Ricci curvature and rigidity

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May 25, 2008

1 Introduction

The aim of this paper is to show the following result.

Theorem 1.1. *Given any integer $n \geq 3$ and $d > 0$, there exists $\varepsilon(n,d) > 0$ such that the following holds. Suppose that (X, g_0) is an n-dimensional hyperbolic compact manifold with diameter $\leq d$ and that Y is a compact manifold which dominates X, that is, there exist a continuous map $f : Y \to X$ of degree one. Then Y has a metric g such that*

$$\begin{aligned} \mathrm{Ric}_g &\geq -(n-1)g & (1)\\ \mathrm{vol}_g(Y) &\leq (1+\varepsilon)\,\mathrm{vol}_{g_0}(X) & (2) \end{aligned}$$

if and only if f is homotopic to a diffeomorphism.

Sketch of the Proof:
We argue by contradiction. Suppose that there is a sequence $(X_k)_{k\in\mathbf{N}}$ of compact hyperbolic manifolds with diameter $\leq d$ and a sequence of compact manifolds Y_k, of degree one continuous maps $f_k : Y_k \to X_k$ and metrics g_k on Y_k fulfilling the hypothesis (1) and (2) for some $\varepsilon_k \to 0$. Since f_k has degree one and X_k is hyperbolic, it is equivalent to say that f_k is homotopic to a diffeomorphism or simply that X_k and Y_k are diffeomorphic. We thus assume that Y_k and X_k are not diffeomorphic. One then shows that up to a subsequence, for large k, Y_k is diffeomorphic to a compact manifold Y, X_k is diffeomorphic to a compact manifold X, and X and Y are diffeomorphic. One argue as follows: by the classical finiteness results we get the sub-convergence of (X_k). Indeed, the curvature is -1, the diameter is bounded by hypothesis, and there is a universal lower bound for the volume of any hyperbolic compact manifold of a given dimension. Cheeger's finiteness theorem then applies. Moreover, on a compact manifold of dimension ≥ 3, there is at most one hyperbolic metric, up to isometry. We can therefore suppose that $X_k = X$ is a fixed hyperbolic manifold. The inequality proved in [BCG] provides a lower bound for the volume of Y_k as it is explained below. We have no a priori bounds on the diameter of (Y_k, g_k), but we can use Cheeger-Colding's theory to obtain sub-convergence in the pointed Gromov-Hausdorff topology to a complete metric space (Z, d) with small singular set. To obtain more geometric control,

the idea is to use the natural maps between Y_k and X (see [BCG]). One can show that they sub-convergence to a limit map between Z and X, which is an isometry. Then Z is compact and diffeomorphic to Y_k for large k.

The paper is organized as follows. The construction and the properties of the natural maps are given in section 2. In section 3, we construct the limit space Z and the limit map $F : Z \to X$. In section 4, we prove that F is an isometry and conclude.

1.1 Some a priori control on (Y, g)

Some a priori control on the metric g will be needed in section 2 and 3. We give here some necessary results.

Let (X, g_0) be an hyperbolic manifold and Y be a manifold satisfying the assumptions of theorem 1.1. For any riemannian metric g on Y satisfying the curvature assumption (1), one has the following inequality

$$\mathrm{vol}_g(Y) \geq \mathrm{vol}_{g_0}(X)\,. \tag{3}$$

It is a consequence of the Besson-Courtois-Gallot inequality [BCG]

$$h(g)^n \,\mathrm{vol}_g(Y) \geq h(g_0)^n \,\mathrm{vol}_{g_0}(X)\,, \tag{4}$$

where $h(g)$ is the volume entropy, or the critical exponent, of the metric g, *i.e.*:

$$h(g) = \lim_{R\to+\infty} \frac{1}{R} \ln(\mathrm{vol}_{\tilde{g}}(B_{\tilde{g}}(x, R)))\,,$$

where $\tilde{g}$ is the lifted metric on $\tilde{Y}$. Indeed, any metric g on Y which satisfies (1), verifies, by Bishop's theorem,

$$h(g) \leq h(g_0) = n - 1\,. \tag{5}$$

One can obtain a local control of the volume by Gromov's isolation theorem (see [Gro2], theorem 0.5). It shows that if the simplicial volume $||Y||$ - a topological invariant - of Y is non-zero, then for any riemannian metric g on Y satisfying the curvature assumption (1), there exists at least one point $y_g \in Y$ such that

$$\mathrm{vol}_g(B(y_g, 1)) \geq v_n > 0. \tag{6}$$

Here $B(y_g, 1)$ is the geodesic ball of radius 1 for the metric g and v_n is a universal constant. This theorem applies in our situation since, by an elementary property of the simplicial volume, $||Y|| \geq ||X||$ if there is a degree one map from Y to X. On the other hand, X has an hyperbolic metric and thus $||X|| > 0$ by Gromov-Thurston's theorem (see [Gro2]).

Given this universal lower bound for the volume of a unit ball $B(y_g, 1)$, the volume of any ball $B(y, r)$ is bounded from bellow in terms of r and $d(y_g, y)$. Indeed, recall

that under the curvature assumption (1), Bishop-Gromov's theorem shows that for any $0 < r \leq R$, one has

$$\frac{\mathrm{vol}_g(B(y,r))}{\mathrm{vol}_g(B(y,R))} \geq \frac{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(\mathrm{r}))}{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}\mathrm{R}))}, \tag{7}$$

where $\mathrm{B}_{\mathbf{H}^n}(\mathrm{r})$ is a r-ball in the hyperbolic space $\mathbf{H}^n$. As $B(y_g,1) \subset B(y, 1+d(y_g,y)+r)$, one deduces from (7) that

$$\begin{aligned} \mathrm{vol}_g(B(y,r)) &\geq \mathrm{vol}_g\left(B(y, 1+d(y_g,y)+r)\right) \frac{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(\mathrm{r}))}{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(1+\mathrm{d}(\mathrm{y_g},\mathrm{y})+\mathrm{r}))} & (8) \\ &\geq v_n \frac{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(\mathrm{r}))}{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(1+\mathrm{d}(\mathrm{y_g},\mathrm{y})+\mathrm{r}))}\,. & (9) \end{aligned}$$

The curvature assumption (1) and the volume estimates (7) or (9) are those required to use the non-collapsing part of Cheeger-Colding's theory, as we shall see in section 3.

2 The natural maps

2.1 Construction of the natural maps

In this section we recall the construction and the main properties of the natural maps defined by Besson-Courtois-Gallot ([BCG],[BCG2]). Suppose that (Y,g) and (X,g_0) are compact riemannian manifolds and that

$$f : Y \to X\,,$$

is a continuous map of degree one. For the sake of simplicity, we assume that g_0 is hyperbolic (the construction holds in more general cases). Then, for any $c > h(g)$ there exists a C^1 map

$$F_c : Y \longrightarrow X\,,$$

homotopic to f, such that for all $y \in Y$,

$$|\,\mathrm{Jac}\,F_c(y)| \leq \left(\frac{c}{h(g_0)}\right)^n\,, \tag{10}$$

with equality for some $y \in Y$ if and only if d_yF_c is an homothety of ratio $\frac{c}{h(g_0)}$.

The inequality (4) is then easily obtained by integration of (10) and by taking a limit when c goes to $h(g)$. To obtain global rigidity properties, one has in general to care about the limit of F_c as c goes to $h(g)$.

We divide the construction of the maps in 4 steps. Let $\tilde{Y}$ and $\tilde{X}$ be the universal coverings of Y and X respectively, and $\tilde{f} : \tilde{Y} \to \tilde{X}$ a lift of f.

Step 1: for each $y \in \tilde{Y}$ and $c > h(g)$, let ν_y^c be the finite measure on $\tilde{Y}$ defined by

$$d\nu_y^c(z) = e^{-c.\rho(y,z)}\mathrm{dv}_{\tilde{\mathrm{g}}}(z)$$

where $z \in \tilde{Y}$, $\tilde{g}$ is the lifted metric on $\tilde{Y}$ and $\rho(.,.)$ is the distance function of $(\tilde{Y}, \tilde{g})$.

Step 2: this measure is pushed forward and gives a finite measure $\tilde{f}_* \nu_y^c$ on $\tilde{X}$. Let us recall that it is defined by

$$\tilde{f}_* \nu_y^c(U) = \nu_y^c(\tilde{f}^{-1}(U)).$$

Step 3: one defines a finite measure μ_y^c on $\partial\tilde{X}$ by convolution of $\tilde{f}_* \nu_y$ with all visual probability measures P_x of $\tilde{X}$. Recall that the visual probability measure P_x at $x \in \tilde{X}$ is defined as follows: the unit tangent bundle $U_x\tilde{X}$ is projected onto the boundary $\partial\tilde{X}$ by the map

$$v \in U_x\tilde{X} \xrightarrow{E_x} \gamma_v(\infty) \in \partial\tilde{X},$$

where $\gamma_v(t) = exp_x(tv)$. The measure P_x is then the push-forward by E_x of the canonical probability measure of $U_x\tilde{X}$, i.e. $P_x(U)$ is the measure of the set of vectors $v \in U_x\tilde{X}$ such that $\gamma_v(+\infty) \in U$.

Then

$$\begin{aligned} \mu_y^c(U) &= \int_{\tilde{X}} P_x(U)\ d\tilde{f}_* \nu_y^c(x) \\ &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(U)\ d\nu_y^c(z). \end{aligned}$$

One can identifies $\partial\tilde{X}$ with the unit sphere in $\mathbf{R}^n$, by choosing an origin $o \in \tilde{X}$ and using E_0. One can then show (REFERENCE) that the density of this measure is given by

$$d\mu_y^c(\theta) = \left(\int_{\tilde{Y}} e^{-h(g_0)B(\tilde{f}(z),\theta)} e^{-c\rho(y,z)} \mathrm{dv}_{\tilde{\mathrm{g}}}(z) \right)\ d\theta,$$

where $\theta \in \partial\tilde{X}$, $d\theta$ is the canonical probability measure on S^{n-1} and $B(x,\theta)$ is a Busemann function on $\tilde{X}$ normalised to vanish at $x = o$. We will use the notation

$$p(x,\theta) = e^{-h(g_0)B(x,\theta)}$$

It is a classical fact that p is the Poisson Kernel of $(\tilde{X}, \tilde{g_0})$.

Step 4: the map

$$F_c : \tilde{Y} \longrightarrow \tilde{X}$$

associates to any $y \in \tilde{Y}$ the unique $x \in \tilde{X}$ which minimizes on $\tilde{X}$ the function

$$\mathcal{B}(x) = \int_{\partial\tilde{X}} B(x,\theta)\ d\mu_y^c(\theta).$$

(see Appendix A in [BCG]).

The maps F_c are shown to be $\mathcal{C}^1$ and equivariant. The quotient map, which is denoted $F_c : Y \to X$, is homotopic to f. Note that F_c depends heavily on the metric g.

2.2 Some technical lemmas

Let us give some definitions.

Definition 2.1. *We call σ_y^c be the probability measure on $\partial\tilde{X}$ defined by*

$$\sigma_y^c = \frac{\mu_y^c}{\mu_y^c(\partial\tilde{X})}.$$

Then

$$||\mu_y^c|| = \mu_y^c(\partial\tilde{X}) = \int_{\tilde{Y}} e^{-c\rho(y,z)} \mathrm{dv}_{\tilde{\mathrm{g}}}(z) = ||\nu_y^c||.$$

We consider two positive definite bilinear forms of trace 1 and the corresponding symmetric endomorphisms.

Definition 2.2. *For any $y \in \tilde{X}$, $u, v \in T_{F_c(y)}\tilde{X}$,*

$$\mathbf{h_y^c}(u,v) = \int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(u) dB_{(F_c(y),\theta)}(v)\ d\sigma_y^c(\theta) = g_0(\mathbf{H_y^c}(u), v).$$

And, for any $y \in \tilde{Y}$, $u, v \in T_y\tilde{Y}$,

$$\mathbf{h'_y}^{\mathbf{c}}(u,v) = \frac{\int_{\tilde{Y}} d\rho_{(y,z)}(u) d\rho_{(y,z)}(v)\ d\nu_y^c(z)}{\mu_y^c(\partial\tilde{X})} = g(\mathbf{H'_y}^{\mathbf{c}}(u), v).$$

Lemma 2.3. *For any $y \in \tilde{Y}$, $u \in T_y\tilde{Y}$, $v \in T_{F(y)}\tilde{X}$, one has*

$$\left|g_0((I - H_y^c)d_yF_c(u), v)\right| \leq c\left(g_0(H_y^c(v), v)\right)^{1/2}\left(g(H_y'^c(u), u)\right)^{1/2}. \tag{11}$$

Proof. since $F_c(y)$ is an extremum of the function $\mathcal{B}$, one has

$$d_{F_c(y)}\mathcal{B}(v) = \int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(v)\ d\mu_y^c(\theta) = 0 \tag{12}$$

for each $v \in T_{F_c(y)}\tilde{X}$. Let V be a parallel vector field on $T\tilde{X}$, then $d_{F_c(y)}\mathcal{B}(V_y) = 0$ for each $y \in \tilde{Y}$. One differentiates this equation in a direction $u \in T_y\tilde{Y}$ thus, using the formula (12) with $v = V_y$, one obtains

$$\int_{\partial\tilde{X}} DdB_{(F_c(y),\theta)}(d_yF_c(u), v)d\mu_y^c(\theta) + \int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(v)\left(\int_{\tilde{Y}} p(\tilde{f}(z),\theta)(-cd\rho_{(y,z)}(u))d\nu_y^c(z)\right)\ d\theta = 0$$

Using Cauchy-Schwarz inequality in the second term, one gets

$$\left|\int_{\partial\tilde{X}} DdB_{(F_c(y),\theta)}(d_yF_c(u), v)d\mu_y^c(\theta)\right| \leq$$
$$\int_{\partial\tilde{X}} |dB_{(F_c(y),\theta)}(v)|\left(\int_{\tilde{Y}} p(\tilde{f}(z),\theta)d\nu_y^c(z)\right)^{1/2}\left(\int_{\tilde{Y}} p(\tilde{f}(z),\theta)|cd\rho_{(y,z)}(u)|^2 d\nu_y^c(z)\right)^{1/2} d\theta$$

which is, using Cauchy-Schwarz inequality again

$$\begin{aligned}
&\leq c\left(\int_{\partial\tilde{X}}|dB_{(F_c(y),\theta)}(v)|^2\int_{\tilde{Y}}p(\tilde{f}(z),\theta)d\nu_y^c(z)d\theta\right)^{1/2}\left(\int_{\partial\tilde{X}}\int_{\tilde{Y}}p(\tilde{f}(z),\theta)|d\rho_{(y,z)}(u)|^2d\nu_y^c(z)d\theta\right)^{1/2}\\
&= c\left(\int_{\partial\tilde{X}}|dB_{(F_c(y),\theta)}(v)|^2d\mu_y^c(\theta)\right)^{1/2}\left(\int_{\tilde{Y}}|d\rho_{(y,z)}(u)|^2\ d\nu_y^c(z)\right)^{1/2}\\
&= c\mu_y^c(\partial\tilde{X})\left(g_0(H_y^c(v),v)\right)^{1/2}\left(g({H'}_y^c(u),u)\right)^{1/2}
\end{aligned}$$

It is shown in [BCG] chapter 5 that $DdB = g_0 - dB\otimes dB$ for an hyperbolic metric. The left term of the inequality is thus $\mu_y^c(\partial\tilde{X})g_0((I-H_y^c)d_yF_c(u),v)$. This proves the lemma. □

Definition 2.4. *Let $0<\lambda_1^c(y)\leq ...\leq\lambda_n^c(y)<1$ be the eigenvalues of H_y^c.*

Proposition 2.5. *There exists a constant $A(n)>0$ such that, for any $y\in Y$,*

$$|\operatorname{Jac}F_c(y)|\leq\left(\frac{c}{h(g_0)}\right)^n\left(1-A\sum_{i=1}^n(\lambda_i^c(y)-\frac{1}{n})^2\right) \tag{13}$$

Proof. The proof is based on the two following lemmas.

Lemma 2.6. *At each $y\in\tilde{Y}$,*

$$|\operatorname{Jac}F_c(y)|\leq\left(\frac{c}{\sqrt{n}}\right)^n\frac{det({H_y}^c)^{1/2}}{det(I-{H_y}^c)}$$

Proof of lemma 2.6. Let (u_i) an orthonormal basis of $T_{F_c(y)}\tilde{X}$ which diagonalizes ${H'_y}^c$. We can suppose that d_yF_c is invertible thus let $v_i'=[(I-{H_y}^c)\circ d_yF_c]^{-1}(u_i)$. Schmidt orthonormalisation process applied to (v_i') gives an orthonormal basis (v_i) at $T_y\tilde{Y}$. The matrix of $(I-{H_y}^c)\circ d_yF_c$ in the base (v_i) and (u_i) is upper triangular, then

$$\det(I-{H_y}^c)\operatorname{Jac}F_c(y)=\prod_{i=1}^n g_0((I-{H_y}^c)\circ d_yF_c(v_i),u_i)$$

which gives, with (11),

$$\begin{aligned}
\det(I-{H_y}^c)|\operatorname{Jac}F_c(y)| &\leq c^n\left(\prod_{i=1}^n g_0({H_y}^c(v_i),v_i)\right)^{1/2}\left(\prod_{i=1}^n g({H'_y}^c(u_i),u_i)\right)^{1/2}\\
&\leq c^n\det({H_y}^c)^{1/2}\left[\frac{1}{n}\sum_{i=1}^n g({H'_y}^c(u_i),u_i)\right]^{n/2}
\end{aligned}$$

and we have the desired inequality with $tr({H'_y}^c)=1$. □

Lemma 2.7. *Let H a symmetric positive definite $n \times n$ matrix whose trace is equal to one then, if $n \geq 3$,*

$$\frac{\det(H^{1/2})}{\det(I-H)} \leq \left(\frac{n}{h(g_0)^2}\right)^{n/2} \left(1 - A\sum_{i=1}^{n}(\lambda_i - \frac{1}{n})^2\right)$$

for a constant $A(n) > 0$.

Proof of lemma 2.7. see Appendix B5 in [BCG]. This is the point where the rigidity of the natural maps fails in dimension 2. This completes the proof of the lemma 2.5. □

□

2.3 Some nice properties

We now show that the natural maps F_c are "nice". In this section, we shall consider F_c as a map from (Y, g) to (X, g_0). We suppose that the metric g satisfies the curvature assumption (1) and the assumption on its volume (2) for some $\varepsilon > 0$. Let us introduce some terminology.

Definition 2.8. *Let $0 < \alpha < 1$. We say that a property holds $\alpha - \text{ae}$ (α-almost everywhere) on a set A if the set A_+ of points of A where the property holds has relative volume bigger than $1 - \alpha$, i. e. $\frac{\text{vol}(A_+)}{\text{vol}(A)} \geq 1 - \alpha$.*

We show that dF_c is α-close to be isometric $\alpha - \text{ae}$ on Y for some $\alpha(\varepsilon, c) > 0$. Moreover $\alpha(\varepsilon, c) \to 0$ as $\varepsilon \to 0$ and $\delta \to h(g)$. On the other hand, given any radius $R > 0$, one shows that $||dF_c||$ is uniformly bounded on balls $B(y_g, R)$, provided c is close enough to $h(g)$. Recall that we have uniform bounds for the volume of g but not for the diameter. The key point is to show that H_y^c is α-close to $\frac{Id}{n}$ on a set of large volume, and to bound it on a ball of fixed radius, with respect to the parameters ε, c.

To control $c - h(g)$ we introduce a parameter $\delta > 0$. We suppose that the volume entropy of g satisfies the inequality

$$h(g) < c \leq h(g) + \delta. \tag{14}$$

Observe that (5), (13) and (14) implies that

$$|\operatorname{Jac} F_c(y)| \leq \left(\frac{h(g)+\delta}{h(g_0)}\right)^n \leq \left(1 + \frac{\delta}{n-1}\right)^{n-1} \tag{15}$$

for all $y \in Y$. The map F_c is thus almost volume decreasing. As $\text{vol}_g(Y)$ is close to $\text{vol}_{g_0}(X)$, the set in Y where F_c is decreases the volume a lot must be small. Equivalently, $|\operatorname{Jac} F_c|$ must be close to 1 in L^1 norm. Now we give a precise statement of this fact.

Lemma 2.9. *There exists $\alpha_1 = \alpha_1(\varepsilon, \delta) > 0$ such that $\alpha -$ ae on Y on has,*

$$1 - \alpha_1 \leq |\operatorname{Jac} F_c(y)|, \tag{16}$$

and for all $y \in Y$ one has

$$|\operatorname{Jac} F_c(y)| \leq 1 + \alpha_1. \tag{17}$$

Moreover, $\alpha_1(\varepsilon, \delta) \to 0$ as ε and $\delta \to 0$.

Proof. Let

$$\alpha = \max\left(\sqrt{\left(1 + \frac{\delta}{n-1}\right)^{n-1} - 1}, \sqrt{\varepsilon}\right).$$

Thus $\left(1 + \frac{\delta}{n-1}\right)^{n-1} \leq 1 + \alpha^2$ and $\varepsilon \leq \alpha^2$. In particular, $|\operatorname{Jac} F_c(y)| \leq 1 + \alpha^2 \leq 1 + \alpha$ for all $y \in Y$.

As F_c has degree one, we have

$$\operatorname{vol}_{g_0}(X) = \int_Y F_c^*(\mathrm{dv}_{g_0}) = \int_Y \operatorname{Jac} F_c(y) \mathrm{dv}_g(y)$$

Denote by Y_{α_1} the set of points $y \in Y$ such that

$$|\operatorname{Jac} F_c(y)| \geq 1 - \alpha.$$

We have

$$\begin{aligned}
\operatorname{vol}_{g_0}(X) &\leq \int_Y |\operatorname{Jac} F_c(y)| \mathrm{dv}_g(y) && (18)\\
&= \int_{Y_{\alpha_1}} |\operatorname{Jac} F_c(y)| \mathrm{dv}_g(y) + \int_{Y - Y_{\alpha_1}} |\operatorname{Jac} F_c(y)| \mathrm{dv}_g(y) && (19)\\
&\leq (1 + \alpha^2) \operatorname{vol}_g(Y_{\alpha_1}) + (1 - \alpha) \operatorname{vol}_g(Y - Y_{\alpha_1}) && (20)\\
&= \operatorname{vol}_g(Y) + \alpha^2 \operatorname{vol}_g(Y_{\alpha_1}) - \alpha \operatorname{vol}_g(Y - Y_{\alpha_1}) && (21)
\end{aligned}$$

Then, using the assumption (2) and the inequality (3) on the volume, we get

$$\begin{aligned}
\operatorname{vol}_g(Y - Y_{\alpha_1}) &\leq \frac{\operatorname{vol}_g(Y) - \operatorname{vol}_{g_0}(X)}{\alpha} + \alpha \operatorname{vol}_g(Y_{\alpha_1}) && (22)\\
&\leq \left(\frac{\varepsilon}{\alpha} + \alpha\right) \operatorname{vol}_g(Y) && (23)\\
&\leq 2\alpha \operatorname{vol}_g(Y). && (24)
\end{aligned}$$

Clearly, $1 - 2\alpha \leq |\operatorname{Jac} F_c(y)|$ on Y_{α_1} and $|\operatorname{Jac} F_c(y)| \leq 1 + 2\alpha$ on Y which proves the lemma with $\alpha_1(\varepsilon, \delta) = 2\alpha$. □

From this lemma, we deduce that F_c is almost injective. Let $x \in X$, one defines $N(F_c, x) \in \mathbf{N} \cup \{\infty\}$ to be the number of preimages of x by F_c. As F_c has degree one, one has $N(F_c, x) \geq 1$ for all $x \in X$. We then define $X_1 := \{x \in X, N(F_c|x) = 1\}$. Observe that $N(F_c, x) \geq 2$ on $X - X_1$.

Lemma 2.10. *There exists $\alpha_2 = \alpha_2(\varepsilon, \delta) > 0$ such that*

$$\mathrm{vol}_{g_0}(X_1) \geq (1 - \alpha_2)\,\mathrm{vol}_{g_0}(X) \tag{25}$$

and

$$\int_{X - X_1} N(F_c|x)\; dV_{g_0}(x) \leq \alpha_2(\varepsilon, \delta)\,\mathrm{vol}_{g_0}(X)\,. \tag{26}$$

Moreover, $\alpha_2(\varepsilon, \delta) \to 0$ as ε and $\delta \to 0$.

In particular, $N(F_c, x) = 1$ α'-ae on X.

Proof. one defines

$$\alpha_2(\varepsilon, \delta) = 2\left(\left(1 + \frac{\delta}{n-1}\right)^n (1 + \varepsilon) - 1\right).$$

From (13) and the area formula (see [Mor] 3.7), we have

$$\begin{aligned}
\left(\frac{c}{h(g_0)}\right)^n \mathrm{vol}_g(Y) &\geq \int_Y |\operatorname{Jac} F_c(y)|\; \mathrm{dv_g}(y) && (27)\\
&= \int_X N(F_c|x)\; \mathrm{dv}_{g_0}(x) && (28)\\
&= \int_{X_1} N(F_c|x)\; \mathrm{dv}_{g_0}(x) + \int_{X - X_1} N(F_c|x) - 1 + 1\; \mathrm{dv}_{g_0}(x) && (29)\\
&= \mathrm{vol}_{g_0}(X) + \int_{X - X_1} N(F_c|x) - 1\; \mathrm{dv}_{g_0}(x). && (30)
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{vol}_{g_0}(X - X_1) &\leq \int_{X - X_1} N(F_c|x) - 1\; \mathrm{dv}_{g_0}(x) && (31)\\
&\leq \left(\frac{c}{h(g_0)}\right)^n \mathrm{vol}_g(Y) - \mathrm{vol}_{g_0}(X) && (32)\\
&\leq \left(\left(\frac{c}{h(g_0)}\right)^n (1 + \varepsilon) - 1\right) \mathrm{vol}_{g_0}(X) && (33)\\
&\leq \frac{\alpha_2(\varepsilon, \delta)}{2}\,\mathrm{vol}_{g_0}(X). && (34)
\end{aligned}$$

Thus

$$\mathrm{vol}_{g_0}(X - X_1) \leq \int_{X - X_1} N(F_c|x)\; \mathrm{dv}_{g_0}(x) \leq \alpha_2(\varepsilon, \delta)\,\mathrm{vol}_{g_0}(X),$$

and this proves the lemma. □

Lemma 2.11. *There exists $\alpha_3 = \alpha_3(\varepsilon, \delta) > 0$ such that the following holds. Let $y \in Y_{\alpha_1}$ and $u \in T_y Y$ then*

$$(1 - \alpha_3)\|u\|_g \leq \|d_y F_c(u)\|_{g_0} \leq (1 + \alpha_3)\|u\|_g\,. \tag{35}$$

Moreover, $\alpha_3(\varepsilon, \delta) \to 0$ as ε,$\delta \to 0$.

Proof. The inequality (13) implies that for all $y \in Y$

$$\| H_y^c - \frac{\mathrm{Id}}{n}\|^2 \leq \frac{1}{A}\left(1 - \frac{|\operatorname{Jac} F_c(y)|}{\left(1+\frac{\delta}{n-1}\right)^n}\right).$$

Let us define

$$\beta_1 = \beta_1(\varepsilon,\delta) = \frac{1}{A^{1/2}}\left(1 - \frac{1-\alpha_1(\varepsilon,\delta)}{\left(1+\frac{\delta}{n-1}\right)^n}\right)^{1/2}.$$

where $\alpha_1(\varepsilon,\delta)$ is the constant from lemma 2.9. Clearly, $\beta_1(\varepsilon,\delta) \to 0$ as ε and $\delta \to 0$. Let Y_{α_1} be the set of points where (16) holds. On Y_{α_1}, one has

$$\| H_y^c - \frac{\mathrm{Id}}{n}\|^2 \leq {\beta_1}^2. \tag{36}$$

Let $(u_i)_{i=1,\dots,n}$ be an orthonormal basis of T_yY and $v_i = d_yF(u_i)$. Writing $\mathrm{Id} - H_y^c = \frac{n-1}{n}\mathrm{Id} + \frac{\mathrm{Id}}{n} - H_y^c$, the left side of (16) gives

$$\begin{aligned}
\left|g_0\left((\mathrm{Id} - H_y^c)d_yF_c(u_i), d_yF_c(u_i)\right)\right| &\geq \left|g_0\left((\frac{n-1}{n}\mathrm{Id})d_yF_c(u_i), d_yF_c(u_i)\right)\right| \\
&\quad - \left|g_0\left((\frac{\mathrm{Id}}{n} - H_y^c)d_yF_c(u_i), d_yF_c(u_i)\right)\right| \qquad (37)\\
&\geq \frac{n-1}{n}||d_yF_c(u_i)||_{g_0}^2 - \| \frac{\mathrm{Id}}{n} - H_y^c\|.||d_yF_c(u_i)||_{g_0}^2 \qquad (38)\\
&\geq \left(\frac{n-1}{n} - \beta\right)||d_yF_c(u_i)||_{g_0}^2. \qquad (39)
\end{aligned}$$

Writing $H_y^c = \frac{\mathrm{Id}}{n} + H_y^c - \frac{\mathrm{Id}}{n}$, one has

$$\begin{aligned}
g_0\left(H_y^c d_yF_c(u_i), d_yF_c(u_i)\right)^{1/2} &\leq g_0\left(\frac{\mathrm{Id}}{n}d_yF_c(u_i), d_yF_c(u_i)\right)^{1/2} \\
&\quad + g_0\left((H_y^c - \frac{\mathrm{Id}}{n})d_yF_c(u_i), d_yF_c(u_i)\right)^{1/2} \qquad (40)\\
&\leq \left(\frac{1}{\sqrt{n}} + \beta^{1/2}\right)||d_yF_c(u_i)||_{g_0}. \qquad (41)
\end{aligned}$$

Taking the trace of the right hand side of (11) and using the Cauchy-Schwarz inequality,

one has

$$\sum_{i=1}^{n} g_0\left(H_y^c d_y F_c(u_i), d_y F_c(u_i)\right)^{1/2} g(H_y'^c(u_i), u_i)^{1/2} \leq \left(\frac{1}{\sqrt{n}} + \beta^{1/2}\right) \left(\sum_{i=1}^{n} ||d_y F_c(u_i)||_{g_0}^2\right)^{1/2} \times \left(\sum_{i=1}^{n} g(H_y'^c(u_i), u_i)\right)^{1/2} \tag{42}$$

$$= \left(\frac{1}{\sqrt{n}} + \beta^{1/2}\right) \left(\sum_{i=1}^{n} ||d_y F_c(u_i)||_{g_0}^2\right)^{1/2} \tag{43}$$

By (11), the trace of (39) is not greater than the right hand side of (43) multiply by c, hence

$$\left(\frac{n-1}{n} - \beta\right) \sum_{i=1}^{n} ||d_y F_c(u_i)||_{g_0}^2 \leq c \left(\frac{1}{\sqrt{n}} + \beta^{1/2}\right) \left(\sum_{i=1}^{n} ||d_y F_c(u_i)||_{g_0}^2\right)^{1/2},$$

thus

$$\left(\sum_{i=1}^{n} ||d_y F_c(u_i)||_{g_0}^2\right)^{1/2} \leq c \frac{\frac{1}{\sqrt{n}} + \beta^{1/1}}{\frac{n-1}{n} - \beta} \leq \sqrt{n}(1 + \frac{\delta}{n-1}) \frac{1 + \sqrt{n}\beta^{1/1}}{1 - \frac{n}{n-1}\beta}.$$

Let us define

$$\beta_2(\varepsilon, \delta) = (1 + \frac{\delta}{n-1})^2 \left(\frac{1 + \sqrt{n}\beta^{1/2}}{1 - \frac{n}{n-1}\beta}\right)^2 - 1.$$

Clearly, $\beta_2(\varepsilon, \delta) \to 0$ as ε and $\delta \to 0$. One has

$$\sum_{i=1}^{n} ||d_y F_c(u_i)||_{g_0}^2 \leq n(1 + \beta_2).$$

Let L be the endomorphism of $T_y Y$ defined by $L = (d_y F_c)^* \circ d_y F_c$. We have

$$\text{trace}(L) = \sum_{i=}^{n} g(L(u_i), u_i) = \sum_{i=}^{n} g(d_y F_c(u_i), d_y F_c(u_i)) \leq n(1 + \beta_2). \tag{44}$$

On the other hand

$$|1 - \alpha|^2 \leq |\operatorname{Jac} F_c(y)|^2 = det(L) \leq \left(\frac{\text{trace}(L)}{n}\right)^n \leq (1 + \beta_2)^n,$$

which shows that there is almost equality in the arithmetic-geometric inequality. We get that there exists some $\alpha_3(\varepsilon, \delta) > 0$, with $\alpha_3(\varepsilon, \delta) \to 0$ as $\varepsilon, \delta \to 0$, such that

$$||L - Id|| \leq \alpha_3(\varepsilon, \delta).$$

Thus for any $y \in Y_{\alpha_1}$ and $u \in T_yY$

$$(1-\alpha_3)\|u\| \leq \|d_yF_c(u)\|_{g_0} \leq (1+\alpha_3)\|u\| \tag{45}$$

and d_yF_c is almost isometric. □

We now prove that given a fixed radius $R > 0$, the natural maps F_c are uniformly bounded on $B(y_g, R)$ if the parameters ε,δ are sufficiently small.

Lemma 2.12. *Let $R > 0$, then there exists $\varepsilon(R) > 0$ and $\delta(R) > 0$ such that for any $0 < \varepsilon < \varepsilon(R)$ and $0 < \delta < \delta(R)$, and for any $y \in B(y_g, R)$,*

$$\|d_yF_c\| \leq 2\sqrt{n} \tag{46}$$

Proof. We first prove that for all $y \in Y$, $\|d_yF_c\|$ is controlled from above by $\lambda_n^c(y)$, the maximal eigenvalue of H_y^c (see definition 2.4). Recall that $0 < \lambda_n^c < 1$. Let u be a unit vector in $T_y\tilde{Y}$ and $v = d_yF_c(u)$. The equation(11) gives

$$(1-\lambda_n^c(y))\,|g_0(d_yF_c(u), d_yF_c(u))| \leq c\lambda_n^c(y)^{1/2} g_0\left(d_yF_c(u), d_yF_c(u)\right)^{1/2} \tag{47}$$

thus

$$\|d_yF_c(u)\|_{g_0} \leq \frac{c\sqrt{\lambda_n^c(y)}}{1-\lambda_n^c(y)}. \tag{48}$$

Hence we have to show that $\lambda_n^c(y)$ is not close to 1. More precisely, let $\beta > 0$ such that $\frac{1}{n} + \beta < 1$, one then defines

$$\gamma(\delta,\beta) = \left(\frac{n-1+\delta}{n-1-n\beta}\right)\sqrt{1+n\beta} - 1 > 0.$$

One can check that if $\lambda_n^c(y) \leq \frac{1}{n} + \beta$, then $\|d_yF_c(u)\|_{g_0} \leq \sqrt{n}(1+\gamma)$. Clearly, $\gamma(\beta,\delta) \to 0$ as $\delta, \beta \to 0$. For our purposes, it is sufficient to suppose that $\gamma \leq 1$. Let $\delta_n > 0$ and $\beta_n > 0$ be such that if $0 < \delta \leq 10\delta_n$ and $0 < \beta \leq 10\beta_n$ then $\gamma(\delta,\beta) \leq 1$. One defines moreover $\varepsilon_n > 0$ such that if $0 < \varepsilon < \varepsilon_n$ then with the notations of lemma 2.11, $\beta_1(\varepsilon,\delta) \leq \beta_n$. In what follows, we suppose ε and δ sufficiently small.

By (36) we have that $|\lambda_n^c(y) - \frac{1}{n}| \leq \beta_1(\varepsilon,\delta)$ on Y_{α_1}. Recall that Y_{α_1} has a large relative volume in Y. The idea is first to estimate λ_n^c on a neighborhood of Y_{α_1} and then to show that this neighbourhood contains $B(y_g, R)$ if the parameters ε, δ are sufficiently small relatively to R.

We begin by estimating the variation of λ_n^c. Recall that H_y^c is defined by

$$g_0(H_y^c(u), v) = \int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(u) dB_{(F_c(y),\theta)}(v)\; d\sigma_y^c(\theta).$$

Let U, V be parallel vector fields near $F_c(y)$ extending unit vectors u, v. We compute the derivative of $g_0(H_y^c(U), V)$ in a direction $w \in T_yY$:

$$w.g_0(H_y^c(U), V) = \int_{\partial\tilde{X}} DdB_{(F(y),\theta)}(d_yF(w), U) dB_{(F_c(y),\theta)}(V) d\sigma_y^c(\theta) +$$
$$\int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(U) DdB_{(F(y),\theta)}(d_yF(w), V) d\sigma_y^c(\theta) + \int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(U) dB_{(F_c(y),\theta)}(V) w.d\sigma_y^c(\theta)$$

Recall that $\|DdB\| \leq 1$ and $\|dB\| \leq 1$ thus

$$|w.g_0(H_y^c(U),V)| \leq 2\|d_yF_c(w)\|_{g_0} + \left|\int_{\partial\tilde{X}} w.d\sigma_y^c(\theta)\right|.$$

Recall now that

$$d\sigma_y^c(\theta) = \frac{d\mu_y^c(\theta)}{\mu_y^c(\partial\tilde{X})} = \frac{\int_{\tilde{Y}} p(\tilde{f}(z),\theta)e^{-c\rho(y,z)}\mathrm{dv}_{\tilde{\mathrm{g}}}(z)}{\int_{\tilde{Y}} e^{-c\rho(y,z)}\mathrm{dv}_{\tilde{\mathrm{g}}}(z)}d\theta\,.$$

Then

$$w.d\sigma_y^c(\theta) = \frac{\int_{\tilde{Y}} p(\tilde{f}(z),\theta)(-c.d\rho_{(y,z)}(w))e^{-c\rho(y,z)}\mathrm{dv}_{\tilde{\mathrm{g}}}(z)}{\mu_y^c(\partial\tilde{X})}d\theta - \tag{49}$$

$$\frac{d\mu_y^c(\theta)}{\mu_y^c(\partial\tilde{X})^2}.\int_{\tilde{Y}}(-c.d\rho_{(y,z)}(w))e^{-c\rho(y,z)}\mathrm{dv}_{\tilde{\mathrm{g}}}(z)\,. \tag{50}$$

As $|d\rho_{(y,z)}(w)| \leq \|w\|_g$, we have

$$\left|\int_{\partial\tilde{X}} w.d\sigma_y^c(\theta)\right| \leq \int_{\partial\tilde{X}} 2c\|w\|_g d\sigma_y^c(\theta) = 2c\|w\|_g\,, \tag{51}$$

we get that, $\left|w.g_0(H_y^c(U),V)\right| \leq 2\|d_yF_c(w)\|_{g_0} + 2c\|w\|_g$. We now suppose that w is a unit vector and we use (48), then

$$\left|w.g_0(H_y^c(U),V)\right| \leq 2c\left(\frac{\sqrt{\lambda_n^c(y)}}{1-\lambda_n^c(y)}+1\right). \tag{52}$$

Let us now consider small constants $\eta > \beta > 0$. One defines

$$r(\delta,\beta,\eta) = \frac{\eta-\beta}{2(n-1+\delta)\left(\frac{\sqrt{\frac{1}{n}+\eta}}{1-(\frac{1}{n}+\eta)}+1\right)} > 0.$$

Our goal is to prove that

$$\inf\{d(y_0,y_1)\ |y_0,y_1 \in Y, \lambda_n^c(y_0)\} \leq \frac{1}{n}+\beta,\ \lambda_n^c(y_1) \geq \frac{1}{n}+\eta \geq r(\delta,\beta,\eta).$$

Let $y_0 \in Y$ so that $\lambda_n^c(y_0) \leq \frac{1}{n}+\beta$. Assume that there exists $y \in Y$ such that $\lambda_n^c(y) \geq \frac{1}{n}+\eta$. One defines

$$r = \inf\{d(y_0,y)\ |y \in Y, \lambda_n^c(y)\} \geq \frac{1}{n}+\eta.$$

Clearly, there exist $y_1 \in Y$ such that $\lambda_n^c(y_1) = \frac{1}{n}+\eta$ and $d(y_0,y_1) = r$. Let $\gamma : [0,r] \longrightarrow Y$ be a minimising geodesic from y_0 to y_1. We easily see that $\lambda_n^c(\gamma(t)) < \frac{1}{n}+\eta$ for any $0 \leq t < r$. Let $U(t)$ be a parallel vector field in X along $F_c(\gamma)$ such that $U(r)$ is a unit

eigenvector of $H^c_{y_1}$. Then using (52) with $\dot\gamma.g_0(H^c_{\gamma(t)}U(t),U(t)) = \frac{d}{dt}g_0(H^c_{\gamma(t)}U(t),U(t))$, one has

$$\begin{aligned}
|\lambda^c_n(y_1) - \lambda^c_n(y_0)| &\leq \left|g_0(H^c_{\gamma(r)}U(r),U(r)) - g_0(H^c_{\gamma(0)}U(0),U(0))\right| &(53)\\
&= \left|\int_0^r \frac{d}{dt} g_0(H^c_{\gamma(t)}U(t),U(t))dt\right| &(54)\\
&\leq 2c.\int_0^r \frac{\sqrt{\lambda^c_n(\gamma(t))}}{1-\lambda^c_n(\gamma(t))} + 1\ dt &(55)\\
&\leq 2cr.\left(\frac{\sqrt{\frac{1}{n}+\eta}}{1-(\frac{1}{n}+\eta)} + 1\right). &(56)
\end{aligned}$$

Thus

$$r \geq \frac{\eta-\beta}{2(n-1+\delta)\left(\frac{\sqrt{\frac{1}{n}+\eta}}{1-(\frac{1}{n}+\eta)}+1\right)} = r(\delta,\beta,\eta).$$

Now we fix $\eta = 2\beta_n$ so that $\gamma(\delta,\eta) \leq 1$ for any $\delta \leq \delta_n$. One defines $r_n = r(\delta_n,\beta_n,2\beta_n)$. Recall that for $\varepsilon \leq \varepsilon_n$ and $\delta \leq \delta_n$, we have $\beta_1(\varepsilon,\delta) \leq \beta_n$. On Y_{α_1}, one has $\lambda^c_n(y) \leq \frac{1}{n} + \beta_1(\varepsilon,\delta) \leq \frac{1}{n} + \beta_n$. Thus if $\lambda^c_n(y_1) \geq \frac{1}{n} + 2\beta_n$, one has

$$d(y_1, Y_{\alpha_1}) \geq r(\delta, \beta_1(\varepsilon,\delta), 2\beta_n) \geq r(\delta_n,\beta_n,2\beta_n) = r_n.$$

We have proved that in the r_n-neighborhood of Y_{α_1}, one has $\lambda^c_n(y) \leq \frac{1}{n} + 2\beta_n$. It implies that

$$||d_yF_c|| \leq (1+\gamma(\delta,2\beta_n))\sqrt{n} \leq 2\sqrt{n}.$$

Let us denote by $V_{r_0}(Y_{\alpha_1})$ the r_0-neighborhood of Y_{α_1}. Il remains to show that $B(y_g,R) \subset V_{r_0}(Y_{\alpha_1})$ if $\varepsilon \leq \varepsilon(R)$ and $\delta \leq \delta(R)$. Recall that $\frac{\mathrm{vol}_g(Y_{\alpha_1})}{\mathrm{vol}_g(Y)} \geq 1-\alpha$, thus

$$\mathrm{vol}_g(Y - Y_{\alpha_1}) \leq \alpha_1 \,\mathrm{vol}_g(Y) \leq \alpha_1(1+\varepsilon)\,\mathrm{vol}_{g_0}(X) = v(\varepsilon,\delta).$$

Clearly, $v(\varepsilon,\delta)\to 0$ when $\varepsilon,\delta\to 0$. On the other hand, by (9) for any $y \in B(y_g,R)$ we have

$$\mathrm{vol}_g(B_g(y,r_0)) \geq v_n \frac{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(\mathrm{r}_0))}{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(1+\mathrm{R}+\mathrm{r}_0))} := v_0(R) > 0. \quad (57)$$

If $v_0(R) > v(\varepsilon,\delta)$, then for any $y \in B(y_g,R)$ one has $B_g(y,r_0) \not\subset Y - Y_{\alpha_1}$, which means that $B_g(y,r_0)$ intersects Y_{α_1}. Thus $d(y,Y_{\alpha_1}) < r_0$ and $y \in V_{r_0}(Y_{\alpha_1})$.

So if we define $\varepsilon(R) > 0$, $\delta(R) > 0$ to be sufficiently small constants such that $v(\varepsilon,\delta) < v_0(R)$, the lemma is proved. □

We now prove that F_c is almost 1-lipschitz.

Lemma 2.13. *For any $R > 0$, there exists $\varepsilon_2(R) > 0$ and $\delta_2(R) > 0$ such that for every $0 < \varepsilon < \varepsilon_2(R)$ and $0 < \delta < \delta_2(R)$, there exists $\kappa = \kappa(\varepsilon, \delta, R) > 0$ such that on $B_g(y_g, R)$:*

$$d_{g_0}(F_c(y_1), F_c(y_2)) \leq (1 + \kappa) d_g(y_1, y_2) + \kappa \,. \tag{58}$$

Moreover, $\kappa(\varepsilon, \delta, R) \to 0$ as $\varepsilon, \delta \to 0$.

Proof. The idea is the following. We have proved that on Y_{α_1} $d_y F_c$ is almost isometric. On the other hand, $d_y F_c$ is uniformly bounded in $B(y_r, R)$ if the parameters ε and δ are sufficiently small. To prove the lemma one computes the lengths of $F_c(\gamma)$ where γ is a minimising geodesic in $B(y_g, R)$ whose intersection with Y_{α_1} is large.

Fix some $R > 0$. We define the following constants :

If $d > 0$,

$$c_1(n, d) = \sup_{0 < s/2 < r < s < d} \frac{\mathrm{vol}_{\mathbf{H}^n}(\partial B_{\mathbf{H}^n}(s))}{\mathrm{vol}_{\mathbf{H}^n}(\partial B_{\mathbf{H}^n}(r))}.$$

If $\tau > 0$, $R > 0$,

$$c_2(n, \tau, R) = c_1(n, 2R)(2\tau vol_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(\tau)).$$

If $\varepsilon > 0$, $\delta > 0$,

$$\theta(\varepsilon, \delta) = 2\alpha_3^2(\varepsilon, \delta)\, \mathrm{vol}_{g_0}(X) + 2(4n + 2\sqrt{n} + 1)\alpha_1(\varepsilon, \delta) vol_{g_0}(X).$$

Clearly, $\theta(\varepsilon, \delta) \to 0$ as $\varepsilon, \delta \to 0$.

Let $\tau(\varepsilon, \delta, R) > 0$ be the function defined by

$$\mathrm{vol}_{\mathbf{H}^n}(\tau)\tau = \theta(\varepsilon, \delta) \frac{2c_1(n, 2R)\, \mathrm{vol}_{\mathbf{H}^n}(1 + R + 1)^2}{v_n^2}.$$

Again we easily see that, $\tau(\varepsilon, \delta, R) \to 0$ as $\varepsilon, \delta \to 0$. One defines $\varepsilon_2(R) > 0$ and $\delta_2(R) > 0$ such that $\varepsilon_2(R) \leq \varepsilon(2R)$, $\delta_2(R) < \delta(2R)$ and if $0 < \varepsilon \leq \varepsilon(R)$ and $0 < \delta < \delta(R)$ then $\tau(\varepsilon, \delta, R) << 1$.

Finally, one defines $\kappa(\varepsilon, \delta, R) = \max(2\sqrt{n}\sqrt{\tau}, 8\sqrt{\tau})$.

There are two cases.

Case i) let y_1, y_2 in $B_g(y_g, R)$ such that $d(y_1, y_2) \leq \sqrt{\tau}$. Using (46), if $0 < \varepsilon < \varepsilon(2R)$, $0 < \delta < \delta(2R)$ one has

$$d(F_c(y_1), F_c(y_2)) \leq 2\sqrt{n}\sqrt{\tau} \leq \kappa. \tag{59}$$

Case ii) : let y_1, y_2 in $B_g(y_g, R)$ such that $d(y_1, y_2) \geq \sqrt{\tau}$. We use [Col] proposition 2.11 with the function

$$e(y) = \sup_{u \in U_y Y} \left(\|d_y F_c.u\| - \|u\| \right)^2 .$$

Let us define $A_1 = B_g(y_1, \tau)$, $A_2 = B_g(y_2, \tau)$ and $W = B_g(y_g, 2R)$. For any $z_1 \in A_1$ and any unit vector $v_1 \in T_{z_1} Y$, the measure $|I(z_1, v_1)|$ of

$$I(z_1, v_1) = \{ t \mid \gamma(t) \in A_2, \gamma_{|[0,t]} \text{ is minimal }, \gamma'(0) = v_1 \}$$

is bounded by 2τ. Thus

$$D(A_1, A_2) := \sup_{z_1, v_1} |I(z_1, v_1)| \leq 2\tau.$$

Similarly, $D(A_2, A_1) \leq 2\tau$. For any $z_1 \in A_1$ and $z_2 \in A_2$, let $\gamma_{z_1 z_2}$ be a minimizing geodesic from z_1 to z_2. Clearly, $\gamma \subset B(y_g, 2R)$. Then by [Col] proposition 2.11 we have

$$\int_{A_1 \times A_2} \int_0^{d(z_1,z_2)} e(\gamma_{z_1,z_2})(s)\ ds \leq c_1(n, 2R)\left(D(A_1, A_2)\,\mathrm{vol}(A_1) + D(A_2, A_1)\,\mathrm{vol}(A_2)\right) \times \int_W e(y)\ \mathrm{dv_g}(y).$$

By Bishop's theorem, for $i = 1{,}2$ we have

$$\mathrm{vol}_g(A_i) \leq vol_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(\tau)),$$

and thus

$$c_1(n, 2R)\left(D(A_1, A_2)\,\mathrm{vol}(A_1) + D(A_2, A_1)\,\mathrm{vol}(A_2)\right) \leq c_2(n, \tau, R).$$

We then have, using (35) on $W \cap Y_{\alpha_1}$ and (46) on $W - Y_{\alpha_1}$,

$$\begin{aligned}
\int_{A_1 \times A_2} \int_0^{d(z_1,z_2)} e(\gamma_{z_1,z_2})(s)\ ds &\leq c_2(n, \tau, R)\left(\int_{W \cap Y_{\alpha_1}} e(y)\ \mathrm{dv_g}(y) + \int_{W - Y_{\alpha_1}} e(y)\ \mathrm{dv_g}(y)\right) \\
&\leq c_2(n, \tau, R)\left(\alpha_3^2.\,\mathrm{vol}_g(Y) + (4n + 2\sqrt{n} + 1)\,\mathrm{vol}_g(Y - Y_{\alpha_1})\right) \\
&\leq c_2(n, \tau, R)\theta(\varepsilon, \delta). \qquad (60)
\end{aligned}$$

Now, if we denote $\gamma = \gamma_{z_1 z_2}$, we have

$$\begin{aligned}
|\ell(F_c \circ \gamma) - \ell(\gamma)| &= \left| \int_0^{d(z_1,z_2)} \|d_{\gamma(s)} F_c(\dot{\gamma})\| - \|\dot{\gamma}\|\ ds \right| \\
&\leq \int_0^{d(z_1,z_2)} \sup_{u \in T_y Y} \left| \|d_{\gamma(s)} F_c(u)\| - \|u\| \right|\ ds.
\end{aligned}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned}
\frac{|\ell(F_c \circ \gamma) - \ell(\gamma)|^2}{d(z_1, z_2)} &\leq \frac{\left(\int_0^{d(z_1,z_2)} sup_u \left| \|d_{\gamma(s)} F_c(u)\| - \|u\| \right|\ ds\right)^2}{d(z_1, z_2)} \\
&\leq \int_0^{d(z_1,z_2)} e(\gamma(s))\ ds.
\end{aligned}$$

Integrating on $A_1 \times A_2$, we deduce from (60) that

$$\int_{A_1 \times A_2} \frac{|\ell(F_c \circ \gamma_{z_1 z_2}) - \ell(\gamma_{z_1 z_2})|^2}{d(z_1, z_2)} \mathrm{dv_g}(z_1)\mathrm{dv_g}(z_2) \leq c_2(n, \tau, R)\theta(\varepsilon, \delta)\,. \qquad (61)$$

By (9), for $i = 1{,}2$ one has

$$\mathrm{vol}_g(A_i) \geq v_n \frac{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(\tau))}{\mathrm{vol}_{\mathbf{H}^n}(\mathrm{B}_{\mathbf{H}^n}(1+\mathrm{R}+\tau))} := v_0(\tau, R) > 0.$$

Thus

$$c_2(n,\tau,R)\theta(\varepsilon,\delta) \leq \frac{1}{v_0(\tau,R)^2} \int_{A_1\times A_2} c_2(n,\tau,R)\theta(\varepsilon,\delta)\ \mathrm{dv_g}(z_1)\mathrm{dv_g}(z_2).$$

We then have

$$\int_{A_1\times A_2} \frac{|\ell(F_c\circ\gamma_{z_1z_2}) - \ell(\gamma_{z_1z_2})|^2}{d(z_1,z_2)} \leq \int_{A_1\times A_2} \frac{c_2(n,\tau,R)\theta(\varepsilon,\delta)}{v_0(\tau,R)^2}. \tag{62}$$

Hence there exists $z_1 \in A_1$, $z_2 \in A_2$ such that

$$|\ell(F_c\circ\gamma_{z_1z_2}) - \ell(\gamma_{z_1z_2})|^2 \leq d(z_1,z_2)\frac{c_2(n,\tau,R)\theta(\varepsilon,\delta)}{v_0(\tau,R)^2}\,.$$

One can check that

$$\frac{c_2(n,\tau,R)\theta(\varepsilon,\delta)}{v_0(\tau,R)^2} \leq \theta(\varepsilon,\delta)\frac{2c_1(n,2R)\,\mathrm{vol}_{\mathbf{H}^n}(1+R+1)^2}{v_n^2\,\mathrm{vol}_{\mathbf{H}^n}(\tau)}\tau \leq \tau^2.$$

Now one has

$$|\ell(F_c\circ\gamma_{z_1z_2}) - \ell(\gamma_{z_1z_2})|^2 \leq d(z_1,z_2)\tau^2,$$

thus

$$d(F_c(z_1),F_c(z_2)) \leq \ell(F_c\circ\gamma_{z_1z_2}) \leq d(z_1z_2) + \tau d(z_1,z_2)^{1/2}.$$

As $d(y_i,z_i) < \tau$ and $d(y_1,y_2) \geq \sqrt{\tau}$, we have

$$d(z_1,z_2) \leq d(y_1,y_2) + 2\tau \leq d(y_1,y_2)(1+2\sqrt{\tau}).$$

On the other hand, as $\tau << 1$ we have

$$d(z_1,z_2) \geq d(y_1,y_2) - 2\tau \geq \frac{\sqrt{\tau}}{2}.$$

We then have

$$\begin{aligned}
d(F_c(y_1),F_c(y_2)) &\leq d(F_c(y_1),F_c(z_1)) + d(F_c(z_1),F_c(z_2)) + d(F_c(z_2),F_c(y_2)) && (63)\\
&\leq 2\sqrt{n}\tau + d(z_1,z_2) + \tau.d(z_1,z_2)^{1/2} + 2\sqrt{n}\tau && (64)\\
&\leq 4\sqrt{n}\tau + d(y_1,y_2)\frac{d(z_1,z_2)}{d(y_1,y_2)}(1+\tau.d(z_1,z_2)^{-1/2}) && (65)\\
&\leq 4\sqrt{n}\tau + d(y_1,y_2)(1+2\sqrt{\tau})(1+\sqrt{2}\tau^{3/4}) && (66)\\
&\leq 4\sqrt{n}\tau + d(y_1,y_2)(1+8\sqrt{\tau})\,. && (67)
\end{aligned}$$

Finally, we get

$$d(F_c(y_1),F_c(y_2)) \leq \kappa + (1+\kappa)d(y_1,y_2)\,, \tag{68}$$

in case ii). □

3 A limit map on the limit space

In this section, we consider a sequence $(Y_k, g_k)_{k\in\mathbf{N}}$ of Riemannian compact n-manifolds satisfying the curvature assumption (1) and the following condition: we suppose that there exists an hyperbolic compact n-manifold (X, g_0), degree one maps $f_k : Y_k \to X$ and a sequence $\varepsilon_K \to 0$ such that

$$\mathrm{vol}_{g_k}(Y_k) \to \mathrm{vol}_{g_0}(X)\,, \tag{69}$$

as $k \to +\infty$. Moreover, for every $k \in \mathbf{N}$, there exists $y_{g_k} \in Y_k$ satisfying the local volume property (6).

We prove that (Y_k, g_k, y_{g_k}) sub-converges in the pointed Gromov-Haudorff topology to a limit space $(Y_\infty, d_\infty, z_\infty)$. Moreover, there exists a sequence of natural maps $F_{c_k} : (Y_k, g_k) \to (X, g_0)$, with suitably chosed parameters, which sub-converges to a "natural map" $F : Y_\infty \longrightarrow X$.

Let us define the Gromov-Hausdorff topology. Recall that for two subsets A, B of a metric space Z the Hausdorff distance between A and B is

$$\mathrm{d_{\mathcal{H}}}^Z(A, B) = \inf\{\varepsilon > 0 \mid B \subset V_\varepsilon(A) \text{ and } A \subset V_\varepsilon(B)\} \in \mathbf{R} \cup \{\infty\}.$$

It is a distance on compact subsets of Z.

Definition 3.1. *Let X_1, X_2 be two metric spaces, then the Gromov-Hausdorff distance $\mathrm{d_{\mathcal{GH}}}(X_1, X_2) \in \mathbf{R} \cup \infty$ is the infimum of the numbers*

$$\mathrm{d_{\mathcal{H}}}^Z(f_1(X_1), f_2(X_2)))$$

for all metric spaces Z and all isometric imbeddings $f_i : X_i \to Z$.

It is a distance on the space of isometry classes of compact metric spaces. One says that a sequence $(X_i)_{i\in\mathbf{N}}$ of metric spaces converges in the Gromov-Hausdorff topology to a metric space X_∞ if $\mathrm{d_{\mathcal{GH}}}(X_i, X_\infty)) \to 0$ as $i \to \infty$. Let $x_i \in X_i$, $x_\infty \in X_\infty$, one says that the sequence $(X_i, x_i)_{i\in\mathbf{N}}$ *converges to (X_∞, x_∞) in the pointed Gromov-Hausdorff topology* if for any $R > 0$, $\mathrm{d_{\mathcal{GH}}}(B_{X_i}(x_i, R), B_{X_\infty}(x_\infty, R)) \to 0$ as $i \to +\infty$ for some sequence (ε_i) tending to 0 (In fact this definition holds only for length spaces, which will be sufficient in our situation).

To deal with the Gromov-Hausdorff distance between X_1 and X_2, it is convenient to avoid the third space Z by using ε-approximations between X_1 and X_2 .

Definition 3.2. *Given two metric spaces X_1,X_2 and $\varepsilon > 0$, an ε-approximation (or ε-isometry) from X_1 to X_2 is a map $f : X_1 \to X_2$ such that*

1. *for any $x, x' \in X_1$, $|d_{X_2}(f(x), f(x') - d_{X_1}(x, x')| < \varepsilon$.*
2. *the ε-neighbourhood of $f(X_1)$ is equal to X_2.*

Then one can show (see [BBI] corollary 7.3.28) that $\mathrm{d}_{\mathcal{GH}}(X_1, X_2) < \varepsilon$ if and only if there exists a 2ε-approximation from X_1 to X_2.

Our goal is to prove the :

Proposition 3.3. *Up to extraction and renumbering, the sequence (Y_k, g_k, y_k) satisfies the following.*

1. *There exists a complete pointed length space $(Y_\infty, d_\infty, y_\infty)$ such that (Y_k, g_k, y_k) converges in the pointed Gromov-Hausdorff topology to $(Y_\infty, d_\infty, y_\infty)$. Moreover, (Y_∞, d_∞) has Hausdorff dimension equal to n.*

2. *there exists sequences of positive numbers (ε_k), (δ_k) tending to 0, (c_k) such that $h(g_k) < c_k < h(g_k) + \delta_k$, (R_k) tending to $+\infty$ such that $\varepsilon_k \leq \varepsilon(R_k)$ and $\delta_k \leq \delta(R_k)$, and α_k-approximations $\psi_k : B_{d_\infty}(y_\infty, R_k) \to B_{g_k}(y_{g_k}, R_k)$ such that the following holds. Let*
$$F_{c_k} : (Y_k, g_k) \to (X, g_0)$$
be the natural map as defined in section 2. Then $F_{c_k} \circ \psi_k$ converges uniformly on compact sets to a map
$$F : Y_\infty \longrightarrow X,$$
which is 1-lipschitz.

Proof of (1) and some properties of the limit

Under the curvature assumption (1) and the local volume assumption (9), the point (1) is a straightforward application of Gromov & Cheeger-Colding compactness theorem. See ([Ch-Co]). Let us make precise some features of the convergence and of the limit space.

The continuity of the volume under the (pointed) Gromov-Hausdorff convergence is crucial for our purposes. For $\ell > 0$, note $\mathcal{H}^\ell$ the ℓ-dimensional Hausdorff measure of a metric space (see [BBI] definition 1.7.7).

Theorem 3.4 ([Ch-Co] 5.9). *Let $p_i \in Y_i$ and $p_\infty \in Y_\infty$ their limit, and let $R > 0$. Then*
$$\lim_{i \to +\infty} \mathrm{vol}_{g_i}(B(p_i, R)) = \mathcal{H}^n(B(p_\infty, R))\,. \tag{70}$$

In particular, Y_∞ satisfies the Bishop-Gromov inequalities (7) and the Bishop inequality. By definition, a *tangent cone* at $p \in Y_\infty$ is a complete pointed Gromov-Hausdorff limit, $\{Y_{\infty,p}, d_\infty, p_\infty\}$ of a sequence of rescaled space, $\{(Y_\infty, r_i^{-1} d, p)\}$, where $\{r_i\}$ is a positive sequence such that $r_i \to 0$. By [GLP] theorem 5.2, every such sequence has a convergent subsequence, but the limit might depend on the choice of the subsequence. The limit is called a tangent *cone* because $(Y_{\infty,p}, \lambda d_\infty)$ is isometric to $(Y_{\infty,p}, d_\infty)$ for any $\lambda > 0$.

Definition 3.5. *The **regular** set $\mathcal{R}$ consists of those points, $p \in Y_\infty$, such that every tangent cone at y is isometric to $\mathbf{R}^n$. The complementary $\mathcal{S} = Y_\infty - \mathcal{R}$ is the **singular** set.*

Let $B_0^n(1) \subset \mathbf{R}^n$ be the unit ball.

Definition 3.6. *The ε**-regular** set $\mathcal{R}_\varepsilon$ consists of those points, $p \in Y_\infty$, such that every tangent cone, $(Y_{\infty,p}, p_\infty)$, satisfies $d_{GH}(B(p_\infty, 1), B_0^n(1)) < \varepsilon$. A point in $Y_\infty - \mathcal{R}_\varepsilon = \mathcal{S}_\varepsilon$ is called ε**-singular**,*

Theorem 3.7 ([Ch-Co] 5.14). *There exists $\varepsilon_n > 0$ such that for $\varepsilon \leq \varepsilon_n$, $\overset{\circ}{\mathcal{R}}_\varepsilon$ has a natural smooth manifold structure. Moreover, for this parametrization, the metric on $\overset{\circ}{\mathcal{R}}_\varepsilon$ is bi-hölder equivalent to a smooth Riemannian metric. The exponent $\alpha(\varepsilon)$ in this bi-hölder equivalence satisfies $\alpha(\varepsilon) \to 1$ as $\varepsilon \to 0$.*

Theorem 3.8 ([Ch-Co]6.1).

$$\mathcal{H}^{n-2}(\mathcal{S}) = 0 \tag{71}$$

Remark 3.9. *Clearly, $\mathcal{R} = \cap_{\varepsilon>0}\mathcal{R}_\varepsilon$. The sets $\mathcal{R}_\varepsilon$, $\mathcal{R}$ are not necessarily open. However, for any $\varepsilon > 0$, there is some $\varepsilon > \delta > 0$ such that $\mathcal{R}_\delta \subset \overset{\circ}{\mathcal{R}}_\varepsilon$ (see Appendix A.1.5 in [Ch-Co]). In [Ch-Co2] section 3, it is proved that $\overset{\circ}{\mathcal{R}}_\varepsilon$ is path connected. This important fact will be used in the last part of this text.*

Let us study the density of this hausdorff measure. A consequence of Bishop's inequality is that

$$\limsup_{r\to 0} \frac{\mathcal{H}^n(B(p,r)}{\mathrm{vol}_{\mathbf{R}^n}(r)} \leq 1.$$

Definition 3.10. *the density at p of Y_∞ is*

$$\theta(p) := \liminf_{r\to 0} \frac{\mathcal{H}^n(B(p,r))}{\mathrm{vol}_{\mathbf{R}^n}(r)}. \tag{72}$$

A consequence of [Ch-Co]A.1.5 is the existence of some positive function $\tau(\varepsilon)$, with $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for every $p \in \mathcal{R}_\varepsilon$,

$$\theta(p) > 1 - \tau(\varepsilon). \tag{73}$$

Conversely, there exists a positive function $\varepsilon(\tau)$, such that $\varepsilon(\tau) \to 0$ as $\tau \to 0$ and such that

$$\theta(p) \geq 1 - \tau \Longrightarrow p \in \mathcal{R}_{\varepsilon(\tau)}\,. \tag{74}$$

Remark 3.11. *A point p is called regular if and only if $\theta(p) = 1$. From now on, we consider $\varepsilon \leq \varepsilon_0$, where $\varepsilon_0 \leq \varepsilon_n$ is sufficiently small so that $\tau(\varepsilon_0) \leq 1/2$, the density is thus $> 1/2$ on $\mathcal{R}_\varepsilon$.*

Existence of the natural map at the limit

For every $k \in \mathbf{N}$ and $h(g_k) < c$, there exists a natural map $F_c : (Y_k, g_k) \to (X, g_0)$, as seen in section 2. To define a good sequence one argues as follows. Given $m \in \mathbf{N}^*$, one chooses positive numbers $\varepsilon_m \leq \varepsilon_2(m)$ and $\delta_m \leq \delta_2(m)$ sufficiently small such that $\kappa(\varepsilon_m, \delta_m, m) \leq \frac{1}{m}$. One defines

$$\alpha_m = \max(\alpha_1(\varepsilon_m, \delta_m), \alpha_2(\varepsilon_m, \delta_m), \alpha_3(\varepsilon_m, \delta_m)\kappa(\varepsilon_m, \delta_m, m)).$$

We check that $\alpha_m \to 0$ as $m \to +\infty$. By the hypothesis (69), there exists $k_1(m) \in \mathbf{N}$ such that for any $k \geq k_1(m)$, $vol_{g_k}(Y_k) \leq (1 + \varepsilon_m) vol_{g_0}(X)$. As $B_{g_k}(y_{g_k}, m)$ tends to $B_\infty(y_\infty, m)$, there exists $k_2(m) \in \mathbf{N}$ such that for any $k \geq k_2(m)$, there exists α_m-approximations from $B_\infty(y_\infty, m)$ to $B_{g_k}(y_{g_k}, m)$. Define $k(m) = \max(k_1(m), k_2(m))$ and let $\psi_m : B_\infty(y_\infty, m) \longrightarrow B_{g_k}(y_{k(m)}, m)$ be an α_m-approximation. One can suppose that $\psi_m(y_\infty) = y_{g_{k(m)}}$. Fix $c_m \in]h(g_k), h(g_k) + \delta_m[$ and consider

$$F_{c_m} \circ \psi_m : B_\infty(y_\infty, m) \longrightarrow X.$$

The lemma (2.13) applies to F_{c_m} on $B_{g_{k(m)}}(y_{g_{k(m)}}, m)$. Thus for any $p, q \in B_\infty(y_\infty, m)$,

$$\begin{aligned} d_{g_0}(F_{c_m} \circ \psi_m(p), F_{c_m} \circ \psi_m(q)) &\leq (1 + \alpha_m) d_{g_k}(\psi_m(p), \psi_m(q)) + \alpha_m \\ &\leq (1 + \alpha_m) d_\infty(p, q) + (1 + \alpha_m)\alpha_m + \alpha_m. \end{aligned}$$

Applying the same reasoning as in Ascoli's theorem, one can show that for any compact $K \subset Y_\infty$, there exists a convergent sub-sequence of F_{c_m} towards a map $F_K : K \to X$. If one uses an exhaustion of Y_∞ by compacts sets and a standard diagonal process, one can extract a sub-sequence of $F_{c_{\psi(m)}} \circ \psi_{\phi m}$ which converges uniformly on any compact set to a map $F : Y_\infty \to X$. It is easy to see that the map h is 1-lipschitz.

Then one renumbers the sub-sequences $Y_{k(\phi(m))}$, $\psi_{\phi(m)}$ and $F_{c_{\phi(m)}}$ such that for any $m \in \mathbf{N}*$, $\mathrm{vol}_{g_m}(Y_m) \leq (1 + \varepsilon_m)\,\mathrm{vol}_{g_0}(X)$, $h(g_m) < c_m < c_m + \delta_m$, and the inequalities of the lemmas (2.5), (2.11) hold with α_1, α_2, α_3 replaced by α_m and those of lemmas (2.12), (2.13) hold on $B(y_m, m) \subset Y_m$ with κ replaced by α_m. For simplicity, the map F_{c_m} will be denoted F_m.

4 The limit map $F : Y_\infty \longrightarrow X$ is isometric

We prove first that F preserves the volume.

Lemma 4.1. *Let $A \subset Y_\infty$ a measurable subset. Then,*

$$\mathrm{vol}_{g_0}(F(A)) = \mathcal{H}^n(A)\,. \tag{75}$$

Proof. As F is 1-lipschitz, it decreases the Hausdorff measure (see [Mor]3.5). Thus, it suffices to prove that for every $B_\infty(p, r) \subset Y_\infty$, $\mathrm{vol}_{g_0}(\overline{F(B_\infty(p, r))}) \geq \mathcal{H}^n(B_\infty(p, r))$. By construction, $\overline{F(B_\infty(p, r))}$ is the Hausdorff limit of $\overline{F_k \circ \psi_k(B_\infty(p, r))}$.

We first show that this is the Hausdorff limit of $\overline{F_k(B_{g_k}(\psi_k(p),r))}$. Let $x \in \overline{F(B_\infty(p,r))}$. There exists $x_k \in F(B_\infty(p,r))$ such that $x_k \to x$. Let $p_k \in B_\infty(p,r)$ such that $F(p_k) = x_k$. By definition of the α_k-approximation, one has $d_{g_k}(\psi_k(p_k),\psi_k(p)) < r+\alpha_k$. There exists $z_k \in B_{g_k}(\psi_k(p),r)$ such that $d_{g_k}(\psi_k(p_k),z_k) < \alpha_k$ (for example z_k may be on the segment $[\psi_k(p_k),\psi_k(p)]$). Note that $d_\infty(p_k,y_\infty) \leq r + d_\infty(p,y_\infty)$ and remember that $\psi_k(y_\infty) = y_{g_k}$. Thus $\psi_k(p_k)$ remains at bounded distance from y_{g_k}. Then, applying the inequality of lemma (2.13) we have

$$\begin{aligned} d_{g_0}(F_k(z_k),F_k(\psi_k(p_k)) &\leq (1+\alpha_k) d_{g_k}(z_k,\psi_k(p_k)) + \alpha_k \\ &\leq (1+\alpha_k)\alpha_k + \alpha_k \\ &\underset{k\to+\infty}{\longrightarrow} 0. \end{aligned}$$

On the other hand, as F_k converges uniformly to F on compact sets, $F_k(\psi_k(p_k)$ has the same limit as $F(p_k) = x_k$, thus $F_k(\psi_k(p_k) \to x$. From the inequality above one deduces that $F_k(z_k) \to x$ thus $x \in \lim_{k\to\infty} \overline{F_k(B_{g_k}(\psi_k(p),r))}$. One has proved that $\overline{F(B_\infty(p,r))} \subset \lim_{k\to\infty} \overline{F_k(B_{g_k}(\psi_k(p),r))}$. To prove the other inclusion one argues in the same way. Given $x \in \lim_{k\to\infty} \overline{F_k(B_{g_k}(\psi_k(p),r))}$, there exists $x_k \in F_k(B_{g_k}(\psi_k(p),r))$ such that $x_k \to x$, thus one can write $x_k = F_k(z_k)$ where $z_k \in B_{g_k}(\psi_k(p),r)$. As ψ_k is an α_k-approximation from $B_\infty(y_\infty,k)$ to $B(y_{g_k},k)$, one has $B_{g_k}(\psi_k(p),r) \subset U_{\alpha_k}\psi_k(B_\infty(p,r+\alpha_k))$ for large k. Thus there exists $q_k \in B_\infty(p,r+\alpha_k)$ such that $d_{g_k}(z_k,\psi_k(q_k)) < r+\alpha_k$. As Y_∞ is a length space, there exists $q'_k \in B_\infty(p,r)$ such that $d_\infty(q'_k,q_k) < \alpha_k$. Then $d_{g_k}(\psi_k(q'_k),z_k) \leq d_{g_k}(\psi_k(q'_k),\psi_k(q_k)) + d_{g_k}(\psi_k(q_k),z_k)) < 3\alpha_k$. Thus

$$\begin{aligned} d_{g_0}(F_k\circ\psi_k(q'_k),x_k) = d_{g_0}(F_k\circ\psi_k(q'_k),F_k(z_k)) &\leq (1+\alpha_k) d_{g_k}(\psi_k(q'_k),z_k) + \alpha_k \\ &\leq (1+\alpha_k)3\alpha_k + \alpha_k \to 0. \end{aligned}$$

Hence $d_{g_0}(F_k\circ\psi_k(q'_k),x) \to 0$. As $F_k\circ\psi_k$ converges uniformly to F on compact sets, one has $d_{g_0}(F(q'_k),x) \to 0$ thus $x \in \overline{F(B_\infty(p,r))}$.

It then suffices to prove that $\lim_{k\to+\infty} \mathrm{vol}_{g_0}(F_k(B_{g_k}(\psi_k(p),r)) \geq \mathcal{H}^n(B_\infty(p,r))$. Remember that $N(F_k|x)$ is the number of preimages of x by F_k. We denote by $X_{k,1}$ the set of $x \in X$ such that $N(F_k|x) = 1$. The construction of the sequence (F_k) and the lemma 2.10 implies that $\mathrm{vol}_{g_0}(X_{k,1}) \geq (1-\alpha_k)\,\mathrm{vol}_{g_0}(X)$. Moreover,

$$\int_{X-X_{k,1}} N(F_k|x)\mathrm{dv}_{g_0}(x) \leq \alpha_k\,\mathrm{vol}_{g_0}(X)\,. \tag{76}$$

We denote by Y_{k,α_k} the set of $y \in Y_k$ such that

$$1-\alpha_k \leq |\,\mathrm{Jac}\,F_k(y)| \leq 1+\alpha_k. \tag{77}$$

Then lemma 2.9 implies that $\mathrm{vol}_{g_k}(Y_{k,1}) \geq (1-\alpha_k)\,\mathrm{vol}_{g_k}(Y_k)$. We then have

$$\begin{aligned}
\mathrm{vol}(F_k(B_{g_k}(\psi_k(p),r)) &= \int_{F_k(B_{g_k}(\psi_k(p),r))} \mathrm{dv}_{g_0} \\
&= \int_{F_k(B_{g_k}(\psi_k(p),r))\cap X_{k,1}} N(F_k|x)\mathrm{dv}_{g_0}(x) + \mathrm{vol}_{g_0}(F_k(B_{g_k}(\psi_k(p),r)) - X_{k,1}) \\
&\geq \int_{B_{g_k}(\psi_k(p),r)\cap F_k^{-1}(X_{k,1})\cap Y_{k,\alpha_k}} |\operatorname{Jac} F_k(y)|\mathrm{dv}_g(y) \\
&\geq (1-\alpha_k)\,\mathrm{vol}_{g_k}\left(B_{g_k}(\psi_k(p),r)\cap F_k^{-1}(X_{k,1})\cap Y_{k,\alpha_k}\right). \qquad (78)
\end{aligned}$$

On the other hand, using (77) and (76) we have

$$\begin{aligned}
\mathrm{vol}(F_k^{-1}(X-X_{k,1})\cap Y_{k,\alpha_k}) &\leq \int_{F_k^{-1}(X-X_{k,1})\cap Y_{k,\alpha_k}} \frac{|\operatorname{Jac} F_k|}{1-\alpha_k}\mathrm{dv}_g \\
&\leq \frac{1}{1-\alpha_k}\int_{X-X_{k,1}} N(F_k|x)\mathrm{dv}_{g_0}(x) \\
&\leq \frac{\alpha_k}{1-\alpha_k}\,\mathrm{vol}_{g_0}(X),
\end{aligned}$$

thus

$$\begin{aligned}
\mathrm{vol}_{g_k}(B_{g_k}(\psi_k(p),r)\cap F_k^{-1}(X_{k,1})\cap Y_{k,\alpha_k}) &= \mathrm{vol}_{g_k}(B_{g_k}(\psi_k(p),r)\cap Y_{k,\alpha_k}) \\
&\quad - \mathrm{vol}_{g_k}(B_{g_k}(\psi_k(p),r)\cap F_k^{-1}(X-X_{k,1})\cap Y_{k,\alpha_k}) \\
&\geq \mathrm{vol}_{g_k}(B_{g_k}(\psi_k(p),r)) - \alpha_k\,\mathrm{vol}_{g_k}(Y_k) - \frac{\alpha_k}{1-\alpha_k}\,\mathrm{vol}_{g_0}(X).
\end{aligned}$$

By putting this inequality in (78) one gets

$$\mathrm{vol}_{g_0}(F_k(B_{g_k}(\psi_k(p),r)) \geq (1-\alpha_k)\,\mathrm{vol}_{g_k}(B_{g_k}(\psi_k(p),r)) - (1-\alpha_k)\alpha_k\,\mathrm{vol}_{g_k}(Y_k) - \alpha_k\,\mathrm{vol}_{g_0}(X).$$

As $B_{g_k}(\psi_k(p),r)$ tends to $B_\infty(p,r)$ in the Gromov-Hausdorff topology, theorem 3.4 implies that $\lim_{k\to\infty}\mathrm{vol}_{g_k}(B_{g_k}(\psi_k(p),r)) = \mathcal{H}^n(B_\infty(p,r))$, hence

$$\lim_{k\to\infty}\mathrm{vol}_{g_0}(F_k(B_{g_k}(\psi_k(p),r)) \geq \mathcal{H}^n(B_\infty(p,r)),$$

which proves the lemma. □

We now prove that F is injective on the set of points where the density is larger than $1/2$.

Lemma 4.2. *The map F is injective on $\mathcal{R}_\varepsilon$.*

Proof. Suppose there are p_1,$p_2 \in \mathcal{R}_\varepsilon$ such that $F(p_1) = F(p_2)$. As F is 1-lipschitz, we have for every $r > 0$,

$$F\left(B_\infty(p_1,r)\cup B_\infty(p_2,r)\right) \subset B_{g_0}(F(p_1),r)\,.$$

By the previous lemma,

$$\mathcal{H}^n\left(B_\infty(p_1,r)\cup B_\infty(p_2,r)\right)=\mathrm{vol}_{g_0}\left(F(B_\infty(p_1,r)\cup B_\infty(p_2,r))\right)\leq \mathrm{vol}_{g_0}\left(B_\infty(F(p_1),r)\right). \tag{79}$$

For $r<\frac{d(p_1,p_2)}{2}$ the balls $B_\infty(p_1,r)$ and $B_\infty(p_2,r)$ are disjoint. Hence, dividing (79) by $\mathrm{vol}_{\mathbf{R}^n}(r)$, we get

$$\frac{\mathcal{H}^n(B_\infty(p_1,r))}{\mathrm{vol}_{\mathbf{R}^n}(r)}+\frac{\mathcal{H}^n(B_\infty(p_2,r))}{\mathrm{vol}_{\mathbf{R}^n}(r)}\leq\frac{\mathrm{vol}_{g_0}\left(B_{g_0}(F(p_1),r)\right)}{\mathrm{vol}_{\mathbf{R}^n}(r)}.$$

By taking the liminf as $r\longrightarrow 0$, we have

$$\theta(p_1)+\theta(p_2)\leq\theta(F(p_1))=1,$$

which is a contradiction, since $\theta\geq 1/2$ on $\mathcal{R}_\varepsilon$ if $\varepsilon<\varepsilon_0$ (see remark 3.11). □

Lemma 4.3. *The map F is open on $\overset{\circ}{\mathcal{R}}_\varepsilon$.*

Proof. Let $p\in\overset{\circ}{\mathcal{R}}_\varepsilon$. We have to prove that there exists $\eta>0$ such that $B_{g_0}(F(p),\eta)\subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$. There exists $r>0$ such that $B_\infty(p,2r)\subset\overset{\circ}{\mathcal{R}}_\varepsilon$. Note that $B:=B_\infty(p,r)$. By the previous lemma, $F(p)\notin F(\partial B)$. Thus, by compactness of ∂B and continuity of F, there exists $\eta>0$ such that $d_{g_0}(F(p),F(\partial B))>\eta$. One could used the theory of local degree but unfortunately, we do not know if (Y_∞,y_∞) is locally lipschitz equivalent to $\mathbf{R}^n$. Let $R>2d(z,z_0)$ be a fixed radius such that $\psi_k(B_\infty(p,2r))\subset B_{g_k}(y_{g_k},R)$ for all large k. Let $z_k=\psi_k(p)$ and $B_k=B(z_k,r)\subset B(y_k,R)$. Consider all integer k large enough such that $\mathrm{d}_{\mathcal{H}}(F_k(\partial B_k),F(\partial B))\leq\frac{\eta}{10}$. Let $\mathcal{C}$ (resp. $\mathcal{C}_k$) be the connected component of $X-F(\partial B)$ (resp. $X-F_k(\partial B_k)$), which contains $F(p)$, (resp $F_k(z_k)$). By the corollary 4.1.26 of [Fed], $\deg(F_k|B_k)$ is constant on $\mathcal{C}_k$, where, for a subset $A\subset Y_k$,

$$\deg(F_k|A)(x)=\sum_{y\in F_k^{-1}(x)\cap A}\ \mathrm{sign}\ \ \mathrm{Jac}\,F_k(y).$$

One sees that $\deg(F_k|B_k)=1$ on $\mathcal{C}_k$ as follows. Denote again by $X_{k,1}\subset X$ the set of $x\in X$ such that $N(F_k|x)=1$, that is x has one preimage by F_k. By the lemma 2.10, $\mathrm{vol}_{g_0}(X_{k,1})\geq(1-\alpha_k)\,\mathrm{vol}_{g_0}(X)$. Clearly, the intersection of $X_{k,1}$ with $\mathcal{C}_k$ has a positive measure for k large enough; indeed, the volume of $B(F_k(z_k),\frac{\eta}{10})\subset\mathcal{C}_k$ is bounded below by (9) and $\mathrm{vol}(B(F_k(z_k),\frac{\eta}{10})-X_{k,1})\longrightarrow 0$ as $k\longrightarrow+\infty$. Now, by lemma 2.12 one has $F_k(B(z_k,\frac{\eta}{20\sqrt{n}}))\subset B(F_k(z_k),\frac{\eta}{10})$ for large k, and the proof of lemma 4.1 shows that the volume of the image is bounded below. It thus intersects $X_{k,1}$ on a set of positive measure for k large enough. This proves that $\deg(F_k|B_k)=1$ on $\mathcal{C}_k$. In particular, any point in $B(F(p),\frac{\eta}{10})$ has a preimage by F_k in B_k. By taking the limit $k\longrightarrow+\infty$, it gives $B(F(p),\frac{\eta}{10})\subset F(\overline{B(p,r)})\subset F(B(p,2r))\subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$. □

Lemma 4.4. *There exists $c(\varepsilon)>0$ such that $F:\overset{\circ}{\mathcal{R}}_\varepsilon\longrightarrow F(\overset{\circ}{\mathcal{R}}_\varepsilon)\subset X$ is locally $(1+c(\varepsilon))$-bilipschitz. Moreover, $c(\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$.*

Proof. The idea is the following: we already know that F is 1-lipschitz and volume preserving. In particular, a ball $B_\infty(p,r) \subset Y_\infty$ is sent into a ball $B_{g_0}(F(p),r) \subset X$. If the ball in Y_∞ is in the almost regular part and has a small radius, its volume is close to the Euclidean one, as is the volume of the hyperbolic ball. One can then estimate how much the image of $B_\infty(p,r)$ is close to fill ${}_{g_0}(F(p),r)$. If one considers the images of two disjoint balls, one can estimate how the hyperbolic balls overlapp, and thus the distance between their centers.

Let $p \in \overset{\circ}{\mathcal{R}}_\varepsilon$. Let $r(p,\varepsilon) > 0$ be a radius such that for every $0 < r \leq r(p,\varepsilon)$,

$$\frac{\mathcal{H}^n(B_\infty(p,r))}{\mathrm{vol}_{\mathbf{R}^n}(r)} \geq 1 - \tau(\varepsilon),$$

and let $r_\varepsilon = min(\varepsilon, r(z,\varepsilon))$. One can suppose that r_ε is smaller than the injectivity radius of X. Let $0 < r < r_\varepsilon^2$ be such that $B_\infty(p,r) \subset \mathcal{R}_\varepsilon$. For every $q \in B_\infty(p,r)$, $B(p, r_\varepsilon - r_\varepsilon^2) \subset B_\infty(q,r_\varepsilon)$. Thus,

$$\begin{aligned}
\mathcal{H}^n(B_\infty(q,r_\varepsilon)) &\geq \mathcal{H}^n(B_\infty(p,r_\varepsilon - r_\varepsilon^2)) && (80)\\
&\geq (1-\tau(\varepsilon))\,\mathrm{vol}_{\mathbf{R}^n}(r_\varepsilon - r_\varepsilon^2) && (81)\\
&\geq (1-\tau(\varepsilon))(1-r_\varepsilon)^n\,\mathrm{vol}_{\mathbf{R}^n}(r_\varepsilon)\,. && (82)
\end{aligned}$$

Suppose that there exists p_1,$p_2 \in B(p,r)$, $p_1 \neq p_2$ and a number $0 < \rho < 1$ such that

$$d_{g_0}(F(p_1),F(p_2)) \leq \rho d_\infty(p_1,p_2).$$

Define $r' = \frac{d(p_1,p_2)}{2} > 0$. By (70) and the Bishop-Gromov inequality (7), for $i = 1, 2$ one has

$$\mathcal{H}^n(B_\infty(p_i,r') \geq \mathcal{H}^n(B(p_i,r_\varepsilon))\frac{\mathrm{vol}_{\mathbf{H}^n}(r')}{\mathrm{vol}_{\mathbf{H}^n}(r_\varepsilon)}.$$

Thus, with lemma 4.1 and (82) we have

$$\begin{aligned}
\mathrm{vol}_{g_0}\left(F(B(p_1,r') \cup B(p_2,r'))\right) &= \mathcal{H}^n(B(p_1,r')) + \mathcal{H}^n(B(p_2,r')) && (83)\\
&\geq 2(1-\tau(\varepsilon))(1-r_\varepsilon)^n\frac{\mathrm{vol}_{\mathbf{H}^n}(r')}{\mathrm{vol}_{\mathbf{H}^n}(r_\varepsilon)}\,\mathrm{vol}_{\mathbf{R}^n}(r_\varepsilon) && (84)\\
&\geq 2(1-\tau(\varepsilon))(1-r_\varepsilon)^n\frac{\mathrm{vol}_{\mathbf{R}^n}(\varepsilon)}{\mathrm{vol}_{\mathbf{H}^n}(\varepsilon)}\,\mathrm{vol}_{\mathbf{R}^n}(r') && (85)\\
&\leq 2\vartheta(\varepsilon)\,\mathrm{vol}_{\mathbf{R}^n}(r') && (86)
\end{aligned}$$

where $\vartheta(\varepsilon) = (1-\tau(\varepsilon))(1-\varepsilon)^n\frac{\mathrm{vol}_{\mathbf{R}^n}(\varepsilon)}{\mathrm{vol}_{\mathbf{H}^n}(\varepsilon)} \to 1$ as $\varepsilon \to 0$.

On the other hand,

$$F(B(p_1,r') \cup B(p_2,r')) \subset B(F(p_1),r') \cup B(F(p_2),r'),$$

Hence

$$\begin{aligned}
\mathrm{vol}_{g_0}\left(F(B(p_1,r') \cup B(p_2,r'))\right) \leq\ & vol_{g_0}(B(F(p_1),r')) + \mathrm{vol}_{g_0}(B(F(p_2),r')) \\
& - \mathrm{vol}_{g_0}(B(F(p_1),r') \cap B(F(p_2),r')).
\end{aligned} \tag{87}$$

For any $x \in X$ and any $s > 0$ lower than the injectivity radius of X one has $\mathrm{vol}_{g_0}(B(x,s)) = \mathrm{vol}_{\mathbf{H}^n}(s)$. Let x be the middle point of the segment $[F(p_1)F(p_2)]$. Then

$$B(x, r'(1-\rho)) \subset B(F(p_1), r') \cap B(F(p_2), r').$$

Indeed, if $x' \in B(x, r'(1-\rho))$ then $d(x', F(p_i)) \leq d(x', x) + d(x, F(p_i)) < r'(1-\rho) + \rho r' = r'$ for $i = 1, 2$. Thus (87) gives

$$\begin{aligned}
\mathrm{vol}_{g_0}\left(F(B(p_1, r') \cup B(p_2, r'))\right) &\leq 2\,\mathrm{vol}_{\mathbf{H}^n}(r') - \mathrm{vol}_{\mathbf{H}^n}(r'(1-\rho)) && (88)\\
&= 2\,\mathrm{vol}_{\mathbf{R}^n}(r')\frac{\mathrm{vol}_{\mathbf{H}^n}(r')}{\mathrm{vol}_{\mathbf{R}^n}(r')} - (1-\rho)^n\,\mathrm{vol}_{\mathbf{R}^n}(r') && (89)\\
&\leq 2\,\mathrm{vol}_{\mathbf{R}^n}(r')\frac{\mathrm{vol}_{\mathbf{H}^n}(\varepsilon)}{\mathrm{vol}_{\mathbf{R}^n}(\varepsilon)} - (1-\rho)^n\,\mathrm{vol}_{\mathbf{R}^n}(r') && (90)\\
&= \left(2\frac{\mathrm{vol}_{\mathbf{H}^n}(\varepsilon)}{\mathrm{vol}_{\mathbf{R}^n}(\varepsilon)} - (1-\rho)^n\right)\mathrm{vol}_{\mathbf{R}^n}(r'). && (91)
\end{aligned}$$

From (86) and (91), we find

$$(1-\rho)^n \leq 2\left(\frac{\mathrm{vol}_{\mathbf{H}^n}(\varepsilon)}{\mathrm{vol}_{\mathbf{R}^n}(\varepsilon)} - \vartheta(\varepsilon)\right) \to 0,$$

therefore

$$\rho \geq 1 - 2^{1/n}\left(\frac{\mathrm{vol}_{\mathbf{H}^n}(\varepsilon)}{\mathrm{vol}_{\mathbf{R}^n}(\varepsilon)} - \vartheta(\varepsilon)\right)^{1/n} := 1 - c(\varepsilon) \to 1,$$

as $\varepsilon \to 0$. One has proved that inside the ball $B(p,r)$,

$$d_{g_0}(F(p_1), F(p_2)) \geq (1 - c(\varepsilon)) d_\infty(p_1, p_2),$$

and the proof of the lemma follows. □

Remark 4.5. *On the connected (see remark 3.9) open set $F(\overset{\circ}{\mathcal{R}}_\varepsilon) \subset X$, the metric g_0 induces a distance ρ_ε. The above lemma shows that $F : (\overset{\circ}{\mathcal{R}}_\varepsilon, d_\infty) \longrightarrow (F(\overset{\circ}{\mathcal{R}}_\varepsilon), \rho_\varepsilon)$ is a $(1 + c(\varepsilon))$-bilipschitz homeomorphism. If one can prove that $\rho_\varepsilon = d_{g_0}$, one deduces that $\mathcal{R}_\varepsilon$ has bounded diameter. One then concludes that $\mathrm{d}_{\mathcal{GH}}(Y_k, Y_\infty) \to 0$ and that $F : Y_\infty \to X$ is isometric.*

More precisely, we prove the

Proposition 4.6. *1. for any $x_1, x_2 \in F(\overset{\circ}{\mathcal{R}}_\varepsilon)$, one has $d_{g_0}(x_1, x_2) = \rho_\varepsilon(x_1, x_2)$.*

2. One has $\overline{F(\overset{\circ}{\mathcal{R}}_\varepsilon)} = X$.

3. $F : (Y_\infty, d_\infty) \longrightarrow (X, d_{g_0})$ is isometric.

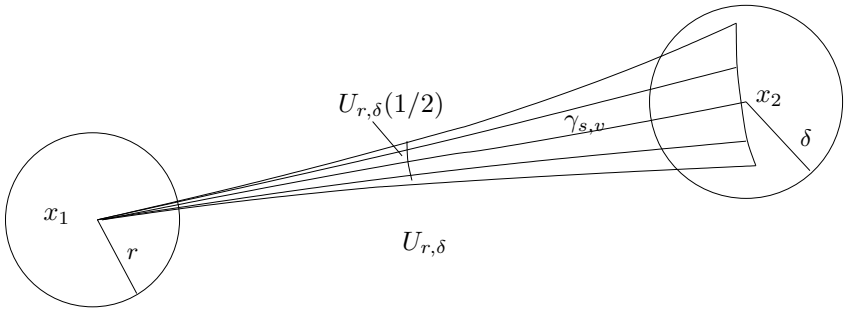
Proof. let x_1,$x_2 \in F(\overset{\circ}{\mathcal{R}}_\varepsilon)$. Without loss of generality, one can suppose that x_2 is not in the image of the cut-locus of x_1. Clearly, $\rho_\varepsilon(x_1, x_2) \geq d_{g_0}(x_1, x_2)$. Let $\gamma : [0,1] \longrightarrow X$ be a g_0-minimal geodesic from x_1 to x_2. It is not clear that γ is in $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ we then prove that there exist paths in $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ arbitrarily close to γ. Let $r > 0$ be a radius such that $B_{g_0}(x_1, r) \subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ and $B_{g_0}(x_2, r) \subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$. We consider geodesics with the origin x_1 and the extremity in $B(x_2, \delta)$, for a small $\delta > 0$. More precisely, let $u = \dot{\gamma}(0)$, then for any $v \in U_{x_1}X$ such that and $u \perp v$, one defines $\gamma_{s,v}(t) = exp_{x_1}(t(u + s.v)d(x_1, x_2))$. Clearly there exists $r(\delta) > 0$ such that $\gamma_{s,v}(1) \in B(x_2, \delta)$ if $|s| \leq r(\delta)$ and one can suppose that $r(\delta) \to 0$ as $\delta \to 0$.

We claim that for every $\delta > 0$, there exists such $\gamma_{s,v}$ which is imbedded in $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$.

We begin to find such $\gamma_{s,v}$ disjoint from $F(\mathcal{S})$, where $\mathcal{S}$ is the singular set of Y_∞ defined in 3.5. The idea is that if any $\gamma_{s,v}$ hits $F(\mathcal{S})$ at least in one point, then the Hausdorff dimension of $F(\mathcal{S})$ will be larger than $n-1$, which is not possible. One considers a truncated cone $U_{r,\delta}$ defined as follows. Let

$$\Gamma :]0, r(\delta)] \times (U_{x_1}X \cap u^\perp) \times [0,1] \to X$$

be defined by $\Gamma(s, v, t) = \gamma_{s,v}(t)$. If δ is sufficiently small, Γ is an embedding. one defines $U_{r,\delta} = \Gamma(]0, r(\delta)]) \times (U_{x_1}X \cap u^\perp) \times [0,1])$. One denotes by $U_{r,\delta}(1/2)$ the hypersurface in $U_{r,\delta}$ defined as $\Gamma(]0, r(\delta)]) \times (U_{x_1}X \cap u^\perp) \times \{1/2\})$.



Let $P : U_{r,\delta} \to U_{r,\delta}(1/2)$ be the projection along geodesics defined by $P(\gamma_{s,v}(t)) = \gamma_{s,v}(1/2)$. Clearly, there exists a constant $C > 0$ such that P is C-lipschitz from $U_{r,\delta}$ to X. In particular, P decreases the Hausdorff dimension, thus

$$\begin{aligned} \dim_{\mathcal{H}}(P(U_{r,\delta} \cap F(\mathcal{S}))) &\leq \dim_{\mathcal{H}}(U_\delta \cap F(\mathcal{S})) \\ &\leq \dim_{\mathcal{H}}(\mathcal{S}) \\ &\leq n-2 \\ &< \dim U_{r,\delta}(1/2) = n-1. \end{aligned}$$

Hence there exists $x \in U_{r,\delta}(1/2)$ such that $x \notin \Pi(F(\mathcal{S}))$. It implies that the geodesic $\gamma_{s,v}$ such that $x = \gamma_{s,v}(1/2)$ does not intersect $F(\mathcal{S})$.

Now we prove that $\gamma_{s,v}$ is imbedded in $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$. Let $t_0 \in [0,1]$ be maximal such that $\gamma_{s,v}([0, t_0[) \subset F(\overset{\circ}{\mathcal{R}}_\varepsilon)$. By lemma 4.4, the path $\beta = F^{-1} \circ \gamma_{s,v}$ is well-defined on $[0, t_0[$ and

has a length bounded by $(1+c(\varepsilon))d(x_1,x_2)$. Thus there exists a limit $p=\lim_{t\to t_0}\beta(t)\in Y_\infty$. As $\gamma_{s,v}(t_0)\notin F(\mathcal{S})$, $p\notin\mathcal{S}$ then $p\in\mathcal{R}=\cap_\varepsilon\mathcal{R}_\varepsilon=\cap_{\varepsilon>0}\overset{\circ}{\mathcal{R}}_\varepsilon$ and $t_0=1$.

Hence

$$\begin{aligned}\rho_\varepsilon(x_1,x_2) &\leq \ell(\gamma_{s,v})+d_0(\gamma_{s,v}(1),x_2)\\ &\leq \sqrt{1+r^2(\delta)}d_0(x_1,x_2)+\delta\end{aligned}$$

As δ was arbitrary, this gives $\rho_\varepsilon(x_1,x_2)\leq d_0(x_1,x_2)$.

The second assertion is proved in a similar way. Suppose there is a ball $B(x,r)\subset X-F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ and consider a geodesic γ from a point x_1 inside $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ to x. Then we find another geodesic from x_1, close to γ, disjoint from $F(\mathcal{S})$ and with extremity in $X-F(\overset{\circ}{\mathcal{R}}_\varepsilon)$. Arguing as above, we find a contradiction.

Now 3) is straightforward. Using the density of $\overset{\circ}{\mathcal{R}}_\varepsilon$ in Y_∞ and $F(\overset{\circ}{\mathcal{R}}_\varepsilon)$ in X, we find that $F:(Y_\infty,d_\infty)\longrightarrow(X,d_0)$ is a $(1+c(\varepsilon))$-bilipschitz homeomorphism for any $0<\varepsilon<\varepsilon_0$ thus is isometric. □

End of Proof of theorem 1.1. Proposition 4.6 implies that the diameter of (Y,g_k) remains bounded. Thus, $\mathrm{d}_{\mathcal{GH}}((Y,g_k),(Y_\infty,d_\infty))\to 0$. As (Y_∞,d_∞) is isometric to $(X;g_0)$, one deduces that $\mathrm{d}_{\mathcal{GH}}((Y,g_k),(X,g_0))\to 0$ as $k\to\infty$. By theorem A.1.12 of [Ch-Co], Y is diffeomorphic to X. The fact that f is homotopic to a diffeomorphism is classic for hyperbolic manifolds. □

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