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# Ricci curvature and rigidity

L. Bessières, G. Besson, G. Courtois and S. Gallot

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## 1 Introduction

The aim of this paper is to show the following result.

**Theorem 1.1.** *Given any integer  $n \geq 3$  and  $d > 0$ , there exists  $\varepsilon(n, d) > 0$  such that the following holds. Suppose that  $(X, g_0)$  is an  $n$ -dimensional hyperbolic compact manifold with diameter  $\leq d$  and that  $Y$  is a compact manifold which dominates  $X$ , that is, there exist a continuous map  $f : Y \rightarrow X$  of degree one. Then  $Y$  has a metric  $g$  such that*

$$\text{Ric}_g \geq -(n-1)g \tag{1}$$

$$\text{vol}_g(Y) \leq (1 + \varepsilon) \text{vol}_{g_0}(X) \tag{2}$$

*if and only if  $f$  is homotopic to a diffeomorphism.*

### Sketch of the Proof:

We argue by contradiction. Suppose that there is a sequence  $(X_k)_{k \in \mathbb{N}}$  of compact hyperbolic manifolds with diameter  $\leq d$  and a sequence of compact manifolds  $Y_k$ , of degree one continuous maps  $f_k : Y_k \rightarrow X_k$  and metrics  $g_k$  on  $Y_k$  fulfilling the hypothesis (1) and (2) for some  $\varepsilon_k \rightarrow 0$ . Since  $f_k$  has degree one and  $X_k$  is hyperbolic, it is equivalent to say that  $f_k$  is homotopic to a diffeomorphism or simply that  $X_k$  and  $Y_k$  are diffeomorphic. We thus assume that  $Y_k$  and  $X_k$  are not diffeomorphic. One then shows that up to a subsequence, for large  $k$ ,  $Y_k$  is diffeomorphic to a compact manifold  $Y$ ,  $X_k$  is diffeomorphic to a compact manifold  $X$ , and  $X$  and  $Y$  are diffeomorphic. One argue as follows: by the classical finiteness results we get the sub-convergence of  $(X_k)$ . Indeed, the curvature is  $-1$ , the diameter is bounded by hypothesis, and there is a universal lower bound for the volume of any hyperbolic compact manifold of a given dimension. Cheeger's finiteness theorem then applies. Moreover, on a compact manifold of dimension  $\geq 3$ , there is at most one hyperbolic metric, up to isometry. We can therefore suppose that  $X_k = X$  is a fixed hyperbolic manifold. The inequality proved in [BCG] provides a lower bound for the volume of  $Y_k$  as it is explained below. We have no a priori bounds on the diameter of  $(Y_k, g_k)$ , but we can use Cheeger-Colding's theory to obtain sub-convergence in the pointed Gromov-Hausdorff topology to a complete metric space  $(Z, d)$  with small singular set. To obtain more geometric control,

the idea is to use the natural maps between  $Y_k$  and  $X$  (see [BCG]). One can show that they sub-converge to a limit map between  $Z$  and  $X$ , which is an isometry. Then  $Z$  is compact and diffeomorphic to  $Y_k$  for large  $k$ .

The paper is organized as follows. The construction and the properties of the natural maps are given in section 2. In section 3, we construct the limit space  $Z$  and the limit map  $F : Z \rightarrow X$ . In section 4, we prove that  $F$  is an isometry and conclude.

## 1.1 Some a priori control on $(Y, g)$

Some a priori control on the metric  $g$  will be needed in section 2 and 3. We give here some necessary results.

Let  $(X, g_0)$  be an hyperbolic manifold and  $Y$  be a manifold satisfying the assumptions of theorem 1.1. For any riemannian metric  $g$  on  $Y$  satisfying the curvature assumption (1), one has the following inequality

$$\text{vol}_g(Y) \geq \text{vol}_{g_0}(X). \quad (3)$$

It is a consequence of the Besson-Courtois-Gallot inequality [BCG]

$$h(g)^n \text{vol}_g(Y) \geq h(g_0)^n \text{vol}_{g_0}(X), \quad (4)$$

where  $h(g)$  is the volume entropy, or the critical exponent, of the metric  $g$ , i.e.:

$$h(g) = \lim_{R \rightarrow +\infty} \frac{1}{R} \ln(\text{vol}_{\tilde{g}}(B_{\tilde{g}}(x, R))),$$

where  $\tilde{g}$  is the lifted metric on  $\tilde{Y}$ . Indeed, any metric  $g$  on  $Y$  which satisfies (1), verifies, by Bishop's theorem,

$$h(g) \leq h(g_0) = n - 1. \quad (5)$$

One can obtain a local control of the volume by Gromov's isolation theorem (see [Gro2], theorem 0.5). It shows that if the simplicial volume  $\|Y\|$  - a topological invariant - of  $Y$  is non-zero, then for any riemannian metric  $g$  on  $Y$  satisfying the curvature assumption (1), there exists at least one point  $y_g \in Y$  such that

$$\text{vol}_g(B(y_g, 1)) \geq v_n > 0. \quad (6)$$

Here  $B(y_g, 1)$  is the geodesic ball of radius 1 for the metric  $g$  and  $v_n$  is a universal constant. This theorem applies in our situation since, by an elementary property of the simplicial volume,  $\|Y\| \geq \|X\|$  if there is a degree one map from  $Y$  to  $X$ . On the other hand,  $X$  has an hyperbolic metric and thus  $\|X\| > 0$  by Gromov-Thurston's theorem (see [Gro2]).

Given this universal lower bound for the volume of a unit ball  $B(y_g, 1)$ , the volume of any ball  $B(y, r)$  is bounded from below in terms of  $r$  and  $d(y_g, y)$ . Indeed, recall

that under the curvature assumption (1), Bishop-Gromov's theorem shows that for any  $0 < r \leq R$ , one has

$$\frac{\text{vol}_g(B(y, r))}{\text{vol}_g(B(y, R))} \geq \frac{\text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(r))}{\text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(R))}, \quad (7)$$

where  $B_{\mathbf{H}^n}(r)$  is a  $r$ -ball in the hyperbolic space  $\mathbf{H}^n$ . As  $B(y_g, 1) \subset B(y, 1 + d(y_g, y) + r)$ , one deduces from (7) that

$$\text{vol}_g(B(y, r)) \geq \text{vol}_g(B(y, 1 + d(y_g, y) + r)) \frac{\text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(r))}{\text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(1 + d(y_g, y) + r))} \quad (8)$$

$$\geq v_n \frac{\text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(r))}{\text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(1 + d(y_g, y) + r))}. \quad (9)$$

The curvature assumption (1) and the volume estimates (7) or (9) are those required to use the non-collapsing part of Cheeger-Colding's theory, as we shall see in section 3.

## 2 The natural maps

### 2.1 Construction of the natural maps

In this section we recall the construction and the main properties of the natural maps defined by Besson-Courtois-Gallot ([BCG],[BCG2]). Suppose that  $(Y, g)$  and  $(X, g_0)$  are compact riemannian manifolds and that

$$f : Y \rightarrow X,$$

is a continuous map of degree one. For the sake of simplicity, we assume that  $g_0$  is hyperbolic (the construction holds in more general cases). Then, for any  $c > h(g)$  there exists a  $C^1$  map

$$F_c : Y \rightarrow X,$$

homotopic to  $f$ , such that for all  $y \in Y$ ,

$$|\text{Jac } F_c(y)| \leq \left( \frac{c}{h(g_0)} \right)^n, \quad (10)$$

with equality for some  $y \in Y$  if and only if  $d_y F_c$  is an homothety of ratio  $\frac{c}{h(g_0)}$ .

The inequality (4) is then easily obtained by integration of (10) and by taking a limit when  $c$  goes to  $h(g)$ . To obtain global rigidity properties, one has in general to care about the limit of  $F_c$  as  $c$  goes to  $h(g)$ .

We divide the construction of the maps in 4 steps. Let  $\tilde{Y}$  and  $\tilde{X}$  be the universal coverings of  $Y$  and  $X$  respectively, and  $\tilde{f} : \tilde{Y} \rightarrow \tilde{X}$  a lift of  $f$ .

**Step 1:** for each  $y \in \tilde{Y}$  and  $c > h(g)$ , let  $\nu_y^c$  be the finite measure on  $\tilde{Y}$  defined by

$$d\nu_y^c(z) = e^{-c \cdot \rho(y, z)} dV_{\tilde{g}}(z)$$

where  $z \in \tilde{Y}$ ,  $\tilde{g}$  is the lifted metric on  $\tilde{Y}$  and  $\rho(\cdot, \cdot)$  is the distance function of  $(\tilde{Y}, \tilde{g})$ .

**Step 2:** this measure is pushed forward and gives a finite measure  $\tilde{f}_* \nu_y^c$  on  $\tilde{X}$ . Let us recall that it is defined by

$$\tilde{f}_* \nu_y^c(U) = \nu_y^c(\tilde{f}^{-1}(U)).$$

**Step 3:** one defines a finite measure  $\mu_y^c$  on  $\partial\tilde{X}$  by convolution of  $\tilde{f}_* \nu_y^c$  with all visual probability measures  $P_x$  of  $\tilde{X}$ . Recall that the visual probability measure  $P_x$  at  $x \in \tilde{X}$  is defined as follows: the unit tangent bundle  $U_x \tilde{X}$  is projected onto the boundary  $\partial\tilde{X}$  by the map

$$v \in U_x \tilde{X} \xrightarrow{E_x} \gamma_v(\infty) \in \partial\tilde{X},$$

where  $\gamma_v(t) = \exp_x(tv)$ . The measure  $P_x$  is then the push-forward by  $E_x$  of the canonical probability measure of  $U_x \tilde{X}$ , i.e.  $P_x(U)$  is the measure of the set of vectors  $v \in U_x \tilde{X}$  such that  $\gamma_v(+\infty) \in U$ .

Then

$$\begin{aligned} \mu_y^c(U) &= \int_{\tilde{X}} P_x(U) d\tilde{f}_* \nu_y^c(x) \\ &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(U) d\nu_y^c(z). \end{aligned}$$

One can identify  $\partial\tilde{X}$  with the unit sphere in  $\mathbf{R}^n$ , by choosing an origin  $o \in \tilde{X}$  and using  $E_o$ . One can then show (REFERENCE) that the density of this measure is given by

$$d\mu_y^c(\theta) = \left( \int_{\tilde{Y}} e^{-h(g_0)B(\tilde{f}(z), \theta)} e^{-c\rho(y, z)} d\nu_{\tilde{g}}(z) \right) d\theta,$$

where  $\theta \in \partial\tilde{X}$ ,  $d\theta$  is the canonical probability measure on  $S^{n-1}$  and  $B(x, \theta)$  is a Busemann function on  $\tilde{X}$  normalised to vanish at  $x = o$ . We will use the notation

$$p(x, \theta) = e^{-h(g_0)B(x, \theta)}$$

It is a classical fact that  $p$  is the Poisson Kernel of  $(\tilde{X}, \tilde{g}_0)$ .

**Step 4:** the map

$$F_c : \tilde{Y} \longrightarrow \tilde{X}$$

associates to any  $y \in \tilde{Y}$  the unique  $x \in \tilde{X}$  which minimizes on  $\tilde{X}$  the function

$$\mathcal{B}(x) = \int_{\partial\tilde{X}} B(x, \theta) d\mu_y^c(\theta).$$

(see Appendix A in [BCG]).

The maps  $F_c$  are shown to be  $\mathcal{C}^1$  and equivariant. The quotient map, which is denoted  $F_c : Y \rightarrow X$ , is homotopic to  $f$ . Note that  $F_c$  depends heavily on the metric  $g$ .

## 2.2 Some technical lemmas

Let us give some definitions.

**Definition 2.1.** We call  $\sigma_y^c$  be the probability measure on  $\partial\tilde{X}$  defined by

$$\sigma_y^c = \frac{\mu_y^c}{\mu_y^c(\partial\tilde{X})}.$$

Then

$$\|\mu_y^c\| = \mu_y^c(\partial\tilde{X}) = \int_{\tilde{Y}} e^{-c\rho(y,z)} d\nu_{\tilde{g}}^c(z) = \|\nu_y^c\|.$$

We consider two positive definite bilinear forms of trace 1 and the corresponding symmetric endomorphisms.

**Definition 2.2.** For any  $y \in \tilde{X}$ ,  $u, v \in T_{F_c(y)}\tilde{X}$ ,

$$\mathbf{h}_y^c(u, v) = \int_{\partial\tilde{X}} dB_{(F_c(y), \theta)}(u) dB_{(F_c(y), \theta)}(v) d\sigma_y^c(\theta) = g_0(\mathbf{H}_y^c(u), v).$$

And, for any  $y \in \tilde{Y}$ ,  $u, v \in T_y\tilde{Y}$ ,

$$\mathbf{h}_y'^c(u, v) = \frac{\int_{\tilde{Y}} d\rho_{(y,z)}(u) d\rho_{(y,z)}(v) d\nu_y^c(z)}{\mu_y^c(\partial\tilde{X})} = g(\mathbf{H}_y'^c(u), v).$$

**Lemma 2.3.** For any  $y \in \tilde{Y}$ ,  $u \in T_y\tilde{Y}$ ,  $v \in T_{F(y)}\tilde{X}$ , one has

$$\left| g_0((I - H_y^c)d_y F_c(u), v) \right| \leq c (g_0(H_y^c(v), v))^{1/2} (g(H_y'^c(u), u))^{1/2}. \quad (11)$$

*Proof.* since  $F_c(y)$  is an extremum of the function  $\mathcal{B}$ , one has

$$d_{F_c(y)}\mathcal{B}(v) = \int_{\partial\tilde{X}} dB_{(F_c(y), \theta)}(v) d\mu_y^c(\theta) = 0 \quad (12)$$

for each  $v \in T_{F_c(y)}\tilde{X}$ . Let  $V$  be a parallel vector field on  $T\tilde{X}$ , then  $d_{F_c(y)}\mathcal{B}(V_y) = 0$  for each  $y \in \tilde{Y}$ . One differentiates this equation in a direction  $u \in T_y\tilde{Y}$  thus, using the formula (12) with  $v = V_y$ , one obtains

$$\int_{\partial\tilde{X}} DdB_{(F_c(y), \theta)}(d_y F_c(u), v) d\mu_y^c(\theta) + \int_{\partial\tilde{X}} dB_{(F_c(y), \theta)}(v) \left( \int_{\tilde{Y}} p(\tilde{f}(z), \theta) (-cd\rho_{(y,z)}(u)) d\nu_y^c(z) \right) d\theta = 0$$

Using Cauchy-Schwarz inequality in the second term, one gets

$$\left| \int_{\partial\tilde{X}} DdB_{(F_c(y), \theta)}(d_y F_c(u), v) d\mu_y^c(\theta) \right| \leq \int_{\partial\tilde{X}} |dB_{(F_c(y), \theta)}(v)| \left( \int_{\tilde{Y}} p(\tilde{f}(z), \theta) d\nu_y^c(z) \right)^{1/2} \left( \int_{\tilde{Y}} p(\tilde{f}(z), \theta) |cd\rho_{(y,z)}(u)|^2 d\nu_y^c(z) \right)^{1/2} d\theta$$

which is, using Cauchy-Schwarz inequality again

$$\begin{aligned}
&\leq c \left( \int_{\partial\tilde{X}} |dB_{(F_c(y),\theta)}(v)|^2 \int_{\tilde{Y}} p(\tilde{f}(z), \theta) d\nu_y^c(z) d\theta \right)^{1/2} \left( \int_{\partial\tilde{X}} \int_{\tilde{Y}} p(\tilde{f}(z), \theta) |d\rho_{(y,z)}(u)|^2 d\nu_y^c(z) d\theta \right)^{1/2} \\
&= c \left( \int_{\partial\tilde{X}} |dB_{(F_c(y),\theta)}(v)|^2 d\mu_y^c(\theta) \right)^{1/2} \left( \int_{\tilde{Y}} |d\rho_{(y,z)}(u)|^2 d\nu_y^c(z) \right)^{1/2} \\
&= c\mu_y^c(\partial\tilde{X}) (g_0(H_y^c(v), v))^{1/2} (g(H_y'^c(u), u))^{1/2}
\end{aligned}$$

It is shown in [BCG] chapter 5 that  $DdB = g_0 - dB \otimes dB$  for an hyperbolic metric. The left term of the inequality is thus  $\mu_y^c(\partial\tilde{X})g_0((I - H_y^c)d_yF_c(u), v)$ . This proves the lemma.  $\square$

**Definition 2.4.** Let  $0 < \lambda_1^c(y) \leq \dots \leq \lambda_n^c(y) < 1$  be the eigenvalues of  $H_y^c$ .

**Proposition 2.5.** There exists a constant  $A(n) > 0$  such that, for any  $y \in Y$ ,

$$|\text{Jac } F_c(y)| \leq \left( \frac{c}{h(g_0)} \right)^n \left( 1 - A \sum_{i=1}^n (\lambda_i^c(y) - \frac{1}{n})^2 \right) \quad (13)$$

*Proof.* The proof is based on the two following lemmas.

**Lemma 2.6.** At each  $y \in \tilde{Y}$ ,

$$|\text{Jac } F_c(y)| \leq \left( \frac{c}{\sqrt{n}} \right)^n \frac{\det(H_y^c)^{1/2}}{\det(I - H_y^c)}$$

*Proof of lemma 2.6.* Let  $(u_i)$  an orthonormal basis of  $T_{F_c(y)}\tilde{X}$  which diagonalizes  $H_y^c$ . We can suppose that  $d_yF_c$  is invertible thus let  $v'_i = [(I - H_y^c) \circ d_yF_c]^{-1}(u_i)$ . Schmidt orthonormalisation process applied to  $(v'_i)$  gives an orthonormal basis  $(v_i)$  at  $T_y\tilde{Y}$ . The matrix of  $(I - H_y^c) \circ d_yF_c$  in the base  $(v_i)$  and  $(u_i)$  is upper triangular, then

$$\det(I - H_y^c) \text{Jac } F_c(y) = \prod_{i=1}^n g_0((I - H_y^c) \circ d_yF_c(v_i), u_i)$$

which gives, with (11),

$$\begin{aligned}
\det(I - H_y^c) |\text{Jac } F_c(y)| &\leq c^n \left( \prod_{i=1}^n g_0(H_y^c(v_i), v_i) \right)^{1/2} \left( \prod_{i=1}^n g(H_y'^c(u_i), u_i) \right)^{1/2} \\
&\leq c^n \det(H_y^c)^{1/2} \left[ \frac{1}{n} \sum_{i=1}^n g(H_y'^c(u_i), u_i) \right]^{n/2}
\end{aligned}$$

and we have the desired inequality with  $\text{tr}(H_y'^c) = 1$ .  $\square$

**Lemma 2.7.** *Let  $H$  a symmetric positive definite  $n \times n$  matrix whose trace is equal to one then, if  $n \geq 3$ ,*

$$\frac{\det(H^{1/2})}{\det(I - H)} \leq \left( \frac{n}{h(g_0)^2} \right)^{n/2} \left( 1 - A \sum_{i=1}^n \left( \lambda_i - \frac{1}{n} \right)^2 \right)$$

for a constant  $A(n) > 0$ .

*Proof of lemma 2.7.* see Appendix B5 in [BCG]. This is the point where the rigidity of the natural maps fails in dimension 2. This completes the proof of the lemma 2.5.  $\square$

$\square$

## 2.3 Some nice properties

We now show that the natural maps  $F_c$  are "nice". In this section, we shall consider  $F_c$  as a map from  $(Y, g)$  to  $(X, g_0)$ . We suppose that the metric  $g$  satisfies the curvature assumption (1) and the assumption on its volume (2) for some  $\varepsilon > 0$ . Let us introduce some terminology.

**Definition 2.8.** *Let  $0 < \alpha < 1$ . We say that a property holds  $\alpha$ -ae ( $\alpha$ -almost everywhere) on a set  $A$  if the set  $A_+$  of points of  $A$  where the property holds has relative volume bigger than  $1 - \alpha$ , i. e.  $\frac{\text{vol}(A_+)}{\text{vol}(A)} \geq 1 - \alpha$ .*

We show that  $dF_c$  is  $\alpha$ -close to be isometric  $\alpha$ -ae on  $Y$  for some  $\alpha(\varepsilon, c) > 0$ . Moreover  $\alpha(\varepsilon, c) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow h(g)$ . On the other hand, given any radius  $R > 0$ , one shows that  $\|dF_c\|$  is uniformly bounded on balls  $B(y_g, R)$ , provided  $c$  is close enough to  $h(g)$ . Recall that we have uniform bounds for the volume of  $g$  but not for the diameter. The key point is to show that  $H_y^c$  is  $\alpha$ -close to  $\frac{Id}{n}$  on a set of large volume, and to bound it on a ball of fixed radius, with respect to the parameters  $\varepsilon, c$ .

To control  $c - h(g)$  we introduce a parameter  $\delta > 0$ . We suppose that the volume entropy of  $g$  satisfies the inequality

$$h(g) < c \leq h(g) + \delta. \tag{14}$$

Observe that (5), (13) and (14) implies that

$$|\text{Jac } F_c(y)| \leq \left( \frac{h(g) + \delta}{h(g_0)} \right)^n \leq \left( 1 + \frac{\delta}{n-1} \right)^{n-1} \tag{15}$$

for all  $y \in Y$ . The map  $F_c$  is thus almost volume decreasing. As  $\text{vol}_g(Y)$  is close to  $\text{vol}_{g_0}(X)$ , the set in  $Y$  where  $F_c$  decreases the volume a lot must be small. Equivalently,  $|\text{Jac } F_c|$  must be close to 1 in  $L^1$  norm. Now we give a precise statement of this fact.



**Lemma 2.9.** *There exists  $\alpha_1 = \alpha_1(\varepsilon, \delta) > 0$  such that  $\alpha$ -ae on  $Y$  one has,*

$$1 - \alpha_1 \leq |\text{Jac } F_c(y)|, \quad (16)$$

and for all  $y \in Y$  one has

$$|\text{Jac } F_c(y)| \leq 1 + \alpha_1. \quad (17)$$

Moreover,  $\alpha_1(\varepsilon, \delta) \rightarrow 0$  as  $\varepsilon$  and  $\delta \rightarrow 0$ .

*Proof.* Let

$$\alpha = \max \left( \sqrt{\left(1 + \frac{\delta}{n-1}\right)^{n-1} - 1}, \sqrt{\varepsilon} \right).$$

Thus  $\left(1 + \frac{\delta}{n-1}\right)^{n-1} \leq 1 + \alpha^2$  and  $\varepsilon \leq \alpha^2$ . In particular,  $|\text{Jac } F_c(y)| \leq 1 + \alpha^2 \leq 1 + \alpha$  for all  $y \in Y$ .

As  $F_c$  has degree one, we have

$$\text{vol}_{g_0}(X) = \int_Y F_c^*(dv_{g_0}) = \int_Y |\text{Jac } F_c(y)| dv_g(y)$$

Denote by  $Y_{\alpha_1}$  the set of points  $y \in Y$  such that

$$|\text{Jac } F_c(y)| \geq 1 - \alpha.$$

We have

$$\text{vol}_{g_0}(X) \leq \int_Y |\text{Jac } F_c(y)| dv_g(y) \quad (18)$$

$$= \int_{Y_{\alpha_1}} |\text{Jac } F_c(y)| dv_g(y) + \int_{Y - Y_{\alpha_1}} |\text{Jac } F_c(y)| dv_g(y) \quad (19)$$

$$\leq (1 + \alpha^2) \text{vol}_g(Y_{\alpha_1}) + (1 - \alpha) \text{vol}_g(Y - Y_{\alpha_1}) \quad (20)$$

$$= \text{vol}_g(Y) + \alpha^2 \text{vol}_g(Y_{\alpha_1}) - \alpha \text{vol}_g(Y - Y_{\alpha_1}) \quad (21)$$

Then, using the assumption (2) and the inequality (3) on the volume, we get

$$\text{vol}_g(Y - Y_{\alpha_1}) \leq \frac{\text{vol}_g(Y) - \text{vol}_{g_0}(X)}{\alpha} + \alpha \text{vol}_g(Y_{\alpha_1}) \quad (22)$$

$$\leq \left(\frac{\varepsilon}{\alpha} + \alpha\right) \text{vol}_g(Y) \quad (23)$$

$$\leq 2\alpha \text{vol}_g(Y). \quad (24)$$

Clearly,  $1 - 2\alpha \leq |\text{Jac } F_c(y)|$  on  $Y_{\alpha_1}$  and  $|\text{Jac } F_c(y)| \leq 1 + 2\alpha$  on  $Y$  which proves the lemma with  $\alpha_1(\varepsilon, \delta) = 2\alpha$ .  $\square$

From this lemma, we deduce that  $F_c$  is almost injective. Let  $x \in X$ , one defines  $N(F_c, x) \in \mathbf{N} \cup \{\infty\}$  to be the number of preimages of  $x$  by  $F_c$ . As  $F_c$  has degree one, one has  $N(F_c, x) \geq 1$  for all  $x \in X$ . We then define  $X_1 := \{x \in X, N(F_c|x) = 1\}$ . Observe that  $N(F_c, x) \geq 2$  on  $X - X_1$ .

**Lemma 2.10.** *There exists  $\alpha_2 = \alpha_2(\varepsilon, \delta) > 0$  such that*

$$\text{vol}_{g_0}(X_1) \geq (1 - \alpha_2) \text{vol}_{g_0}(X) \quad (25)$$

and

$$\int_{X-X_1} N(F_c|x) dV_{g_0}(x) \leq \alpha_2(\varepsilon, \delta) \text{vol}_{g_0}(X). \quad (26)$$

Moreover,  $\alpha_2(\varepsilon, \delta) \rightarrow 0$  as  $\varepsilon$  and  $\delta \rightarrow 0$ .

In particular,  $N(F_c, x) = 1$   $\alpha'$ -ae on  $X$ .

*Proof.* one defines

$$\alpha_2(\varepsilon, \delta) = 2 \left( \left( 1 + \frac{\delta}{n-1} \right)^n (1 + \varepsilon) - 1 \right).$$

From (13) and the area formula (see [Mor] 3.7), we have

$$\left( \frac{c}{h(g_0)} \right)^n \text{vol}_g(Y) \geq \int_Y |\text{Jac } F_c(y)| dv_g(y) \quad (27)$$

$$= \int_X N(F_c|x) dv_{g_0}(x) \quad (28)$$

$$= \int_{X_1} N(F_c|x) dv_{g_0}(x) + \int_{X-X_1} N(F_c|x) - 1 + 1 dv_{g_0}(x) \quad (29)$$

$$= \text{vol}_{g_0}(X) + \int_{X-X_1} N(F_c|x) - 1 dv_{g_0}(x). \quad (30)$$

And

$$\text{vol}_{g_0}(X - X_1) \leq \int_{X-X_1} N(F_c|x) - 1 dv_{g_0}(x) \quad (31)$$

$$\leq \left( \frac{c}{h(g_0)} \right)^n \text{vol}_g(Y) - \text{vol}_{g_0}(X) \quad (32)$$

$$\leq \left( \left( \frac{c}{h(g_0)} \right)^n (1 + \varepsilon) - 1 \right) \text{vol}_{g_0}(X) \quad (33)$$

$$\leq \frac{\alpha_2(\varepsilon, \delta)}{2} \text{vol}_{g_0}(X). \quad (34)$$

Thus

$$\text{vol}_{g_0}(X - X_1) \leq \int_{X-X_1} N(F_c|x) dv_{g_0}(x) \leq \alpha_2(\varepsilon, \delta) \text{vol}_{g_0}(X),$$

and this proves the lemma.  $\square$

**Lemma 2.11.** *There exists  $\alpha_3 = \alpha_3(\varepsilon, \delta) > 0$  such that the following holds. Let  $y \in Y_{\alpha_1}$  and  $u \in T_y Y$  then*

$$(1 - \alpha_3) \|u\|_g \leq \|d_y F_c(u)\|_{g_0} \leq (1 + \alpha_3) \|u\|_g. \quad (35)$$

Moreover,  $\alpha_3(\varepsilon, \delta) \rightarrow 0$  as  $\varepsilon, \delta \rightarrow 0$ .

*Proof.* The inequality (13) implies that for all  $y \in Y$

$$\|H_y^c - \frac{\text{Id}}{n}\|^2 \leq \frac{1}{A} \left(1 - \frac{|\text{Jac } F_c(y)|}{\left(1 + \frac{\delta}{n-1}\right)^n}\right).$$

Let us define

$$\beta_1 = \beta_1(\varepsilon, \delta) = \frac{1}{A^{1/2}} \left(1 - \frac{1 - \alpha_1(\varepsilon, \delta)}{\left(1 + \frac{\delta}{n-1}\right)^n}\right)^{1/2}.$$

where  $\alpha_1(\varepsilon, \delta)$  is the constant from lemma 2.9. Clearly,  $\beta_1(\varepsilon, \delta) \rightarrow 0$  as  $\varepsilon$  and  $\delta \rightarrow 0$ . Let  $Y_{\alpha_1}$  be the set of points where (16) holds. On  $Y_{\alpha_1}$ , one has

$$\|H_y^c - \frac{\text{Id}}{n}\|^2 \leq \beta_1^2. \quad (36)$$

Let  $(u_i)_{i=1, \dots, n}$  be an orthonormal basis of  $T_y Y$  and  $v_i = d_y F(u_i)$ . Writing  $\text{Id} - H_y^c = \frac{n-1}{n}\text{Id} + \frac{\text{Id}}{n} - H_y^c$ , the left side of (16) gives

$$|g_0((\text{Id} - H_y^c)d_y F_c(u_i), d_y F_c(u_i))| \geq \left|g_0\left(\left(\frac{n-1}{n}\text{Id}\right)d_y F_c(u_i), d_y F_c(u_i)\right)\right| \quad (37)$$

$$- \left|g_0\left(\left(\frac{\text{Id}}{n} - H_y^c\right)d_y F_c(u_i), d_y F_c(u_i)\right)\right|$$

$$\geq \frac{n-1}{n} \|d_y F_c(u_i)\|_{g_0}^2 - \left\|\frac{\text{Id}}{n} - H_y^c\right\| \cdot \|d_y F_c(u_i)\|_{g_0}^2 \quad (38)$$

$$\geq \left(\frac{n-1}{n} - \beta\right) \|d_y F_c(u_i)\|_{g_0}^2. \quad (39)$$

Writing  $H_y^c = \frac{\text{Id}}{n} + H_y^c - \frac{\text{Id}}{n}$ , one has

$$g_0(H_y^c d_y F_c(u_i), d_y F_c(u_i))^{1/2} \leq g_0\left(\frac{\text{Id}}{n} d_y F_c(u_i), d_y F_c(u_i)\right)^{1/2} \quad (40)$$

$$+ g_0\left(\left(H_y^c - \frac{\text{Id}}{n}\right) d_y F_c(u_i), d_y F_c(u_i)\right)^{1/2}$$

$$\leq \left(\frac{1}{\sqrt{n}} + \beta^{1/2}\right) \|d_y F_c(u_i)\|_{g_0}. \quad (41)$$

Taking the trace of the right hand side of (11) and using the Cauchy-Schwarz inequality,

one has

$$\begin{aligned} \sum_{i=1}^n g_0(H_y^c d_y F_c(u_i), d_y F_c(u_i))^{1/2} g(H_y'^c(u_i), u_i)^{1/2} &\leq \left(\frac{1}{\sqrt{n}} + \beta^{1/2}\right) \left(\sum_{i=1}^n \|d_y F_c(u_i)\|_{g_0}^2\right)^{1/2} \\ &\times \left(\sum_{i=1}^n g(H_y'^c(u_i), u_i)\right)^{1/2} \end{aligned} \quad (42)$$

$$= \left(\frac{1}{\sqrt{n}} + \beta^{1/2}\right) \left(\sum_{i=1}^n \|d_y F_c(u_i)\|_{g_0}^2\right)^{1/2} \quad (43)$$

By (11), the trace of (39) is not greater than the right hand side of (43) multiply by  $c$ , hence

$$\left(\frac{n-1}{n} - \beta\right) \sum_{i=1}^n \|d_y F_c(u_i)\|_{g_0}^2 \leq c \left(\frac{1}{\sqrt{n}} + \beta^{1/2}\right) \left(\sum_{i=1}^n \|d_y F_c(u_i)\|_{g_0}^2\right)^{1/2},$$

thus

$$\left(\sum_{i=1}^n \|d_y F_c(u_i)\|_{g_0}^2\right)^{1/2} \leq c \frac{\frac{1}{\sqrt{n}} + \beta^{1/2}}{\frac{n-1}{n} - \beta} \leq \sqrt{n} \left(1 + \frac{\delta}{n-1}\right) \frac{1 + \sqrt{n}\beta^{1/2}}{1 - \frac{n}{n-1}\beta}.$$

Let us define

$$\beta_2(\varepsilon, \delta) = \left(1 + \frac{\delta}{n-1}\right)^2 \left(\frac{1 + \sqrt{n}\beta^{1/2}}{1 - \frac{n}{n-1}\beta}\right)^2 - 1.$$

Clearly,  $\beta_2(\varepsilon, \delta) \rightarrow 0$  as  $\varepsilon$  and  $\delta \rightarrow 0$ . One has

$$\sum_{i=1}^n \|d_y F_c(u_i)\|_{g_0}^2 \leq n(1 + \beta_2).$$

Let  $L$  be the endomorphism of  $T_y Y$  defined by  $L = (d_y F_c)^* \circ d_y F_c$ . We have

$$\text{trace}(L) = \sum_{i=1}^n g(L(u_i), u_i) = \sum_{i=1}^n g(d_y F_c(u_i), d_y F_c(u_i)) \leq n(1 + \beta_2). \quad (44)$$

On the other hand

$$|1 - \alpha|^2 \leq |\text{Jac } F_c(y)|^2 = \det(L) \leq \left(\frac{\text{trace}(L)}{n}\right)^n \leq (1 + \beta_2)^n,$$

which shows that there is almost equality in the arithmetic-geometric inequality. We get that there exists some  $\alpha_3(\varepsilon, \delta) > 0$ , with  $\alpha_3(\varepsilon, \delta) \rightarrow 0$  as  $\varepsilon, \delta \rightarrow 0$ , such that

$$\|L - Id\| \leq \alpha_3(\varepsilon, \delta).$$

Thus for any  $y \in Y_{\alpha_1}$  and  $u \in T_y Y$

$$(1 - \alpha_3)\|u\| \leq \|d_y F_c(u)\|_{g_0} \leq (1 + \alpha_3)\|u\| \quad (45)$$

and  $d_y F_c$  is almost isometric.  $\square$

We now prove that given a fixed radius  $R > 0$ , the natural maps  $F_c$  are uniformly bounded on  $B(y_g, R)$  if the parameters  $\varepsilon, \delta$  are sufficiently small.

**Lemma 2.12.** *Let  $R > 0$ , then there exists  $\varepsilon(R) > 0$  and  $\delta(R) > 0$  such that for any  $0 < \varepsilon < \varepsilon(R)$  and  $0 < \delta < \delta(R)$ , and for any  $y \in B(y_g, R)$ ,*

$$\|d_y F_c\| \leq 2\sqrt{n} \quad (46)$$

*Proof.* We first prove that for all  $y \in Y$ ,  $\|d_y F_c\|$  is controlled from above by  $\lambda_n^c(y)$ , the maximal eigenvalue of  $H_y^c$  (see definition 2.4). Recall that  $0 < \lambda_n^c < 1$ . Let  $u$  be a unit vector in  $T_y \tilde{Y}$  and  $v = d_y F_c(u)$ . The equation(11) gives

$$(1 - \lambda_n^c(y)) |g_0(d_y F_c(u), d_y F_c(u))| \leq c\lambda_n^c(y)^{1/2} g_0(d_y F_c(u), d_y F_c(u))^{1/2} \quad (47)$$

thus

$$\|d_y F_c(u)\|_{g_0} \leq \frac{c\sqrt{\lambda_n^c(y)}}{1 - \lambda_n^c(y)}. \quad (48)$$

Hence we have to show that  $\lambda_n^c(y)$  is not close to 1. More precisely, let  $\beta > 0$  such that  $\frac{1}{n} + \beta < 1$ , one then defines

$$\gamma(\delta, \beta) = \left( \frac{n - 1 + \delta}{n - 1 - n\beta} \right) \sqrt{1 + n\beta} - 1 > 0.$$

One can check that if  $\lambda_n^c(y) \leq \frac{1}{n} + \beta$ , then  $\|d_y F_c(u)\|_{g_0} \leq \sqrt{n}(1 + \gamma)$ . Clearly,  $\gamma(\beta, \delta) \rightarrow 0$  as  $\delta, \beta \rightarrow 0$ . For our purposes, it is sufficient to suppose that  $\gamma \leq 1$ . Let  $\delta_n > 0$  and  $\beta_n > 0$  be such that if  $0 < \delta \leq 10\delta_n$  and  $0 < \beta \leq 10\beta_n$  then  $\gamma(\delta, \beta) \leq 1$ . One defines moreover  $\varepsilon_n > 0$  such that if  $0 < \varepsilon < \varepsilon_n$  then with the notations of lemma 2.11,  $\beta_1(\varepsilon, \delta) \leq \beta_n$ . In what follows, we suppose  $\varepsilon$  and  $\delta$  sufficiently small.

By (36) we have that  $|\lambda_n^c(y) - \frac{1}{n}| \leq \beta_1(\varepsilon, \delta)$  on  $Y_{\alpha_1}$ . Recall that  $Y_{\alpha_1}$  has a large relative volume in  $Y$ . The idea is first to estimate  $\lambda_n^c$  on a neighborhood of  $Y_{\alpha_1}$  and then to show that this neighbourhood contains  $B(y_g, R)$  if the parameters  $\varepsilon, \delta$  are sufficiently small relatively to  $R$ .

We begin by estimating the variation of  $\lambda_n^c$ . Recall that  $H_y^c$  is defined by

$$g_0(H_y^c(u), v) = \int_{\partial \tilde{X}} dB_{(F_c(y), \theta)}(u) dB_{(F_c(y), \theta)}(v) d\sigma_y^c(\theta).$$

Let  $U, V$  be parallel vector fields near  $F_c(y)$  extending unit vectors  $u, v$ . We compute the derivative of  $g_0(H_y^c(U), V)$  in a direction  $w \in T_y Y$ :

$$\begin{aligned} w \cdot g_0(H_y^c(U), V) &= \int_{\partial \tilde{X}} DdB_{(F(y), \theta)}(d_y F(w), U) dB_{(F_c(y), \theta)}(V) d\sigma_y^c(\theta) + \\ &\int_{\partial \tilde{X}} dB_{(F_c(y), \theta)}(U) DdB_{(F(y), \theta)}(d_y F(w), V) d\sigma_y^c(\theta) + \int_{\partial \tilde{X}} dB_{(F_c(y), \theta)}(U) dB_{(F_c(y), \theta)}(V) w \cdot d\sigma_y^c(\theta) \end{aligned}$$

Recall that  $\|DdB\| \leq 1$  and  $\|dB\| \leq 1$  thus

$$|w \cdot g_0(H_y^c(U), V)| \leq 2\|d_y F_c(w)\|_{g_0} + \left| \int_{\partial \tilde{X}} w \cdot d\sigma_y^c(\theta) \right|.$$

Recall now that

$$d\sigma_y^c(\theta) = \frac{d\mu_y^c(\theta)}{\mu_y^c(\partial \tilde{X})} = \frac{\int_{\tilde{Y}} p(\tilde{f}(z), \theta) e^{-c\rho(y,z)} dv_{\tilde{g}}(z)}{\int_{\tilde{Y}} e^{-c\rho(y,z)} dv_{\tilde{g}}(z)} d\theta.$$

Then

$$w \cdot d\sigma_y^c(\theta) = \frac{\int_{\tilde{Y}} p(\tilde{f}(z), \theta) (-c \cdot d\rho_{(y,z)}(w)) e^{-c\rho(y,z)} dv_{\tilde{g}}(z)}{\mu_y^c(\partial \tilde{X})} d\theta - \quad (49)$$

$$\frac{d\mu_y^c(\theta)}{\mu_y^c(\partial \tilde{X})^2} \cdot \int_{\tilde{Y}} (-c \cdot d\rho_{(y,z)}(w)) e^{-c\rho(y,z)} dv_{\tilde{g}}(z). \quad (50)$$

As  $|d\rho_{(y,z)}(w)| \leq \|w\|_g$ , we have

$$\left| \int_{\partial \tilde{X}} w \cdot d\sigma_y^c(\theta) \right| \leq \int_{\partial \tilde{X}} 2c\|w\|_g d\sigma_y^c(\theta) = 2c\|w\|_g, \quad (51)$$

we get that,  $|w \cdot g_0(H_y^c(U), V)| \leq 2\|d_y F_c(w)\|_{g_0} + 2c\|w\|_g$ . We now suppose that  $w$  is a unit vector and we use (48), then

$$|w \cdot g_0(H_y^c(U), V)| \leq 2c \left( \frac{\sqrt{\lambda_n^c(y)}}{1 - \lambda_n^c(y)} + 1 \right). \quad (52)$$

Let us now consider small constants  $\eta > \beta > 0$ . One defines

$$r(\delta, \beta, \eta) = \frac{\eta - \beta}{2(n-1+\delta) \left( \frac{\sqrt{\frac{1}{n} + \eta}}{1 - (\frac{1}{n} + \eta)} + 1 \right)} > 0.$$

Our goal is to prove that

$$\inf\{d(y_0, y_1) \mid y_0, y_1 \in Y, \lambda_n^c(y_0) \leq \frac{1}{n} + \beta, \lambda_n^c(y_1) \geq \frac{1}{n} + \eta \geq r(\delta, \beta, \eta)\}.$$

Let  $y_0 \in Y$  so that  $\lambda_n^c(y_0) \leq \frac{1}{n} + \beta$ . Assume that there exists  $y \in Y$  such that  $\lambda_n^c(y) \geq \frac{1}{n} + \eta$ . One defines

$$r = \inf\{d(y_0, y) \mid y \in Y, \lambda_n^c(y) \geq \frac{1}{n} + \eta\}.$$

Clearly, there exist  $y_1 \in Y$  such that  $\lambda_n^c(y_1) = \frac{1}{n} + \eta$  and  $d(y_0, y_1) = r$ . Let  $\gamma : [0, r] \rightarrow Y$  be a minimising geodesic from  $y_0$  to  $y_1$ . We easily see that  $\lambda_n^c(\gamma(t)) < \frac{1}{n} + \eta$  for any  $0 \leq t < r$ . Let  $U(t)$  be a parallel vector field in  $X$  along  $F_c(\gamma)$  such that  $U(r)$  is a unit

eigenvector of  $H_{y_1}^c$ . Then using (52) with  $\dot{\gamma} \cdot g_0(H_{\gamma(t)}^c U(t), U(t)) = \frac{d}{dt} g_0(H_{\gamma(t)}^c U(t), U(t))$ , one has

$$|\lambda_n^c(y_1) - \lambda_n^c(y_0)| \leq |g_0(H_{\gamma(r)}^c U(r), U(r)) - g_0(H_{\gamma(0)}^c U(0), U(0))| \quad (53)$$

$$= \left| \int_0^r \frac{d}{dt} g_0(H_{\gamma(t)}^c U(t), U(t)) dt \right| \quad (54)$$

$$\leq 2c \cdot \int_0^r \frac{\sqrt{\lambda_n^c(\gamma(t))}}{1 - \lambda_n^c(\gamma(t))} + 1 dt \quad (55)$$

$$\leq 2cr \cdot \left( \frac{\sqrt{\frac{1}{n} + \eta}}{1 - (\frac{1}{n} + \eta)} + 1 \right). \quad (56)$$

Thus

$$r \geq \frac{\eta - \beta}{2(n-1+\delta) \left( \frac{\sqrt{\frac{1}{n} + \eta}}{1 - (\frac{1}{n} + \eta)} + 1 \right)} = r(\delta, \beta, \eta).$$

Now we fix  $\eta = 2\beta_n$  so that  $\gamma(\delta, \eta) \leq 1$  for any  $\delta \leq \delta_n$ . One defines  $r_n = r(\delta_n, \beta_n, 2\beta_n)$ . Recall that for  $\varepsilon \leq \varepsilon_n$  and  $\delta \leq \delta_n$ , we have  $\beta_1(\varepsilon, \delta) \leq \beta_n$ . On  $Y_{\alpha_1}$ , one has  $\lambda_n^c(y) \leq \frac{1}{n} + \beta_1(\varepsilon, \delta) \leq \frac{1}{n} + \beta_n$ . Thus if  $\lambda_n^c(y_1) \geq \frac{1}{n} + 2\beta_n$ , one has

$$d(y_1, Y_{\alpha_1}) \geq r(\delta, \beta_1(\varepsilon, \delta), 2\beta_n) \geq r(\delta_n, \beta_n, 2\beta_n) = r_n.$$

We have proved that in the  $r_n$ -neighborhood of  $Y_{\alpha_1}$ , one has  $\lambda_n^c(y) \leq \frac{1}{n} + 2\beta_n$ . It implies that

$$\|d_y F_c\| \leq (1 + \gamma(\delta, 2\beta_n))\sqrt{n} \leq 2\sqrt{n}.$$

Let us denote by  $V_{r_0}(Y_{\alpha_1})$  the  $r_0$ -neighborhood of  $Y_{\alpha_1}$ . It remains to show that  $B(y_g, R) \subset V_{r_0}(Y_{\alpha_1})$  if  $\varepsilon \leq \varepsilon(R)$  and  $\delta \leq \delta(R)$ . Recall that  $\frac{\text{vol}_g(Y_{\alpha_1})}{\text{vol}_g(Y)} \geq 1 - \alpha$ , thus

$$\text{vol}_g(Y - Y_{\alpha_1}) \leq \alpha_1 \text{vol}_g(Y) \leq \alpha_1(1 + \varepsilon) \text{vol}_{g_0}(X) = v(\varepsilon, \delta).$$

Clearly,  $v(\varepsilon, \delta) \rightarrow 0$  when  $\varepsilon, \delta \rightarrow 0$ . On the other hand, by (9) for any  $y \in B(y_g, R)$  we have

$$\text{vol}_g(B_g(y, r_0)) \geq v_n \frac{\text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(r_0))}{\text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(1 + R + r_0))} := v_0(R) > 0. \quad (57)$$

If  $v_0(R) > v(\varepsilon, \delta)$ , then for any  $y \in B(y_g, R)$  one has  $B_g(y, r_0) \not\subset Y - Y_{\alpha_1}$ , which means that  $B_g(y, r_0)$  intersects  $Y_{\alpha_1}$ . Thus  $d(y, Y_{\alpha_1}) < r_0$  and  $y \in V_{r_0}(Y_{\alpha_1})$ .

So if we define  $\varepsilon(R) > 0$ ,  $\delta(R) > 0$  to be sufficiently small constants such that  $v(\varepsilon, \delta) < v_0(R)$ , the lemma is proved.  $\square$

We now prove that  $F_c$  is almost 1-lipschitz.

**Lemma 2.13.** *For any  $R > 0$ , there exists  $\varepsilon_2(R) > 0$  and  $\delta_2(R) > 0$  such that for every  $0 < \varepsilon < \varepsilon_2(R)$  and  $0 < \delta < \delta_2(R)$ , there exists  $\kappa = \kappa(\varepsilon, \delta, R) > 0$  such that on  $B_g(y_g, R)$ :*

$$d_{g_0}(F_c(y_1), F_c(y_2)) \leq (1 + \kappa)d_g(y_1, y_2) + \kappa. \quad (58)$$

Moreover,  $\kappa(\varepsilon, \delta, R) \rightarrow 0$  as  $\varepsilon, \delta \rightarrow 0$ .

*Proof.* The idea is the following. We have proved that on  $Y_{\alpha_1}$   $d_y F_c$  is almost isometric. On the other hand,  $d_y F_c$  is uniformly bounded in  $B(y_r, R)$  if the parameters  $\varepsilon$  and  $\delta$  are sufficiently small. To prove the lemma one computes the lengths of  $F_c(\gamma)$  where  $\gamma$  is a minimising geodesic in  $B(y_g, R)$  whose intersection with  $Y_{\alpha_1}$  is large.

Fix some  $R > 0$ . We define the following constants :

If  $d > 0$ ,

$$c_1(n, d) = \sup_{0 < s/2 < r < s < d} \frac{\text{vol}_{\mathbf{H}^n}(\partial B_{\mathbf{H}^n}(s))}{\text{vol}_{\mathbf{H}^n}(\partial B_{\mathbf{H}^n}(r))}.$$

If  $\tau > 0, R > 0$ ,

$$c_2(n, \tau, R) = c_1(n, 2R)(2\tau \text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(\tau))).$$

If  $\varepsilon > 0, \delta > 0$ ,

$$\theta(\varepsilon, \delta) = 2\alpha_3^2(\varepsilon, \delta) \text{vol}_{g_0}(X) + 2(4n + 2\sqrt{n} + 1)\alpha_1(\varepsilon, \delta) \text{vol}_{g_0}(X).$$

Clearly,  $\theta(\varepsilon, \delta) \rightarrow 0$  as  $\varepsilon, \delta \rightarrow 0$ .

Let  $\tau(\varepsilon, \delta, R) > 0$  be the function defined by

$$\text{vol}_{\mathbf{H}^n}(\tau)\tau = \theta(\varepsilon, \delta) \frac{2c_1(n, 2R) \text{vol}_{\mathbf{H}^n}(1 + R + 1)^2}{v_n^2}.$$

Again we easily see that,  $\tau(\varepsilon, \delta, R) \rightarrow 0$  as  $\varepsilon, \delta \rightarrow 0$ . One defines  $\varepsilon_2(R) > 0$  and  $\delta_2(R) > 0$  such that  $\varepsilon_2(R) \leq \varepsilon(2R)$ ,  $\delta_2(R) < \delta(2R)$  and if  $0 < \varepsilon \leq \varepsilon_2(R)$  and  $0 < \delta < \delta_2(R)$  then  $\tau(\varepsilon, \delta, R) \ll 1$ .

Finally, one defines  $\kappa(\varepsilon, \delta, R) = \max(2\sqrt{n}\sqrt{\tau}, 8\sqrt{\tau})$ .

There are two cases.

**Case i)** let  $y_1, y_2$  in  $B_g(y_g, R)$  such that  $d(y_1, y_2) \leq \sqrt{\tau}$ . Using (46), if  $0 < \varepsilon < \varepsilon_2(R)$ ,  $0 < \delta < \delta_2(R)$  one has

$$d(F_c(y_1), F_c(y_2)) \leq 2\sqrt{n}\sqrt{\tau} \leq \kappa. \quad (59)$$

**Case ii) :** let  $y_1, y_2$  in  $B_g(y_g, R)$  such that  $d(y_1, y_2) \geq \sqrt{\tau}$ . We use [Col] proposition 2.11 with the function

$$e(y) = \sup_{u \in U_y Y} (\|d_y F_c \cdot u\| - \|u\|)^2.$$

Let us define  $A_1 = B_g(y_1, \tau)$ ,  $A_2 = B_g(y_2, \tau)$  and  $W = B_g(y_g, 2R)$ . For any  $z_1 \in A_1$  and any unit vector  $v_1 \in T_{z_1} Y$ , the measure  $|I(z_1, v_1)|$  of

$$I(z_1, v_1) = \{t \mid \gamma(t) \in A_2, \gamma|_{[0,t]} \text{ is minimal}, \gamma'(0) = v_1\}$$



is bounded by  $2\tau$ . Thus

$$D(A_1, A_2) := \sup_{z_1, v_1} |I(z_1, v_1)| \leq 2\tau.$$

Similarly,  $D(A_2, A_1) \leq 2\tau$ . For any  $z_1 \in A_1$  and  $z_2 \in A_2$ , let  $\gamma_{z_1 z_2}$  be a minimizing geodesic from  $z_1$  to  $z_2$ . Clearly,  $\gamma \subset B(y_g, 2R)$ . Then by [Col] proposition 2.11 we have

$$\begin{aligned} \int_{A_1 \times A_2} \int_0^{d(z_1, z_2)} e(\gamma_{z_1, z_2})(s) ds &\leq c_1(n, 2R) (D(A_1, A_2) \text{vol}(A_1) + D(A_2, A_1) \text{vol}(A_2)) \\ &\quad \times \int_W e(y) dv_g(y). \end{aligned}$$

By Bishop's theorem, for  $i = 1, 2$  we have

$$\text{vol}_g(A_i) \leq \text{vol}_{\mathbf{H}^n}(B_{\mathbf{H}^n}(\tau)),$$

and thus

$$c_1(n, 2R) (D(A_1, A_2) \text{vol}(A_1) + D(A_2, A_1) \text{vol}(A_2)) \leq c_2(n, \tau, R).$$

We then have, using (35) on  $W \cap Y_{\alpha_1}$  and (46) on  $W - Y_{\alpha_1}$ ,

$$\begin{aligned} \int_{A_1 \times A_2} \int_0^{d(z_1, z_2)} e(\gamma_{z_1, z_2})(s) ds &\leq c_2(n, \tau, R) \left( \int_{W \cap Y_{\alpha_1}} e(y) dv_g(y) + \int_{W - Y_{\alpha_1}} e(y) dv_g(y) \right) \\ &\leq c_2(n, \tau, R) (\alpha_3^2 \cdot \text{vol}_g(Y) + (4n + 2\sqrt{n} + 1) \text{vol}_g(Y - Y_{\alpha_1})) \\ &\leq c_2(n, \tau, R) \theta(\varepsilon, \delta). \end{aligned} \quad (60)$$

Now, if we denote  $\gamma = \gamma_{z_1 z_2}$ , we have

$$\begin{aligned} |\ell(F_c \circ \gamma) - \ell(\gamma)| &= \left| \int_0^{d(z_1, z_2)} \|d_{\gamma(s)} F_c(\dot{\gamma})\| - \|\dot{\gamma}\| ds \right| \\ &\leq \int_0^{d(z_1, z_2)} \sup_{u \in T_y Y} \| \|d_{\gamma(s)} F_c(u)\| - \|u\| \| ds. \end{aligned}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned} \frac{|\ell(F_c \circ \gamma) - \ell(\gamma)|^2}{d(z_1, z_2)} &\leq \frac{\left( \int_0^{d(z_1, z_2)} \sup_u \| \|d_{\gamma(s)} F_c(u)\| - \|u\| \| ds \right)^2}{d(z_1, z_2)} \\ &\leq \int_0^{d(z_1, z_2)} e(\gamma(s)) ds. \end{aligned}$$

Integrating on  $A_1 \times A_2$ , we deduce from (60) that

$$\int_{A_1 \times A_2} \frac{|\ell(F_c \circ \gamma_{z_1 z_2}) - \ell(\gamma_{z_1 z_2})|^2}{d(z_1, z_2)} dv_g(z_1) dv_g(z_2) \leq c_2(n, \tau, R) \theta(\varepsilon, \delta). \quad (61)$$

By (9), for  $i = 1, 2$  one has

$$\text{vol}_g(A_i) \geq v_n \frac{\text{vol}_{\mathbf{H}^n}(\mathbf{B}_{\mathbf{H}^n}(\tau))}{\text{vol}_{\mathbf{H}^n}(\mathbf{B}_{\mathbf{H}^n}(1 + R + \tau))} := v_0(\tau, R) > 0.$$

Thus

$$c_2(n, \tau, R)\theta(\varepsilon, \delta) \leq \frac{1}{v_0(\tau, R)^2} \int_{A_1 \times A_2} c_2(n, \tau, R)\theta(\varepsilon, \delta) \, \text{d}v_g(z_1)\text{d}v_g(z_2).$$

We then have

$$\int_{A_1 \times A_2} \frac{|\ell(F_c \circ \gamma_{z_1 z_2}) - \ell(\gamma_{z_1 z_2})|^2}{d(z_1, z_2)} \leq \int_{A_1 \times A_2} \frac{c_2(n, \tau, R)\theta(\varepsilon, \delta)}{v_0(\tau, R)^2}. \quad (62)$$

Hence there exists  $z_1 \in A_1, z_2 \in A_2$  such that

$$|\ell(F_c \circ \gamma_{z_1 z_2}) - \ell(\gamma_{z_1 z_2})|^2 \leq d(z_1, z_2) \frac{c_2(n, \tau, R)\theta(\varepsilon, \delta)}{v_0(\tau, R)^2}.$$

One can check that

$$\frac{c_2(n, \tau, R)\theta(\varepsilon, \delta)}{v_0(\tau, R)^2} \leq \theta(\varepsilon, \delta) \frac{2c_1(n, 2R) \text{vol}_{\mathbf{H}^n}(1 + R + 1)^2}{v_n^2 \text{vol}_{\mathbf{H}^n}(\tau)} \tau \leq \tau^2.$$

Now one has

$$|\ell(F_c \circ \gamma_{z_1 z_2}) - \ell(\gamma_{z_1 z_2})|^2 \leq d(z_1, z_2)\tau^2,$$

thus

$$d(F_c(z_1), F_c(z_2)) \leq \ell(F_c \circ \gamma_{z_1 z_2}) \leq d(z_1 z_2) + \tau d(z_1, z_2)^{1/2}.$$

As  $d(y_i, z_i) < \tau$  and  $d(y_1, y_2) \geq \sqrt{\tau}$ , we have

$$d(z_1, z_2) \leq d(y_1, y_2) + 2\tau \leq d(y_1, y_2)(1 + 2\sqrt{\tau}).$$

On the other hand, as  $\tau \ll 1$  we have

$$d(z_1, z_2) \geq d(y_1, y_2) - 2\tau \geq \frac{\sqrt{\tau}}{2}.$$

We then have

$$d(F_c(y_1), F_c(y_2)) \leq d(F_c(y_1), F_c(z_1)) + d(F_c(z_1), F_c(z_2)) + d(F_c(z_2), F_c(y_2)) \quad (63)$$

$$\leq 2\sqrt{n}\tau + d(z_1, z_2) + \tau \cdot d(z_1, z_2)^{1/2} + 2\sqrt{n}\tau \quad (64)$$

$$\leq 4\sqrt{n}\tau + d(y_1, y_2) \frac{d(z_1, z_2)}{d(y_1, y_2)} (1 + \tau \cdot d(z_1, z_2)^{-1/2}) \quad (65)$$

$$\leq 4\sqrt{n}\tau + d(y_1, y_2)(1 + 2\sqrt{\tau})(1 + \sqrt{2}\tau^{3/4}) \quad (66)$$

$$\leq 4\sqrt{n}\tau + d(y_1, y_2)(1 + 8\sqrt{\tau}). \quad (67)$$

Finally, we get

$$d(F_c(y_1), F_c(y_2)) \leq \kappa + (1 + \kappa)d(y_1, y_2), \quad (68)$$

in case ii).  $\square$

### 3 A limit map on the limit space

In this section, we consider a sequence  $(Y_k, g_k)_{k \in \mathbf{N}}$  of Riemannian compact  $n$ -manifolds satisfying the curvature assumption (1) and the following condition: we suppose that there exists an hyperbolic compact  $n$ -manifold  $(X, g_0)$ , degree one maps  $f_k : Y_k \rightarrow X$  and a sequence  $\varepsilon_K \rightarrow 0$  such that

$$\text{vol}_{g_k}(Y_k) \rightarrow \text{vol}_{g_0}(X), \quad (69)$$

as  $k \rightarrow +\infty$ . Moreover, for every  $k \in \mathbf{N}$ , there exists  $y_{g_k} \in Y_k$  satisfying the local volume property (6).

We prove that  $(Y_k, g_k, y_{g_k})$  sub-converges in the pointed Gromov-Haudorff topology to a limit space  $(Y_\infty, d_\infty, z_\infty)$ . Moreover, there exists a sequence of natural maps  $F_{c_k} : (Y_k, g_k) \rightarrow (X, g_0)$ , with suitably chosed parameters, which sub-converges to a "natural map"  $F : Y_\infty \rightarrow X$ .

Let us define the Gromov-Hausdorff topology. Recall that for two subsets  $A, B$  of a metric space  $Z$  the Hausdorff distance between  $A$  and  $B$  is

$$d_{\mathcal{H}}^Z(A, B) = \inf\{\varepsilon > 0 \mid B \subset V_\varepsilon(A) \text{ and } A \subset V_\varepsilon(B)\} \in \mathbf{R} \cup \{\infty\}.$$

It is a distance on compact subsets of  $Z$ .

**Definition 3.1.** *Let  $X_1, X_2$  be two metric spaces, then the Gromov-Hausdorff distance  $d_{\mathcal{GH}}(X_1, X_2) \in \mathbf{R} \cup \infty$  is the infimum of the numbers*

$$d_{\mathcal{H}}^Z(f_1(X_1), f_2(X_2))$$

*for all metric spaces  $Z$  and all isometric imbeddings  $f_i : X_i \rightarrow Z$ .*

It is a distance on the space of isometry classes of compact metric spaces. One says that a sequence  $(X_i)_{i \in \mathbf{N}}$  of metric spaces converges in the Gromov-Hausdorff topology to a metric space  $X_\infty$  if  $d_{\mathcal{GH}}(X_i, X_\infty) \rightarrow 0$  as  $i \rightarrow \infty$ . Let  $x_i \in X_i, x_\infty \in X_\infty$ , one says that the sequence  $(X_i, x_i)_{i \in \mathbf{N}}$  converges to  $(X_\infty, x_\infty)$  in the pointed Gromov-Hausdorff topology if for any  $R > 0$ ,  $d_{\mathcal{GH}}(B_{X_i}(x_i, R), B_{X_\infty}(x_\infty, R)) \rightarrow 0$  as  $i \rightarrow +\infty$  for some sequence  $(\varepsilon_i)$  tending to 0 (In fact this definition holds only for length spaces, which will be sufficient in our situation).

To deal with the Gromov-Hausdorff distance between  $X_1$  and  $X_2$ , it is convenient to avoid the third space  $Z$  by using  $\varepsilon$ -approximations between  $X_1$  and  $X_2$ .

**Definition 3.2.** *Given two metric spaces  $X_1, X_2$  and  $\varepsilon > 0$ , an  $\varepsilon$ -approximation (or  $\varepsilon$ -isometry) from  $X_1$  to  $X_2$  is a map  $f : X_1 \rightarrow X_2$  such that*

1. *for any  $x, x' \in X_1$ ,  $|d_{X_2}(f(x), f(x')) - d_{X_1}(x, x')| < \varepsilon$ .*
2. *the  $\varepsilon$ -neighbourhood of  $f(X_1)$  is equal to  $X_2$ .*

Then one can show (see [BBI] corollary 7.3.28) that  $d_{\mathcal{GH}}(X_1, X_2) < \varepsilon$  if and only if there exists a  $2\varepsilon$ -approximation from  $X_1$  to  $X_2$ .

Our goal is to prove the :

**Proposition 3.3.** *Up to extraction and renumbering, the sequence  $(Y_k, g_k, y_k)$  satisfies the following.*

1. *There exists a complete pointed length space  $(Y_\infty, d_\infty, y_\infty)$  such that  $(Y_k, g_k, y_k)$  converges in the pointed Gromov-Hausdorff topology to  $(Y_\infty, d_\infty, y_\infty)$ . Moreover,  $(Y_\infty, d_\infty)$  has Hausdorff dimension equal to  $n$ .*
2. *there exists sequences of positive numbers  $(\varepsilon_k)$ ,  $(\delta_k)$  tending to 0,  $(c_k)$  such that  $h(g_k) < c_k < h(g_k) + \delta_k$ ,  $(R_k)$  tending to  $+\infty$  such that  $\varepsilon_k \leq \varepsilon(R_k)$  and  $\delta_k \leq \delta(R_k)$ , and  $\alpha_k$ -approximations  $\psi_k : B_{d_\infty}(y_\infty, R_k) \rightarrow B_{g_k}(y_{g_k}, R_k)$  such that the following holds. Let*

$$F_{c_k} : (Y_k, g_k) \rightarrow (X, g_0)$$

*be the natural map as defined in section 2. Then  $F_{c_k} \circ \psi_k$  converges uniformly on compact sets to a map*

$$F : Y_\infty \longrightarrow X,$$

*which is 1-lipschitz.*

## Proof of (1) and some properties of the limit

Under the curvature assumption (1) and the local volume assumption (9), the point (1) is a straightforward application of Gromov & Cheeger-Colding compactness theorem. See ([Ch-Co]). Let us make precise some features of the convergence and of the limit space.

The continuity of the volume under the (pointed) Gromov-Hausdorff convergence is crucial for our purposes. For  $\ell > 0$ , note  $\mathcal{H}^\ell$  the  $\ell$ -dimensional Hausdorff measure of a metric space (see [BBI] definition 1.7.7).

**Theorem 3.4 ([Ch-Co] 5.9).** *Let  $p_i \in Y_i$  and  $p_\infty \in Y_\infty$  their limit, and let  $R > 0$ . Then*

$$\lim_{i \rightarrow +\infty} \text{vol}_{g_i}(B(p_i, R)) = \mathcal{H}^n(B(p_\infty, R)). \quad (70)$$

In particular,  $Y_\infty$  satisfies the Bishop-Gromov inequalities (7) and the Bishop inequality. By definition, a *tangent cone* at  $p \in Y_\infty$  is a complete pointed Gromov-Hausdorff limit,  $\{Y_{\infty,p}, d_\infty, p_\infty\}$  of a sequence of rescaled space,  $\{(Y_\infty, r_i^{-1}d, p)\}$ , where  $\{r_i\}$  is a positive sequence such that  $r_i \rightarrow 0$ . By [GLP] theorem 5.2, every such sequence has a convergent subsequence, but the limit might depend on the choice of the subsequence. The limit is called a *tangent cone* because  $(Y_{\infty,p}, \lambda d_\infty)$  is isometric to  $(Y_{\infty,p}, d_\infty)$  for any  $\lambda > 0$ .

**Definition 3.5.** The **regular** set  $\mathcal{R}$  consists of those points,  $p \in Y_\infty$ , such that every tangent cone at  $y$  is isometric to  $\mathbf{R}^n$ . The complementary  $\mathcal{S} = Y_\infty - \mathcal{R}$  is the **singular** set.

Let  $B_0^n(1) \subset \mathbf{R}^n$  be the unit ball.

**Definition 3.6.** The  $\varepsilon$ -**regular** set  $\mathcal{R}_\varepsilon$  consists of those points,  $p \in Y_\infty$ , such that every tangent cone,  $(Y_{\infty,p}, p_\infty)$ , satisfies  $d_{GH}(B(p_\infty, 1), B_0^n(1)) < \varepsilon$ . A point in  $Y_\infty - \mathcal{R}_\varepsilon = \mathcal{S}_\varepsilon$  is called  $\varepsilon$ -**singular**,

**Theorem 3.7** ([Ch-Co] 5.14). There exists  $\varepsilon_n > 0$  such that for  $\varepsilon \leq \varepsilon_n$ ,  $\mathring{\mathcal{R}}_\varepsilon$  has a natural smooth manifold structure. Moreover, for this parametrization, the metric on  $\mathring{\mathcal{R}}_\varepsilon$  is bi-hölder equivalent to a smooth Riemannian metric. The exponent  $\alpha(\varepsilon)$  in this bi-hölder equivalence satisfies  $\alpha(\varepsilon) \rightarrow 1$  as  $\varepsilon \rightarrow 0$ .

**Theorem 3.8** ([Ch-Co]6.1).

$$\mathcal{H}^{n-2}(\mathcal{S}) = 0 \tag{71}$$

**Remark 3.9.** Clearly,  $\mathcal{R} = \bigcap_{\varepsilon > 0} \mathcal{R}_\varepsilon$ . The sets  $\mathcal{R}_\varepsilon, \mathcal{R}$  are not necessarily open. However, for any  $\varepsilon > 0$ , there is some  $\delta > 0$  such that  $\mathcal{R}_\delta \subset \mathring{\mathcal{R}}_\varepsilon$  (see Appendix A.1.5 in [Ch-Co]). In [Ch-Co2] section 3, it is proved that  $\mathring{\mathcal{R}}_\varepsilon$  is path connected. This important fact will be used in the last part of this text.

Let us study the density of this hausdorff measure. A consequence of Bishop's inequality is that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^n(B(p, r))}{\text{vol}_{\mathbf{R}^n}(r)} \leq 1.$$

**Definition 3.10.** the density at  $p$  of  $Y_\infty$  is

$$\theta(p) := \liminf_{r \rightarrow 0} \frac{\mathcal{H}^n(B(p, r))}{\text{vol}_{\mathbf{R}^n}(r)}. \tag{72}$$

A consequence of [Ch-Co]A.1.5 is the existence of some positive function  $\tau(\varepsilon)$ , with  $\tau(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , such that for every  $p \in \mathcal{R}_\varepsilon$ ,

$$\theta(p) > 1 - \tau(\varepsilon). \tag{73}$$

Conversely, there exists a positive function  $\varepsilon(\tau)$ , such that  $\varepsilon(\tau) \rightarrow 0$  as  $\tau \rightarrow 0$  and such that

$$\theta(p) \geq 1 - \tau \implies p \in \mathcal{R}_{\varepsilon(\tau)}. \tag{74}$$

**Remark 3.11.** A point  $p$  is called regular if and only if  $\theta(p) = 1$ . From now on, we consider  $\varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0 \leq \varepsilon_n$  is sufficiently small so that  $\tau(\varepsilon_0) \leq 1/2$ , the density is thus  $> 1/2$  on  $\mathcal{R}_\varepsilon$ .

## Existence of the natural map at the limit

For every  $k \in \mathbf{N}$  and  $h(g_k) < c$ , there exists a natural map  $F_c : (Y_k, g_k) \rightarrow (X, g_0)$ , as seen in section 2. To define a good sequence one argues as follows. Given  $m \in \mathbf{N}^*$ , one chooses positive numbers  $\varepsilon_m \leq \varepsilon_2(m)$  and  $\delta_m \leq \delta_2(m)$  sufficiently small such that  $\kappa(\varepsilon_m, \delta_m, m) \leq \frac{1}{m}$ . One defines

$$\alpha_m = \max(\alpha_1(\varepsilon_m, \delta_m), \alpha_2(\varepsilon_m, \delta_m), \alpha_3(\varepsilon_m, \delta_m)\kappa(\varepsilon_m, \delta_m, m)).$$

We check that  $\alpha_m \rightarrow 0$  as  $m \rightarrow +\infty$ . By the hypothesis (69), there exists  $k_1(m) \in \mathbf{N}$  such that for any  $k \geq k_1(m)$ ,  $\text{vol}_{g_k}(Y_k) \leq (1 + \varepsilon_m)\text{vol}_{g_0}(X)$ . As  $B_{g_k}(y_{g_k}, m)$  tends to  $B_\infty(y_\infty, m)$ , there exists  $k_2(m) \in \mathbf{N}$  such that for any  $k \geq k_2(m)$ , there exists  $\alpha_m$ -approximations from  $B_\infty(y_\infty, m)$  to  $B_{g_k}(y_{g_k}, m)$ . Define  $k(m) = \max(k_1(m), k_2(m))$  and let  $\psi_m : B_\infty(y_\infty, m) \rightarrow B_{g_k}(y_{g_k}, m)$  be an  $\alpha_m$ -approximation. One can suppose that  $\psi_m(y_\infty) = y_{g_k(m)}$ . Fix  $c_m \in ]h(g_k), h(g_k) + \delta_m[$  and consider

$$F_{c_m} \circ \psi_m : B_\infty(y_\infty, m) \rightarrow X.$$

The lemma (2.13) applies to  $F_{c_m}$  on  $B_{g_k(m)}(y_{g_k(m)}, m)$ . Thus for any  $p, q \in B_\infty(y_\infty, m)$ ,

$$\begin{aligned} d_{g_0}(F_{c_m} \circ \psi_m(p), F_{c_m} \circ \psi_m(q)) &\leq (1 + \alpha_m)d_{g_k}(\psi_m(p), \psi_m(q)) + \alpha_m \\ &\leq (1 + \alpha_m)d_\infty(p, q) + (1 + \alpha_m)\alpha_m + \alpha_m. \end{aligned}$$

Applying the same reasoning as in Ascoli's theorem, one can show that for any compact  $K \subset Y_\infty$ , there exists a convergent sub-sequence of  $F_{c_m}$  towards a map  $F_K : K \rightarrow X$ . If one uses an exhaustion of  $Y_\infty$  by compact sets and a standard diagonal process, one can extract a sub-sequence of  $F_{c_{\psi(m)}} \circ \psi_{\phi(m)}$  which converges uniformly on any compact set to a map  $F : Y_\infty \rightarrow X$ . It is easy to see that the map  $h$  is 1-lipschitz.

Then one rennumbers the sub-sequences  $Y_{k(\phi(m))}$ ,  $\psi_{\phi(m)}$  and  $F_{c_{\phi(m)}}$  such that for any  $m \in \mathbf{N}^*$ ,  $\text{vol}_{g_m}(Y_m) \leq (1 + \varepsilon_m)\text{vol}_{g_0}(X)$ ,  $h(g_m) < c_m < c_m + \delta_m$ , and the inequalities of the lemmas (2.5), (2.11) hold with  $\alpha_1, \alpha_2, \alpha_3$  replaced by  $\alpha_m$  and those of lemmas (2.12), (2.13) hold on  $B(y_m, m) \subset Y_m$  with  $\kappa$  replaced by  $\alpha_m$ . For simplicity, the map  $F_{c_m}$  will be denoted  $F_m$ .

## 4 The limit map $F : Y_\infty \rightarrow X$ is isometric

We prove first that  $F$  preserves the volume.

**Lemma 4.1.** *Let  $A \subset Y_\infty$  a measurable subset. Then,*

$$\text{vol}_{g_0}(F(A)) = \mathcal{H}^n(A). \tag{75}$$

*Proof.* As  $F$  is 1-lipschitz, it decreases the Hausdorff measure (see [Mor]3.5). Thus, it suffices to prove that for every  $B_\infty(p, r) \subset Y_\infty$ ,  $\text{vol}_{g_0}(F(B_\infty(p, r))) \geq \mathcal{H}^n(B_\infty(p, r))$ . By construction,  $\overline{F(B_\infty(p, r))}$  is the Hausdorff limit of  $\overline{F_k \circ \psi_k(B_\infty(p, r))}$ .

We first show that this is the Hausdorff limit of  $\overline{F_k(B_{g_k}(\psi_k(p), r))}$ . Let  $x \in \overline{F(B_\infty(p, r))}$ . There exists  $x_k \in F(B_\infty(p, r))$  such that  $x_k \rightarrow x$ . Let  $p_k \in B_\infty(p, r)$  such that  $F(p_k) = x_k$ . By definition of the  $\alpha_k$ -approximation, one has  $d_{g_k}(\psi_k(p_k), \psi_k(p)) < r + \alpha_k$ . There exists  $z_k \in B_{g_k}(\psi_k(p), r)$  such that  $d_{g_k}(\psi_k(p_k), z_k) < \alpha_k$  (for example  $z_k$  may be on the segment  $[\psi_k(p_k), \psi_k(p)]$ ). Note that  $d_\infty(p_k, y_\infty) \leq r + d_\infty(p, y_\infty)$  and remember that  $\psi_k(y_\infty) = y_{g_k}$ . Thus  $\psi_k(p_k)$  remains at bounded distance from  $y_{g_k}$ . Then, applying the inequality of lemma (2.13) we have

$$\begin{aligned} d_{g_0}(F_k(z_k), F_k(\psi_k(p_k))) &\leq (1 + \alpha_k)d_{g_k}(z_k, \psi_k(p_k)) + \alpha_k \\ &\leq (1 + \alpha_k)\alpha_k + \alpha_k \\ &\xrightarrow[k \rightarrow +\infty]{} 0. \end{aligned}$$

On the other hand, as  $F_k$  converges uniformly to  $F$  on compact sets,  $F_k(\psi_k(p_k))$  has the same limit as  $F(p_k) = x_k$ , thus  $\overline{F_k(\psi_k(p_k))} \rightarrow x$ . From the inequality above one deduces that  $\overline{F_k(z_k)} \rightarrow x$  thus  $x \in \lim_{k \rightarrow \infty} \overline{F_k(B_{g_k}(\psi_k(p), r))}$ . One has proved that  $\overline{F(B_\infty(p, r))} \subset \lim_{k \rightarrow \infty} \overline{F_k(B_{g_k}(\psi_k(p), r))}$ . To prove the other inclusion one argues in the same way. Given  $x \in \lim_{k \rightarrow \infty} \overline{F_k(B_{g_k}(\psi_k(p), r))}$ , there exists  $x_k \in F_k(B_{g_k}(\psi_k(p), r))$  such that  $x_k \rightarrow x$ , thus one can write  $x_k = F_k(z_k)$  where  $z_k \in B_{g_k}(\psi_k(p), r)$ . As  $\psi_k$  is an  $\alpha_k$ -approximation from  $B_\infty(y_\infty, k)$  to  $B(y_{g_k}, k)$ , one has  $B_{g_k}(\psi_k(p), r) \subset U_{\alpha_k} \psi_k(B_\infty(p, r + \alpha_k))$  for large  $k$ . Thus there exists  $q_k \in B_\infty(p, r + \alpha_k)$  such that  $d_{g_k}(z_k, \psi_k(q_k)) < r + \alpha_k$ . As  $Y_\infty$  is a length space, there exists  $q'_k \in B_\infty(p, r)$  such that  $d_\infty(q'_k, q_k) < \alpha_k$ . Then  $d_{g_k}(\psi_k(q'_k), z_k) \leq d_{g_k}(\psi_k(q'_k), \psi_k(q_k)) + d_{g_k}(\psi_k(q_k), z_k) < 3\alpha_k$ . Thus

$$\begin{aligned} d_{g_0}(F_k \circ \psi_k(q'_k), x_k) = d_{g_0}(F_k \circ \psi_k(q'_k), F_k(z_k)) &\leq (1 + \alpha_k)d_{g_k}(\psi_k(q'_k), z_k) + \alpha_k \\ &\leq (1 + \alpha_k)3\alpha_k + \alpha_k \rightarrow 0. \end{aligned}$$

Hence  $d_{g_0}(F_k \circ \psi_k(q'_k), x) \rightarrow 0$ . As  $\overline{F_k \circ \psi_k}$  converges uniformly to  $F$  on compact sets, one has  $d_{g_0}(F(q'_k), x) \rightarrow 0$  thus  $x \in \overline{F(B_\infty(p, r))}$ .

It then suffices to prove that  $\lim_{k \rightarrow +\infty} \text{vol}_{g_0}(F_k(B_{g_k}(\psi_k(p), r))) \geq \mathcal{H}^n(B_\infty(p, r))$ . Remember that  $N(F_k|x)$  is the number of preimages of  $x$  by  $F_k$ . We denote by  $X_{k,1}$  the set of  $x \in X$  such that  $N(F_k|x) = 1$ . The construction of the sequence  $(F_k)$  and the lemma 2.10 implies that  $\text{vol}_{g_0}(X_{k,1}) \geq (1 - \alpha_k) \text{vol}_{g_0}(X)$ . Moreover,

$$\int_{X - X_{k,1}} N(F_k|x) d\nu_{g_0}(x) \leq \alpha_k \text{vol}_{g_0}(X). \quad (76)$$

We denote by  $Y_{k, \alpha_k}$  the set of  $y \in Y_k$  such that

$$1 - \alpha_k \leq |\text{Jac } F_k(y)| \leq 1 + \alpha_k. \quad (77)$$

Then lemma 2.9 implies that  $\text{vol}_{g_k}(Y_{k,1}) \geq (1 - \alpha_k) \text{vol}_{g_k}(Y_k)$ . We then have

$$\begin{aligned}
\text{vol}(F_k(B_{g_k}(\psi_k(p), r))) &= \int_{F_k(B_{g_k}(\psi_k(p), r))} \text{d}v_{g_0} \\
&= \int_{F_k(B_{g_k}(\psi_k(p), r)) \cap X_{k,1}} N(F_k|x) \text{d}v_{g_0}(x) + \text{vol}_{g_0}(F_k(B_{g_k}(\psi_k(p), r)) - X_{k,1}) \\
&\geq \int_{B_{g_k}(\psi_k(p), r) \cap F_k^{-1}(X_{k,1}) \cap Y_{k,\alpha_k}} |\text{Jac } F_k(y)| \text{d}v_g(y) \\
&\geq (1 - \alpha_k) \text{vol}_{g_k}(B_{g_k}(\psi_k(p), r) \cap F_k^{-1}(X_{k,1}) \cap Y_{k,\alpha_k}). \tag{78}
\end{aligned}$$

On the other hand, using (77) and (76) we have

$$\begin{aligned}
\text{vol}(F_k^{-1}(X - X_{k,1}) \cap Y_{k,\alpha_k}) &\leq \int_{F_k^{-1}(X - X_{k,1}) \cap Y_{k,\alpha_k}} \frac{|\text{Jac } F_k|}{1 - \alpha_k} \text{d}v_g \\
&\leq \frac{1}{1 - \alpha_k} \int_{X - X_{k,1}} N(F_k|x) \text{d}v_{g_0}(x) \\
&\leq \frac{\alpha_k}{1 - \alpha_k} \text{vol}_{g_0}(X),
\end{aligned}$$

thus

$$\begin{aligned}
\text{vol}_{g_k}(B_{g_k}(\psi_k(p), r) \cap F_k^{-1}(X_{k,1}) \cap Y_{k,\alpha_k}) &= \text{vol}_{g_k}(B_{g_k}(\psi_k(p), r) \cap Y_{k,\alpha_k}) \\
&\quad - \text{vol}_{g_k}(B_{g_k}(\psi_k(p), r) \cap F_k^{-1}(X - X_{k,1}) \cap Y_{k,\alpha_k}) \\
&\geq \text{vol}_{g_k}(B_{g_k}(\psi_k(p), r)) - \alpha_k \text{vol}_{g_k}(Y_k) - \frac{\alpha_k}{1 - \alpha_k} \text{vol}_{g_0}(X).
\end{aligned}$$

By putting this inequality in (78) one gets

$$\text{vol}_{g_0}(F_k(B_{g_k}(\psi_k(p), r))) \geq (1 - \alpha_k) \text{vol}_{g_k}(B_{g_k}(\psi_k(p), r)) - (1 - \alpha_k) \alpha_k \text{vol}_{g_k}(Y_k) - \alpha_k \text{vol}_{g_0}(X).$$

As  $B_{g_k}(\psi_k(p), r)$  tends to  $B_\infty(p, r)$  in the Gromov-Hausdorff topology, theorem 3.4 implies that  $\lim_{k \rightarrow \infty} \text{vol}_{g_k}(B_{g_k}(\psi_k(p), r)) = \mathcal{H}^n(B_\infty(p, r))$ , hence

$$\lim_{k \rightarrow \infty} \text{vol}_{g_0}(F_k(B_{g_k}(\psi_k(p), r))) \geq \mathcal{H}^n(B_\infty(p, r)),$$

which proves the lemma.  $\square$

We now prove that  $F$  is injective on the set of points where the density is larger than  $1/2$ .

**Lemma 4.2.** *The map  $F$  is injective on  $\mathcal{R}_\varepsilon$ .*

*Proof.* Suppose there are  $p_1, p_2 \in \mathcal{R}_\varepsilon$  such that  $F(p_1) = F(p_2)$ . As  $F$  is 1-lipschitz, we have for every  $r > 0$ ,

$$F(B_\infty(p_1, r) \cup B_\infty(p_2, r)) \subset B_{g_0}(F(p_1), r).$$



By the previous lemma,

$$\mathcal{H}^n(B_\infty(p_1, r) \cup B_\infty(p_2, r)) = \text{vol}_{g_0}(F(B_\infty(p_1, r) \cup B_\infty(p_2, r))) \leq \text{vol}_{g_0}(B_\infty(F(p_1), r)). \quad (79)$$

For  $r < \frac{d(p_1, p_2)}{2}$  the balls  $B_\infty(p_1, r)$  and  $B_\infty(p_2, r)$  are disjoint. Hence, dividing (79) by  $\text{vol}_{\mathbf{R}^n}(r)$ , we get

$$\frac{\mathcal{H}^n(B_\infty(p_1, r))}{\text{vol}_{\mathbf{R}^n}(r)} + \frac{\mathcal{H}^n(B_\infty(p_2, r))}{\text{vol}_{\mathbf{R}^n}(r)} \leq \frac{\text{vol}_{g_0}(B_{g_0}(F(p_1), r))}{\text{vol}_{\mathbf{R}^n}(r)}.$$

By taking the liminf as  $r \rightarrow 0$ , we have

$$\theta(p_1) + \theta(p_2) \leq \theta(F(p_1)) = 1,$$

which is a contradiction, since  $\theta \geq 1/2$  on  $\mathcal{R}_\varepsilon$  if  $\varepsilon < \varepsilon_0$  (see remark 3.11).  $\square$

**Lemma 4.3.** *The map  $F$  is open on  $\overset{\circ}{\mathcal{R}}_\varepsilon$ .*

*Proof.* Let  $p \in \overset{\circ}{\mathcal{R}}_\varepsilon$ . We have to prove that there exists  $\eta > 0$  such that  $B_{g_0}(F(p), \eta) \subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ . There exists  $r > 0$  such that  $B_\infty(p, 2r) \subset \overset{\circ}{\mathcal{R}}_\varepsilon$ . Note that  $B := B_\infty(p, r)$ . By the previous lemma,  $F(p) \notin F(\partial B)$ . Thus, by compactness of  $\partial B$  and continuity of  $F$ , there exists  $\eta > 0$  such that  $d_{g_0}(F(p), F(\partial B)) > \eta$ . One could use the theory of local degree but unfortunately, we do not know if  $(Y_\infty, y_\infty)$  is locally lipschitz equivalent to  $\mathbf{R}^n$ . Let  $R > 2d(z, z_0)$  be a fixed radius such that  $\psi_k(B_\infty(p, 2r)) \subset B_{g_k}(y_{g_k}, R)$  for all large  $k$ . Let  $z_k = \psi_k(p)$  and  $B_k = B(z_k, r) \subset B(y_k, R)$ . Consider all integer  $k$  large enough such that  $d_{\mathcal{H}}(F_k(\partial B_k), F(\partial B)) \leq \frac{\eta}{10}$ . Let  $\mathcal{C}$  (resp.  $\mathcal{C}_k$ ) be the connected component of  $X - F(\partial B)$  (resp.  $X - F_k(\partial B_k)$ ), which contains  $F(p)$ , (resp.  $F_k(z_k)$ ). By the corollary 4.1.26 of [Fed],  $\deg(F_k|B_k)$  is constant on  $\mathcal{C}_k$ , where, for a subset  $A \subset Y_k$ ,

$$\deg(F_k|A)(x) = \sum_{y \in F_k^{-1}(x) \cap A} \text{sign Jac } F_k(y).$$

One sees that  $\deg(F_k|B_k) = 1$  on  $\mathcal{C}_k$  as follows. Denote again by  $X_{k,1} \subset X$  the set of  $x \in X$  such that  $N(F_k|x) = 1$ , that is  $x$  has one preimage by  $F_k$ . By the lemma 2.10,  $\text{vol}_{g_0}(X_{k,1}) \geq (1 - \alpha_k) \text{vol}_{g_0}(X)$ . Clearly, the intersection of  $X_{k,1}$  with  $\mathcal{C}_k$  has a positive measure for  $k$  large enough; indeed, the volume of  $B(F_k(z_k), \frac{\eta}{10}) \subset \mathcal{C}_k$  is bounded below by (9) and  $\text{vol}(B(F_k(z_k), \frac{\eta}{10}) - X_{k,1}) \rightarrow 0$  as  $k \rightarrow +\infty$ . Now, by lemma 2.12 one has  $F_k(B(z_k, \frac{\eta}{20\sqrt{n}})) \subset B(F_k(z_k), \frac{\eta}{10})$  for large  $k$ , and the proof of lemma 4.1 shows that the volume of the image is bounded below. It thus intersects  $X_{k,1}$  on a set of positive measure for  $k$  large enough. This proves that  $\deg(F_k|B_k) = 1$  on  $\mathcal{C}_k$ . In particular, any point in  $B(F(p), \frac{\eta}{10})$  has a preimage by  $F_k$  in  $B_k$ . By taking the limit  $k \rightarrow +\infty$ , it gives  $B(F(p), \frac{\eta}{10}) \subset F(\overline{B(p, r)}) \subset F(B(p, 2r)) \subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ .  $\square$

**Lemma 4.4.** *There exists  $c(\varepsilon) > 0$  such that  $F : \overset{\circ}{\mathcal{R}}_\varepsilon \rightarrow F(\overset{\circ}{\mathcal{R}}_\varepsilon) \subset X$  is locally  $(1+c(\varepsilon))$ -bilipschitz. Moreover,  $c(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .*

*Proof.* The idea is the following: we already know that  $F$  is 1-lipschitz and volume preserving. In particular, a ball  $B_\infty(p, r) \subset Y_\infty$  is sent into a ball  $B_{g_0}(F(p), r) \subset X$ . If the ball in  $Y_\infty$  is in the almost regular part and has a small radius, its volume is close to the Euclidean one, as is the volume of the hyperbolic ball. One can then estimate how much the image of  $B_\infty(p, r)$  is close to fill  $_{g_0}(F(p), r)$ . If one considers the images of two disjoint balls, one can estimate how the hyperbolic balls overlapp, and thus the distance between their centers.

Let  $p \in \overset{\circ}{\mathcal{R}}_\varepsilon$ . Let  $r(p, \varepsilon) > 0$  be a radius such that for every  $0 < r \leq r(p, \varepsilon)$ ,

$$\frac{\mathcal{H}^n(B_\infty(p, r))}{\text{vol}_{\mathbf{R}^n}(r)} \geq 1 - \tau(\varepsilon),$$

and let  $r_\varepsilon = \min(\varepsilon, r(z, \varepsilon))$ . One can suppose that  $r_\varepsilon$  is smaller than the injectivity radius of  $X$ . Let  $0 < r < r_\varepsilon^2$  be such that  $B_\infty(p, r) \subset \mathcal{R}_\varepsilon$ . For every  $q \in B_\infty(p, r)$ ,  $B(p, r_\varepsilon - r_\varepsilon^2) \subset B_\infty(q, r_\varepsilon)$ . Thus,

$$\mathcal{H}^n(B_\infty(q, r_\varepsilon)) \geq \mathcal{H}^n(B_\infty(p, r_\varepsilon - r_\varepsilon^2)) \quad (80)$$

$$\geq (1 - \tau(\varepsilon)) \text{vol}_{\mathbf{R}^n}(r_\varepsilon - r_\varepsilon^2) \quad (81)$$

$$\geq (1 - \tau(\varepsilon))(1 - r_\varepsilon)^n \text{vol}_{\mathbf{R}^n}(r_\varepsilon). \quad (82)$$

Suppose that there exists  $p_1, p_2 \in B(p, r)$ ,  $p_1 \neq p_2$  and a number  $0 < \rho < 1$  such that

$$d_{g_0}(F(p_1), F(p_2)) \leq \rho d_\infty(p_1, p_2).$$

Define  $r' = \frac{d(p_1, p_2)}{2} > 0$ . By (70) and the Bishop-Gromov inequality (7), for  $i = 1, 2$  one has

$$\mathcal{H}^n(B_\infty(p_i, r')) \geq \mathcal{H}^n(B(p_i, r_\varepsilon)) \frac{\text{vol}_{\mathbf{H}^n}(r')}{\text{vol}_{\mathbf{H}^n}(r_\varepsilon)}.$$

Thus, with lemma 4.1 and (82) we have

$$\text{vol}_{g_0}(F(B(p_1, r') \cup B(p_2, r'))) = \mathcal{H}^n(B(p_1, r')) + \mathcal{H}^n(B(p_2, r')) \quad (83)$$

$$\geq 2(1 - \tau(\varepsilon))(1 - r_\varepsilon)^n \frac{\text{vol}_{\mathbf{H}^n}(r')}{\text{vol}_{\mathbf{H}^n}(r_\varepsilon)} \text{vol}_{\mathbf{R}^n}(r_\varepsilon) \quad (84)$$

$$\geq 2(1 - \tau(\varepsilon))(1 - r_\varepsilon)^n \frac{\text{vol}_{\mathbf{R}^n}(\varepsilon)}{\text{vol}_{\mathbf{H}^n}(\varepsilon)} \text{vol}_{\mathbf{R}^n}(r') \quad (85)$$

$$\leq 2\vartheta(\varepsilon) \text{vol}_{\mathbf{R}^n}(r') \quad (86)$$

where  $\vartheta(\varepsilon) = (1 - \tau(\varepsilon))(1 - \varepsilon)^n \frac{\text{vol}_{\mathbf{R}^n}(\varepsilon)}{\text{vol}_{\mathbf{H}^n}(\varepsilon)} \rightarrow 1$  as  $\varepsilon \rightarrow 0$ .

On the other hand,

$$F(B(p_1, r') \cup B(p_2, r')) \subset B(F(p_1), r') \cup B(F(p_2), r'),$$

Hence

$$\begin{aligned} \text{vol}_{g_0}(F(B(p_1, r') \cup B(p_2, r'))) &\leq \text{vol}_{g_0}(B(F(p_1), r')) + \text{vol}_{g_0}(B(F(p_2), r')) \\ &\quad - \text{vol}_{g_0}(B(F(p_1), r') \cap B(F(p_2), r')). \end{aligned} \quad (87)$$

For any  $x \in X$  and any  $s > 0$  lower than the injectivity radius of  $X$  one has  $\text{vol}_{g_0}(B(x, s)) = \text{vol}_{\mathbf{H}^n}(s)$ . Let  $x$  be the middle point of the segment  $[F(p_1)F(p_2)]$ . Then

$$B(x, r'(1 - \rho)) \subset B(F(p_1), r') \cap B(F(p_2), r').$$

Indeed, if  $x' \in B(x, r'(1 - \rho))$  then  $d(x', F(p_i)) \leq d(x', x) + d(x, F(p_i)) < r'(1 - \rho) + \rho r' = r'$  for  $i = 1, 2$ . Thus (87) gives

$$\text{vol}_{g_0}(F(B(p_1, r') \cup B(p_2, r'))) \leq 2 \text{vol}_{\mathbf{H}^n}(r') - \text{vol}_{\mathbf{H}^n}(r'(1 - \rho)) \quad (88)$$

$$= 2 \text{vol}_{\mathbf{R}^n}(r') \frac{\text{vol}_{\mathbf{H}^n}(r')}{\text{vol}_{\mathbf{R}^n}(r')} - (1 - \rho)^n \text{vol}_{\mathbf{R}^n}(r') \quad (89)$$

$$\leq 2 \text{vol}_{\mathbf{R}^n}(r') \frac{\text{vol}_{\mathbf{H}^n}(\varepsilon)}{\text{vol}_{\mathbf{R}^n}(\varepsilon)} - (1 - \rho)^n \text{vol}_{\mathbf{R}^n}(r') \quad (90)$$

$$= \left( 2 \frac{\text{vol}_{\mathbf{H}^n}(\varepsilon)}{\text{vol}_{\mathbf{R}^n}(\varepsilon)} - (1 - \rho)^n \right) \text{vol}_{\mathbf{R}^n}(r'). \quad (91)$$

From (86) and (91), we find

$$(1 - \rho)^n \leq 2 \left( \frac{\text{vol}_{\mathbf{H}^n}(\varepsilon)}{\text{vol}_{\mathbf{R}^n}(\varepsilon)} - \vartheta(\varepsilon) \right) \rightarrow 0,$$

therefore

$$\rho \geq 1 - 2^{1/n} \left( \frac{\text{vol}_{\mathbf{H}^n}(\varepsilon)}{\text{vol}_{\mathbf{R}^n}(\varepsilon)} - \vartheta(\varepsilon) \right)^{1/n} := 1 - c(\varepsilon) \rightarrow 1,$$

as  $\varepsilon \rightarrow 0$ . One has proved that inside the ball  $B(p, r)$ ,

$$d_{g_0}(F(p_1), F(p_2)) \geq (1 - c(\varepsilon))d_\infty(p_1, p_2),$$

and the proof of the lemma follows.  $\square$

**Remark 4.5.** *On the connected (see remark 3.9) open set  $F(\overset{\circ}{\mathcal{R}}_\varepsilon) \subset X$ , the metric  $g_0$  induces a distance  $\rho_\varepsilon$ . The above lemma shows that  $F : (\overset{\circ}{\mathcal{R}}_\varepsilon, d_\infty) \rightarrow (F(\overset{\circ}{\mathcal{R}}_\varepsilon), \rho_\varepsilon)$  is a  $(1 + c(\varepsilon))$ -bilipschitz homeomorphism. If one can prove that  $\rho_\varepsilon = d_{g_0}$ , one deduces that  $\mathcal{R}_\varepsilon$  has bounded diameter. One then concludes that  $d_{g_{\mathcal{H}}}(Y_k, Y_\infty) \rightarrow 0$  and that  $F : Y_\infty \rightarrow X$  is isometric.*

More precisely, we prove the

**Proposition 4.6.** 1. *for any  $x_1, x_2 \in F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ , one has  $d_{g_0}(x_1, x_2) = \rho_\varepsilon(x_1, x_2)$ .*

2. *One has  $\overline{F(\overset{\circ}{\mathcal{R}}_\varepsilon)} = X$ .*

3.  *$F : (Y_\infty, d_\infty) \rightarrow (X, d_{g_0})$  is isometric.*

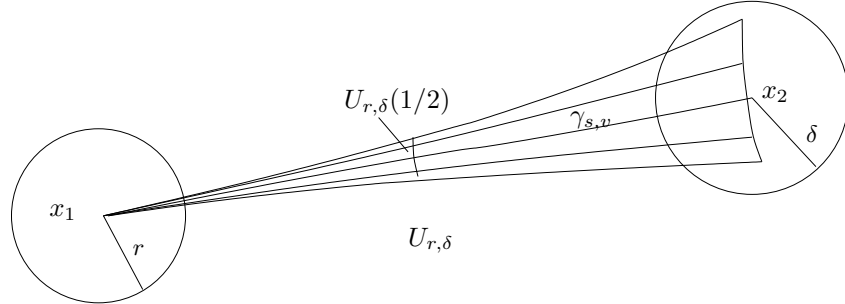
*Proof.* let  $x_1, x_2 \in F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ . Without loss of generality, one can suppose that  $x_2$  is not in the image of the cut-locus of  $x_1$ . Clearly,  $\rho_\varepsilon(x_1, x_2) \geq d_{g_0}(x_1, x_2)$ . Let  $\gamma : [0, 1] \rightarrow X$  be a  $g_0$ -minimal geodesic from  $x_1$  to  $x_2$ . It is not clear that  $\gamma$  is in  $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$  we then prove that there exist paths in  $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$  arbitrarily close to  $\gamma$ . Let  $r > 0$  be a radius such that  $B_{g_0}(x_1, r) \subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$  and  $B_{g_0}(x_2, r) \subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ . We consider geodesics with the origin  $x_1$  and the extremity in  $B(x_2, \delta)$ , for a small  $\delta > 0$ . More precisely, let  $u = \dot{\gamma}(0)$ , then for any  $v \in U_{x_1}X$  such that  $u \perp v$ , one defines  $\gamma_{s,v}(t) = \exp_{x_1}(t(u + s.v)d(x_1, x_2))$ . Clearly there exists  $r(\delta) > 0$  such that  $\gamma_{s,v}(1) \in B(x_2, \delta)$  if  $|s| \leq r(\delta)$  and one can suppose that  $r(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

We claim that for every  $\delta > 0$ , there exists such  $\gamma_{s,v}$  which is imbedded in  $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ .

We begin to find such  $\gamma_{s,v}$  disjoint from  $F(\mathcal{S})$ , where  $\mathcal{S}$  is the singular set of  $Y_\infty$  defined in 3.5. The idea is that if any  $\gamma_{s,v}$  hits  $F(\mathcal{S})$  at least in one point, then the Hausdorff dimension of  $F(\mathcal{S})$  will be larger than  $n - 1$ , which is not possible. One considers a truncated cone  $U_{r,\delta}$  defined as follows. Let

$$\Gamma : ]0, r(\delta)] \times (U_{x_1}X \cap u^\perp) \times [0, 1] \rightarrow X$$

be defined by  $\Gamma(s, v, t) = \gamma_{s,v}(t)$ . If  $\delta$  is sufficiently small,  $\Gamma$  is an embedding. one defines  $U_{r,\delta} = \Gamma(]0, r(\delta)] \times (U_{x_1}X \cap u^\perp) \times [0, 1])$ . One denotes by  $U_{r,\delta}(1/2)$  the hypersurface in  $U_{r,\delta}$  defined as  $\Gamma(]0, r(\delta)] \times (U_{x_1}X \cap u^\perp) \times \{1/2\})$ .



Let  $P : U_{r,\delta} \rightarrow U_{r,\delta}(1/2)$  be the projection along geodesics defined by  $P(\gamma_{s,v}(t)) = \gamma_{s,v}(1/2)$ . Clearly, there exists a constant  $C > 0$  such that  $P$  is  $C$ -lipschitz from  $U_{r,\delta}$  to  $X$ . In particular,  $P$  decreases the Hausdorff dimension, thus

$$\begin{aligned} \dim_{\mathcal{H}}(P(U_{r,\delta} \cap F(\mathcal{S}))) &\leq \dim_{\mathcal{H}}(U_{r,\delta} \cap F(\mathcal{S})) \\ &\leq \dim_{\mathcal{H}}(\mathcal{S}) \\ &\leq n - 2 \\ &< \dim U_{r,\delta}(1/2) = n - 1. \end{aligned}$$

Hence there exists  $x \in U_{r,\delta}(1/2)$  such that  $x \notin \Pi(F(\mathcal{S}))$ . It implies that the geodesic  $\gamma_{s,v}$  such that  $x = \gamma_{s,v}(1/2)$  does not intersect  $F(\mathcal{S})$ .

Now we prove that  $\gamma_{s,v}$  is imbedded in  $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ . Let  $t_0 \in [0, 1]$  be maximal such that  $\gamma_{s,v}([0, t_0]) \subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ . By lemma 4.4, the path  $\beta = F^{-1} \circ \gamma_{s,v}$  is well-defined on  $[0, t_0[$  and

has a length bounded by  $(1+c(\varepsilon))d(x_1, x_2)$ . Thus there exists a limit  $p = \lim_{t \rightarrow t_0} \beta(t) \in Y_\infty$ . As  $\gamma_{s,v}(t_0) \notin F(\mathcal{S})$ ,  $p \notin \mathcal{S}$  then  $p \in \mathcal{R} = \bigcap_\varepsilon \mathcal{R}_\varepsilon = \bigcap_{\varepsilon > 0} \overset{\circ}{\mathcal{R}}_\varepsilon$  and  $t_0 = 1$ .

Hence

$$\begin{aligned} \rho_\varepsilon(x_1, x_2) &\leq \ell(\gamma_{s,v}) + d_0(\gamma_{s,v}(1), x_2) \\ &\leq \sqrt{1+r^2(\delta)}d_0(x_1, x_2) + \delta \end{aligned}$$

As  $\delta$  was arbitrary, this gives  $\rho_\varepsilon(x_1, x_2) \leq d_0(x_1, x_2)$ .

The second assertion is proved in a similar way. Suppose there is a ball  $B(x, r) \subset X - F(\overset{\circ}{\mathcal{R}}_\varepsilon)$  and consider a geodesic  $\gamma$  from a point  $x_1$  inside  $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$  to  $x$ . Then we find another geodesic from  $x_1$ , close to  $\gamma$ , disjoint from  $F(\mathcal{S})$  and with extremity in  $X - F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ . Arguing as above, we find a contradiction.

Now 3) is straightforward. Using the density of  $\overset{\circ}{\mathcal{R}}_\varepsilon$  in  $Y_\infty$  and  $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$  in  $X$ , we find that  $F : (Y_\infty, d_\infty) \rightarrow (X, d_0)$  is a  $(1+c(\varepsilon))$ -bilipschitz homeomorphism for any  $0 < \varepsilon < \varepsilon_0$  thus is isometric.  $\square$

*End of Proof of theorem 1.1.* Proposition 4.6 implies that the diameter of  $(Y, g_k)$  remains bounded. Thus,  $d_{\mathcal{GH}}((Y, g_k), (Y_\infty, d_\infty)) \rightarrow 0$ . As  $(Y_\infty, d_\infty)$  is isometric to  $(X; g_0)$ , one deduces that  $d_{\mathcal{GH}}((Y, g_k), (X, g_0)) \rightarrow 0$  as  $k \rightarrow \infty$ . By theorem A.1.12 of [Ch-Co],  $Y$  is diffeomorphic to  $X$ . The fact that  $f$  is homotopic to a diffeomorphism is classic for hyperbolic manifolds.  $\square$

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