in case (b).

It is clear that $(A,\sigma)_{F_Q}$ is hyperbolic in case (a). In case (b) the algebra A carries a hyperbolic symplectic involution τ and $\Delta_\tau(\sigma)=(\lambda)\cup[Q']$ vanishes over F_Q. Therefore, $(A,\sigma)_{F_Q}$ is hyperbolic. □

4. F_Q-minimal quadratic forms of dimension 5

A quadratic form φ over F is called *F_Q-minimal* if φ_{F_Q} is isotropic and ψ_{F_Q} is anisotropic for every proper subform $\psi\subset\varphi$. In this section, we show that Theorem 3.6 can be used to recover (and is in fact equivalent to) the description of F_Q-minimal forms of dimension 5 due to Hoffmann, Lewis, and Van Geel [9, Prop. 4.1].

A general procedure to construct central simple algebras with involution that become hyperbolic over F_Q uses Clifford algebras. Recall from [12, V(1.9), V(2.4)] that for any quadratic form φ of odd dimension $2m+1$ over F the even Clifford algebra $C_0(\varphi)$ is central simple over F of degree 2^m. It carries a canonical involution τ_0, which is the restriction of the involution on the full Clifford algebra that leaves invariant every vector in the underlying vector space of φ. The involution τ_0 is orthogonal if $m\equiv 0$ or 3 mod 4 and symplectic otherwise, see [11, (8.4)].

Proposition 4.1. *Let φ be a quadratic form of odd dimension over F.*

1. *If φ_{F_Q} is isotropic, then $(C_0(\varphi),\tau_0)_{F_Q}$ is hyperbolic. The converse holds if* $\dim\varphi=5$.
2. *If φ contains a subform similar to $\langle 1,-a,-b\rangle$, then $(C_0(\varphi),\tau_0)$ contains $(Q,\overline{})$. The converse holds if* $\dim\varphi=5$.

Proof. (1) The first statement readily follows from [11, (8.5)] and the second from [11, (15.21)].

(2) Suppose the underlying vector space of φ contains orthogonal vectors e_0, e_1, e_2 satisfying for some $\lambda\in F^\times$

$$\varphi(e_0)=\lambda,\qquad \varphi(e_1)=-\lambda a,\qquad \varphi(e_2)=-\lambda b.$$

Then the products e_0e_1 and e_0e_2 generate a τ_0-stable subalgebra of $C_0(\varphi)$ isomorphic to $(Q,\overline{})$.

For the rest of the proof, suppose $\dim\varphi = 5$ and $(C_0(\varphi),\tau_0)$ contains $(Q,\bar{\ })$. The centralizer of $(Q,\bar{\ })$ is a quaternion algebra with orthogonal involution (Q',ρ) such that

$$(C_0(\varphi),\tau_0) = (Q,\bar{\ })\otimes(Q',\rho).$$

Let $V\subset \mathrm{Sym}(C_0(\varphi),\tau_0)$ be the vector space of τ_0-symmetric elements of trace 0. The map $x\mapsto x^2$ defines a quadratic form $s\colon V\to F$ that is similar to φ by the equivalence $\mathsf{B}_2\equiv\mathsf{C}_2$, see [11, (15.16)]. Let $y\in Q'$ be a ρ-skew-symmetric unit and let $Q^0\subset Q$ be the vector space of pure quaternions. The restriction of s to the subspace $Q^0\otimes y\subset V$ is similar to $\langle 1,-a,-b\rangle$, hence the proof is complete. □

Corollary 4.2 (Hoffmann–Lewis–Van Geel [9, Prop. 4.1])**.** *A 5-dimensional quadratic form φ over F is F_Q-minimal if and only if the following conditions hold:*

(a) *φ is similar to a Pfister neighbour of the 3-fold Pfister form $\langle\langle a,b,\lambda\rangle\rangle$ for some $\lambda\in F^\times$, and*
(b) *$C_0(\varphi)\simeq M_2(Q')$ for some quaternion F-algebra Q' such that $Q\otimes_F Q'$ is a division algebra.*

Proof. Proposition 4.1 shows that φ is F_Q-minimal if and only if $(C_0(\varphi),\tau_0)$ is hyperbolic but $(C_0(\varphi),\tau_0)$ does not contain $(Q,\bar{\ })$. By Theorem 3.6, this condition is equivalent to

$$(C_0(\varphi),\tau_0)\simeq \mathrm{Ad}_{\langle\langle\lambda\rangle\rangle}\otimes(Q',\bar{\ }) \tag{9}$$

for some quaternion F-algebra $Q'=(a',b')_F$ and some $\lambda\in F^\times$ with $Q\otimes Q'$ a division algebra and $\langle\langle a,b,\lambda\rangle\rangle\simeq\langle\langle a',b',\lambda\rangle\rangle$. It follows from the isomorphism (9) that φ is similar to a Pfister neighbour of $\langle\langle a',b',\lambda\rangle\rangle$, by [11, p. 271]. Thus, (a) and (b) hold if φ is F_Q-minimal. Conversely, if $C_0(\varphi)\simeq M_2(Q')$ for some quaternion F-algebra $Q'=(a',b')_F$, then as observed in the proof of Proposition 2.1(2) we have

$$(C_0(\varphi),\tau_0)\simeq \mathrm{Ad}_{\langle\langle\mu\rangle\rangle}\otimes(Q',\bar{\ })\qquad\text{for some }\mu\in F^\times.$$

It then follows from [11, p. 271] that φ is similar to a Pfister neighbour of $\langle\langle a',b',\mu\rangle\rangle$. If (a) holds, then $\langle\langle a,b,\lambda\rangle\rangle\simeq\langle\langle a',b',\mu\rangle\rangle$ and by the common slot lemma we may assume $\lambda=\mu$. Thus, φ is F_Q-minimal if (a) and (b) hold. □

5. Totally decomposable orthogonal involutions of degree 8

In this section, A denotes a central simple F-algebra of degree 8 and σ is an orthogonal involution on A. The algebra with involution (A,σ) is called *totally decomposable* if there are σ-stable quaternion subalgebras Q_1, Q_2, Q_3 in A such that $A=Q_1\otimes Q_2\otimes Q_3$. Denoting by σ_i the restriction of σ to Q_i for $i=1$, 2, 3, we then have

$$(A,\sigma)=(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(Q_3,\sigma_3).$$

The totally decomposable algebras of degree 8 are characterized by the property that $\operatorname{disc}\sigma=1$ and one of the components of the Clifford algebra $C(A,\sigma)$ is split, see [11, (42.11)]. Proposition 5.1 below shows how to use this criterion to relate totally decomposable algebras to quadratic forms.

Recall that for any quadratic form φ of dimension 8 with $\operatorname{disc}\varphi=1$ the even Clifford algebra decomposes into a direct product of central simple F-algebras of degree 8,

$$C_0(\varphi)\simeq C_+(\varphi)\times C_-(\varphi).$$

The canonical involution τ_0 on $C_0(\varphi)$ restricts to orthogonal involutions τ_+, τ_- on $C_+(\varphi)$ and $C_-(\varphi)$, and we have

$$(C_+(\varphi), \tau_+) \simeq (C_-(\varphi), \tau_-).$$

It is easily checked that $(C_+(\varphi), \tau_+)$ is totally decomposable.

Proposition 5.1. *For every central simple algebra of degree* 8 *with totally decomposable orthogonal involution* (A, σ)*, there is a quadratic form* φ *with* $\dim \varphi = 8$ *and* $\operatorname{disc} \sigma = 1$ *such that*

$$(A, \sigma) \simeq (C_+(\varphi), \tau_+).$$

The form φ *is uniquely determined by* (A, σ) *up to similarity. Moreover,*

- *the algebra* A *is split if and only if* φ *is a multiple of a* 3*-fold Pfister form; in that case* $(A, \sigma) \simeq \operatorname{Ad}_\varphi$*;*
- *the algebra* (A, σ) *contains* $(Q, \overline{})$ *if and only if* φ *contains a subform similar to* $\langle 1, -a, -b\rangle$*.*

Furthermore, the following conditions are equivalent:

(a) (A, σ) *is isotropic;*
(b) (A, σ) *is hyperbolic;*
(c) φ *is isotropic.*

Proof. Since (A, σ) is totally decomposable, it follows from [11, (42.11)] that one of the components $C_+(A, \sigma)$ of the Clifford algebra is split. The canonical involution σ_+ on $C_+(A, \sigma)$ is orthogonal, hence there is a quadratic form φ of dimension 8 such that

$$(C_+(A, \sigma), \sigma_+) \simeq \operatorname{Ad}_\varphi.$$

By triality (see [11, (42.3)]) we have

$$(C_0(\varphi), \tau_0) \simeq (A, \sigma) \times (A, \sigma).$$

Therefore, $\operatorname{disc} \varphi = 1$ and $(A, \sigma) \simeq (C_+(\varphi), \tau_+)$. Conversely, triality also shows that if $(A, \sigma) \simeq (C_+(\varphi), \tau_+)$, then the canonical involution $\underline{\sigma}$ on $C(A, \sigma)$ satisfies

$$(C(A, \sigma), \underline{\sigma}) \simeq \operatorname{Ad}_\varphi \times (A, \sigma),$$

hence the form φ is uniquely determined up to similarity.

The Clifford algebra of φ splits if and only if φ is a multiple of a 3-fold Pfister form, by [16, Ch. 2, Th. 14.4] and [12, X(5.6)]. When that condition holds we have $(A, \sigma) \simeq \operatorname{Ad}_\varphi$ by [11, (35.1)].

If φ contains a multiple of $\langle 1, -a, -b\rangle$, then the same argument as in the proof of Proposition 4.1(2) shows that $(C_0(\varphi), \tau_0)$ contains $(Q, \overline{})$. Projecting on each component, it follows that $(C_+(\varphi), \tau_+)$ contains $(Q, \overline{})$. Conversely, if (A, σ) contains $(Q, \overline{})$, then we have

$$(A, \sigma) = (Q, \overline{}) \otimes (A_1, \sigma_1) \tag{10}$$

for some central simple algebra with symplectic involution (A_1, σ_1) of degree 4. By [11, (15.19)] there is a 5-dimensional quadratic form ψ such that $\operatorname{disc} \psi = 1$ and $(A_1, \sigma_1) \simeq (C_0(\psi), \tau_0)$. Letting τ_0' be the canonical involution on $C_0(\langle ab, -a, -b\rangle)$, we may rewrite (10) as

$$(A, \sigma) \simeq (C_0(\langle ab, -a, -b\rangle), \tau_0') \otimes (C_0(\psi), \tau_0). \tag{11}$$

In view of the canonical embedding

$$C_0(\langle ab, -a, -b\rangle) \otimes_F C_0(\psi) \hookrightarrow C_0(\langle ab, -a, -b\rangle \perp \psi),$$

which is compatible with the canonical involutions, (11) yields

$$(A,\sigma)\simeq (C_+(\langle ab,-a,-b\rangle\perp\psi),\tau_+).$$

Uniqueness of φ shows that φ is similar to $\langle ab,-a,-b\rangle\perp\psi$, hence it contains a subform similar to $\langle 1,-a,-b\rangle$.

The equivalence of (a), (b), (c) is clear if A is split, since then $(A,\sigma)\simeq \mathrm{Ad}_\varphi$ and φ is a 3-fold Pfister form, hence it is isotropic if and only if it is hyperbolic. For the rest of the proof, we may thus assume A is not split. If (A,σ) is hyperbolic, then it follows from [6] that the split component of $(C(A,\sigma),\underline{\sigma})$ is isotropic, hence (b)⇒(c). Conversely, if (c) holds, then [11, (8.5)] shows that $(C_0(\varphi),\tau_0)$ is hyperbolic, hence $(C_+(\varphi),\tau_+)$ also is hyperbolic, proving (c)⇒(b). The equivalence of (a) and (b) readily follows from [3, Prop. 2.10]. □

To give an example of a division F-algebra of degree 8 with a totally decomposable orthogonal involution that is hyperbolic over F_Q but does not contain $(Q,\bar{\ })$, we use an example of F_Q-minimal quadratic form of dimension 7 due to Hoffmann and Van Geel [10]. For the rest of this section, we fix the following notation: $F_1=F_0(t,u)$ is the function field in two independent indeterminates over an arbitrary field F_0 of characteristic different from 2, and $F=F_1((a))((b))$ is the iterated Laurent series field in two indeterminates a, b. In accordance with our running notation, Q denotes the quaternion algebra $(a,b)_F$. Let

$$\varphi_0=\langle 1+t,u\rangle\perp\langle -a\rangle\langle 1,u\rangle\perp\langle -b\rangle\langle 1,t+u\rangle\perp\langle ab\rangle\langle t\rangle$$

(see [10, p. 43]) and

$$\varphi=\varphi_0\perp\langle ab\rangle\langle t(1+t)(t+u)\rangle,$$

so $\dim\varphi=8$ and $\operatorname{disc}\varphi=1$. Let also

$$(A,\sigma)=(C_+(\varphi),\tau_+)\quad (=(C_0(\varphi_0),\tau_0)),$$

a central simple F-algebra of degree 8 with a totally decomposable involution.

Theorem 5.2. *The algebra with involution (A,σ) does not contain $(Q,\bar{\ })$, yet $(A,\sigma)_{F_Q}$ is hyperbolic. Moreover, A is a division algebra.*

Proof. Since $\langle 1,t\rangle\simeq\langle 1+t,t(1+t)\rangle$ and $\langle t,u\rangle\simeq\langle t+u,tu(t+u)\rangle$, we have

$$\varphi_0\perp\langle t(1+t),-at,-btu(t+u),ab,abu\rangle\simeq\langle 1,t,u\rangle\langle\langle a,b\rangle\rangle. \tag{12}$$

Since the right side is hyperbolic over F_Q, it follows that $(\varphi_0)_{F_Q}$ is isotropic, and therefore $(A,\sigma)_{F_Q}$ is hyperbolic by Proposition 4.1(1) or 5.1.

To show (A,σ) does not contain $(Q,\bar{\ })$, we prove φ does not contain any subform similar to $\langle 1,-a,-b\rangle$. As in [10, p. 43], we decompose φ as

$$\varphi=\alpha\perp\langle -a\rangle\beta\perp\langle -b\rangle\gamma\perp\langle ab\rangle\delta, \tag{13}$$

where $\alpha=\langle 1+t,u\rangle$, $\beta=\langle 1,u\rangle$, $\gamma=\langle 1,t+u\rangle$, and $\delta=\langle t,t(1+t)(t+u)\rangle$. By Springer's theorem, the isometry classes of α, β, γ, and δ over F_1 are uniquely determined by φ. If three of those quadratic forms represent a common value λ, then φ contains a 3-dimensional subform of $\langle\lambda\rangle\langle\langle a,b\rangle\rangle$, hence a subform similar to $\langle 1,-a,-b\rangle$. We claim that the converse also holds. Indeed, assume first that φ contains $\langle\lambda\rangle\langle 1,-a,-b\rangle$ for some $\lambda\in F_1^\times$. Writing $\varphi=\langle\lambda\rangle\langle 1,-a,-b\rangle\perp\varphi'$ as in (13), we get, by uniqueness of the forms α, β and γ up to isometry, that all three represent λ. Consider now the general situation, where $\lambda\in F^\times$ need not be in F_1;

modifying it by a square if necessary, we may write it as λ_0, $-a\lambda_0$, $-b\lambda_0$ or $ab\lambda_0$ for some $\lambda_0 \in F_1^\times$. The same argument as above then proves the claim.

Thus, to prove that φ does not contain any subform similar to $\langle 1,-a,-b\rangle$, we have to show that no three of the quadratic forms α, β, γ, and δ have any common value over F_1. This can be checked after some scalar extension. For instance, by [10, Lemma(4.4)(iii)], the only common value of α and β over $F_0(t)((u))$ is the square class of u. On the other hand, applying [12, VI(1.3)], one may check that γ and δ are both isomorphic over $F_0((t))((u))$ to $\langle 1,t\rangle$, which does not represent u. On the other hand, over $F_0(t)((t+u))$ the form γ only represents the square classes of 1 and $t+u$, whereas δ only represents the square classes of t and $t(1+t)(t+u)$. Therefore, γ and δ have no common value in F_1.

To complete the proof, we show A is a division algebra. Taking the Clifford invariant of each side of (12) and applying [12, V(3.13)], we obtain the following equality in the Brauer group of F:

$$[A] + [C_0(\langle t(1+t), -at, -btu(t+u), ab, abu\rangle)] = [Q].$$

The even Clifford algebra of the 5-dimensional form is easily computed:

$$C_0(\langle t(1+t), -at, -btu(t+u), ab, abu\rangle) \simeq (-u, bt)_F \otimes (ab(t+u), -au(1+t))_F,$$

hence

$$\begin{aligned} A &\simeq (a,b)_F \otimes (-u,bt)_F \otimes (ab(t+u), -au(1+t))_F \\ &\simeq (-u,t)_F \otimes (a(t+u), u(1+t)(t+u))_F \otimes (b, 1+t)_F. \end{aligned}$$

Since b is a uniformizing parameter for the b-adic valuation on F, it follows from [15, §19.6, Prop.] that the right side is a division algebra if (and only if) the algebra

$$B = (-u,t)_{F_1((a))} \otimes (a(t+u), u(1+t)(t+u))_{F_1((a))} \otimes F_1\big(\sqrt{1+t}\big)((a))$$

is division. (Alternatively, one may view A as a ring of twisted Laurent series over B in an indeterminate whose square is b.) Now, $a(t+u)$ is a uniformizing parameter for the a-adic valuation on $F_1\big(\sqrt{1+t}\big)((a))$, hence the same argument shows that B is a division algebra if (and only if) the algebra

$$C = (-u,t)_{F_1} \otimes F_1\big(\sqrt{1+t}, \sqrt{u(1+t)(t+u)}\big)$$

is division. Since $1+t$ and $u(t+u)$ are squares in $F_0(u)((t))$, we may embed C in the quaternion algebra $(-u,t)_{F_0(u)((t))}$, which is clearly division. Therefore, A is a division algebra. $\square$

6. Non-totally decomposable orthogonal involutions of degree 8

In this section, we consider the case of central simple algebras with orthogonal involution (A,σ) of degree 8 that are not totally decomposable. These algebras do not contain any quaternion algebra with canonical involution $(H, \bar{\ })$, since the centralizer of H would be an algebra of degree 4 with symplectic involution, hence decomposable by [11, (16.16)]; the algebra (A,σ) would then be totally decomposable.

We start with a couple of lemmas of independent interest related to triality. Let Q_1, Q_2, Q_3 be quaternion F-algebras such that $Q_1 \otimes_F Q_2 \otimes_F Q_3$ is split. By a well-known result due to Albert and to Pfister, this condition implies that Q_1, Q_2, and Q_3 have a common maximal subfield, see [11, (16.30)]. Therefore, we may write

$$Q_1 = (c, d_1)_F, \qquad Q_2 = (c, d_2)_F, \qquad Q_3 = (c, d_3)_F$$

for some c, d_1, d_2, $d_3 \in F^\times$ such that the quaternion algebra $(c, d_1 d_2 d_3)_F$ is split. For $\alpha = 1$, 2, 3, let ρ_α be the orthogonal involution on Q_α with $\operatorname{disc} \rho_\alpha = c$. The involution ρ_α is uniquely determined up to conjugation by [11, (7.4)].

Lemma 6.1. *For $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ we have*

$$(Q_\alpha, \bar{\ }) \otimes (Q_\beta, \bar{\ }) \simeq \operatorname{Ad}_{\langle\langle d_\alpha \rangle\rangle} \otimes (Q_\gamma, \rho_\gamma) \simeq \operatorname{Ad}_{\langle\langle d_\beta \rangle\rangle} \otimes (Q_\gamma, \rho_\gamma).$$

Proof. By Tao's computation of the Clifford algebra of a decomposable involution [17] we have

$$\begin{aligned}
&C\big((Q_\alpha, \bar{\ }) \otimes (Q_\beta, \bar{\ })\big) \simeq Q_\alpha \times Q_\beta \simeq (c, d_\alpha)_F \times (c, d_\beta)_F, \\
&C\big(\operatorname{Ad}_{\langle\langle d_\alpha \rangle\rangle} \otimes (Q_\gamma, \rho_\gamma)\big) \simeq (c, d_\alpha)_F \times (c, d_\alpha d_\gamma)_F \simeq (c, d_\alpha)_F \times (c, d_\beta)_F, \\
&C\big(\operatorname{Ad}_{\langle\langle d_\beta \rangle\rangle} \otimes (Q_\gamma, \rho_\gamma)\big) \simeq (c, d_\beta)_F \times (c, d_\beta d_\gamma)_F \simeq (c, d_\beta)_F \times (c, d_\alpha)_F.
\end{aligned}$$

Since central simple algebras with orthogonal involutions of degree 4 are classified by their Clifford algebra (see [11, (15.7)]), the lemma follows. □

Now, for $\alpha = 1$, 2, 3, let $(A_\alpha, \sigma_\alpha)$ be a central simple F-algebra with orthogonal involution of degree 8 such that for β, γ with $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$,

$$(A_\alpha, \sigma_\alpha) \simeq \operatorname{Ad}_{\langle 1, -1, 1, -d_\beta \rangle} \otimes (Q_\alpha, \rho_\alpha) \simeq \operatorname{Ad}_{\langle 1, -1, 1, -d_\gamma \rangle} \otimes (Q_\alpha, \rho_\alpha).$$

Thus, $(A_\alpha, \sigma_\alpha)$ is Witt-equivalent to $(Q_\beta, \bar{\ }) \otimes (Q_\gamma, \bar{\ })$ by Lemma 6.1.

Lemma 6.2. *The triple $\big((A_1, \sigma_1), (A_2, \sigma_2), (A_3, \sigma_3)\big)$ is trialitarian, in the sense that for $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ we have*

$$C(A_\alpha, \sigma_\alpha) \simeq (A_\beta, \sigma_\beta) \otimes (A_\gamma, \sigma_\gamma).$$

(See [11, p. 548]*.)*

Proof. By triality, it suffices to prove the isomorphism for $\alpha = 1$, $\beta = 2$, and $\gamma = 3$. By definition, (A_1, σ_1) is an orthogonal sum of the algebra $M_2(Q_1)$ with a hyperbolic involution and of $\operatorname{Ad}_{\langle\langle d_2 \rangle\rangle} \otimes (Q_1, \rho_1)$, so by Lemma 6.1

$$(A_1, \sigma_1) \simeq \big((M_2(F), \bar{\ }) \otimes (Q_1, \bar{\ })\big) \boxplus \big((Q_2, \bar{\ }) \otimes (Q_3, \bar{\ })\big).$$

By [11, (15.12)] we have

$$C\big((M_2(F), \bar{\ }) \otimes (Q_1, \bar{\ })\big) \simeq (M_2(F), \bar{\ }) \times (Q_1, \bar{\ })$$

and

$$C\big((Q_2, \bar{\ }) \otimes (Q_3, \bar{\ })\big) \simeq (Q_2, \bar{\ }) \times (Q_3, \bar{\ }).$$

Arguing as in Garibaldi's "Orthogonal Sum Lemma" [6, Lemma 3.2], we get

$$(C(A_1, \sigma_1), \underline{\sigma_1}) \simeq (C_+(A_1, \sigma_1), \sigma_+) \times (C_-(A_1, \sigma_1), \sigma_-)$$

with

$$(C_+(A_1, \sigma_1), \sigma_+) \simeq \big((M_2(F), \bar{\ }) \otimes (Q_2, \bar{\ })\big) \boxplus \big((Q_1, \bar{\ }) \otimes (Q_3, \bar{\ })\big)$$

and

$$(C_-(A_1, \sigma_1), \sigma_-) \simeq \big((M_2(F), \bar{\ }) \otimes (Q_3, \bar{\ })\big) \boxplus \big((Q_1, \bar{\ }) \otimes (Q_2, \bar{\ })\big).$$

Thus, $(C_+(A_1, \sigma_1), \sigma_+)$ is Witt-equivalent to $(Q_1, \bar{\ }) \otimes (Q_3, \bar{\ })$, hence it is isomorphic to (A_2, σ_2). Likewise, $(C_-(A_1, \sigma_1), \sigma_-)$ is isomorphic to (A_3, σ_3). □

Theorem 6.3. *Let (A,σ) be a central simple F-algebra with orthogonal involution of degree 8. Assume (A,σ) is not totally decomposable. Then $(A,\sigma)_{F_Q}$ is hyperbolic if and only if there is a quaternion F-algebra Q' with the following properties:*

- $\operatorname{ind}(Q\otimes_F Q')\leq 2$, *and*
- (A,σ) *is Witt-equivalent to* $(Q,\overline{})\otimes(Q',\overline{})$.

When these equivalent properties hold, we can find c, d, $d'\in F^\times$ such that

$$Q\simeq(c,d)_F,\qquad Q'\simeq(c,d')_F,$$

and

$$(A,\sigma)\simeq\operatorname{Ad}_{\langle 1,-1,1,-d\rangle}\otimes(Q'',\rho'')$$

where $Q''=(c,dd')_F$ and ρ'' is an orthogonal involution on Q'' with $\operatorname{disc}\rho''=c$.

Proof. Clearly, $(A,\sigma)_{F_Q}$ is hyperbolic if (A,σ) is Witt-equivalent to an algebra containing $(Q,\overline{})$. Conversely, suppose $(A,\sigma)_{F_Q}$ is hyperbolic. Since (A,σ) is not totally decomposable, it is not hyperbolic. If A is split, Proposition 2.1 shows that the anisotropic kernel of (A,σ) is $\operatorname{Ad}_{n_Q}\simeq(Q,\overline{})\otimes(Q,\overline{})$, hence

$$(A,\sigma)\simeq\operatorname{Ad}_{\langle 1,-1,1,-a\rangle}\otimes\operatorname{Ad}_{\langle\langle b\rangle\rangle}.$$

For the rest of the proof, we may thus assume A is not split. By Proposition 1.3, one of the components of the Clifford algebra, $C_+(A,\sigma)$ say, is split by F_Q. However, $C_+(A,\sigma)$ is not split since (A,σ) is not totally decomposable, hence $C_+(A,\sigma)$ is Brauer-equivalent to Q. As was observed in Proposition 1.3, $(C_+(A,\sigma),\sigma_+)_{F_Q}$ is isotropic. If it is hyperbolic, then (A,σ) is hyperbolic by the main theorem of [6], a contradiction. Therefore, the anisotropic kernel of $(C_+(A,\sigma),\sigma_+)$ has degree 4. It has discriminant 1 by triality, hence $(C_+(A,\sigma),\sigma_+)$ is Witt-equivalent to a product $(Q',\overline{})\otimes(Q'',\overline{})$ for some quaternion F-algebras Q', Q'' such that $Q'\otimes Q''$ is Brauer-equivalent to Q. We may therefore find c, d, d', $d''\in F^\times$ such that

$$Q\simeq(c,d)_F,\qquad Q'\simeq(c,d')_F,\qquad Q''\simeq(c,d'')_F.$$

Letting ρ (resp. ρ', resp. ρ'') be an orthogonal involution on Q (resp. Q', resp. Q'') with discriminant c, we have by Lemma 6.1

$$(C_+(A,\sigma),\sigma_+)\simeq\operatorname{Ad}_{\langle 1,-1,1,-d'\rangle}\otimes(Q,\rho)\simeq\operatorname{Ad}_{\langle 1,-1,1,-d''\rangle}\otimes(Q,\rho).$$

By Lemma 6.2, it follows that (A,σ) is isomorphic to

$$\operatorname{Ad}_{\langle 1,-1,1,-d\rangle}\otimes(Q',\rho')\qquad\text{or}\qquad\operatorname{Ad}_{\langle 1,-1,1,-d\rangle}\otimes(Q'',\rho''),$$

hence it is Witt-equivalent to

$$(Q,\overline{})\otimes(Q',\overline{})\qquad\text{or}\qquad(Q,\overline{})\otimes(Q'',\overline{}).$$

Interchanging Q' and Q'' if necessary, we thus obtain the stated description of (A,σ). □

7. Examples of arbitrarily large degree

Let (A,σ) be a central simple F-algebra with involution of orthogonal or symplectic type. Consider the (iterated) Laurent series fields $F_1=F((x))$, $F_2=F((x))((y))$, and the quaternion F_2-algebra $H=(x,y)_{F_2}$. Let ρ be any involution of orthogonal or symplectic type on H, and let

$$(A_1,\sigma_1)=(A,\sigma)\otimes_F\operatorname{Ad}_{\langle\langle x\rangle\rangle},\qquad(A_2,\sigma_2)=(A,\sigma)\otimes_F(H,\rho).$$

If $(A,\sigma)_{F_Q}$ is hyperbolic, then $(A_1,\sigma_1)_{F_Q}$ and $(A_2,\sigma_2)_{F_Q}$ also are hyperbolic, since they contain a hyperbolic factor.

Theorem 7.1. *Assume (A,σ) is anisotropic. Then (A_1,σ_1) and (A_2,σ_2) are anisotropic. Moreover, the following conditions are equivalent:*

(i) *(A,σ) contains $(Q,\overline{})$;*
(ii) *(A_1,σ_1) contains $(Q,\overline{})$;*
(iii) *(A_2,σ_2) contains $(Q,\overline{})$.*

Proof. Let $\xi_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $\eta_1 = \left(\begin{smallmatrix}0 & x\\ 1 & 0\end{smallmatrix}\right) \in M_2(F_1)$, so

$$\mathrm{ad}_{\langle\langle x\rangle\rangle}(\xi_1) = \xi_1 \qquad \text{and} \qquad \mathrm{ad}_{\langle\langle x\rangle\rangle}(\eta_1) = -\eta_1.$$

Let also $(a_i)_{i\in I}$ be an F-basis of A, so $(a_i\otimes 1, a_i\otimes\xi_1, a_i\otimes\eta_1, a_i\otimes\xi_1\eta_1)_{i\in I}$ is an F_1-basis of A_1. We extend the x-adic valuation v_1 on F_1 to a map

$$g_1\colon A_1 \to \left(\tfrac{1}{2}\mathbb{Z}\right)\cup\{\infty\}$$

defined by

$$g_1\Big(\sum_{i\in I} a_i\otimes(\alpha_i+\beta_i\xi_1+\gamma_i\eta_1+\delta_i\xi_1\eta_1)\Big) = \min_{i\in I}\big(v_1(\alpha_i), v_1(\beta_i), v_1(\gamma_i)+\tfrac{1}{2}, v_1(\delta_i)+\tfrac{1}{2}\big)$$

for α_i, β_i, γ_i, $\delta_i \in F_1$. It is readily verified that the map g_1 satisfies the following conditions for s, $t\in A_1$ and $\alpha\in F_1$:

- $g_1(1)=0$ and $g_1(s)=\infty$ if and only if $s=0$;
- $g_1(s+t)\geq \min\big(g_1(s),g_1(t)\big)$ and $g_1(s\alpha)=g_1(s)+v_1(\alpha)$;
- $g_1(st)\geq g_1(s)+g_1(t)$.

(It suffices to prove the last inequality for s, t in the above F_1-base of A_1, see [20, Lemma 1.2].) The map g_1 defines a filtration of A_1, and the associated graded ring $\mathrm{gr}(A_1)$ is

$$\mathrm{gr}(A_1)\simeq A\otimes_F M_2(F[x,x^{-1}])$$

with the grading defined by

$$\mathrm{gr}(A_1)_\lambda = A\otimes\begin{pmatrix} x^\lambda & 0\\ 0 & x^\lambda\end{pmatrix} \qquad \text{for } \lambda\in\mathbb{Z},$$

$$\mathrm{gr}(A_1)_\lambda = A\otimes\begin{pmatrix} 0 & x^{\lambda+\frac{1}{2}}\\ x^{\lambda-\frac{1}{2}} & 0\end{pmatrix} \qquad \text{for } \lambda\in\left(\tfrac{1}{2}\mathbb{Z}\right)\setminus\mathbb{Z}.$$

Therefore, $\mathrm{gr}(A_1)$ is a graded simple algebra, and g_1 is a v_1-gauge in the sense of [20]. The involution σ_1 preserves g_1. On $\mathrm{gr}(A_1)_0$, the induced involution $\widetilde{\sigma_1}$ is $\sigma\otimes\mathrm{Id}$, hence it is anisotropic. Therefore, σ_1 is anisotropic by [21, Cor. 2.3], g_1 is the unique v_1-gauge that is preserved by σ_1 by [21, Th. 2.2], and we have

$$g_1(\sigma_1(s)s) = 2g_1(s) \qquad \text{for all } s\in A_1. \tag{14}$$

Now, suppose (A_1,σ_1) contains $(Q,\overline{})$; it then contains elements i, j such that

$$i^2=a,\quad j^2=b,\quad ji=-ij,\quad \sigma_1(i)=-i,\quad \sigma_1(j)=-j. \tag{15}$$

Then by (14) we have $g_1(i)=\frac{1}{2}g_1(-a)=0$ and, similarly, $g_1(j)=0$. The images $\widetilde{i}$, $\widetilde{j}$ of i, j in $\mathrm{gr}_1(A_1)_0$ satisfy conditions similar to (15). Since $\mathrm{gr}(A_1)_0\simeq A\times A$ we may consider a projection $\mathrm{gr}(A)_0\to A$, which is a homomorphism of algebras with involution $\pi\colon(\mathrm{gr}(A_1)_0,\widetilde{\sigma_1})\to(A,\sigma)$. The images $\pi(\widetilde{i})$, $\pi(\widetilde{j})$ generate a copy of $(Q,\overline{})$ in (A,σ). Thus, (A,σ) contains $(Q,\overline{})$ if (A_1,σ_1) contains $(Q,\overline{})$. The converse is clear.

The argument for (A_2,σ_2) follows the same lines. Let ξ_2, $\eta_2\in H$ be such that

$$\xi_2^2=x,\qquad \eta_2^2=y,\qquad \eta_2\xi_2=-\xi_2\eta_2.$$

Note that if ρ is orthogonal its discriminant is represented by the quadratic form $\langle x,y,-xy\rangle$, hence it is the square class of x, y, or $-xy$. Therefore, we may assume $\rho=\operatorname{Int}(\xi_2)\circ\overline{}$, $\operatorname{Int}(\eta_2)\circ\overline{}$, or $\operatorname{Int}(\xi_2\eta_2)\circ\overline{}$. In each case (and also if ρ is symplectic) we have $\rho(\xi_2)=\pm\xi_2$ and $\rho(\eta_2)=\pm\eta_2$.

Let $v_2\colon F_2\to\mathbb{Z}^2\cup\{\infty\}$ be the (x,y)-adic valuation such that $v_2(x^\lambda y^\mu)=(\lambda,\mu)$ for λ, $\mu\in\mathbb{Z}$, where $\mathbb{Z}^2$ is endowed with the right-to-left lexicographic ordering. Considering again an F-basis $(a_i)_{i\in I}$ of A, we extend v_2 to a map

$$g_2\colon A_2\to\left(\tfrac12\mathbb{Z}\right)^2\cup\{\infty\}$$

defined by

$$g_2\Big(\sum_{i\in I}a_i\otimes(\alpha_i+\beta_i\xi_2+\gamma_i\eta_2+\delta_i\xi_2\eta_2)\Big)=$$
$$\min_{i\in I}\big(v_2(\alpha_i),v_2(\beta_i)+(\tfrac12,0),v_2(\gamma_i)+(0,\tfrac12),v_2(\delta_i)+(\tfrac12,\tfrac12)\big)$$

for α_i, β_i, γ_i, $\delta_i\in F_2$. The map g_2 is a v_2-gauge on A_2 with associated graded ring

$$\operatorname{gr}(A_2)=A\otimes(x,y)_{F[x^{-1},y,y^{-1}]}.$$

The involution σ_2 preserves g_2 and the induced involution $\widetilde{\sigma_2}$ on $\operatorname{gr}(A_2)_0=A$ is σ. Therefore, the same arguments as for (A_1,σ_1) show that (A_2,σ_2) is anisotropic, and that (A_2,σ_2) contains $(Q,\overline{})$ if and only if (A,σ) contains $(Q,\overline{})$. □

Theorem 7.1 applies in particular to the division algebra with orthogonal involution (A,σ) of Theorem 5.2, and yields central simple algebras with anisotropic involution (A_1,σ_1) and (A_2,σ_2) of degree 16 that do not contain $(Q,\overline{})$, even though they are hyperbolic over F_Q. The involution σ_1 is orthogonal and $\operatorname{ind}A_1=8$, while the involution σ_2 may be of orthogonal or symplectic type and A_2 is division. Of course, these constructions can be iterated to obtain examples of algebras with anisotropic involution of arbitrarily large degree that become hyperbolic over F_Q and do not contain $(Q,\overline{})$. Such examples can also be derived from the central simple algebras of degree 4 with symplectic involution in case (b) of Theorem 3.6, although no division algebra can be obtained in this way since the algebras in case (b) of Theorem 3.6 have index 2.

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