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ABSTRACT. The basic theme of this paper is to explore connections between places of function fields over a base field F of characteristic zero and birational morphisms between smooth F-varieties. This is done by considering various localised categories involving function fields or varieties as objects, and constructing functors between these categories. The main result is that in the localised category $S_b^{-1}\mathbf{Sm}(F)$, where $\mathbf{Sm}(F)$ denotes the usual category of smooth varieties over F and S_b is the set of birational morphisms, the set of morphisms between two objects X and Y with Y proper is the set of R-equivalence classes $Y(F(X))/R$. We also explore the relation between smooth proper varieties isomorphic to $\operatorname{Spec} F$ in $S_b^{-1}\mathbf{Sm}(F)$ and special varieties like rationally connected varieties and retract rational varieties.

With appendices by Jean-Louis Colliot-Thélène and Ofer Gabber

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1

INTRODUCTION

Birational geometry over a field F is the study of function fields over F, viewed as generic points of algebraic varieties[1], or alternately the study of algebraic F-varieties "up to proper closed subsets". In this context, two ideas seem related:

- places between function fields;
- rational maps.

The main motivation of this paper has been to understand the precise relationship between them. We have done this by defining two rather different "birational categories" and comparing them, at least when F is of characteristic 0.

The first idea gives the category $\mathbf{place}(F)$ (objects: function fields; morphisms: F-places), that we like to call the *coarse birational category*. For the second idea, one has to be a little careful: the naïve attempt at taking as objects smooth varieties and as morphisms rational maps does not work because, as was pointed out to us by Hélène Esnault, one cannot compose rational maps in general. On the other hand, one can certainly start from the category $\mathbf{Sm}(F)$ of smooth F-varieties and *localise* it (in the sense of Gabriel-Zisman [13]) with respect to the set S_b of birational morphisms. We like to call the resulting category $S_b^{-1}\mathbf{Sm}(F)$ the *fine birational category*. By hindsight, the problem mentioned by Esnault can be understood as a problem of calculus of fractions of S_b in $\mathbf{Sm}(F)$.

In spite of the lack of calculus of fractions, the category $S_b^{-1}\mathbf{Sm}(F)$ was studied in [26] and we were able to show that, under resolution of singularities, the natural functor $S_b^{-1}\mathbf{Sm}^{\mathrm{proj}}(F) \to S_b^{-1}\mathbf{Sm}(F)$ is an equivalence of categories, where $\mathbf{Sm}^{\mathrm{proj}}(F)$ denotes the full subcategory of smooth projective varieties (*loc. cit.*, Prop. 8.5).

What was not done in [26] was the computation of Hom sets in $S_b^{-1}\mathbf{Sm}(F)$. This is the first main result of this paper:

Theorem 1 (cf. Th. 5.4.14). *Assume F of characteristic 0. Let X, Y be two smooth F-varieties, with Y proper. Then, in $S_b^{-1}\mathbf{Sm}(F)$, we have an isomorphism*

$$\mathrm{Hom}(X, Y) \simeq Y(F(X))/R$$

where the right hand side is the set of R-equivalence classes in the sense of Manin.

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[1]By convention all varieties are irreducible here, although not necessarily geometrically irreducible.

Since $S_b^{-1} \mathbf{Sm}(F)$ is by definition the universal target category for functors on smooth varieties that invert birational morphisms, Theorem 1 says that R-equivalence is in some sense the universal such functor. It also implies that one can define a composition law on classes of R-equivalence (for smooth proper varieties, say), a fact which is not obvious a priori.

The second main task of this paper is to relate the coarse and fine birational categories, as there is no obvious comparison functor between them. In order to solve this issue, we introduce in Definition 3.1.1 an "incidence category" $\mathbf{SmP}(F)$, whose objects are smooth F-varieties and morphisms from X to Y are given by pairs (f, v), where f is a morphism $X \to Y$, v is a place $F(Y) \rightsquigarrow F(X)$ and f, v are *compatible* in an obvious sense. This category maps to both $\mathbf{place}(F)^{\mathrm{op}}$ and $\mathbf{Sm}(F)$ by obvious forgetful functors. Note that the set S_b lifts naturally to $\mathbf{SmP}(F)$, so that $S_b^{-1} \mathbf{SmP}(F)$ maps both to $S_b^{-1} \mathbf{Sm}(F)$ and to $\mathbf{place}(F)^{\mathrm{op}}$.

Replacing $\mathbf{Sm}(F)$ by $\mathbf{SmP}(F)$ turns out to have a strong rigidifying effect. In order to describe our results, let us denote by S_b^p the subset of S_b consisting of proper birational maps and by S_o the subset of S_b consisting of open immersions. Note that, to localise with respect to S_b, we may first localise with respect to S_b^p and then localise with respect to S_o. We have:

Theorem 2. *Assume F of characteristic* 0*. Then:*

(1) *(cf. Prop. 4.2.1) The category* $\mathbf{SmP}(F)$ *admits a calculus of right fractions with respect to* S_b^p*.*
(2) *(cf. Prop. 4.2.3) The category* $(S_b^p)^{-1} \mathbf{SmP}(F)$ *admits a calculus of left fractions with respect to* S_o*.*
(3) *(cf. Theorem 4.2.5) The functor* $S_b^{-1} \mathbf{SmP}(F) \to \mathbf{place}(F)^{\mathrm{op}}$ *is an equivalence of categories.*

To go further, we introduce the set S_r of *stable birational morphisms*: by definition, a morphism $s : X \to Y$ is in S_r if it is dominant and the function field extension $F(X)/F(Y)$ is purely transcendental. Analogously, a morphism of function fields in $\mathbf{place}(F)$ is in S_r if it is an inclusion and if the corresponding extension is purely transcendental. We then get an obvious diagram of categories

(0.1)
$$S_r^{-1} \mathbf{SmP}(F) \longrightarrow S_r^{-1} \mathbf{Sm}(F)$$
$$\downarrow \wr$$
$$S_r^{-1} \mathbf{place}(F)^{\mathrm{op}}.$$

We wondered about the nature of the localisation functor $S_b^{-1} \mathbf{Sm}(F)$ $\to S_r^{-1} \mathbf{Sm}(F)$ for a long time, until the answer was given us by Colliot-Thélène through a wonderfully simple geometric argument (see Appendix A). This is the first part of

Theorem 3. *Assume F of characteristic 0. Then:*
  (1) *(cf. Theorem 1.7.7) The functor $S_b^{-1} \mathbf{Sm}(F) \to S_r^{-1} \mathbf{Sm}(F)$ is an equivalence of categories.*
  (2) Hom *sets in $S_r^{-1} \mathbf{place}(F)$ may be described as quotients of the* Hom *sets in $\mathbf{place}(F)$ by the relation of "homotopy of places" (see Def. 5.1.1 and Prop. 5.1.3).*
  (3) *In (0.1), the horizontal functor is full (surjective on morphisms) and essentially surjective (surjective on isomorphism classes of objects); the "fibres" of this functor on* Hom *sets may be described to some extent (see Lemma 6.2.1 and Theorem 6.2.3).*

Part 2 of Theorem 3 may be seen as an easier analogue of Theorem 1: in fact, we first found Theorem 3 (2) and then guessed Theorem 1 by analogy.

In particular, we obtain a functor $\mathbf{place}(F)^{\mathrm{op}} \to S_b^{-1} \mathbf{Sm}(F)$ which is full and essentially surjective, justifying the terminology of "coarse" and "fine" birational categories.

In order to simplify the exposition, we have restricted ourselves in this introduction to smooth varieties over a field of characteristic 0. This is because our main theorems rely on Hironaka's resolution of singularities. In reality, just as in [26], we work in a slightly greater generality, considering a subcategory $\mathcal{C}$ of the category $\mathbf{Var}(F)$ of all F-varieties, where $\mathcal{C}$ is supposed to have good permanence properties with respect to the set S_b or S_b^p. This allows us to obtain at least some results which are independent of resolution of singularities. One should be aware, however, that the natural functor $S_b^{-1} \mathbf{Sm}(F) \to S_b^{-1} \mathbf{Var}(F)$ is *not* an equivalence of categories, even in characteristic 0: this was already observed in [26, Rk. 8.11].

Let us now describe the contents of this paper in more detail. Most of our arguments are categorical in nature, hence the geometrical properties used here are mostly basic and formal (basic does not mean elementary, as resolution of singularities plays an essential rôle!). We start by setting up our notation in Section 1, which ends with a nontrivial result (Theorem 1.7.7). In Section 2, we put on the Zariski-Samuel "abstract Riemann surface of a field" a structure of locally ringed space, and prove that it is a cofiltered inverse limit of proper models viewed as schemes (we could not find this done in the literature). For a precise statement, see Theorem 2.1.8. This provides a first relationship

between places and morphisms of varieties (Proposition 2.2.3). In Section 3, we introduce the incidence category $\mathbf{SmP}(F)$ sitting in the larger category $\mathbf{VarP}(F)$, the forgetful functors $\mathbf{VarP}(F) \to \mathbf{Var}(F)$ and $\mathbf{VarP}(F) \to \mathbf{place}(F)^{\mathrm{op}}$, and prove elementary results on these functors (see Lemmas 3.2.2 and 3.2.4). In Section 4 we prove the results of Theorem 2, in a greater generality: the main result is Theorem 4.2.5.

In Section 5, we first define the relation of homotopy of places (Def. 5.1.2) and prove easily part 2 of Theorem 3. We then set up in Subsection 5.4 a categorical machine which allows us eventually to prove Theorem 1. Before Colliot-Thélène's result leading to Theorem 3 (1), we were proving Theorem 1 with S_b replaced by S_r. In Section 6, we give a somewhat more concrete description of the composition of R-equivalence classes stemming from Theorem 1 (in a slightly more general form, see Proposition 6.1.2) and prove part 3 of Theorem 3.

Section 7 is probably the most geometrical and concrete of the whole paper: we give there some applications of our results. One of them is that, if X is a smooth proper variety over a field of characteristic 0, then the set $X(F(X))/R$ has a natural structure of a monoid (for the composition of R-equivalence classes given by Theorem 1). This seems especially striking when X is a smooth compactification of a linear algebraic group G: in this case, there are *two* composition laws on this set, the second one being induced by the product of G (unfortunately, the former is only distributive on the left with respect to the latter, see comment after Lemma 7.2.2). We also relate, albeit weakly, our theory with Kollár's notion of rationally connected varieties in Theorem 7.3.1, and with Saltman's notion of retract-rational varieties in Proposition 7.4.2. In Subsection 7.6, we define the "function field of BG" for G a linear algebraic group, as an easy application of the "no name lemma", and put forth some questions about it. Finally we list a few open questions/problems in Subsection 7.7.

We have tried our best to avoid using resolution of singularities to prove at least some significant results, but unfortunately we have not been very successful (see however Corollary 4.4.3). Nevertheless, the reader should definitely have a look at Subsection 5.3, where a collection of nontrivial theorems avoiding resolution of singularities is presented, the latest being recent results of Gabber. We hope that these theorems can be used to improve the scope of our main results (see already Corollary 5.3.5 and Remark 6.1.3).

This paper grew out of the preprint [25], where some of its results were initially proven. Indeed [25] contained a mix of results on birational categories, pure birational motives and triangulated birational

motives; after a long period of gestation, we decided that the best was to separate the present results, which have little to do with motives, from the rest of the work. Meanwhile, a better understanding of the localisation techniques that had been used in [25] led to [26] on abstract localisation theorems.

Meanwhile again, results of [25] have started to be used, notably in [21, 24, 23, 22]. We now plan to provide a final version of [25] soon. However, the localisation of the Morel-Voevodsky $\mathbf{A}^1$-homotopy category of schemes $\mathcal{H}(F)$ [34] with respect to S_r should also be studied, as well as that of Morel-Voevodsky's effective stable $\mathbf{A}^1$-homotopy category of schemes.

For the latter, this is very likely any "chunk" of the slice filtration of [44]; this analogy was pointed out to us early on by Chunck Weibel. Actually, recent results of Voevodsky, Levine and Østvær-Röndigs probably imply that the functor

$$S_r^{-1}\mathcal{SH}^{S^1}(F) \to S_r^{-1}DM^{\mathrm{eff}}(F)$$

induced by the "Suslin complex" functor $C_* : \mathcal{SH}^{S^1}(F) \to DM^{\mathrm{eff}}(F)$ from the Morel-Voevodsky homotopy category of S^1-spectra to Voevodsky's triangulated category of effective motivic complexes is an equivalence of categories if F is of characteristic 0 (the above-mentioned results imply at the very least that this functor is essentially surjective, see [37, Th. 3.5.15]).

On the other hand, the study of the natural functor $S_r^{-1}\mathbf{Sm}(F) \to S_r^{-1}\mathcal{H}(F)$ promises to be intriguing, if one compares Theorem 1 with the following conjecture of F. Morel:

Conjecture 1 ([33, p. 386]). *If X is a smooth variety, the natural map*

$$X(F) \to \mathrm{Hom}_{\mathcal{H}(F)}(\mathrm{Spec}\,F, X)$$

is surjective and identifies the right hand side with the quotient of the set $X(F)$ by the equivalence relation generated by

$$(x \sim y) \iff \exists h : \mathbf{A}^1 \to X \mid h(0) = x \text{ and } h(1) = y.$$

Note that this "$\mathbf{A}^1$-equivalence" coincides with R-equivalence if X is proper.

1. Preliminaries and notation

In this section, we collect some basic material that will be used in the paper. This allows us to fix our notation. We start with:

Notation. F is the base field. All varieties are (irreducible) F-varieties and all morphisms are F-morphisms. If X is a variety, η_X denotes its generic point.

1.1. **Localisation of categories and calculus of fractions.** We refer to Gabriel-Zisman [13, Chapter I] for the necessary background. Recall [13, I.1] that if $\mathcal{C}$ is a small category and S is a collection of morphisms in $\mathcal{C}$, there is a category $\mathcal{C}[S^{-1}]$ and a functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ which is universal among functors from $\mathcal{C}$ which invert the elements of S. When S satisfies *calculus of fractions* [13, I.2] the category $\mathcal{C}[S^{-1}]$ is equivalent to another one, denoted $S^{-1}\mathcal{C}$ by Gabriel and Zisman, in which the Hom sets are more explicit.

This definition raises set-theoretic problems: we have chosen to ignore them in this paper, and hence to be rather informal with sets, classes or collections of morphisms.

In this paper, we shall encounter situations where calculus of fractions is satisfied, as well as others where it is not. We shall take the practice to abuse notation and write $S^{-1}\mathcal{C}$ rather than $\mathcal{C}[S^{-1}]$ even when calculus of fractions is not verified.

We shall occasionally use the following notion:

1.1.1. **Definition.** Let $\mathcal{C}$ be a category and S a family of morphisms of $\mathcal{C}$. An object $X \in \mathcal{C}$ is *local* relatively to S or S-*local* (*left closed* in the terminology of [13, Ch. 1, Def. 4.1 p. 19]) if, for any $s : Y \to Z$ in S, the map

$$\mathcal{C}(Z, X) \xrightarrow{s^*} \mathcal{C}(Y, X)$$

is bijective.

1.2. Equivalence relations.

1.2.1. **Definition.** Let $\mathcal{C}$ be a category. An *equivalence relation* on $\mathcal{C}$ consists, for all $X, Y \in \mathcal{C}$, of an equivalence relation $\sim_{X,Y}=\sim$ on $\mathcal{C}(X, Y)$ such that $f \sim g \Rightarrow fh \sim gh$ and $kf \sim kg$ whenever it makes sense.

In [30, p. 52], the above notion is called a 'congruence'. Given an equivalence relation $\sim$ on $\mathcal{C}$, we may form the *factor category* $\mathcal{C}/\sim$, with the same objects as $\mathcal{C}$ and such that $\mathcal{C}/\sim (X, Y) = \mathcal{C}(X, Y)/\sim$. This category and the projection functor $\mathcal{C} \to \mathcal{C}/\sim$ are universal for functors from $\mathcal{C}$ which equalise equivalent morphisms.

1.2.2. **Example.** Let $\mathcal{A}$ be an Ab-category (sets of morphisms are abelian groups and composition is bilinear). An *ideal* $\mathcal{I}$ in $\mathcal{A}$ is given by a subgroup $\mathcal{I}(X, Y) \subseteq \mathcal{A}(X, Y)$ for all $X, Y \in \mathcal{A}$ such that $\mathcal{I}\mathcal{A} \subseteq \mathcal{I}$ and $\mathcal{A}\mathcal{I} \subseteq \mathcal{I}$. Then the ideal $\mathcal{I}$ defines an equivalence relation on $\mathcal{A}$, compatible with the additive structure.

Let $\sim$ be an equivalence relation on the category $\mathcal{C}$. We have the collection $S_{\sim} = \{f \in \mathcal{C} \mid f \text{ is invertible in } \mathcal{C}/\sim\}$. The functor $\mathcal{C} \to \mathcal{C}/\sim$ factors as a functor $S_{\sim}^{-1}\mathcal{C} \to \mathcal{C}/\sim$. Conversely, let $S \subset \mathcal{C}$ be a set of morphisms. We have the equivalence relation $\sim_S$ on $\mathcal{C}$ such that $f \sim_S g$ if $f = g$ in $S^{-1}\mathcal{C}$, and the localisation functor $\mathcal{C} \to S^{-1}\mathcal{C}$ factors as $\mathcal{C}/\sim_S \to S^{-1}\mathcal{C}$. Neither of these two factorisations is an equivalence of categories in general; however, [17, Prop. 1.3.3] remarks that if $f \sim g$ implies $f = g$ in $S_{\sim}^{-1}\mathcal{C}$, then $S_{\sim}^{-1}\mathcal{C} \to \mathcal{C}/\sim$ is an isomorphism of categories.

1.2.3. **Exercise.** Let A be a commutative ring and $I \subseteq A$ an ideal.
a) Assume that the set of minimal primes of A that do not contain I is finite (*e.g.* that A is noetherian). Show that the following two conditions are equivalent:

(i)   There exists a multiplicative subset S of A such that $A/I \simeq S^{-1}A$ (compatibly with the maps $A \to A/I$ and $A \to S^{-1}A$).
(ii) I is generated by an idempotent.

(Hint: show first that, without any hypothesis, (i) is equivalent to

(iii) For any $a \in I$, there exists $b \in I$ such that $ab = a$.)

b) Give a counterexample to (i) $\Rightarrow$ (ii) in the general case (hint: take $A = k^{\mathbf{N}}$, where k is a field).

1.3. Places, valuations and centres [45, Ch. VI], [3, Ch. 6].

Recall [3, Ch. 6, §2, Def. 3] that a *place* from a field K to a field L is a map $\lambda : K \cup \{\infty\} \to L \cup \{\infty\}$ such that $\lambda(1) = 1$ and λ preserves sum and

product whenever they are defined. We shall usually denote places by screwdriver arrows:

$$\lambda : K \rightsquigarrow L.$$

Then $\mathcal{O}_\lambda = \lambda^{-1}(L)$ is a valuation ring of K and $\lambda_{|\mathcal{O}_\lambda}$ factors as

$$\mathcal{O}_\lambda \twoheadrightarrow \kappa(\lambda) \hookrightarrow L$$

where $\kappa(\lambda)$ is the residue field of $\mathcal{O}_\lambda$. Conversely, the data of a valuation ring $\mathcal{O}$ of K with residue field κ and of a field homomorphism $\kappa \to L$ uniquely defines a place from K to L (*loc. cit.* , Prop. 2). It is easily checked that the composition of two places is a place.

If K and L are extensions of a field F, we say that λ is an *F-place* if $\lambda_{|F} = Id$ and then write $F(\lambda)$ rather than $\kappa(\lambda)$.

In this situation, let X be an integral F-scheme of finite type with function field K. A point $x \in X$ is a *centre* of a valuation ring $\mathcal{O} \subset K$ if $\mathcal{O}$ dominates the local ring $\mathcal{O}_{X,x}$. If $\mathcal{O}$ has a centre on X, we sometimes say that $\mathcal{O}$ is *finite* on X. As a special case of the valuative criterion of separatedness (*resp.* of the valuative criterion of properness), x is unique (*resp.* exists) for all $\mathcal{O}$ if and only if X is separated (*resp.* proper) [19, Ch. 2, Th. 4.3 and 4.7].

On the other hand, if $\lambda : K \rightsquigarrow L$ is an F-place, then a point $x \in X(L)$ is a *centre* of λ if there is a map $\varphi : \operatorname{Spec} \mathcal{O}_\lambda \to X$ letting the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} \mathcal{O}_\lambda & \longleftarrow & \operatorname{Spec} K \\
{\scriptstyle\lambda^*}\uparrow & \nwarrow{\scriptstyle\varphi} & \downarrow \\
\operatorname{Spec} L & \xrightarrow{\ x\ } & X
\end{array}
$$

commute. Note that the image of the closed point by φ is then a centre of the valuation ring $\mathcal{O}_\lambda$ and that φ uniquely determines x.

In this paper, when X is separated we shall denote by $c(\mathcal{O}) \in X$ the centre of a valuation ring $\mathcal{O}$ and by $c(\lambda) \in X(L)$ the centre of a place λ, and carefully distinguish the two notions (one being a scheme-theoretic point and the other a rational point).

1.4. **Rational maps.** Let X, Y be two F-schemes of finite type. Recall that a *rational map* from X to Y is a pair (U, f) where U is a dense open subset of X and $f : U \to Y$ is a morphism. Two rational maps (U, f) and (U', f') are equivalent if there exists a dense open subset U'' contained in U and U' such that $f_{|U''} = f'_{|U''}$. We denote by $\mathbf{Rat}(X, Y)$ the set of equivalence classes of rational maps, so that

$$\mathbf{Rat}(X, Y) = \varinjlim \operatorname{Map}_F(U, Y)$$

where the limit is taken over the open dense subsets of X.

Suppose X integral for simplicity. Then there is a largest open subset U of X on which a given rational map $f : X \dashrightarrow Y$ is defined [19, Ch. I, Ex. 4.2]. The (reduced) closed complement $X - U$ is called the *fundamental set* of f (notation: $\mathrm{Fund}(f)$). We say that f is *dominant* if $f(U)$ is dense in Y.

Similarly, let $f : X \to Y$ be a birational morphism. The complement of the largest open subset of X on which f is an isomorphism is called the *exceptional locus* of f and is denoted by $\mathrm{Exc}(f)$.

Note that the sets $\mathbf{Rat}(X, Y)$ only define a *precategory* (or diagram, or diagram scheme, or quiver) $\mathbf{Rat}(F)$, because rational maps cannot be composed in general. To clarify this, let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be two rational maps, where X, Y, Z are varieties. We say that f and g are *composable* if $f(\eta_X) \notin \mathrm{Fund}(g)$, where η_X is the generic point of X. Then there exists an open subset $U \subseteq X$ such that f is defined on U and $f(U) \cap \mathrm{Fund}(g) = \emptyset$, and $g \circ f$ makes sense as a rational map. This happens in two important cases:

- f is dominant;
- g is a morphism.

This composition law is associative wherever it makes sense. In particular, we do have the category $\mathbf{Rat}_{\mathrm{dom}}(F)$ with objects F-varieties and morphisms dominant rational maps. Similarly, the category $\mathbf{Var}(F)$ of 1.7 acts on $\mathbf{Rat}(F)$ on the left.

1.5. **The graph trick.** We shall often use this well-known and basic device, which allows us to replace a rational map by a morphism.

Let U, Y be two F-varieties. Let $j : U \to X$ be an open immersion (X a variety) and $g : U \to Y$ a morphism. Consider the graph $\Gamma_g \subset U \times Y$. By the first projection, $\Gamma_g \xrightarrow{\sim} U$. Let $\bar{\Gamma}_g$ be the closure of Γ_g in $X \times Y$, viewed as a reduced scheme. Then the rational map $g : X \dashrightarrow Y$ has been replaced by $g' : \bar{\Gamma}_g \to Y$ (second projection) through the birational map $p : \bar{\Gamma}_g \to X$ (first projection). Clearly, if Y is proper then p is proper.

1.6. **Structure theorems on varieties.** Here we collect some well-known results, for future reference.

1.6.1. **Theorem** (Nagata [36]). *Any variety X can be embedded into a proper variety $\bar{X}$. We shall sometimes call $\bar{X}$ a* compactification *of X.*

1.6.2. **Theorem** (Relative Chow lemma, [EGA 2, Th. 5.6.1]). *For any morphism $f : Y \to X$, there exists a projective birational morphism $p : \widetilde{Y} \to Y$ such that $f \circ p$ is quasi-projective.*

1.6.3. Theorem (Hironaka [20]). *If* char $F = 0$,
a) For any variety X there exists a projective birational morphism $f : \widetilde{X} \to X$ with $\widetilde{X}$ smooth. (Such a morphism is sometimes called a modification.) Moreover, f may be chosen such that it is an isomorphism away from the inverse image of the singular locus of X. In particular, any smooth variety X may be embedded as an open subset of a smooth proper variety (projective if X is quasi-projective).
b) For any proper birational morphism $p : Y \to X$ between smooth varieties, there exists a proper birational morphism $\widetilde{p} : \widetilde{Y} \to X$ which factors through p and is a composition of blow-ups with smooth centres.

In several places we shall assume characteristic 0 in order to use resolution of singularities. We shall specify this by putting an asterisk to the statement of the corresponding result (so, the asterisk will mean that the characteristic 0 assumption is due to the use of Theorem 1.6.3).

1.7. A zoo of categories and multiplicative systems. In this paper, we shall mainly work within the category of F-*varieties* $\mathbf{Var}(F) = \mathbf{Var}$: objects are F-varieties (*i.e.* integral separated F-schemes of finite type) and morphisms are all F-morphisms. We shall also consider subcategories of $\mathbf{Var}$ as in [26, §8], *e.g.* smooth, quasi-projective, etc. We retain the notation used in *loc. cit.* Moreover, for any subcategory $\mathcal{C}$ of $\mathbf{Var}$, we shall write $\mathcal{C}_{\mathrm{dom}}$ for the subcategory of $\mathcal{C}$ with the same objects, and morphisms the dominant morphisms of $\mathcal{C}$.

As in [26], we shall use various collections of morphisms of $\mathbf{Var}(F)$ that are to be inverted. The main ones are

- *Birational morphisms S_b:* $s \in S_b$ if s is dominant and induces an isomorphism of function fields
- *Strict birational morphisms:*

  $\underline{S}_b = \{s \in S_b \mid s \text{ induces an equality of function fields}\}$

- *Stably birational morphisms S_r:* $s \in S_r$ if s is dominant and induces a purely transcendental extension of function fields

In fact, the difference between S_b and $\underline{S}_b$ is immaterial in view of the following

1.7.1. Lemma. *Any birational morphism of separated varieties is the composition of a strict birational morphism and an isomorphism.*

Proof. Let $s : X \to Y$ be a birational morphism. First assume X and Y affine, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Let $K = F(X)$ and $L = F(Y)$, so that K is the quotient field of A and L is the quotient field of B. Let $s^* : K \xrightarrow{\sim} L$ be the isomorphism induced by s. Then

$A \xrightarrow{\sim} A' = s^*(A)$, hence s may be factored as $X \xrightarrow{s'} X' \xrightarrow{u} Y$ with $X' = \operatorname{Spec} A'$, where s' is strict birational and u is an isomorphism. In the general case, we may patch the above construction (which is canonical) over an affine open cover (U_i) of Y and an affine open cover of X refining $(s^{-1}(U_i))$. $\qquad\square$

In addition, we shall occasionally encounter the following subsets of S_b:

- S_o: open immersions
- S_b^p: proper birational morphisms, $\underline{S}_b^p = S_b^p \cap \underline{S}_b$
- S_b^w: the multiplicative subset of S_b^p generated by blow-ups with smooth centres (the exponent w is meant to recall "weak factorisation")

and of S_r:

- S_r^p: proper stably birational morphisms
- S_h: the multiplicative subset of S_r^p generated by morphisms of the form $pr_2 : X \times \mathbf{P}^1 \to X$.
- S_r^w: the multiplicative subset of S_r^p generated by S_b^w and S_h.

Recall the following well-known lemma:

1.7.2. Lemma ([26, Lemma 8.2]). *Let $f, g : X \to Y$ be two morphisms, with X integral and Y separated. Then $f = g$ if and only if $f(\eta_X) = g(\eta_X) =: y$ and f, g induce the same map $F(y) \to F(X)$ on the residue fields.* $\qquad\square$

1.7.3. Remark. In view of this lemma, let us reinterpret Theorems 1.6.2 and 1.6.3 in categorical language: this is basically the only way in which we shall use these theorems. Consider the subcategory $\mathbf{Var}_b$ of $\mathbf{Var}$ with the same objects, but with morphisms restricted to $\underline{S}_b^p$. The graph trick (§1.5) shows that for an object X in $\mathbf{Var}_b$, the category $\mathbf{Var}_b/X$ (*cf.* [26, 1.1]) is cofiltering (the opposite of filtering, in the sense of [SGA4-I, Exp. 1, Def. 2.7]). For a full subcategory $\mathcal{C} \subseteq \mathbf{Var}$, let $\mathcal{C}_b = \mathcal{C} \cap \mathbf{Var}_b$. Then Chow's lemma (*resp.* Hironaka's theorem) implies that $\mathcal{C}_b$ is final (the opposite of cofinal, in the sense of [1, p. 149] or [SGA4-I, Exp. 1, Prop. 8.1.3]) in $\mathbf{Var}_b$ for $\mathcal{C} = \mathbf{Var}^{\mathrm{qp}}$ (*resp.* $\mathcal{C} = \mathbf{Sm}$ in characteristic 0). Here we use the fact that, thanks to Lemma 1.7.2, $\mathbf{Var}_b$ is an ordered category, *i.e.* there is at most one morphism between two given objects. Also note that the notion of 'final' in [30] is the opposite of that in [SGA4-I]. This was pointed out to us by Maltsiniotis; in [26], we use the convention of [30] whilst here we use that of [SGA4-I]!

In practice, we shall not use the categories $\mathcal{C}_b$ but will prefer the following working definition in the sequel:

1.7.4. Definition. Let $\mathcal{C}$ be a full subcategory of **Var**, and $S \subset$ **Var** be a multiplicative class of dominant morphisms.
a) We say that $\mathcal{C}$ is *S-final in* **Var** if, given $X \in$ **Var**, there exists $Y \xrightarrow{s} X$ with $Y \in \mathcal{C}$ and $s \in S$.
b) We say that $\mathcal{C}$ is *properly final* in **Var** if it is S_b^p-final in **Var** and if, moreover, any $X \in \mathcal{C}$ is an open subset of an $\bar{X} \in \mathcal{C}$ which is *proper*.

1.7.5. Examples. a) S-final: For $S = S_b^p$, we may take $\mathcal{C} =$ **Var**, **Var**$^{\mathrm{qp}}$ and, *in characteristic 0, $\mathcal{C} =$ **Sm**, **Sm**$^{\mathrm{qp}}$. Recall that the superscript qp denotes quasi-projective varieties. For $S = S_b$ or S_o, we may take the same examples, with F only perfect in the case of **Sm** and **Sm**$^{\mathrm{qp}}$.
b) Properly final: the same examples as in a) for $S = S_b^p$.
We leave it to the reader to consider other interesting examples (normal varieties...).

We have the following elementary lemma.

1.7.6. Lemma. *Let $\mathcal{C}$ be a strictly full subcategory of* **Var**(F).
a) If $\mathcal{C}$ is S_o-final, then the sets S_b, $\underline{S}_b$ and S_o have the same saturation in $\mathcal{C}_{\mathrm{dom}}(F) := \mathcal{C} \cap$ **Var**$_{\mathrm{dom}}(F)$ *(hence they give isomorphic localisations).*
b) If $\mathcal{C}$ is S_b^p-final (resp. S_b^p-final and S_o-final) and stable under product with $\mathbf{P}^1$, the sets S_r^p and $S_b^p \cup S_h$ (resp. S_r and $S_b \cup S_h$) have the same saturation in $\mathcal{C}$.
c) *(Weak weak factorisation.) If $\mathrm{char}\, F = 0$, the sets S_b^p and S_b^w have the same saturation in **Sm**(F).

Proof. a) It suffices to observe that, by definition, a birational morphism $s : X \to Y$ for $X, Y \in \mathcal{C}$ sits in a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ j\ } & X \\
\wr \downarrow & & \downarrow s \\
V & \xrightarrow{\ j'\ } & Y
\end{array}
$$

where $j, j' \in S_o \subset \underline{S}_b$. By hypothesis, we may choose $U, V \in \mathcal{C}$.

b) For a morphism $s : Y \to X$ in S_r^p with $X, Y \in \mathcal{C}$, it suffices to consider a commutative diagram

(1.1)

$$
\begin{array}{ccc}
 & \widetilde{Y} & \\
 {}^{t}\swarrow & & \searrow^{u} \\
Y & & X \times (\mathbf{P}^1)^n \\
 {}_{s}\searrow & & \swarrow_{\pi} \\
 & X &
\end{array}
$$

obtained by the graph trick, with $t, u \in S_b^p$; by S_b^p-finality, we may dominate $\widetilde{Y}$ by an object of $\mathcal{C}$.

For a morphism $s : Y \to X$ in S_r, use the graph trick to produce a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{j'} & \bar{Y} \\
{}_{s}\downarrow & & \downarrow^{\bar{s}} \\
X & \xrightarrow{j} & \bar{X}
\end{array}
$$

with j, j' two open immersions and $\bar{X}, \bar{Y}$ proper (Theorem 1.6.1). Note that, then, $\bar{s} \in S_r^p$. Pulling back $\bar{s}$ via j, we find $\bar{s}' : Y' \to X$ with $\bar{s}' \in S_r^p$ and Y open in Y'. By S_b^p-finality, we may find $t : Y'' \to Y'$ with $t \in S_b^p$ and $Y'' \in \mathcal{C}$. Then we have a rational map $Y \dashrightarrow Y''$. By S_o-finality, this rational map is defined on some common open subset $U \in \mathcal{C}$.

c) (*cf.* [16, p. 47, (iii)]). Let $p : Y \to X$ be in S_b^p. By Theorem 1.6.3 b), there exists $q : \widetilde{Y} \to Y$ in S_b^p such that $pq \in S_b^w$. Applying the theorem a second time, there exists $r : Z \to \widetilde{Y}$ in S_b^p such that $qr \in S_b^w$. So q is invertible in $(S_b^w)^{-1} \mathbf{Sm}(F)$ and therefore p is invertible in $(S_b^w)^{-1} \mathbf{Sm}(F)$. $\qquad\square$

Here is now the main result of this section.

1.7.7. Theorem. *In* **Var***, the sets S_b^p and S_r^p (resp. S_b and S_r) have the same saturation. This is also true in a strictly full subcategory $\mathcal{C} \subseteq$* **Var** *provided it is S_b^p-final in* **Var** *(resp. S_b^p and S_o-final) (cf. the examples 1.7.5), and stable under product by the varieties appearing in the proof of the lemma in Appendix A. *In particular, it is true for $\mathcal{C} =$* **Sm** *in characteristic* 0.

Proof. By Lemma 1.7.6 b), it suffices to prove that S_h is contained in the saturation of S_b^p. Let $f : Y \times \mathbf{P}^1 \to Y$ be the first projection for a $Y \in$ **Var**. We have to show that f becomes invertible in $(S_b^p)^{-1}$ **Var**.

By Yoneda's lemma, it suffices to show that $F(f)$ is invertible for any (representable) functor $F : (S_b^p)^{-1} \mathbf{Var}^{\mathrm{op}} \to \mathbf{Sets}$. This follows by taking the proof of Appendix A and "multiplying" it by Y. $\square$

2. Places, valuations and the Riemann varieties

In this section, we give a first categorical relationship between the idea of places and that of algebraic varieties. This leads us to consider Zariski's "abstract Riemann surface of a field" as a locally ringed space. As we could not find this elaborated in the literature (except for a terse allusion in [20, 0.6, p. 146]: we thank Bernard Teissier for pointing out this reference), we start by giving the details.

2.1. **The Riemann-Zariski variety as a locally ringed space.**

2.1.1. **Definition.** We denote by $\mathcal{R}(F) = \mathcal{R}$ the full subcategory of the category of locally ringed spaces such that $(X, \mathcal{O}_X) \in \mathcal{R}$ if and only if $\mathcal{O}_X$ is a sheaf of local F-algebras.

(Here, we understand by "local ring" a commutative ring whose non-invertible elements form an ideal but we don't require it to be Noetherian.)

2.1.2. **Lemma.** *Cofiltering inverse limits exist in $\mathcal{R}$. More precisely, if $(X_i, \mathcal{O}_{X_i})_{i \in I}$ is a cofiltering inverse system of objects of $\mathcal{R}$, its inverse limit is represented by $(X, \mathcal{O}_X)$ with $X = \varprojlim X_i$ and $\mathcal{O}_X = \varinjlim p_i^* \mathcal{O}_{X_i}$, where $p_i : X \to X_i$ is the natural projection.*

Sketch. Since a filtering direct limit of local rings for local homomorphisms is local, the object of the lemma belongs to $\mathcal{R}$ and we are left to show that it satisfies the universal property of inverse limits in $\mathcal{R}$. This is clear on the space level, while on the sheaf level it follows from the fact that inverse images of sheaves commute with direct limits. $\square$

Recall from Zariski-Samuel [45, Ch. VI, §17] the *abstract Riemann surface S_K* of a function field K/F: as a set, it consists of all nontrivial valuations on K which are trivial on F. It is topologised by the following basis $\mathcal{E}$ of open sets: if R is a subring of K, finitely generated over F, $E(R) \in \mathcal{E}$ consists of all valuations v such that $\mathcal{O}_v \supseteq R$.

As has become common practice, we shall slightly modify this definition:

2.1.3. **Definition.** The *Riemann variety* Σ_K of K is the following ringed space:

- As a topological space, $\Sigma_K = S_K \cup \{\eta_K\}$ where η_K is the trivial valuation of K. (The topology is defined as for S_K.)
- The sections over $E(R)$ of the structural sheaf of Σ_K are the intersection $\bigcap_{v \in E(R)} \mathcal{O}_v$, *i.e.* the integral closure of R.

2.1.4. Lemma. *The stalk at $v \in \Sigma_K$ of the structure sheaf is $\mathcal{O}_v$. In particular, $\Sigma_K \in \mathcal{R}$.*

Proof. Let $x \in \mathcal{O}_v$. The ring $R = F[x]$ is finitely generated and contained in $\mathcal{O}_v$, thus $\mathcal{O}_v$ is the filtering direct limit of the R such that $v \in E(R)$. $\square$

Let R be a finitely generated F-subalgebra of K. We have a canonical morphism of locally ringed spaces $c_R : E(R) \to \operatorname{Spec} R$ defined as follows: on points we map $v \in E(R)$ to its centre $c_R(v)$ on $\operatorname{Spec} R$. On the sheaf level, the map is defined by the inclusions $\mathcal{O}_{X,c_X(v)} \subset \mathcal{O}_v$.

We now reformulate [45, p. 115 ff] in scheme-theoretic language. Let $X \in \mathbf{Var}$ (separated) be provided with a dominant morphism $\operatorname{Spec} K \to X$ such that the corresponding field homomorphism $F(X) \to K$ is an inclusion (as opposed to a monomorphism). We call such an X a *Zariski-Samuel model* of K; X is a *model* of K if, moreover, $F(X) = K$. Note that Zariski-Samuel models of K form a cofiltering ordered set. Generalising $E(R)$, we may define $E(X) = \{v \in \Sigma_K \mid v \text{ is finite on } X\}$ for a Zariski-Samuel model of K; this is still an open subset of Σ_K, as the union of the $E(U_i)$, where (U_i) is some finite affine open cover of X. We still have a morphism of locally ringed spaces $c_X : E(X) \to X$ defined by glueing the affine ones. If X is proper, $E(X) = \Sigma_X$ by the valuative criterion of properness. Then:

2.1.5. Theorem (Zariski-Samuel)**.** *a) The induced morphism of ringed spaces*

$$\Sigma_K \to \varprojlim X$$

where X runs through the proper Zariski-Samuel models of K, is an isomorphism in $\mathcal{R}$. The generic point η_K is dense in Σ_K.
b) This statement remains true if we replace proper Zariski models by proper models or by models in $\mathcal{C}$ where $\mathcal{C} \subseteq \mathbf{Var}^{\mathrm{prop}}$ is $\underline{S}_b$-final.

Proof. a) Zariski and Samuel's theorem [45, th. VI.41 p. 122] says that the underlying morphism of topological spaces is a homeomorphism; thus, by Lemma 2.1.2, we only need to check that the structure sheaf of Σ_K is the direct limit of the pull-backs of those of the X. This amounts to showing that, for $v \in \Sigma_K$, $\mathcal{O}_v$ is the direct limit of the $\mathcal{O}_{X,c_X(v)}$.

We argue essentially as in [45, pp. 122–123] (or as in the proof of Lemma 2.1.4). Let $x \in \mathcal{O}_v$, and let X be the projective Zariski-Samuel model determined by $\{1, x\}$ as in *loc. cit.* , bottom p. 119, so that either $X \simeq \mathbf{P}_F^1$ or $X = \operatorname{Spec} F'$ where F' is a finite extension of F contained in K. In both cases, $c = c_X(v)$ actually belongs to $\operatorname{Spec} F[x]$ and $x \in \mathcal{O}_{X,c} \subset \mathcal{O}_v$.

Finally, η_K is contained in every basic open set, therefore is dense in Σ_K.

b) This is obvious. $\square$

2.1.6. **Definition.** Let $\mathcal{C}$ be a full subcategory of **Var**. We denote by $\hat{\mathcal{C}}$ the full subcategory of $\mathcal{R}$ whose objects are cofiltered inverse limits of objects of $\mathcal{C}$ under morphisms of $\underline{S}_b$ (*cf.* §1.7).

Note that $\mathcal{C} \subset \hat{\mathcal{C}}$ and, for any function field K/F, $\Sigma_K \in \widehat{\mathbf{Var}}^{\text{prop}}$ by Theorem 2.1.5. Also, for any $X \in \widehat{\mathbf{Var}}$, the function field $F(X)$ is well-defined.

2.1.7. **Lemma.** *Let $X \in \widehat{\mathbf{Var}}$ and $K = F(X)$.*
a) For a finitely generated F-algebra $R \subset K$, the set

$$E_X(R) = \{x \in X \mid R \subset O_{X,x}\}$$

is an open subset of X. These open subsets form a basis for the topology of X.
b) The generic point $\eta_K \in X$ is dense in X, and X is quasi-compact.

Proof. a) If X is a variety, then $E_X(R)$ is open, being the set of definition of the rational map $X \dashrightarrow \operatorname{Spec} R$ induced by the inclusion $R \subset K$. In general, let $(X, \mathcal{O}_X) = \varprojlim_\alpha (X_\alpha, \mathcal{O}_{X_\alpha})$ with the X_α varieties and let $p_\alpha : X \to X_\alpha$ be the projection. Since R is finitely generated, we have

$$E_X(R) = \bigcup_\alpha p_\alpha^{-1}(E_{X_\alpha}(R))$$

which is open in X.

Let $x \in X$: using Lemma 2.1.2, we can find an α and an affine open $U \subset X_\alpha$ such that $x \in p_\alpha^{-1}(U)$. Writing $U = \operatorname{Spec} R$, we see that $x \in E_X(R)$, thus the $E_X(R)$ form a basis of the topology of X.

In b), the density follows from a) since clearly $\eta_K \in E_X(R)$ for every R, and X is quasi-compact, being the inverse limit of the quasi-compact spaces X_α. $\square$

2.1.8. Theorem. *Let* $X = \varprojlim X_\alpha$, $Y = \varprojlim Y_\beta$ *be two objects of* $\widehat{\mathbf{Var}}$. *Then we have a canonical isomorphism*

$$\widehat{\mathbf{Var}}(X, Y) \simeq \varprojlim_\beta \varinjlim_\alpha \mathbf{Var}(X_\alpha, Y_\beta).$$

Proof. Suppose first that Y is constant. We then have an obvious map

$$\varinjlim_\alpha \mathbf{Var}(X_\alpha, Y) \to \widehat{\mathbf{Var}}(X, Y).$$

Injectivity follows from Lemma 1.7.2. For surjectivity, let $f : X \to Y$ be a morphism. Let $y = f(\eta_K)$. Since η_K is dense in X by Lemma 2.1.7 b), $f(X) \subseteq \overline{\{y\}}$. This reduces us to the case where f is *dominant*.

Let $x \in X$ and $y = f(x)$. Pick an affine open neighbourhood $\operatorname{Spec} R$ of y in Y. Then $R \subset \mathcal{O}_{X,x}$, hence $R \subset \mathcal{O}_{X_\alpha, x_\alpha}$ for some α, where $x_\alpha = p_\alpha(x)$, $p_\alpha : X \to X_\alpha$ being the canonical projection. This shows that the rational map $f_\alpha : X_\alpha \dashrightarrow Y$ induced by restricting f to the generic point is defined at x_α for α large enough.

Let U_α be the set of definition of f_α. We have just shown that X is the increasing union of the open sets $p_\alpha^{-1}(U_\alpha)$. Since X is quasi-compact, this implies that $X = p_\alpha^{-1}(U_\alpha)$ for some α, *i.e.* that f factors through X_α for this value of α.

In general we have

$$\widehat{\mathbf{Var}}(X, Y) \xrightarrow{\sim} \varprojlim_\beta \widehat{\mathbf{Var}}(X, Y_\beta)$$

by the universal property of inverse limits, which completes the proof. $\square$

2.1.9. Remark. Let $\mathrm{pro}_{\underline{S}_b} - \mathbf{Var}$ be the full subcategory of the category of pro-objects of $\mathbf{Var}$ consisting of the (X_α) in which the transition maps $X_\alpha \to X_\beta$ are strict birational morphisms. Then Theorem 2.1.8 may be reinterpreted as saying that the functor

$$\varprojlim : \mathrm{pro}_{\underline{S}_b} - \mathbf{Var} \to \widehat{\mathbf{Var}}$$

is an *equivalence of categories*.

2.2. **Riemann varieties and places.**

2.2.1. Definition. We denote by $\mathbf{place}(F) = \mathbf{place}$ the category with objects finitely generated extensions of F and morphisms F-places. We denote by $\mathbf{field}(F) = \mathbf{field}$ the subcategory of $\mathbf{place}(F)$ with the same objects, but in which morphisms are F-homomorphisms of fields. We shall sometimes call the latter *trivial places*.

2.2.2. **Remark.** If $\lambda : K \rightsquigarrow L$ is a morphism in $\mathbf{place}(F)$, then its residue field $F(\lambda)$ is finitely generated over F, as a subfield of the finitely generated field L. On the other hand, given a finitely generated extension K/F, there exist valuation rings of K/F with infinitely generated residue fields as soon as $\operatorname{trdeg}(K/F) > 1$, *cf.* [45, Ch. VI, §15, Ex. 4].

We are going to study two functors

$$\operatorname{Spec} : \mathbf{field}^{\mathrm{op}} \to \widehat{\mathbf{Var}}$$

$$\Sigma : \mathbf{place}^{\mathrm{op}} \to \widehat{\mathbf{Var}}$$

and a natural transformation $\eta : \operatorname{Spec} \Rightarrow \Sigma$.

The first functor is simply $K \mapsto \operatorname{Spec} K$. The second one maps K to the Riemann variety Σ_K. Let $\lambda : K \rightsquigarrow L$ be an F-place. We define $\lambda^* : \Sigma_L \to \Sigma_K$ as follows: if $w \in \Sigma_L$, we may consider the associated place $\widetilde{w} : L \rightsquigarrow F(w)$; then $\lambda^* w$ is the valuation underlying $\widetilde{w} \circ \lambda$.

Let $E(R)$ be a basic open subset of Σ_K. Then

$$(\lambda^*)^{-1}(E(R)) = \begin{cases} \emptyset & \text{if } R \nsubseteq \mathcal{O}_\lambda \\ E(\lambda(R)) & \text{if } R \subseteq \mathcal{O}_\lambda. \end{cases}$$

Moreover, if $R \subseteq \mathcal{O}_\lambda$, then λ maps $\mathcal{O}_{\lambda^* w}$ to $\mathcal{O}_w$ for any valuation $w \in (\lambda^*)^{-1} E(R)$. This shows that λ^* is continuous and defines a morphism of locally ringed spaces. We leave it to the reader to check that $(\mu \circ \lambda)^* = \lambda^* \circ \mu^*$.

Note that we have for any K a morphism of ringed spaces

$$\eta_K : \operatorname{Spec} K \to \Sigma_K$$

with image the trivial valuation of Σ_K (which is its generic point). This defines the natural transformation η we alluded to.

2.2.3. **Proposition.** *The functors* Spec *and* Σ *are fully faithful; moreover, for any* K, L, *the map*

$$\widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K) \xrightarrow{\eta_L^*} \widehat{\mathbf{Var}}(\operatorname{Spec} L, \Sigma_K)$$

is bijective.

Proof. The case of Spec is obvious. For the rest, let $K, L \in \mathbf{place}(F)$ and consider the composition

$$\mathbf{place}(K, L) \xrightarrow{\Sigma} \widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K) \xrightarrow{\eta_L^*} \widehat{\mathbf{Var}}(\operatorname{Spec} L, \Sigma_K).$$

It suffices to show that η_L^* is injective and $\eta_L^* \circ \Sigma$ is surjective.

Let $\psi_1, \psi_2 \in \widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K)$ be such that $\eta_L^* \psi_1 = \eta_L^* \psi_2$. Pick a proper model X of K; by Theorem 2.1.8, $c_X \circ \psi_1$ and $c_X \circ \psi_2$ factor through

morphisms $f_1, f_2 : Y \to X$ for some model Y of L. By Lemma 1.7.2, $f_1 = f_2$, hence $c_X \circ \psi_1 = c_X \circ \psi_2$ and finally $\psi_1 = \psi_2$ by Theorem 2.1.5. Thus η_L^* is injective.

On the other hand, let $\varphi \in \widehat{\mathbf{Var}}(\operatorname{Spec} L, \Sigma_K)$ and $v = \varphi(\operatorname{Spec} L)$: then φ induces a homomorphism $\mathcal{O}_v \to L$, hence a place $\lambda : K \rightsquigarrow L$ and clearly $\varphi = \eta_L^* \circ \Sigma(\lambda)$. This shows that the composition $\eta_L^* \circ \Sigma$ is surjective, which concludes the proof. $\square$

3. Places and morphisms

In this section, we give a second relationship between the categories **place** and **Var**. We start with the main tool, which is the notion of compatibility between a place and a morphism.

3.1. A compatibility condition.

3.1.1. Definition. Let X, Y be two integral F-schemes of finite type, with Y separated, $f : X \dashrightarrow Y$ a rational map and $v : F(Y) \rightsquigarrow F(X)$ a place. We say that f and v are *compatible* if

- v is finite on Y (*i.e.* has a centre in Y).
- The corresponding diagram

$$
\begin{array}{ccc}
\eta_X & \xrightarrow{\;v^*\;} & \operatorname{Spec} \mathcal{O}_v \\
\downarrow & & \downarrow \\
U & \xrightarrow{\;f\;} & Y
\end{array}
$$

commutes, where U is an open subset of X on which f is defined.

3.1.2. Proposition. *Let X, Y, v be as in Definition 3.1.1. Suppose that v is finite on Y, and let $y \in Y$ be its centre. Then a rational map $f : X \dashrightarrow Y$ is compatible with v if and only if*

- *$y = f(\eta_X)$ and*
- *the diagram of fields*

$$
\begin{array}{ccc}
F(v) & & \\
\big\uparrow & \searrow{\scriptstyle v} & \\
F(y) & \xrightarrow{\;f^*\;} & F(X)
\end{array}
$$

commutes.

In particular, there is at most one such f.

Proof. Suppose v and f compatible. Then $y = f(\eta_X)$ because $v^*(\eta_X)$ is the closed point of $\operatorname{Spec} \mathcal{O}_v$. The commutativity of the diagram then follows from the one in Definition 3.1.1. Conversely, if f verifies the two conditions, then it is obviously compatible with v. The last assertion follows from Lemma 1.7.2. $\square$

3.1.3. Corollary. *a) Let Y be an integral F-scheme of finite type, and let $\mathcal{O}$ be a valuation ring of $F(Y)/F$ with residue field K and centre $y \in Y$. Assume that $F(y) \xrightarrow{\sim} K$. Then, for any rational map $f : X \dashrightarrow Y$ with X integral, such that $f(\eta_X) = y$, there exists a unique place $v : F(Y) \rightsquigarrow F(X)$ with valuation ring $\mathcal{O}$ which is compatible with f.*
b) If f is an immersion, the condition $F(y) \xrightarrow{\sim} K$ is also necessary for the existence of v.
c) In particular, let $f : X \dashrightarrow Y$ be a dominant rational map. Then f is compatible with the trivial place $F(Y) \hookrightarrow F(X)$, and this place is the only one with which f is compatible.

Proof. This follows immediately from Proposition 3.1.2. $\square$

3.1.4. Proposition. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of varieties. Let $v : F(Y) \rightsquigarrow F(X)$ and $w : F(Z) \rightsquigarrow F(Y)$ be two places. Suppose that f and v are compatible and that g and w are compatible. Then $g \circ f$ and $v \circ w$ are compatible.*

Proof. We first show that $v \circ w$ is finite on Z. By definition, the diagram

$$
\begin{array}{ccc}
\eta_Y & \xrightarrow{\ w^*\ } & \operatorname{Spec} \mathcal{O}_w \\
\downarrow & & \downarrow \\
\operatorname{Spec} \mathcal{O}_v & \longrightarrow & \operatorname{Spec} \mathcal{O}_{v \circ w}
\end{array}
$$

is cocartesian. Since the two compositions

$$
\eta_Y \xrightarrow{\ w^*\ } \operatorname{Spec} \mathcal{O}_w \to Z
$$

and

$$
\eta_Y \to \operatorname{Spec} \mathcal{O}_v \to Y \xrightarrow{\ g\ } Z
$$

coincide (by the compatibility of g and w), there is a unique induced (dominant) map $\operatorname{Spec} \mathcal{O}_{v \circ w} \to Z$. In the diagram

$$
\begin{array}{ccccc}
\eta_X & \xrightarrow{\ v^*\ } & \operatorname{Spec} \mathcal{O}_v & \longrightarrow & \operatorname{Spec} \mathcal{O}_{v \circ w} \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z
\end{array}
$$

the left square commutes by compatibility of f and v, and the right square commutes by construction. Therefore the big rectangle commutes, which means that $g \circ f$ and $v \circ w$ are compatible. $\qquad\square$

3.2. The category $\mathbf{VarP}(F)$.

3.2.1. Definition. We denote by $\mathbf{VarP}(F)$ the following category:

- Objects are integral separated F-schemes of finite type, *i.e.* F-varieties.
- Let $X, Y \in \mathbf{VarP}(F)$. A morphism $\varphi \in \mathbf{VarP}(X, Y)$ is a pair (v, f) with $f : X \to Y$ a morphism, $v : F(Y) \rightsquigarrow F(X)$ a place and v, f compatible.
- The composition of morphisms is given by Proposition 3.1.4.

If $\mathcal{C}$ is one of the categories introduced in Subsection 1.7, we shall denote by $\mathcal{C}P(F)$ the subcategory of $\mathbf{VarP}(F)$ whose objects are the objects of $\mathcal{C}$ and whose morphisms are the pairs (v, f) where $f \in \mathcal{C}$.

We now want to do an elementary study of the two forgetful functors appearing in the diagram below:

(3.1)
$$\begin{array}{ccc} \mathbf{VarP}(F) & \xrightarrow{\ \Phi_1\ } & \mathbf{Var}(F) \\ {\scriptstyle \Phi_2}\downarrow & & \\ \mathbf{place}(F)^{\mathrm{op}}. & & \end{array}$$

Clearly, Φ_1 and Φ_2 are essentially surjective. Concerning Φ_1, we have the following partial result on its fullness:

3.2.2. Lemma. *Let $f : X \dashrightarrow Y$ be a rational map, with X integral and Y separated. Assume that $y = f(\eta_X)$ is a regular point (i.e. $A = \mathcal{O}_{Y,y}$ is regular). Then there is a place $v : F(Y) \rightsquigarrow F(X)$ compatible with f.*

Proof. By Corollary 3.1.3 a), it is sufficient to produce a valuation ring $\mathcal{O}$ containing A and with the same residue field as A.

The following construction is certainly classical. Let $\mathfrak{m}$ be the maximal ideal of A and let $(a_1, \ldots, a_d)$ be a regular sequence generating $\mathfrak{m}$, with $d = \dim A = \operatorname{codim}_Y y$. For $0 \le i < j \le d+1$, let

$$A_{i,j} = (A/(a_j, \ldots, a_d))_{\mathfrak{p}}$$

where $\mathfrak{p} = (a_{i+1}, \ldots, a_{j-1})$ (for $i = 0$ we invert no a_k, and for $j = d+1$ we mod out no a_k). Then, for any (i, j), $A_{i,j}$ is a regular local ring of dimension $j - i - 1$. In particular, $F_i = A_{i,i+1}$ is the residue field of $A_{i,j}$

for any $j \geq i + 1$. We have $A_{0,d+1} = A$ and there are obvious maps

$$A_{i,j} \to A_{i+1,j} \quad \text{(injective)}$$
$$A_{i,j} \to A_{i,j-1} \quad \text{(surjective)}.$$

Consider the discrete valuation v_i associated to the discrete valuation ring $A_{i,i+2}$: it defines a place, still denoted by v_i, from F_{i+1} to F_i. The composition of these places is a place v from $F_d = F(Y)$ to $F_0 = F(y)$, whose valuation ring dominates A and whose residue field is clearly $F(y)$. $\qquad \square$

3.2.3. Remark. In Lemma 3.2.2, the assumption that y is a regular point is necessary. Indeed, take for f a closed immersion. By [3, Ch. 6, §1, Th. 2], there exists a valuation ring $\mathcal{O}$ of $F(Y)$ which dominates $\mathcal{O}_{Y,y}$ and whose residue field κ is an algebraic extension of $F(y) = F(X)$. However we cannot choose $\mathcal{O}$ such that $\kappa = F(y)$ in general. The same counterexamples as in [26, Remark 8.11] apply (singular curves, the point $(0,0,\dots,0)$ on the affine cone $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ over $\mathbf{R}$ for $n \geq 3$).

Now concerning Φ_2, we have:

3.2.4. Lemma. *Let X, Y be two varieties and $\lambda : F(Y) \rightsquigarrow F(X)$ a place. Assume that λ is finite on Y. Then there exists a unique rational map $f : X \dashrightarrow Y$ compatible with λ.*

Proof. Let y be the centre of $\mathcal{O}_\lambda$ on Y and $V = \operatorname{Spec} R$ an affine neighbourhood of y, so that $R \subset \mathcal{O}_\lambda$, and let S be the image of R in $F(\lambda)$. Choose a finitely generated F-subalgebra T of $F(X)$ containing S, with quotient field $F(X)$. Then $X' = \operatorname{Spec} T$ is an affine model of $F(X)/F$. The composition $X' \to \operatorname{Spec} S \to V \to Y$ is then compatible with v. Its restriction to a common open subset U of X and X' defines the desired map f. The uniqueness of f follows from Proposition 3.1.2. $\square$

3.2.5. Remark. Let Z be a third variety and $\mu : F(Z) \rightsquigarrow F(Y)$ be another place, finite on Z; let $g : Y \dashrightarrow Z$ be the rational map compatible with μ. If f and g are composable, then $g \circ f$ is compatible with $\lambda \circ \mu$: this follows easily from Proposition 3.1.4. However it may well happen that f and g are not composable. For example, assume Y smooth. Given μ, hence g (that we suppose not to be a morphism), choose $y \in \operatorname{Fund}(g)$ and find a λ with centre y, for example by the method in the proof of Lemma 3.2.2. Then the rational map f corresponding to λ has image contained in $\operatorname{Fund}(g)$.

We conclude this section with a useful lemma which shows that places rigidify the situation very much.

3.2.6. Lemma. *a) Let Z, Z' be two models of a function field L, with Z' separated, and v a valuation of L with centres z, z' respectively on Z and Z'. Assume that there is a birational morphism $g : Z \to Z'$. Then $g(z) = z'$.*
b) Consider a diagram

$$
\begin{array}{ccc}
 & & Z \\
 & \overset{f}{\nearrow} & \Big\downarrow g \\
X & & \\
 & \underset{f'}{\searrow} & \\
 & & Z'
\end{array}
$$

with g a birational morphism. Let $K = F(X)$, $L = F(Z) = F(Z')$ and suppose given a place $v : L \rightsquigarrow K$ compatible both with f and f'. Then $f' = g \circ f$.

Proof. a) Let $f : \operatorname{Spec} \mathcal{O}_v \to Z$ be the dominant map determined by z. Then $f' = g \circ f$ is a dominant map $\operatorname{Spec} \mathcal{O}_v \to Z'$. By the valuative criterion of separatedness, it must correspond to z'. b) This follows from a) and Proposition 3.1.2. $\qquad\square$

4. Equivalences of categories

In this section we prove that, given a full subcategory $\mathcal{C}$ of **Var** satisfying certain hypotheses and the multiplicative system S_b of birational morphisms as in Subsection 1.7, the functor

$$S_b^{-1}\Phi_2 : \mathcal{C}\mathrm{P}(F) \to \mathbf{place}(F)^{\mathrm{op}}$$

induced by the functor Φ_2 of Diagram (3.1) is an equivalence of categories, except in the case of dominant morphisms where $\mathbf{place}(F)$ is replaced by $\mathbf{field}(F)$.

These results are of a similar nature to the localisation theorems proven in [26, §8], although the proofs are completely different; we draw the link with them in the last subsection.

As a warm-up, we start with the easier case of dominant maps. Here, resolution of singularities only comes in to prove this fact in the case of smooth proper varieties. (Clearly we need resolution of singularities to know that every function field has a smooth proper model.)

4.1. **Dominant maps.** Let X, Y be two F-schemes of finite type, with X irreducible. Recall from Subsection 1.4 the set $\mathbf{Rat}(X, Y)$ of rational maps from X to Y. There is a well-defined map

$$(4.1) \qquad \mathbf{Rat}(X, Y) \to Y(F(X))$$
$$(U, f) \mapsto f_{|\eta_X}$$

where η_X is the generic point of X.

4.1.1. **Lemma.** *If X is integral, the map (4.1) is surjective. If moreover Y is separated, it is bijective.*

Proof. The first statement is clear, and the second one follows from Lemma 1.7.2. □

On the other hand, let $\mathbf{field}(F)$ be the category defined in Definition 2.2.1 (finitely generated fields and inclusions of fields). Recall [19, Ch. I, Th. 4.4] that there is an anti-equivalence of categories

$$(4.2) \qquad \mathbf{Rat}_{\mathrm{dom}}(F) \xrightarrow{\sim} \mathbf{field}(F)^{\mathrm{op}}$$
$$X \mapsto F(X).$$

Actually this follows easily from Lemma 4.1.1. We want to revisit this theorem from the point of view of the previous section. Recall from Subsection 1.4 the precategory $\mathbf{Rat}(F)$ and its subcategory $\mathbf{Rat}_{\mathrm{dom}}(F)$, and from Subsection 1.7 the category $\mathbf{Var}_{\mathrm{dom}}(F)$. We have the corresponding categories $\mathbf{Rat}_{\mathrm{dom}}\mathrm{P}(F)$ and $\mathbf{Var}_{\mathrm{dom}}\mathrm{P}(F)$ (*cf.* Definition 3.2.1). We have an obvious faithful functor

$$(4.3) \qquad \rho : \mathbf{Var}_{\mathrm{dom}}(F) \to \mathbf{Rat}_{\mathrm{dom}}(F)$$

which is the identity on objects. It is clear that ρ sends a birational morphism to an isomorphism. Hence ρ factors into a functor

$$(4.4) \qquad \bar{\rho} : S_b^{-1} \mathbf{Var}_{\mathrm{dom}}(F) \to \mathbf{Rat}_{\mathrm{dom}}(F).$$

and the same is true of the functor $\mathbf{Var}_{\mathrm{dom}}\mathrm{P}(F) \to \mathbf{Rat}_{\mathrm{dom}}\mathrm{P}(F)$.

4.1.2. **Theorem.** *Let $\mathcal{C}$ be a full subcategory of $\mathbf{Var}$ and $S = S_o, S_b$ or S_b^p. We assume that $\mathcal{C}$ is S-final in $\mathbf{Var}$ in the sense of Definition 1.7.4. Let $\mathcal{C}_{\mathrm{dom}} = \mathcal{C} \cap \mathbf{Var}_{\mathrm{dom}}$.*
a) The forgetful functors $\mathcal{C}_{\mathrm{dom}}\mathrm{P}(F) \to \mathcal{C}_{\mathrm{dom}}(F)$ and $\mathbf{Rat}_{\mathrm{dom}}\mathrm{P}(F) \to \mathbf{Rat}_{\mathrm{dom}}(F)$ are isomorphisms of categories.
b) S admits a calculus of right fractions within $\mathcal{C}_{\mathrm{dom}}(F)$ and $\mathcal{C}_{\mathrm{dom}}\mathrm{P}(F)$.

c) In the commutative diagram

$$S^{-1}\mathcal{C}_{\mathrm{dom}}\mathrm{P}(F) \longrightarrow S^{-1}\mathcal{C}_{\mathrm{dom}}(F)$$

$$\downarrow \bar{\rho} \qquad\qquad\qquad \downarrow \bar{\rho}$$

$$\mathbf{Rat}_{\mathrm{dom}}\mathrm{P}(F) \longrightarrow \mathbf{Rat}_{\mathrm{dom}}(F)$$

$$\downarrow$$

$$\mathbf{field}(F)^{\mathrm{op}}$$

all functors are equivalences of categories, except possibly the two functors $\bar{\rho}$ which are fully faithful. These functors are equivalences if and only if any $K \in \mathbf{field}(F)$ has a model in $\mathcal{C}$. (Note that this is much weaker than the condition in Definition 4.2.4.)

Proof. a) This follows immediately from Corollary 3.1.3 c).

b) In view of a) it suffices to deal with $\mathcal{C}_{\mathrm{dom}}(F)$. For any pair (u, s) of morphisms as in the diagram

(4.5)
$$\begin{array}{ccc} & & Y' \\ & & \downarrow s \\ X & \xrightarrow{\ u\ } & Y \end{array}$$

with $s \in S$ and u dominant, the pull-back of s by u exists and is in S. Using S-finality, we may then replace this pull-back by one starting from an object of $\mathcal{C}$. Moreover, if $sf = sg$ with f and g dominant and $s \in S$, then $f = g$.

c) In view of a) and (4.2), it remains to see that the composite functor $S^{-1}\mathcal{C}_{\mathrm{dom}}\mathrm{P}(F) \to \mathbf{field}(F)^{\mathrm{op}}$ is fully faithful. The graph trick and the S-finality of $\mathcal{C}$ show that it is full. It remains to show faithfulness. Let $\varphi, \psi \in \mathrm{Hom}(X, Y)$ inducing the same map $F(Y) \hookrightarrow F(X)$. By calculus of right fractions, we may write

$$\varphi = fs^{-1}, \quad \psi = gs^{-1}$$

where $s : X' \to X$ is in S_b. Since f and g are dominant, we have $f = g$ by Lemma 1.7.2, hence $\varphi = \psi$. $\qquad\square$

4.1.3. Remarks. 1) By calculus of right fractions, we have

$$\mathrm{Hom}(X, Y) = \varinjlim \mathrm{Hom}(U, Y)$$

in $S_o^{-1}\mathbf{Var}_{\mathrm{dom}}(F)$, where $U \to X$ runs through the members of S_o [13, I.2.3]. Thus we recover the formula for morphisms in $\mathbf{Rat}_{\mathrm{dom}}(F)$.

2) The functor $(S_b^p)^{-1}\mathbf{Var}_{\mathrm{dom}}\mathrm{P}(F) \to \mathbf{field}(F)^{\mathrm{op}}$ is not full (hence is not an equivalence of categories). For example, let X be a proper

variety and Y an affine open subset of X, and let K be their common function field. Then the identity map $K \to K$ is not in the image of the above functor. Indeed, if it were, then by calculus of fractions it would be represented by a map of the form fs^{-1} where $s : X' \to X$ is proper birational. But then X' would be proper and $f : X' \to Y$ should be constant, a contradiction. It can be shown that the localisation functor

$$(S_b^p)^{-1} \mathbf{Var}_{\mathrm{dom}} \mathrm{P}(F) \to S_b^{-1} \mathbf{Var}_{\mathrm{dom}} \mathrm{P}(F)$$

has a right adjoint-right inverse given by

$$(S_b^p)^{-1} \mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \mathrm{P}(F) \to (S_b^p)^{-1} \mathbf{Var}_{\mathrm{dom}} \mathrm{P}(F)$$

via the equivalence $(S_b^p)^{-1} \mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \mathrm{P}(F) \xrightarrow{\sim} S_b^{-1} \mathbf{Var}_{\mathrm{dom}} \mathrm{P}(F)$ given by Theorem 4.1.2. The proof is similar to that of Theorem 4.2.5 (ii) below.

4.2. **Localising VarP(F) and some subcategories.** We now fix in this subsection a full subcategory $\mathcal{C}$ of $\mathbf{Var}$ which is S_b^p-final in $\mathbf{Var}$ (*cf.* Definition 1.7.4). We set $\mathcal{C}^{\mathrm{prop}} = \mathcal{C} \cap \mathbf{Var}^{\mathrm{prop}}$ and recall the notation $\mathcal{C}P$, $\mathcal{C}^{\mathrm{prop}}P$ from Definition 3.2.1.

4.2.1. **Proposition.** *The category $\mathcal{C}P(F)$ admits a calculus of right fractions with respect to S_b^p. In particular, in $(S_b^p)^{-1}\mathcal{C}P(F)$, any morphism may be written in the form fp^{-1} with $p \in S_b^p$.*

Proof. Consider a diagram (4.5) in $\mathcal{C}P(F)$, with $s \in S_b^p$. Let $\lambda : F(Y) \rightsquigarrow F(X)$ be the place compatible with u which is implicit in the statement. By Proposition 3.1.2, λ has centre $z = u(\eta_X)$ on Y. Since s is proper, λ therefore has also a centre z' on Y'. By Lemma 3.2.6 a), $s(z') = z$. By Lemma 3.2.4, there exists a unique rational map $\varphi : X \dashrightarrow Y'$ compatible with λ, and $s \circ \varphi = u$ by Lemma 3.2.6 b). By the graph trick, we get a commutative diagram

$$(4.6) \qquad \begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

in which $X' \subset X \times_Y Y'$ is the closure of the graph of φ, $s' \in S_b$ and u' is compatible with λ. Moreover, since s is proper, the projection $X \times_Y Y' \to Y'$ is proper and s' is proper. Since $\mathcal{C}$ is S_b^p-final in $\mathbf{Var}$, we may find some $X'' \xrightarrow{t} X'$ with $t \in S_b^p$ and $X'' \in \mathcal{C}$.

Let now

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Y'$$

be a diagram in $\mathcal{C}P$ with $s \in S_b^p$, such that $sf = sg$. By Corollary 3.1.3 c), the place underlying s is the identity. Hence the two places

underlying f and g must be equal. But then $f = g$ by Proposition 3.1.2. $\qquad\square$

4.2.2. Proposition. *Consider a diagram in $\mathcal{C}P(F)$*

$$
\begin{array}{ccc}
 & Z & \\
f \swarrow & & \searrow p \\
X & & Y \\
f' \nwarrow & & \nearrow p' \\
 & Z' &
\end{array}
$$

where $p, p' \in S_b^p$. Let $K = F(Z) = F(Z') = F(Y)$, $L = F(X)$ and suppose given a place $v : L \rightsquigarrow K$ compatible both with f and f'. Then $(v, fp^{-1}) = (v, f'p'^{-1})$ in $(S_b^p)^{-1}\mathcal{C}P(F)$.

Proof. By the graph trick and finality, complete the diagram as follows:

$$
\begin{array}{ccc}
 & Z & \\
f \swarrow \; {\scriptstyle p_1}\uparrow & & \searrow p \\
X & Z'' & Y \\
f' \nwarrow \; {\scriptstyle p_1'}\downarrow & & \nearrow p' \\
 & Z' &
\end{array}
$$

with $p_1, p_1' \in S_b^p$ and $Z'' \in \mathcal{C}$. Then we have

$$pp_1 = p'p_1', \quad fp_1 = f'p_1'$$

(the latter by Lemma 3.2.6 b)), hence the claim. $\qquad\square$

4.2.3. Proposition. *Suppose $\mathcal{C}$ properly final in **Var** (Definitiion 1.7.4 b)). Then, in $(S_b^p)^{-1}\mathcal{C}P(F)$, S_o admits a calculus of left fractions. In particular (cf. Proposition 4.2.1), any morphism in $S_b^{-1}\mathcal{C}P(F)$ can be written as $j^{-1}fq^{-1}$, with $j \in S_o$ and $q \in S_b^p$.*

Proof. a) Consider a diagram in $(S_b^p)^{-1}\mathcal{C}P(F)$

$$
\begin{array}{ccc}
X & \xrightarrow{\;j\;} & X' \\
{\scriptstyle\varphi}\downarrow & & \\
Y & &
\end{array}
$$

with $j \in S_o$. By Proposition 4.2.1, we may write $\varphi = fp^{-1}$ with $p \in S_b^p$ and f a morphism of $\mathcal{C}P$. Since $\mathcal{C}$ is properly final in **Var**, we may embed Y as an open subset of a proper $\bar{Y}$, with $\bar{Y} \in \mathcal{C}$. All

this gives us a rational map $X' \dashrightarrow \bar{Y}$. Using the graph trick and S_b^p-finality, we may "resolve" φ into a morphism $g : \widetilde{X}' \to \bar{Y}$, with $\widetilde{X}' \in \mathcal{C}$ provided with a proper birational morphism $q : \widetilde{X}' \to X$. Let $\psi = gq^{-1} \in (S_b^p)^{-1}\mathcal{C}P$. Then the diagram of $(S_b^p)^{-1}\mathcal{C}P$

$$
\begin{array}{ccc}
X & \xrightarrow{\ j\ } & X' \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\
Y & \xrightarrow{\ j_1\ } & \bar{Y}
\end{array}
$$

commutes because the following bigger diagram commutes in $\mathcal{C}P$:

thanks to Lemma 1.7.2, where $\widetilde{X}'' \in \mathcal{C}$ and $r, r' \in S_b^p$ ($\widetilde{X}'', r, r'$ exist by S_b^p-finality of $\mathcal{C}$ in **Var**).

b) Consider a diagram

$$X' \xrightarrow{\ j\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $(S_b^p)^{-1}\mathcal{C}P$, where $j \in S_o$ and $fj = gj$. By Proposition 4.2.1 and S_b^p-finality, we may write $f = \widetilde{f}p^{-1}$ and $g = \widetilde{g}p^{-1}$, where $\widetilde{f}, \widetilde{g}$ are morphisms in $\mathcal{C}P$ and $p : \widetilde{X} \to X$ is in S_b^p. Let U be a common open subset to X' and $\widetilde{X}$: then the equality $fj = gj$ implies that the restrictions of $\widetilde{f}$ and $\widetilde{g}$ to U coincide as morphisms of $(S_b^p)^{-1}\mathcal{C}P$. Hence the places underlying $\widetilde{f}$ and $\widetilde{g}$ are equal, which implies that $\widetilde{f} = \widetilde{g}$ (Proposition 3.1.2), and thus $f = g$. $\square$

We shall also need the following definition, which will allow us to state some unconditional results:

4.2.4. Definition. Let $\mathcal{C}$ be a full subcategory of **Var** and $\mathcal{C}^{\mathrm{prop}} = \mathcal{C} \cap \mathbf{Var}^{\mathrm{prop}}$. We denote by $\mathbf{place}_{\mathcal{C}}(F) = \mathbf{place}_{\mathcal{C}}$ the full subcategory of $\mathbf{place}(F)$ whose objects are function fields K/F which admit a *final system* of models in $\mathcal{C}^{\mathrm{prop}}$.

(The condition "final" means that any proper model of K can be dominated by a model in $\mathcal{C}^{\mathrm{prop}}$.)

Note that $\mathbf{place}_{\mathcal{C}} = \mathbf{place}$ in all the examples 1.7.5.

4.2.5. Theorem. *Let $\mathcal{C}$ be properly final in* **Var** *(Definition 1.7.4 b)). Consider the string of functors induced by the functor Φ_2 of Diagram (3.1)*

$$(S_b^p)^{-1}\mathcal{C}^{\mathrm{prop}}\mathrm{P}(F) \xrightarrow{S} (S_b^p)^{-1}\mathcal{C}P(F) \xrightarrow{T} S_b^{-1}\mathcal{C}P(F) \xrightarrow{U} \mathbf{place}(F)^{\mathrm{op}}$$

and let $V = TS$. Then

(i) *S is fully faithful and T is faithful.*

(ii) *Through V, T is left adjoint/left inverse to S: for objects $X \in (S_b^p)^{-1}\mathcal{C}P(F)$ and $Y \in (S_b^p)^{-1}\mathcal{C}^{\mathrm{prop}}\mathrm{P}(F)$, the map*

$$T : \mathrm{Hom}(X, S(Y)) \to \mathrm{Hom}(T(X), V(Y))$$

is an isomorphism, and V is an equivalence of categories.

(iii) *Suppose that open immersions are final in $\mathcal{C}$ in the sense that any $X \in \mathcal{C}$ has a final collection of (nonempty) open subsets belonging to $\mathcal{C}$. Then U is fully faithful, with $\mathbf{place}_{\mathcal{C}}(F)^{\mathrm{op}}$ as its essential image.*

(iv) *Let $X, Y \in (S_b^p)^{-1}\mathcal{C}P(F)$. Then the image of $\mathrm{Hom}(X, Y)$ in $\mathrm{Hom}(UT(Y), UT(X))$ via UT is contained in the set of places which are finite on Y. In particular, T is not full.*

Proof. In 6 steps:

A) Thanks to Propositions 4.2.1 and 4.2.2, UT and UTS are faithful, hence S and T are faithful. Let $X, Y \in (S_b^p)^{-1}\mathcal{C}^{\mathrm{prop}}\mathrm{P}(F)$ and $\varphi : S(X) \to S(Y)$. By Proposition 4.2.1 we may write $\varphi = fp^{-1}$ with $p \in S_b^p$. But then the source of p is proper and φ comes from $(S_b^p)^{-1}\mathcal{C}^{\mathrm{prop}}\mathrm{P}(F)$. So S is full, which proves (i).

B) In (ii), we already know that T is injective. Let $\varphi \in \mathrm{Hom}(T(X), V(Y))$. By Proposition 4.2.3, $\varphi = j^{-1}fp^{-1}$ with $j \in S_o$ and $p \in S_b^p$. Since Y is proper, j is necessarily an isomorphism, which proves surjectivity.

C) It follows from A) and B) that V is fully faithful. Essential surjectivity follows from Theorem 1.6.1 and the S_b^p-finality hypothesis on $\mathcal{C}$. This completes the proof of (ii).

D) We come to the proof of (iii). Since UTS is faithful and TS is an equivalence, U is faithful. To show that it is full, let $X, Y \in \mathbf{VarP}(F)$ and $\lambda : F(Y) \rightsquigarrow F(X)$ a place. Let $Y \to \bar{Y}$ be a compactification of Y (Theorem 1.6.1). By S_b-finality of $\mathcal{C}$, choose $\bar{Y}' \xrightarrow{s} \bar{Y}$ with $\bar{Y}' \in \mathcal{C}$ and $s \in S_b^p$: then $\bar{Y}' \in \mathcal{C}^{\mathrm{prop}}$ and λ is finite over $\bar{Y}'$. By Lemma 3.2.4, there is a rational map $f : X \dashrightarrow \bar{Y}'$ compatible with λ. By the additional hypothesis, Y and $\bar{Y}'$ have a common open subset $V \in \mathcal{C}$ and X has an open subset $U \in \mathcal{C}$ on which f is defined. This gives us $\varphi : X \to Y$ in $S_b^{-1}\mathcal{C}P$ such that $U(\varphi) = \lambda$.

E) Since $V = TS$ is an equivalence of categories, the essential image of U is the same as that of UTS, which is $\mathbf{place}_{\mathcal{C}}^{\mathrm{op}}$ by definition.

F) (iv) is obvious. $\qquad\square$

4.2.6. Definition. Let $S_r \subset \mathbf{place}(F)$ denote the multiplicative system of maps of the form $K \hookrightarrow L$, where L/K is purely transcendental.

4.2.7. Corollary. *We have equivalences of categories*

$$(S_r^p)^{-1}\mathcal{C}^{\mathrm{prop}}\mathrm{P}(F) \overset{TS}{\to} S_r^{-1}\mathcal{C}P(F) \overset{U}{\to} S_r^{-1}\,\mathbf{place}_{\mathcal{C}}(F)^{\mathrm{op}}$$

induced by the functors of Theorem 4.2.5. $\qquad\square$

Let $\mathcal{C}$ be as in Example 1.7.5. As a byproduct of Theorem 4.2.5, the functors

$$(S_b^p)^{-1}\mathcal{C}^{\mathrm{prop}}P \to (S_b^p)^{-1}\,\mathbf{Var}^{\mathrm{prop}}\,P, \quad (S_b)^{-1}\mathcal{C}P \to (S_b)^{-1}\,\mathbf{Var}\,P$$

are equivalences of categories. This does not formally follow for the functor $(S_b^p)^{-1}\mathcal{C}P \to (S_b^p)^{-1}\,\mathbf{Var}\,P$; nevertheless, we do have:

4.2.8. Proposition. *The functor*

$$(S_b^p)^{-1}\mathcal{C}\mathrm{P}(F) \to (S_b^p)^{-1}\,\mathbf{Var}\mathbf{P}(F)$$

is an equivalence of categories.

Proof. The functor is faithful by the faithfulness of T and U in Theorem 4.2.5 ((i) for U and (iii) for T). Its essential surjectivity follows from S_b^p-finality, while its fullness follows from this finality and Proposition 4.2.1: if $X, Y \in (S_b^p)^{-1}\mathcal{C}\mathrm{P}(F)$ and $\varphi : X \to Y$ is a morphism in $(S_b^p)^{-1}\,\mathbf{Var}\mathbf{P}(F)$, then by 4.2.1 we may write $\varphi = fp^{-1}$, with $p : \widetilde{X} \to X$ proper birational. But by S_b^p-finality, we may find $p' : \widetilde{X}' \to \widetilde{X}$ proper birational with $\widetilde{X}' \in \mathcal{C}$, and replace fp^{-1} by $fp'(pp')^{-1}$. $\qquad\square$

4.2.9. Remarks. 1) Even in $(S_b^p)^{-1}\,\mathbf{Var}\mathbf{P}(F)$, S_o does not admit a calculus of right fractions. Indeed, consider a diagram in $(S_b^p)^{-1}\,\mathbf{Var}\mathbf{P}(F)$

$$\begin{array}{ccc} & & Y' \\ & & \downarrow{\scriptstyle j} \\ X & \overset{f}{\longrightarrow} & Y \end{array}$$

where $j \in S_o$ and, for simplicity, f comes from $\mathbf{Var}\mathbf{P}(F)$. Suppose that we can complete this diagram into a commutative diagram in

$(S_b^p)^{-1} \mathbf{VarP}(F)$

$$
\begin{array}{ccc}
\widetilde{X'} & & \\
{\scriptstyle p}\downarrow & \searrow^{g} & \\
X' & \xrightarrow{\varphi} & Y' \\
{\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\
X & \xrightarrow{f} & Y
\end{array}
$$

with $p \in S_b^p$ and g comes from $\mathbf{VarP}(F)$. By Proposition 3.1.2 the localisation functor $\mathbf{VarP}(F) \to (S_b^p)^{-1} \mathbf{VarP}(F)$ is faithful, so the diagram (without φ) must already commute in $\mathbf{VarP}(F)$. If $f(X) \cap Y' = \emptyset$, this is impossible.

2) The category $(S_b^p)^{-1} \mathbf{VarP}(F)$ is equivalent to a nonfull subcategory of $\mathbf{place}(F)^{\mathrm{op}}$ via the localisation functor UT of Theorem 4.2.5, but it seems difficult to describe exactly the image of UT. For example, if X and Y are proper, then the image is all of $\mathrm{Hom}(UT(Y), UT(X))$. On the other hand, if X is proper and Y is affine, then for any map $\varphi = fp^{-1} : X \to Y$, the source X' of p is proper hence $f(X')$ is a closed point of Y, so that $\mathrm{Hom}(UT(Y), UT(X))$ is contained in the set of places from $F(Y)$ to $F(X)$ whose centre on Y is a closed point (and one sees easily that this inclusion is an equality). In general, the description seems to depend heavily on the nature of X and Y.

Let us summarise what we have done so far. We have realised $\mathbf{place}(F)^{\mathrm{op}}$ as the localisation of $\mathbf{VarP}(F)$ with respect to S_b in two steps: first localising $\mathbf{VarP}(F)$ with respect to S_b^p and then localising the resulting category with respect to S_o. The first step enjoys calculus of right fractions but the second one does not by Remark 4.2.9. However, the localisation functor

$$(S_b^p)^{-1} \mathbf{VarP}(F) \to S_b^{-1} \mathbf{VarP}(F) \xrightarrow{\sim} \mathbf{place}(F)^{\mathrm{op}}$$

has a right adjoint-right inverse. The corresponding local objects of $(S_b^p)^{-1} \mathbf{VarP}(F)$ (cf. Definition 1.1.1) are proper varieties. We get the same localisations when restricting to quasi-projective varieties, and also to smooth or smooth quasi-projective varieties but only in characteristic 0 (using resolution of singularities).

4.3. **Localising $\mathbf{Sm}(F)$.** We now recall some of the results of [26, §8]: *in characteristic 0, we have the following equivalences of categories:

$$S_b^{-1} \mathbf{Sm}^{\mathrm{proj}} \simeq S_b^{-1} \mathbf{Sm}^{\mathrm{prop}} \simeq S_b^{-1} \mathbf{Sm}^{\mathrm{qp}} \simeq S_b^{-1} \mathbf{Sm}$$

induced by the obvious inclusion functors.

The proofs are completely different from those of the previous subsection: in particular they do not use any calculus of fractions. In fact, S_b^p *does not admit any calculus of fractions within* $\mathbf{Var}(F)$, contrary to the case of $\mathbf{VarP}(F)$ (*cf.* Proposition 4.2.1). This is shown by the same examples as in Remark 3.2.3.

If we restrict to $\mathbf{Sm}(F)$, we can use Proposition 4.2.1 and Lemma 3.2.2 to prove part of calculus of fractions, but we have then to resolve the singularities of the variety X' in the proof of Proposition 4.2.1. The following lemma will yield a slightly different proof in Proposition 4.3.3 below. It does not require resolution of singularities, but the proof of Proposition 4.3.3 will require it.

4.3.1. Lemma. *Let* $s : Y \to X$ *be in* S_r^p, *with* X *smooth. Then* s *is an envelope* [11]: *for any extension* K/F, *the map* $Y(K) \to X(K)$ *is surjective.*

Proof. It suffices to deal with $K = F$.

a) We first handle the case where $s \in S_b^p$. Let $x \in X(F)$. We propose two proofs:

First proof. By lemma 3.2.2, there is a place λ of $F(X)$ with centre x and residue field F. The valuative criterion for properness implies that λ has a centre y on Y; then $s(y) = x$ by Lemma 3.2.6 and $F(y) \subseteq F(\lambda) = F$.

Second proof (*cf.* Kollár-Szabó [29, Prop. 6]). Consider the blow-up $Bl_x(X)$ of X. By the graph trick we have a commutative diagram

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
{\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\
Bl_x(X) & \xrightarrow{\;p\;} & X
\end{array}
$$

with $s' \in S_b^p$. Let E be the exceptional locus of s' (see §1.4). Since $Bl_x(X)$ is smooth, $s'(E)$ has codimension at least 2 (*cf.* [8, 1.40]), hence it does not contain $p^{-1}(x)$. If F is infinite we stop here since $p^{-1}(x) \simeq \mathbf{P}^{d-1}$, where $d = \dim X$. Otherwise we conclude by induction on d, applying the induction hypothesis to the restriction of s' to a suitable irreducible component of $s'^{-1}(p^{-1}(x))$.

b) Suppose now that $s \in S_r^p$. Then Y is birational to $X \times (\mathbf{P}^1)^n$ for some n. By a) applied to u in Diagram (1.1), we get that $\widetilde{Y}(F) \to X(F)$ is surjective and therefore that $Y(F) \to X(F)$ is surjective. $\square$

4.3.2. Remark. It follows from [40, Lemma 5.8] that an envelope in the sense of Gillet is the same as a proper cdh covering in the sense of Voevodsky. Lemma 4.3.1 also appears as Lemma 5.10 of [40] (for a

proper birational map): the above proof shows that the resolution of singularities assuption of *loc. cit.* is not necessary.

4.3.3. *Proposition. If $\operatorname{char} F = 0$, the multiplicative set S_b^p verifies the second axiom of calculus of right fractions within $\mathbf{Sm}(F)$.*

Proof. We consider a diagram (4.5) in $\mathbf{Sm}(F)$, with $s \in S_b^p$. By Lemma 4.3.1, $z = u(\eta_X)$ has a preimage $z' \in Y'$ with same residue field. Let $Z = \overline{\{z\}}$ and $Z' = \overline{\{z'\}}$: the map $Z' \to Z$ is birational. Since the map $\bar{u} : X \to Z$ factoring u is dominant, we get by Theorem 4.1.2 b) a commutative diagram like (4.6), with s' proper birational. We then need to resolve the singularities of X' to finish the proof. $\qquad\square$

4.3.4. *Corollary. If $\operatorname{char} F = 0$, any morphism in $S_b^{-1}\mathbf{Sm}(F)$ may be represented as $j^{-1}fp^{-1}$, where $j \in S_o$ and $p \in S_b^p$.*

Proof. As that of Proposition 4.2.3. $\qquad\square$

4.3.5. Remarks. 1) Even in characteristic 0, S_b^p is far from admitting a calculus of right fractions within $\mathbf{Sm}(F)$. Indeed, let $s : Y \to X$ be a proper birational morphism that contracts some closed subvariety $i : Z \subset Y$ to a point. Then, given any two morphisms $f, g : Y' \rightrightarrows Z$, we have $sif = sig$. But if $ift = igt$ for some $t \in S_b^p$, then $if = ig$ (hence $f = g$) since t is dominant.
2) Lemma 4.3.1, Proposition 4.3.3 and Corollary 4.3.4 extend to a strictly full subcategory $\mathcal{C}$ of $\mathbf{Var}$ provided it is S_b^p and S_o-final and, moreover, any object $X \in \mathcal{C}$ has the following equivalent properties:

- For any $x \in X$, there exists a valuation of $F(X)$ with centre x and with the same residue field as x.
- For any morphism $f : Y \to X$ with $Y \in \mathbf{Var}$, there exists a place compatible with f.

The only non-regular example we can think of is that of a variety with isolated singularities over an algebraically closed field. In particular, it is not clear to us whether minimal models in the sense of the minimal model programme have this property in general (*cf.* the affine cone of Remark 3.2.3).

4.4. Comparing $\mathbf{place}(F)$ and $\mathbf{Sm}(F)$ after localisation. Putting together the results of the last two subsections, we get via the functor Φ_1 of Diagram (3.1), a functor

$$\mathbf{place}(F)^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}^{\mathrm{proj}}(F) \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}(F)$$

*in characteristic 0. Now, by Theorem 1.7.7, the functor $S_b^{-1}\mathbf{Sm}^{\mathrm{proj}}(F) \to S_r^{-1}\mathbf{Sm}^{\mathrm{proj}}(F)$ is an equivalence of categories. Thus, finally, we get

a canonical functor

$$(4.7) \qquad S_r^{-1} \mathbf{place}(F)^{\mathrm{op}} \to S_b^{-1} \mathbf{Sm}^{\mathrm{proj}}(F)$$

that we shall study in more detail in §6.2.

The drawback of this approach is that it rests on resolution of singularities. We now give an alternative approach via the results of Section 2, which will yield unconditional results.

Let $\mathcal{D}$ be a full subcategory of $\mathbf{Var}^{\mathrm{prop}}$. Recall the subcategory $\hat{\mathcal{D}} \subset \mathcal{R}$ of Definition 2.1.6. We have a commutative diagram of categories

(4.8)

where the two lower functors are fully faithful (by Proposition 2.2.3 for Σ).

4.4.1. Definition (*cf.* Theorem 2.1.8). Let $X, Y \in \hat{\mathcal{D}}$, with $X = \varprojlim X_\alpha$, $Y = \varprojlim Y_\beta$. A morphism $s : X \to Y$ is *birational* if, for each β, the projection $X \xrightarrow{s} Y \to Y_\beta$ factors through a birational map $s_{\alpha,\beta} : X_\alpha \to Y_\beta$ for some α (this does not depend on the choice of α). We denote by $S_b \in \hat{\mathcal{D}}$ the collection of these morphisms.

4.4.2. Theorem. *The functor J induces an equivalence of categories*

$$\bar{J} : S_b^{-1}\mathcal{D} \xrightarrow{\sim} S_b^{-1}\hat{\mathcal{D}}.$$

Proof. To lighten notation we drop the functor J. We apply Proposition 5.10 b) of [26]. We have to check Conditions (b1), (b2) and (b3), namely:

(b1) Given two maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{D}$ and a map $s : Z = \varprojlim Z_\alpha \to X$ in $S_b \subset \hat{\mathcal{D}}$, $fs = gs \Rightarrow f = g$. This is clear by Lemma 1.7.2, since by Theorem 2.1.8 s factors through some Z_α, with $Z_\alpha \to X$ birational.

(b2) For any $X = \varprojlim X_\alpha \in \hat{\mathcal{D}}$, there exists a birational morphism $s : X \to X'$ with $X' \in \mathcal{D}$. It suffices to take $X' = X_\alpha$ for some α.

(b3) Given a diagram

$$
\begin{array}{c}
X_1 \\
s_1 \uparrow \\
X = \varprojlim X_\alpha \xrightarrow{\ \ f\ \ } Y
\end{array}
$$

with $X \in \hat{\mathcal{D}}$, $X', Y \in \mathcal{D}$ and $s_1 \in S_b$, there exists $s_2 : X \to X_2$ in S_b, with $X_2 \in \mathcal{D}$, covering both s_1 and f. Again, it suffices to take $X_2 = X_\alpha$ for α large enough (use Theorem 2.1.8).

$\square$

To summarize, localising Diagram (4.8) with respect to S_b yields a new diagram

$$
\begin{array}{ccc}
 & S_b^{-1}\mathcal{D}\mathrm{P} & \\
{}^{\Phi_2^{\mathcal{D}}}\swarrow & & \searrow \\
\mathbf{place}_{\mathcal{D}}^{\mathrm{op}} & & S_b^{-1}\mathcal{D} \\
{}_{\bar{\Sigma}}\searrow & & \swarrow{}_{\bar{J}} \\
 & S_b^{-1}\hat{\mathcal{D}}. &
\end{array}
$$

In this diagram, $\Phi_2^{\mathcal{D}}$ is induced by the functor Φ_2 of (3.1); $\bar{J}$ is an equivalence of categories without any hypothesis on $\mathcal{D}$. If $\mathcal{D}$ is of the form $\mathcal{C}^{\mathrm{prop}}$ with $\mathcal{C}_b$ final in $\mathbf{Var}_b$ and open immersions final in $\mathcal{C}$, then $\Phi_2^{\mathcal{D}}$ is also an equivalence of categories.

In particular, we get

4.4.3. Corollary. *For any F, there are canonical functors*

$$
\mathbf{place}_{\mathbf{Sm}^{\mathrm{prop}}}(F)^{\mathrm{op}} \to S_b^{-1}\,\mathbf{Sm}^{\mathrm{prop}}(F)
$$
$$
\mathbf{place}_{\mathbf{Sm}^{\mathrm{proj}}}(F)^{\mathrm{op}} \to S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F)
$$
$$
S_r^{-1}\,\mathbf{place}_{\mathbf{Sm}^{\mathrm{prop}}}(F)^{\mathrm{op}} \to S_r^{-1}\,\mathbf{Sm}^{\mathrm{prop}}(F)
$$
$$
S_r^{-1}\,\mathbf{place}_{\mathbf{Sm}^{\mathrm{proj}}}(F)^{\mathrm{op}} \to S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F).
$$

5. Stable birationality and R-equivalence

In this section, we give alternate descriptions of $S_r^{-1}\,\mathbf{place}(F)$ and (in characteristic 0) $S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) = S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F)$, involving Manin's R-equivalence and an analogous notion of "homotopy" between places. In particular, we explicitly compute the Hom sets in both categories.

5.1. **Homotopy of places.**

5.1.1. Definition. Let $K, L \in \mathbf{place}(F)$. Two places $\lambda_0, \lambda_1 : K \rightsquigarrow L$ are *elementarily homotopic* if there exists a place $\mu : K \rightsquigarrow L(t)$ such that $s_i \circ \mu = \lambda_i, i = 0, 1$, where $s_i : L(t) \rightsquigarrow L$ denotes the place corresponding to specialisation at i.

The property of two places being elementarily homotopic is preserved under composition on the right. Indeed if λ_0 and λ_1 are elementarily homotopic and if $\mu : M \rightsquigarrow K$ is another place, then obviously so are $\lambda_0 \circ \mu$ and $\lambda_1 \circ \mu$. If on the other hand $\tau : L \rightsquigarrow M$ is another place, then $\tau \circ \lambda_0$ and $\tau \circ \lambda_1$ are not in general elementarily homotopic (we are indebted to Gabber for pointing this out), as one can see for example from the uniqueness of factorisation of places.

Consider the equivalence relation h generated by elementary homotopy (*cf.* Definition 1.2.1). So h is the coarsest equivalence relation on morphisms in $\mathbf{place}(F)$ which is compatible with left and right composition and such that two elementarily homotopic places are equivalent with respect to h.

5.1.2. Definition (*cf.* Def. 1.2.1). We denote by $\mathbf{place}(F)/\mathrm{h}$ the factor category of $\mathbf{place}(F)$ by the homotopy relation h.

Thus the objects of $\mathbf{place}(F)/\mathrm{h}$ are function fields, while the set of morphisms consists of equivalence classes of homotopic places between the function fields. There is an obvious full surjective functor

$$\Pi : \mathbf{place}(F) \to \mathbf{place}(F)/\mathrm{h}.$$

The following lemma provides a first, more elementary, description of $S_r^{-1} \mathbf{place}(F)$ and of the localisation functor. It is valid in all characteristics.

5.1.3. Proposition. *There is a unique isomorphism of categories*

$$\mathbf{place}(F)/\mathrm{h} \to S_r^{-1} \mathbf{place}(F)$$

which makes the diagram of categories and functors

commutative. In particular, the localisation functor S_r^{-1} is full and its fibres are the equivalence classes for h.

Proof. [2] We first note that any two homotopic places become equal in $S_r^{-1}\textbf{place}(F)$. Clearly it suffices to prove this when they are elementarily homotopic. But then s_0 and s_1 are left inverses of the natural inclusion $i : L \to L(t)$, which becomes an isomorphism in $S_r^{-1}\textbf{place}(F)$. Thus s_0 and s_1 become equal in $S_r^{-1}\textbf{place}(F)$. So the localisation functor $\textbf{place}(F) \to S_r^{-1}\textbf{place}(F)$ canonically factors through Π into a functor $\textbf{place}(F)/\mathrm{h} \longrightarrow S_r^{-1}\textbf{place}(F)$.

On the other hand we claim that, with the above notation, $i \circ s_0 : L(t) \rightsquigarrow L(t)$ is homotopic to $1_{L(t)}$ in $\textbf{place}(F)$. Indeed they are elementarily homotopic via the place $L(t) \rightsquigarrow L(t, s)$ (in this case an inclusion) that is the identity on L and maps t to st. Hence the projection functor Π factors as $S_r^{-1}\textbf{place}(F) \to \textbf{place}(F)/\mathrm{h}$, and it is plain that this functor is inverse to the previous one. $\square$

5.2. **Review of R-equivalence.** We have the following definitions:

5.2.1**. Definition.** a) (Manin) Two rational points x_0, x_1 of an F-scheme X of finite type are *directly R-equivalent* if there is a rational map $f : \mathbf{P}^1 \dashrightarrow X$ defined at 0 and 1 and such that $f(0) = x_0$, $f(1) = x_1$.
b) (Manin) R-equivalence on $X(F)$ is the equivalence relation generated by direct R-equivalence.
c) (Chow) X is *linearly connected* if any two points of X may be joined (over a universal domain) by a chain of rational curves.
d) X is *strongly linearly connected* (or R-trivial) if $X(K)/R$ is reduced to a point for any extension K/F (in particular, $X(F) \neq \emptyset$).

5.2.2**. Remark.** Thus, "linearly connected" is closely related to the notion of "rationally chain-connected" of Kollár et al. More precisely, according to [8, p. 99, Def. 4.21], a rationally chain connected F-scheme is a proper variety by definition. Then, if X is a proper F-variety, X is linearly connected if and only if it is rationally chain-connected. Indeed, this is true if F is uncountable by ibid., p. 100, Remark 4.22 (2), and on the other hand the property of rational chain-connectedness is invariant under algebraically closed extension by ibid., p. 100, Remark 4.22 (3).

We shall discuss the well-known relationship with rationally connected varieties in §7.3.

Recall that, for any X, Y, we have an isomorphism

(5.1) $(X \times Y)(K)/R \xrightarrow{\sim} X(K)/R \times Y(K)/R.$

The proof is easy. We also have the certainly well-known

---

[2] See also [17, Remark 1.3.4] for a closely related statement.

5.2.3. Lemma. *Suppose that X is linearly connected. Then, for any algebraically closed extension K/F, $X(K)/R$ is reduced to a point.*

Proof. Let $x_0, x_1 \in X(K)$. Then x_0 and x_1 are defined over some finitely generated subextension E/F. By assumption, there exists a universal domain $\Omega \supset E$ such that x_0 and x_1 are R-equivalent in $X(\Omega)$. If the transcendence degree of K over F is larger than that of Ω, then Ω embeds in K over E and we are done. Thus we may assume that K has finite transcendence degree over F, hence can be embedded into Ω.

Let $\gamma_1, \ldots, \gamma_n : \mathbf{P}^1_\Omega \dashrightarrow X_\Omega$ be a chain of rational curves linking x_0 and x_1 over Ω. Pick a finitely generated extension L of K over which all the γ_i are defined.

We may write $L = K(U)$ for some smooth K-variety U. Then the γ_i define rational maps $\widetilde{\gamma}_i : U \times \mathbf{P}^1 \dashrightarrow X$. For each i, let $W_i \subseteq U \times \mathbf{P}^1$ be the locus of definition of γ_i. By Chevalley's constructibility theorem, the projections of the W_i on U contain a common nonempty open subset U'. Pick a rational point $u \in U'(K)$: then the fibres of the γ_i at u are rational curves defined over K that link x_0 to x_1. $\square$

The following theorem completes Lemma 4.3.1 under resolution of singularities (but see also Corollary 5.3.5 below). Even though F is supposed to be of characteristic 0, we write "regular variety" instead of "smooth variety" in order to state Corollary 5.3.5 conveniently below. (Gabber pointed out that a resolution of singularities of a general scheme X, say of dimension 2, yields a regular scheme $\widetilde{X}$, but that the composite $\widetilde{X} \to X \to \operatorname{Spec} k$ may not be smooth if X is of finite type over some [nonperfect] base field k.)

5.2.4. *Theorem. *Assume* $\operatorname{char} F = 0$.
a) Let $s : Y \to X$ be in S_r^p, with X, Y regular. Then the induced map $Y(K)/R \to X(K)/R$ is bijective for any field extension K/F.
b) Let $f : Y \dashrightarrow Z$ be a rational map with Y regular and Z proper. Then there is an induced map $f_* : Y(K)/R \to Z(K)/R$, which depends functorially on K/F.

Proof. For smooth proper varieties and $s \in S_b$ in a), this is the content of [5, Prop. 10]. The proofs in this more general case are exactly the same as those of *loc. cit.* (in a), reduce to blow-ups with smooth centres by using Lemma 1.7.6 c); in b), use the graph trick). It remains to pass from the case $s \in S_b^p$ to $s \in S_r^p$ in a): this is done using Diagram (1.1), just as in the proof of Lemma 4.3.1, but this time using resolution. $\square$

5.3. Theorems of Murre, Chow, van der Waerden and Gabber.
This subsection is not seriously used in the rest of the paper and may be skipped at first reading: we explain how the assumption of resolution of singularities may be relaxed to some extent. We start with the following not so well-known but nevertheless basic theorem of Murre [35], which was later rediscovered by Chow and van der Waerden [4, 43].

5.3.1. Theorem (Murre, Chow, van der Waerden). *Let $f : X \to Y$ be a projective birational morphism of F-varieties and $y \in Y$ be a smooth rational point. Then the fibre $f^{-1}(y)$ is linearly connected. In particular, by Lemma 5.2.3, $f^{-1}(y)(K)/R$ is reduced to a point for any algebraically closed extension K/F.*

For the sake of completeness, we give the general statement of Chow, which does not require a base field:

5.3.2. Theorem (Chow). *Let A be a regular local ring and $f : X \to \operatorname{Spec} A$ be a projective birational morphism. Let s be the closed point of $\operatorname{Spec} A$ and F its residue field. Then the special fibre $f^{-1}(s)$ is linearly connected (over F).*

Gabber has recently refined these theorems:

5.3.3. Theorem (Gabber). *Let A, X, f, s, F be as in Theorem 5.3.2, but assume only that f is proper. Let X_{reg} be the regular locus of X and $f^{-1}(s)^{\mathrm{reg}} = f^{-1}(s) \cap X_{\mathrm{reg}}$, which is known to be open in $f^{-1}(s)$. Then, for any extension K/F, any two points of $f^{-1}(s)^{\mathrm{reg}}(K)$ become R-equivalent in $f^{-1}(s)(K)$.*
In particular, if X is regular, then $f^{-1}(s)$ is strongly linearly connected.

See Appendix B for a proof of Theorem 5.3.3.

5.3.4. Theorem (Gabber [12]). *If F is a field, X is a regular irreducible F-scheme of finite type and K/F a field extension, then the map*

$$\varprojlim X'(K)/R \to X(K)/R$$

has a section, which is contravariant in X and covariant in K. The limit is over the proper birational $X' \to X$.

5.3.5. Corollary. *In Theorem 5.2.4 the hypothesis of characteristic 0 is not necessary (at least for $s \in S_b^p$ in a)). Moreover, if K is algebraically closed, the hypothesis that Y is regular in a) is not necessary.*

Proof. 1) We first consider Theorem 5.2.4 a). As in the proof of Lemma 4.3.1, it suffices to deal with $K = F$. By this lemma, we have to show injectivity.

We assume that $s \in S_b^p$. Let $y_0, y_1 \in Y(F)$. Suppose that $s(y_0)$ and $s(y_1)$ are R-equivalent. We want to show that y_0 and y_1 are then R-equivalent. By definition, $s(y_0)$ and $s(y_1)$ are connected by a chain of direct R-equivalences. Applying Lemma 4.3.1, the intermediate rational points lift to $Y(F)$. This reduces us to the case where $s(y_0)$ and $s(y_1)$ are directly R-equivalent.

Let $\gamma : \mathbf{P}^1 \dashrightarrow X$ be a rational map defined at 0 and 1 such that $\gamma(i) = s(y_i)$. Applying Lemma 4.3.1 with $K = F(t)$, we get that γ lifts to a rational map $\widetilde{\gamma} : \mathbf{P}^1 \dashrightarrow Y$. Since s is proper, $\widetilde{\gamma}$ is still defined at 0 and 1. Let $y_i' = \widetilde{\gamma}(i) \in Y(F)$: then $y_i, y_i' \in s^{-1}(s(y_i))$. If F is algebraically closed, they are R-equivalent by Theorem 5.3.1, thus y_0 and y_1 are R-equivalent. If F is arbitrary but Y is regular, then we appeal to Theorem 5.3.3.

2) We now consider Theorem 5.2.4 b). By the usual graph trick, as Z is proper, we can resolve f to get a morphism

$$
\begin{array}{ccc}
 & \widetilde{Y} & \\
p \swarrow & & \searrow \tilde{f} \\
Y & & Z
\end{array}
$$

such that p is a proper birational morphism. By Theorem 5.3.4, the map $p_* : \widetilde{Y}(K)/R \to Y(K)/R$ has a section which is "natural" in p (*i.e.* when we take a finer p, the two sections are compatible). The statement follows. $\square$

5.3.6. Remark. Concerning Theorem 5.3.3, Fakhruddin pointed out that $f^{-1}(s)$ is in general not strongly linearly connected, while Gabber pointed out that $f^{-1}(s)^{\mathrm{reg}}(F)$ may be empty even if X is normal, when F is not algebraically closed. Here is Gabber's example: in dimension 2, blow-up the maximal ideal of A and then a non-rational point of the special fiber, then contract the proper transform of the special fiber. Gabber also gave examples covering Fakhruddin's remark: suppose $\dim A = 2$ and start from $X_0 =$ the blow-up of $\mathrm{Spec}\, A$ at s. Using [10], one can "pinch" X_0 so as to convert a non-rational closed point of the special fibre into a rational point. The special fibre of the resulting $X \to \mathrm{Spec}\, A$ is then a singular quotient of $\mathbf{P}_F^1$, with two R-equivalence classes. He also gave a normal example [12].

5.4. R-equivalence and R-equivalences. The main purpose of this subsection is to compute the Hom sets in the category $S_b^{-1} \mathbf{Sm}^{\mathrm{proj}}(F)$: we achieve this if char $F = 0$ in Theorem 5.4.14. Our main tool is to compare it with a rather natural category $\underline{R}^{-1} \mathbf{Sch}(F)$ that we introduce in Definition 5.4.2.

To put things in a general context, suppose we have a category $\mathcal{C}$ together with a collection of covariant functors $\Phi_\alpha : \mathcal{C} \to \mathbf{Set}$ from $\mathcal{C}$ to the category of sets. We may then introduce the collection $\underline{R} = \underline{R}(\Phi_\alpha)$ of those morphisms of $\mathcal{C}$ which are inverted by all Φ_α and consider the localisation $\underline{R}^{-1}\mathcal{C}$, so that the Φ_α induce a conservative collection of functors $\Phi_\alpha : \underline{R}^{-1}\mathcal{C} \to \mathbf{Set}$. One may then wonder whether these new functors are co-representable. In the situation studied here, we shall see that this is partly the case (Proposition 5.4.12).

5.4.1. Definition. Let $f : X \to Y$ be a morphism of F-schemes of finite type. We say that f is an *R-equivalence* if, for any extension K/F, the induced map

$$X(K)/R \to Y(K)/R$$

is bijective.

Clearly it is sufficient in this definition to let K run through all finitely generated extensions of F.

5.4.2. Definition. Let $\mathbf{Sch}(F) = \mathbf{Sch}$ be the category of separated F-schemes of finite type with all F-morphisms. We denote by $\underline{R}$ the collection of R-equivalences in $\mathbf{Sch}$ and by $\underline{R}^{-1}\mathbf{Sch}$ the corresponding localised category.

For any K/F and any $X \in \mathbf{Sch}$, let us write $\Phi_K(X)$ for $X(K)/R$. Clearly, Φ_K defines a functor from $\mathbf{Sch}$ to the category of sets $\mathbf{Set}$. Since Φ_K inverts the morphisms of $\underline{R}$, we have

5.4.3. Lemma. *The functor Φ_K induces a functor $\underline{R}^{-1}\mathbf{Sch} \to \mathbf{Set}$, still denoted by Φ_K.* $\square$

Let now $X \in \mathbf{Sch}$ be a variety, and let $K = F(X)$. For any $Y \in \mathbf{Sch}$, we have an obvious map

$$\mathbf{Sch}(X, Y) \xrightarrow{\tilde{\alpha}} \Phi_K(Y)$$
$$\tilde{\alpha}(f) = f(\eta_X)$$

which is clearly a natural transformation. Lemma 5.4.3 implies:

5.4.4. Lemma. *The natural transformation $\tilde{\alpha}$ induces a natural transformation (in Y)*

$$\underline{R}^{-1}\mathbf{Sch}(X, Y) \xrightarrow{\alpha} \Phi_K(Y).$$

Note that by the (covariant) Yoneda lemma, α is characterised by its value η_X on $1_X \in \underline{R}^{-1}\mathbf{Sch}(X, X)$.

Our aim is to prove that α is an isomorphism for proper Y when X is regular and proper, using resolution of singularities.[3] This is achieved in Proposition 5.4.12. To do this, we need to develop a series of lemmas.

5.4.5. Lemma. α *induces a map*

$$\underline{R}^{-1}\mathbf{Sch}(\eta_X, Y) := \varinjlim_{U \subset X} \underline{R}^{-1}\mathbf{Sch}(U, Y) \xrightarrow{\alpha} \Phi_K(Y).$$

Proof. It suffices to remark that α commutes with restriction to open subsets of X, which is obvious from its definition. $\square$

5.4.6. Definition. Let $\underline{S} \subset \mathbf{Sch}$ be the collection of the following morphisms:

(i) Proper birational maps $s : X \to Y$, with X and Y regular.
(ii) Projections $U \times X \to X$, where U is an open subset of $\mathbf{P}^1$.

By definition, $S_b^p \cap \mathbf{Sm} \subset \underline{S}$. Note that $\underline{S} \subset \underline{R}$: for the first type of morphisms this follows from Theorem 5.2.4 a) and for the second type it is obvious in view of (5.1). Hence the localisation functor

$$P_R : \mathbf{Sch} \to \underline{R}^{-1}\mathbf{Sch}$$

factors as a composition of two functors:

$$\mathbf{Sch} \xrightarrow{P_S} \underline{S}^{-1}\mathbf{Sch} \xrightarrow{P} \underline{R}^{-1}\mathbf{Sch}.$$

We also get a canonical functor, that we display for future reference:

$$(5.2) \qquad (S_b \cup S_h)^{-1}\mathbf{Sm}^{\mathrm{prop}} \to \underline{R}^{-1}\mathbf{Sch}$$

because $\underline{S} \cap \mathbf{Sm}^{\mathrm{prop}} = S_b \cup S_h$.

For X, Y, K as above, we have a commutative diagram

$$(5.3)$$

$$
\begin{array}{ccc}
\underline{R}^{-1}\mathbf{Sch}(X,Y) & \xrightarrow{\ \alpha\ } & \Phi_K(Y) = Y(K)/R \\
\big\uparrow{\scriptstyle P} & & \big\uparrow \\
\underline{S}^{-1}\mathbf{Sch}(X,Y) & & {\scriptstyle \lambda} \\
\big\uparrow{\scriptstyle P_S} & & \big\uparrow \\
\mathbf{Sch}(X,Y) & \longrightarrow & \mathbf{Rat}(X,Y) = Y(K)
\end{array}
$$

where λ is surjective.

---

[3]The reader willing to use Theorem 5.3.3 rather than Theorem 5.2.4 will check that we "only" need X to verify the following form of resolution of singularities: for any proper birational morphism $X' \to X$, there exists a proper birational morphism $X'' \to X'$ with X'' regular (*cf.* Definition 4.2.4).

5.4.7. *Lemma. *Suppose Y proper and let*

$$\underline{S}^{-1}\mathbf{Sch}(\eta_X, Y) = \varinjlim_{U \subset X} \underline{S}^{-1}\mathbf{Sch}(U, Y).$$

Then the obvious map $\mathbf{Rat}(X,Y) \to \underline{S}^{-1}\mathbf{Sch}(\eta_X, Y)$ factors through λ into a map $\beta_0 : \Phi_K(Y) \to \underline{S}^{-1}\mathbf{Sch}(\eta_X, Y)$, such that the diagram

$$
\begin{array}{ccc}
\underline{R}^{-1}\mathbf{Sch}(X,Y) & \xrightarrow{\;\;\alpha\;\;} & \Phi_K(Y) \\
{\scriptstyle P}\big\uparrow & & \big\downarrow{\scriptstyle \beta_0} \\
\underline{S}^{-1}\mathbf{Sch}(X,Y) & \longrightarrow & \underline{S}^{-1}\mathbf{Sch}(\eta_X, Y)
\end{array}
$$

commutes.

Proof. Let $y_0, y_1 \in Y(K)$ be two directly R-equivalent points. Let $\gamma : \mathbf{P}^1_K \dashrightarrow Y_K$ be a rational map linking them at 0 and 1. Since Y is proper, γ is a morphism. Then there is an open subset $U \subset X$ and a map $f : \mathbf{P}^1 \times U \to Y$ such that $f \circ s_i = f_i$, where s_i corresponds to the inclusion of the point i in $\mathbf{P}^1$ and $f_i : U \to Y$ corresponds to y_i. Since s_0 and s_1 are sections of the projection $\mathbf{P}^1 \times U \to U$ which belongs to $\underline{S}$, they become equal in $\underline{S}^{-1}\mathbf{Sch}(U, Y)$, hence the existence of β_0. The last assertion is obvious. $\qquad\square$

Suppose now X regular and Y proper, and let us still write $K = F(X)$. We are going to refine Lemma 5.4.7, under resolution of singularities:

5.4.8. *Lemma. *Suppose that $\operatorname{char} F = 0$ or that $\dim X \leq 2$. Then there is a map $\beta_1 : \mathbf{Rat}(X,Y) \to \underline{S}^{-1}\mathbf{Sch}(X,Y)$ such that the diagram*

$$
\begin{array}{ccc}
\underline{S}^{-1}\mathbf{Sch}(X,Y) & \longrightarrow & \underline{S}^{-1}\mathbf{Sch}(\eta_X, Y) \\
{\scriptstyle P_S}\big\uparrow & {\scriptstyle \beta_1}\nwarrow & \big\uparrow \\
\mathbf{Sch}(X,Y) & \longrightarrow & \mathbf{Rat}(X,Y)
\end{array}
$$

commutes.

Proof. Let $f : X \dashrightarrow Y$ be a rational map. Since Y is proper, the graph trick provides us with a proper birational morphism $p : \widetilde{X} \to X$ and a map $\widetilde{f} : \widetilde{X} \to Y$ resolving f. By resolution, we may assume that $\widetilde{X}$ is regular. We set

$$\beta_1(f) = \widetilde{f} \circ p^{-1}.$$

Let us show that this does not depend on the choice of $\widetilde{X}$. If $(\widetilde{X}', p', \widetilde{f}')$ is another choice, we may dominate it by a third one, which reduces us to the case where there is a map $q : \widetilde{X}' \to \widetilde{X}$. Then

$pq = p'$ and $\widetilde{f}q = \widetilde{f}'$ by Lemma 1.7.2, hence $\widetilde{f}p^{-1} = \widetilde{f}'p'^{-1}$. Thus β_1 is well-defined and clearly extends P_S: in other words, the lower triangle commutes. The commutativity of the upper one is obvious.   □

5.4.9. *Lemma. *Under the assumptions of Lemma 5.4.8, the map β_1 factors further through a map*

$$\beta : \Phi_K(Y) \to \underline{S}^{-1}\mathbf{Sch}(X, Y)$$

which lifts the map β_0 of Lemma 5.4.7.

Proof. Since λ is surjective, the lifting claim will follow from the existence of β: thus it suffices to show that two R-equivalent rational maps f_0, f_1 have the same image under β_1. We may clearly assume that f_0 and f_1 are directly R-equivalent.

Let U be an open subset of X on which f_0 and f_1 are defined, and let $p_0 : \widetilde{X} \to X$ be a proper birational morphism covering U and such that f_0 and f_1 extend to $\widetilde{f}_0$ and $\widetilde{f}_1$ on $\widetilde{X}$, with $\widetilde{X}$ regular. To show that $\beta_1(f_0) = \beta_1(f_1)$, it is sufficient to show that $P_S(\widetilde{f}_0) = P_S(\widetilde{f}_1)$.

Since f_0 and f_1 are directly R-equivalent, there is a rational map $f : \mathbf{P}^1_K \to Y_K$ linking them. Since Y is proper, f is actually a morphism. Hence, after possibly shrinking U, we get a morphism

$$f : \mathbf{P}^1 \times U \to Y$$

such that $f \circ s_i = f_i$, where $s_i : U \to \mathbf{P}^1 \times U$ is given by the inclusion of i in $\mathbf{P}^1$ $(i = 0, 1)$. We may resolve this rational map from $\mathbf{P}^1 \times \widetilde{X}$ via a proper birational morphism $p : Z \to \mathbf{P}^1 \times \widetilde{X}$: let us denote by $\widetilde{f} : Z \to Y$ the corresponding morphism.

Now let Z_i be the graph of the rational map

$$\widetilde{X} \supset U \xrightarrow{s_i} \mathbf{P}^1 \times U \to Z$$

and Z' be a regular resolution of the closure of U embedded diagonally into $Z_0 \times Z_1$: we get two diagrams (for $i = 0, 1$)

which commute thanks to Lemma 1.7.2, with p' proper birationnal.

Note that $p_0 \in S_b^p$ and $p_0 p' \in S_b^p$, hence $P_S(p')$ is invertible. Also, in the images in $\underline{S}^{-1}\mathbf{Sch}$ of the commutative diagrams

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\widetilde{s}_i} & \mathbf{P}^1 \times \widetilde{X} \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle 1_{\mathbf{P}^1} \times p_0} \\
X & \xrightarrow{s_i} & \mathbf{P}^1 \times X
\end{array}
$$

we have $P_S(s_0) = P_S(s_1)$ as s_0 and s_1 are both right-inverse to the projection $\mathbf{P}^1 \times X \to X$ which belongs to $\underline{S}$. Since $P_S((1_{\mathbf{P}^1} \times p_0)p)$ is invertible, it follows that $P_S(q_0) = P_S(q_1)$ as well. Finally:

$$
P_S(\widetilde{f}_0 p') = P_S(\widetilde{f} q_0) = P_S(\widetilde{f} q_1) = P_S(\widetilde{f}_1 p')
$$

and $P_S(\widetilde{f}_0) = P_S(\widetilde{f}_1)$, as required.   □

Summarising the above lemmas, we have refined the commutative diagram (5.3) to a commutative diagram

$$
(5.4) \qquad
\begin{array}{ccc}
\underline{R}^{-1}\mathbf{Sch}(X,Y) & \xrightarrow{\ \alpha\ } & \Phi_K(Y) \\
{\scriptstyle P}\uparrow \quad \swarrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \lambda} \\
\underline{S}^{-1}\mathbf{Sch}(X,Y) & & \\
{\scriptstyle P_S}\uparrow & & \\
\mathbf{Sch}(X,Y) & \longrightarrow & \mathbf{Rat}(X,Y)
\end{array}
$$

when Y is proper and X is regular, and under a resolution assumption on X.

5.4.10. *Lemma. β is a natural transformation on $\underline{S}^{-1}\mathbf{Sch}$ and $P\beta$ is a natural transformation on $\underline{R}^{-1}\mathbf{Sch}$.*

Proof. It suffices to show that the map β_1 of Lemma 5.4.8 is a natural transformation on $\mathbf{Sch}$, which is obvious.   □

5.4.11. *Lemma. We have $\beta\alpha P_R = P_S$ and $\alpha P\beta = 1$.*

Proof. The first identity is trivial. For the second it suffices to check that $\alpha P\beta_1 = \lambda$. Let $f \in \mathbf{Rat}(X,Y)$ be a rational map and $(\widetilde{X}, p, \widetilde{f})$ be as in the proof of Lemma 5.4.8, so that $\beta_1(f) = \widetilde{f}p^{-1}$. We then have

$$
\alpha P\beta_1(f) = \alpha P(\widetilde{f}p^{-1}) = \widetilde{f}(\eta_X) = f(\eta_X) = \lambda(f).
$$

    □

5.4.12. ***Proposition.** *Suppose X regular and proper (and Y proper). Then $P\beta\alpha = 1$ and $\beta\alpha P = 1$. In particular, α, β and P are isomorphisms.*

Proof. By Lemma 5.4.10 and Yoneda's lemma, both equalities are tested on 1_X (which is possible because X is proper); then they follow from the first identity of Lemma 5.4.11. $\square$

Summarising, we get:

5.4.13. ***Theorem.** *If X is regular and Y is proper, the maps α and $P\alpha$ of Diagram (5.3) have canonical sections $P\beta$ and β. If moreover X is proper, then α, P and β are isomorphisms. In particular, the restriction of P to the full subcategories consisting of regular proper varieties is fully faithful and Hom sets are computed as R-equivalence sets.*

5.4.14. ***Theorem.** *If $\operatorname{char} F = 0$, the functors*

$$S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}} \to S_b^{-1}\,\mathbf{Sm}^{\mathrm{prop}} \to \underline{S}^{-1}\,\mathbf{Sch} \to \underline{R}^{-1}\,\mathbf{Sch}$$

are all fully faithful, and the Hom sets in the first two categories are given by R-equivalence sets.

Proof. Indeed, by Theorem 1.6.3, projective birational morphisms from smooth varieties are cofinal among proper birational morphisms to a fixed (smooth) variety. It follows that all varieties appearing in the proofs of lemmas 5.4.8–5.4.11 may be replaced by smooth varieties. For X, Y smooth and proper (*resp.* smooth and projective), these proofs refine the map β into a map to $(S_b \cup S_h)^{-1}\,\mathbf{Sm}^{\mathrm{prop}}(X, Y)$ (*resp.* to $(S_b \cup S_h)^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(X, Y)$) via the functor (5.2), and the analogues of Lemma 5.4.10 – Prop. 5.4.12 remain valid for this refinement, with the same proofs. We get the conclusion by applying Theorem 1.7.7. $\square$

5.4.15. **Remarks.**
  (1) In Subsection 6.1, we shall give a more concrete description of the composition of R-equivalence classes.
  (2) We don't know an example where proposition 5.4.12 fails if we relax the properness condition on Y.
  (3) In view of the definition of rational chain connectedness and of Lemma 5.2.3, it would be natural to study the localisation of $\mathbf{Sch}(F)$ with respect to those morphisms which are inverted by all functors of the form Φ_K, with $K \supseteq F$ *algebraically closed*. In particular, in characteristic 0, all rationally connected varieties are equivalent to the point in this category. But we don't know

how to compute Hom sets, say, between two smooth projective varieties. It is tempting to consider the sets, for two smooth proper varieties X, Y

$$(Y(\overline{F(X)})/R)^G$$

where $\overline{F(X)}$ is an algebraic closure of $F(X)$ and G is its absolute Galois group. However, we don't know how to compose such Galois-invariant R-equivalence classes.

6. More on R-equivalence classes

In this section, we come back to the previous results for a less abstract and more detailed viewpoint. In Subsection 6.1, we extend the composition of R-equivalence classes from regular proper varieties to somewhat more general situations. In Subsection 6.2, we study the surjection $\mathbf{place}(F(X), F(Y))/\,\mathrm{h} \to Y(F(X))/R$ for two smooth projective varieties, and say as much as we can about its fibres.

6.1. **Composing R-equivalence classes.** As a by-product of Theorem 5.4.13, one gets for three regular proper varieties X, Y, Z over a field of characteristic 0 a composition law

$$Y(F(X))/R \times Z(F(Y))/R \to Z(F(X))/R$$

which is by no means obvious.

We are going to generalise this composition law a little bit. We still assume char $F = 0$. To start with let X, Y be two varieties, with Y regular, and Z a proper scheme. We then get a pairing

$$(6.1) \qquad \underline{R}^{-1}\,\mathbf{Sch}(X, Y) \times Z(F(Y))/R \to Z(F(X))/R$$
$$(f, z) \mapsto f^*z = \alpha(P\beta(z) \circ f).$$

This is easily seen to pass to $\underline{R}^{-1}\,\mathbf{Sch}(\eta_X, Y)$. In particular, any rational map $f : X \dashrightarrow Y$ induces an action $z \mapsto f^*z$. The same kind of arguments as before give:

6.1.1. ***Lemma.** *a) Suppose moreover that Y is proper. Then f^* only depends on the R-equivalence class of f. In particular, (6.1) factors through a composition*

$$Y(F(X))/R \times Z(F(Y))/R \to Z(F(X))/R.$$

b) Let $X, Y, Z, T \in \mathbf{Sch}$ with X a variety, Y, Z smooth and proper and T proper. Then the composition law of a) is associative in an obvious sense.

(In b), the associativity follows from a straightforward calculation using Proposition 5.4.12.)

We now relax the hypothesis "proper" on Y in the previous lemma.

6.1.2. *Proposition. *Assume* char $F = 0$.
a) Suppose $X, Y, Z \in \mathbf{Sch}$ *are such that* X *is a variety,* Y *is a regular variety and* Z *is proper. Then there is a composition map*

$$(6.2) \qquad Y(F(X))/R \times Z(F(Y))/R \to Z(F(X))/R,$$

denoted by $z \circ y$ *for* $z \in Z(F(Y))/R$ *and* $y \in Y(F(X))/R$. *This composition law is the same as that in Lemma 6.1.1 a) when* Y *is proper.*
b) Let $X, Y, Z, T \in \mathbf{Sch}$ *be such that* X *is a variety,* Y *is a regular variety,* Z *is a regular proper variety and* T *is proper. Then the above composition law is associative.*

Proof. a) Let $K = F(X)$ and $L = F(Y)$. Given $z \in Z(L)$, defining a rational map $f_z \in \mathbf{Rat}(Y, Z)$, let $\beta(z) : Y(K)/R \to Z(K)/R$ be the map induced by f_z via Theorem 5.2.4 b). Let $z' \in Z(L)$ be R-equivalent to z: we claim that $\beta(z) = \beta(z')$. For this, we may assume that z and z' are directly R-equivalent. Let $\varphi : \mathbf{P}^1_L \dashrightarrow Z_L$ be a rational map (actually a morphism) such that $\varphi(0) = z$, $\varphi(1) = z'$, and let $\widetilde{z} \in Z(M)$ be the image by φ of the generic point of $\mathbf{P}^1_L$, where $M = L(t)$. Then $\widetilde{z}$ induces a map $\beta(\widetilde{z}) : (Y \times \mathbf{P}^1)(K)/R \to Z(K)/R$, corresponding to the rational map $Y \times \mathbf{P}^1 \dashrightarrow Z$ extending φ. Via the isomorphisms $Y(K)/R \xrightarrow{\sim} (Y \times \mathbf{P}^1)(K)/R$ induced by the inclusions of 0 and 1 into $\mathbf{P}^1$, we have $\beta(\widetilde{z}) = \beta(z) = \beta(z')$.

For $y \in Y(K)/R$ and $z \in Z(L)/R$, we may now define

$$z \circ y = \beta(z)(y).$$

When Y is proper, it clearly is the map defined in Lemma 6.1.1 a).

b) Given $y \in Y(F(X))$, $z \in Z(F(Y))$ and $t \in T(F(Z))$, we need to show that $t \circ (z \circ y) = (t \circ z) \circ y$. We have

$$t \circ (z \circ y) = t \circ (\beta(z)(y)) = \beta(t)\beta(z)(y)$$

and

$$(t \circ z) \circ y = \beta(t \circ z)(y).$$

Choose a morphism $Z' \to T$ "resolving" t, with $Z' \to Z$ proper birational. We may similarly "resolve" the rational map $Y \dashrightarrow Z'$ defined by z by a proper birational morphism $Y' \to Y$. We may further assume that Z', and then Y', is regular. This shows that $\beta(t \circ z) = \beta(t)\beta(z)$, hence $\circ$ is associative. $\qquad\square$

6.1.3. **Remark.** One can replace Theorem 5.2.4 by Corollary 5.3.5 in the proof of Proposition 6.1.2 a), thus removing the resolution of singularities assumption. On the other hand, adapting the proof that $\beta(t \circ z) = \beta(t)\beta(z)$ in b) seems to require more work.

6.2. **Comparing morphisms in S_r^{-1} place and $S_b^{-1} \mathbf{Sm}^{\mathrm{proj}}$.** Let X and Y be two smooth projective varieties over F with function fields K and L respectively. Consider the map

$$c : \mathbf{place}(F)(K, L) \to X(L)$$
$$\lambda \mapsto c(\lambda)$$

where $c(\lambda)$ is the centre of the place λ considered as an element of $X(L)$ (see subsection 1.3).

6.2.1. ***Lemma.** *a) The above map is surjective and induces a surjection*

$$\bar{c} : \mathbf{place}(F)(K, L)/\mathrm{h} \longrightarrow\!\!\!\!\!\rightarrow X(L)/R$$

where h *denotes homotopy equivalence.*
b) Let $\lambda_0, \lambda_1 \in \mathbf{place}(F)(K, L)$ be such that $c(\lambda_0)$ is directly R-equivalent to $c(\lambda_1)$ in $X(L)$. Then there exist $\mu_0, \mu_1 \in \mathbf{place}(F)(K, L)$ such that

  (i)   *μ_0 and μ_1 are elementarily homotopic;*
  (ii) *$c(\mu_0) = c(\lambda_0)$ and $c(\mu_1) = c(\lambda_1)$ in $X(L)$.*

6.2.2. **Remark.** Part b) of this lemma tries to elucidate the fibres of the surjection in part a); see also Theorem 6.2.3 below.

Proof. a) As X is smooth, the surjectivity follows from Lemma 3.2.2 (given $P \in X(L)$, spread it to an F-morphism $\theta_P : V \to X$ where V is open in Y). For the second assertion, it clearly suffices to prove that if λ_0 and λ_1 are elementarily homotopic, then their centres in $X(L)$ are R-equivalent. Let $\nu : K \rightsquigarrow L(t)$ be a place such that $s_i \circ \nu = \lambda_i$ for $i = 0$, 1. Let i_0 and i_1 be the morphisms from Y to $Y \times_F \mathbf{P}^1$ which are respectively 0 and 1 on the second co-ordinate. Then i_0 and i_1 are compatible respectively with s_0 and s_1, whose centres are clearly R-equivalent in $Y \times_F \mathbf{P}^1$. By Lemma 3.2.4, the graph trick and Theorem 1.6.3, there is a smooth projective variety Z over F with a birational morphism to $Y \times_F \mathbf{P}^1$ and an F-morphism $g : Z \to X$ such that g is compatible with ν. But R-equivalence is invariant under birational maps, and birational morphisms being compatible with the identity place, we see that the centres c_0 and c_1 of s_0 and s_1 in Z are R-equivalent. As g is compatible with ν and $s_i \circ \nu = \lambda_i$, it is clear that the centres of λ_0 and λ_1 are $g(c_0)$ and $g(c_1)$, hence are R-equivalent in $X(L)$.

b) The centres $c(\lambda_0)$ and $c(\lambda_1)$ induce F-morphisms

$$\theta_0 : V_0 \to X, \quad \theta_1 : V_1 \to X,$$

where V_0 and V_1 are open sets in Y, such that θ_i maps the generic point of Y to $c(\lambda_i)$ for $i = 1, 2$ respectively. Further, by Corollary 3.1.3, θ_i is compatible with λ_i for $i = 0, 1$ respectively. By R-equivalence, we have an L-morphism $g : \mathbf{P}_L^1 \to X_L$ such that $g(0) = c(\lambda_0)$ and $g(1) = c(\lambda_1)$. Spreading this morphism, we get an F-morphism $g' : V' \times_F \mathbf{P}^1 \to V' \times_F X$ where V' is open in Y. Further the images of the generic point of $V' \times \{i\}$ under g' for $i = 0$, 1 are respectively the centres of λ_i. By Corollary 3.1.3 we get a place $\nu : K \rightsquigarrow L(t)$ compatible with $p_2 \circ g'$, where $p_2 : V' \times X \to X$ is the second projection. Shrinking the open sets if necessary, we may assume that $V_0 = V_1 = V'$. The natural inclusions $i_0 : V' \to V' \times_F \mathbf{P}^1$ and $i_1 : V' \to V' \times_F \mathbf{P}^1$ which map respectively to 0 and 1 in the second co-ordinate are clearly compatible with the "specialising" places s_0 and s_1 respectively. Then $\mu_i = s_i \circ \nu$ and λ_i have the same centre for $i = 0, 1$. $\qquad\square$

Let X be a smooth projective model of K and λ a place $K \rightsquigarrow L$. If Y is a smooth projective model of L, then by Lemma 3.2.4, there is a unique rational map $Y \dashrightarrow X$ compatible with λ. This rational map is determined by the centre $c(\lambda) \in X(L)$. Let $\mu : K \rightsquigarrow L$ be another place. Write

$$\lambda \, \mathrm{C}_X \, \mu$$

if $c(\lambda) = c(\mu) \in X(L)$, or equivalently if λ and μ are compatible with the same rational map to X. The relation C_X has the following permanence properties:

- C_X is an equivalence relation.
- If $\lambda \, \mathrm{C}_X \, \mu$ and $p : X \to X'$ is in S_b, then $\lambda \, \mathrm{C}_{X'} \, \mu$.
- If $\lambda \, \mathrm{C}_X \, \mu$ and $\nu : L \rightsquigarrow M$ is another place, then $(\nu \circ \lambda) \, \mathrm{C}_X \, (\nu \circ \mu)$. To see this, choose a model Y of L such that the common rational map $f : Y \dashrightarrow X$ compatible to λ and μ is in fact a morphism; if Z is a smooth model of M and $g : Z \dashrightarrow Y$ is a rational map compatible with ν, then $f \circ g$ is defined and, by Proposition 3.1.4, it is compatible with $\nu \circ \lambda$ and $\nu \circ \mu$.

On the other hand, if $\lambda \, \mathrm{C}_X \, \mu$ and $\nu : M \rightsquigarrow K$ is another place, then it is not true in general that $(\lambda \circ \nu) \, \mathrm{C}_Z \, (\mu \circ \nu)$ for a suitable model Z of M. As a counterexample, take $L = F$, $K = F(t_1, t_2)$, $M = F(t)$, for λ and μ the composite places

$$\lambda : (t_1, t_2) \to (0, t) \to (0, 0)$$
$$\mu : (t_1, t_2) \to (t, 0) \to (0, 0)$$

and for ν the place given by

$$\nu(t) = \frac{t_1}{t_1 - t_2}.$$

Under the functor

$$\Phi^b : \mathbf{place}(F)^{\mathrm{op}} \xrightarrow{U^{-1}} S_b^{-1} \mathbf{Sm}^{\mathrm{proj}} \mathrm{P}(F) \to S_b^{-1} \mathbf{Sm}^{\mathrm{proj}}(F)$$

from Theorem 4.2.5, $\lambda\, \mathrm{C}_X\, \mu$ implies $\Phi^b(\lambda) = \Phi^b(\mu)$: this is obvious. Hence, if we denote by C the equivalence relation on $\mathbf{place}(F)$ generated by all C_X, Φ^b factors through a functor

$$\bar{\Phi}^b : \mathbf{place}(F)^{\mathrm{op}}/\,\mathrm{C} \to S_b^{-1} \mathbf{Sm}^{\mathrm{proj}}(F).$$

We have the following diagram of categories and functors, in which all functors are full and [essentially] surjective:

(6.3)
$$
\begin{array}{ccc}
\mathbf{place}(F)^{\mathrm{op}} & \longrightarrow & \mathbf{place}(F)^{\mathrm{op}}/\,\mathrm{C} \\
\downarrow{\scriptstyle S_r^{-1}} & & \downarrow{\scriptstyle \bar{\Phi}^b} \\
S_r^{-1}\,\mathbf{place}(F)^{\mathrm{op}} & \xrightarrow{\Phi^r} & S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) \\
\downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} \\
\mathbf{place}(F)/\mathrm{h} & \xrightarrow{\bar{c}} & \mathbf{Sm}^{\mathrm{proj}}(F)/R
\end{array}
$$

where the (unmarked) top left horizontal functor is the natural projection, the (unmarked) top right vertical functor is the composition $S_r^{-1} \circ \Phi^b$, the two vertical isomorphisms of categories are those of Propositions 5.1.3 and Theorem 5.4.14 respectively, and $\bar{c}$ is induced by the maps of Lemma 6.2.1 a).

From Lemma 6.2.1 b), we get:

6.2.3. ***Theorem.** *In* (6.3)*, the top square is cocartesian, that is, the functors induce cocartesian squares of sets on morphisms.* □

7. Examples, applications and open questions

In this section, we put together some concrete applications of the above results and list some open questions.

7.1. **Places, R-equivalence and birational functors.** To start with, here is a more concrete reformulation of part of Theorem 5.4.14:

7.1.1. ***Corollary.** *Suppose F of characteristic 0. Let*

$$P : \mathbf{Sm}^{\mathrm{proj}}(F) \to \mathcal{A}$$

be a functor to some category $\mathcal{A}$. Suppose that P is a birational functor. Then R-equivalence classes act on P: if X, Y are two smooth projective varieties, any class $x \in X(F(Y))/R$ induces a morphism $x_* : P(Y) \to P(X)$. This assignment is compatible with the composition of R-equivalence classes given by Theorem 5.4.14.
In particular, for two morphisms $f, g : X \to Y$, $P(f) = P(g)$ as soon as $f(\eta_X)$ and $g(\eta_X)$ are R-equivalent.

In fact, Theorem 5.4.14 further says that R-equivalence is "universal" among birational functors.

As another application, we get:

7.1.2. *Corollary (*cf.* Madore [31, Prop. 3.1]). *Suppose F of characteristic 0. Let $Y \in \mathbf{Sch}(F)$ be proper, and let $\lambda : K \rightsquigarrow L$ be an F-place between function fields. Then λ induces a map*

$$\lambda_* : Y(K)/R \to Y(L)/R.$$

Proof. Consider λ as a morphism in $\mathbf{place_{Sm^{prop}}}(F)$. Pick a smooth proper model X of K and a smooth proper model Z of L. Composing the first functor of Corollary 4.4.3 with (5.2), we get a morphism $\lambda^* : Z \to X$ in $\underline{R}^{-1}\mathbf{Sch}$, hence a map

$$\lambda_* : \underline{R}^{-1}\mathbf{Sch}(X, Y) \to \underline{R}^{-1}\mathbf{Sch}(Z, Y).$$

Now, by Theorem 5.4.13, these two Hom sets are isomorphic to the corresponding sets of R-equivalence classes. (We leave it to the reader to check that this map is indeed the one induced by the map $\lambda_* : Y(K) \to Y(L)$ given by the valuative criterion of properness.) $\qquad\square$

Let us point out the difference between Madore's result and proof and ours. Madore's theorem concerns a projective scheme $\mathcal{Y}$ over a discrete valuation ring $\mathcal{O}$ with quotient field K and residue field L. Denoting by Y the generic fibre of $\mathcal{Y}$ and $\bar{Y}$ its special fibre, he proves that the map $\mathcal{Y}(\mathcal{O}) = Y(K) \to \bar{Y}(L)$ factors through R-equivalence by an explicit computation in the projective space. Thus his result does not cover arbitrary proper schemes and places, but on the other hand it does not require any resolution of singularities and is valid "in families", even in unequal characteristic.

7.2. Algebraic groups and R-equivalence. We have the following obvious corollary of the existence of the composition law on R-equivalence classes (§6.1):

7.2.1. *Corollary. *Let X be a smooth proper variety with function field K. Then $X(K)/R$ has a structure of a monoid with η_X as the identity element.* $\qquad\square$

As a special case of the above, we consider a connected algebraic group G defined over F. Recall that for any extension K/F, the set $G(K)/R$ is in fact a group. Let $\bar{G}$ denote a smooth compactification of G over F (we assume that there is one). It is known (P. Gille, [14]) that the natural map $G(F)/R \to \bar{G}(F)/R$ is an isomorphism if F has characteristic zero and G is reductive.

Let K denote the function field $F(G)$. By the above corollary, there is a composition law $\circ$ on $\bar{G}(K)/R$. On the other hand, the multiplication morphism

$$m : G \times G \to G$$

considered as a rational map on $\bar{G} \times \bar{G}$ induces a product map (Corollary 5.3.5)

$$\bar{G}(K)/R \times \bar{G}(K)/R \to \bar{G}(K)/R$$

which we denote by $(g, h) \mapsto g \cdot h$; this is clearly compatible with the corresponding product map on $G(K)/R$ obtained using the multiplication homomorphism on G. Thus we have two composition laws on $\bar{G}(K)/R$.

The following lemma is a formal consequence of Yoneda's lemma:

7.2.2. Lemma. *Let* $g_1, g_2, h \in \bar{G}(K)/R$. *Then we have* $(g_1 \cdot g_2) \circ h = (g_1 \circ h) \cdot (g_2 \circ h)$. $\qquad\square$

In particular, let us take $G = SL_{1,A}$, where A is a central simple algebra over F. It is then known that $G(K)/R \simeq SK_1(A_K)$ for any function field K. If char $F = 0$, we may use Gille's theorem and find that, for $K = F(G)$, $SK_1(A_K)$ admits a second composition law with unit element the generic element, which is distributive on the right with respect to the multiplication law. However, it is not distributive on the left in general:

Note that the natural map $\mathrm{Hom}(\mathrm{Spec}\, F, \bar{G}) = \bar{G}(F)/R \to \bar{G}(K)/R = \mathrm{Hom}(\bar{G}, \bar{G})$ is split injective, a retraction being induced by the unit section $\mathrm{Spec}\, F \to G \to \bar{G}$. Now let $g \in G(F)$; for any $\varphi \in G(K) = \mathbf{Rat}(G, G)$, we clearly have $[g] \circ [\varphi] = [g]$. In particular, $[g] \circ ([\varphi] \cdot [\varphi']) \neq ([g] \circ [\varphi]) \cdot ([g] \circ [\varphi'])$ unless $[g] = 1$. (This argument works for any group object in a category with a final object.)

7.3. Strongly linearly connected smooth proper varieties.
One natural question that arises is the following: characterise morphisms $f : X \to Y$ between smooth proper varieties which become invertible in the category $S_b^{-1} \mathbf{Sm}^{\mathrm{prop}}$. Here we shall study this question only in the simplest case, where $Y = \mathrm{Spec}\, F$.

7.3.1. *Theorem. a) Let X be a smooth proper variety over F, which is assumed to be of characteristic 0. Consider the following conditions:*

(1) *$p : X \to \operatorname{Spec} F$ is an isomorphism in $S_b^{-1} \mathbf{Sm}^{\mathrm{prop}}$.*
(2) *p is an isomorphism in $S_r^{-1} \mathbf{Sm}^{\mathrm{prop}}$.*
(3) *For any extension E/F, $X(E)/R$ has one element (i.e. X is strongly linearly connected according to Definition 5.2.1 d)).*
(4) *Same, for E/F of finite type.*
(5) *$X(F) \neq \emptyset$ and $X(K)/R$ has one element for $K = F(X)$.*
(6) *$X(F) \neq \emptyset$ and, given $x_0 \in X(F)$, there exists a chain of rational curves $(f_i : \mathbf{P}_K^1 \to X_K)_{i=1}^n$ such that $f_1(0) = \eta_X$, $f_{i+1}(0) = f_i(1)$ and $f_n(1) = x_0$. Here $K = F(X)$ and η_X is the generic point of X.*
(7) *Same as (6), but with $n = 1$.*

Then $(1) \iff (2) \iff (3) \iff (4) \iff (5) \iff (6) \Leftarrow (7)$.
b) If X satisfies Conditions $(1) - (6)$ and is projective, it is rationally connected.

Proof. a) $(1) \Rightarrow (2)$ is trivial and the converse follows from Theorem 1.7.7. Thanks to Theorem 5.4.14, $(2) \iff (4)$ is an easy consequence of the Yoneda lemma. The implications $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Leftarrow (7)$ are trivial and $(4) \Rightarrow (3)$ is easy by a direct limit argument. To see $(6) \Rightarrow (1)$, note that by Theorem 5.4.14 (6) implies that $1_X = x_0 \circ p$ in $S_b^{-1} \mathbf{Sm}^{\mathrm{prop}}(X, X)$, hence p is an isomorphism.

b) This is well-known, since strongly linearly connected implies linearly connected, which is equivalent to rationally chain-connected (see Remark 5.2.2), the latter being equivalent to rationally connected for smooth projective varieties in characteristic 0 [27], [8, p. 107, Cor. 4.28]. □

7.3.2. Remarks. a) The example of an anisotropic conic shows that, in (5), the assumption $X(F) \neq \emptyset$ does not follow from the next one.

b) J.-L. Colliot-Thélène pointed out that $(6) \Rightarrow (3)$ actually holds in any characteristic provided X is projective, thanks to Madore's theorem explained after Corollary 7.1.2. Indeed, we may clearly assume $E = F$. Let $x \in X(F)$, A its local ring and take the same notation as in the proof of Lemma 3.2.2. For $i \in [0, d-1]$, let $y_i \in X$ be the point image of $\operatorname{Spec} F_i$, so that $y_{d-1} = \eta_X$, $y_0 = x$ and y_{i-1} is a specialisation of y_i for each i. Applying Madore's theorem, we find inductively that, for all i, y_i is R-equivalent to x_0 over F_i, hence x is R-equivalent to x_0 over F.

This suggests an alternative way to define the composition of R-equivalence classes, at least for regular projective varieties. Here is

how it should work (we place ourselves in the framework of Proposition 6.1.2): consider 3 F-varieties X, Y, Z with X integral, Y regular and Z projective; let $K = F(X), L = F(Y)$ and finally $y \in Y(K)/R$, $z \in Z(L)/R$. Then y lifts to a rational map $f : X \dashrightarrow Y$. By Lemma 3.2.2, there exists a place $\lambda : L \rightsquigarrow K$ compatible with f, and moreover, λ may be chosen as a composition of a trivial place and a sequence of discrete valuations of rank 1. By Madore's theorem, one may define $\lambda_* z \in Z(K)/R$. One should then prove that $\lambda_* z$ only depends on y.

c) Theorem 7.3.1 b) implies that if (6) is true, then (7) is true at least over the algebraic closure of K [8, p. 107, cor. 4.28]. It is not clear whether (6) $\iff$ (7) on the nose. What follows is a variation on the ideas of Kollár.

7.3.3. Definition. Let X, Y be two F-varieties, $E = F(Y)$ and $x \in X(E)$. Let $\widetilde{x} : Y \dashrightarrow X$ be the corresponding rational map. We set

$$\dim_F(x) = \dim \overline{\widetilde{x}(Y)}.$$

We say that x is *general* if $\dim_F(x) = \dim(X)$ and the corresponding extension $E/F(X)$ is separably generated.

Recall [8, p. 90, Def. 4.5 ff] that a rational curve $f : \mathbf{P}^1 \to X$ on a smooth variety is *free* (*resp. very free*) if f^*T_X (*resp.* $f^*T_X \otimes \mathcal{O}_{P^1}(-1)$) is generated by its global sections.

7.3.4. Proposition. *Let X be a smooth proper F-variety and $f : \mathbf{P}^1_K \to X_K$ a rational curve, where $K = F(X)$.*
a) There exists an open subset $U \subseteq X$ such that f extends to a morphism $\varphi : U \times \mathbf{P}^1 \to X$.
b) Let $\mathcal{F}$ be the cokernel of the composition

$$(7.1) \qquad \beta : \varphi^*\Omega_X \to \Omega_{U \times \mathbf{P}^1} \simeq p^*\Omega_U \oplus q^*\Omega_{\mathbf{P}^1} \to p^*\Omega_U$$

where p, q are the projections of $U \times \mathbf{P}^1$ on U and $\mathbf{P}^1$. The following conditions are equivalent:

(1) *$\mathcal{F}$ is torsion.*
(2) *$\mathrm{Supp}(\mathcal{F}) \neq U \times \mathbf{P}^1$.*
(3) *There exists $t \in \mathbf{P}^1(K)$ such that $f(t)$ is general.*
(4) *There exists a proper closed subset $B \subset \mathbf{P}^1_K$ such that $f(t)$ is general for any $t \notin B(K)$.*

They imply that f is free.
c) Suppose that F is perfect and that $(1) - (4)$ are verified. If $r = h^0(\mathbf{P}^1_K, f^*\Omega_{X_K})$, then for any $t \in \mathbf{P}^1(K)$, $\dim_F f(t) \geq r$.
In particular, if there exists $t \in \mathbf{P}^1(K)$ such that $f(t) \in \mathbf{P}^1(F)$ (note that this is true in Condition (7) of Theorem 7.3.1), then f is very free.

Proof. a) is clear. b) (1) $\iff$ (2) is obvious. Let $t \in \mathbf{P}^1(K)$. Spread t to $\widetilde{t} : U' \to \mathbf{P}^1$ (U' open subset of U). Up to shrinking U, we may assume $U' = U$. Pulling back (7.1), we get a composition

$$\gamma_{\widetilde{t}}^* \varphi^* \Omega_X \to \gamma_{\widetilde{t}}^* (p^* \Omega_U \oplus q^* \Omega_{\mathbf{P}^1}) \to \Omega_U$$

where $\gamma_{\widetilde{t}} : U \to U \times \mathbf{P}^1$ is the graph map associated to $\widetilde{t}$. Note that $p\gamma_{\widetilde{t}} = 1_U$, hence the second map can be identified with the projection onto the first summand of the middle term. It follows that the cokernel of the composition is $\gamma_{\widetilde{t}}^* \mathcal{F}$. Its vanishing means that $\varphi \circ \gamma_{\widetilde{t}}$ is unramified, i.e. that $f(t)$ is general. Thus: $f(t)$ is general $\iff \gamma_{\widetilde{t}}(U) \not\subset \mathrm{Supp}(\mathcal{F})$.

This shows that (3) $\Rightarrow$ (2). Clearly, (4) $\Rightarrow$ (3). Finally, assuming (2), $\mathrm{Supp}(\mathcal{F})$ is a proper closed subset, hence can contain only finitely many irreducible subvarieties of $X \times \mathbf{P}^1$ of codimension 1, so that (2) $\Rightarrow$ (4).

It remains to see that f is free under these conditions. Let $j : \mathbf{P}_K^1 \to \mathbf{P}^1 \times U$ be the immersion corresponding to the generic point of U. Pulling back (7.1) by j, we get an exact sequence

$$f^* \Omega_{X_K} \xrightarrow{\beta_K} j^* p^* \Omega_U \to j^* \mathcal{F} \to 0.$$

The sheaf $j^* \mathcal{F}$ is torsion, and since $f^* \Omega_{X_K}$ and $j^* p^* \Omega_U$ are locally free of the same rank, the first map is injective. Note that $j^* p^* \Omega_U \simeq \mathcal{O}_{P_K^1}^d$, where $d = \dim U$. This implies that the twists of $f^* \Omega_{X_K}$ are all ≤ 0, which by definition means that f is free.

c) Let $t \in \mathbf{P}^1(K)$ and let $\mathcal{F}_t$ be the fibre of $j^* \mathcal{F}$ at t. Then $\dim_K \mathcal{F}_t \leq d - r$: indeed, if $f^* \Omega_{X_K} \simeq \mathcal{O}_{\mathbf{P}_K^1}^r \oplus \mathcal{G}$, the restriction of β_K to $\mathcal{O}_{\mathbf{P}_K^1}^r$ is given by an injective matrix with constant coefficients, hence remains injective after taking the fibre. On the other hand, with the notation of the proof of a), the map β of (7.1) factors through a pull-back of Ω_W where W the closure of the image of $\varphi \circ \gamma_{\widetilde{t}} : U' \to X$. Since F is perfect, this sheaf is generically locally free of rank $\dim W$. Thus $\dim_K \mathcal{F}_t \geq d - \dim W$, which gives c). $\qquad\square$

7.4. Retract-rational varieties.

Recall that, following Saltman, X (smooth but not necessarily proper) is *retract-rational* if it contains an open subset U such that U is a retract of an open subset of $\mathbf{A}^n$. When F is infinite, this includes the case where there exists Y such that $X \times Y$ is rational, as in [5, Ex. A. pp. 222/223].

We have a similar notion for function fields:

7.4.1. **Definition.** A function field K/F is *retract-rational* if there exists an integer $n \geq 0$ and two places $\lambda : K \rightsquigarrow F(t_1, \ldots, t_n)$, $\mu : F(t_1, \ldots, t_n) \rightsquigarrow K$ such that $\mu\lambda = 1_K$.

Note that this forces λ to be a trivial place (*i.e.* an inclusion of fields). Using Lemma 3.2.2, we easily see that X is retract-rational if and only if $F(X)$ is retract-rational.

7.4.2. **Proposition.** *If X is a retract-rational smooth variety, then $F(X) \simeq F$ in $S_r^{-1} \mathbf{place}(F)^{\mathrm{op}}$. If moreover X is proper and F is infinite, then X verifies Condition (7) of Theorem 7.3.1 for a Zariski dense set of points x_0.*

Proof. The first statement is obvious by Yoneda's lemma. Let us prove the second: by hypothesis, there exist open subsets $U \subseteq X$ and $V \subseteq \mathbf{A}^n$ and morphisms $f : U \to V$ and $g : V \to U$ such that $gf = 1_U$. This already shows that $U(F)$ is Zariski-dense in X. Let now $x_0 \in U(F)$, and let $K = F(X)$. Consider the straight line $\gamma : \mathbf{A}^1_K \to \mathbf{A}^n_K$ such that $\gamma(0) = f(x_0)$ and $\gamma(1) = f(\eta_X)$. Then $g \circ \gamma$ links x_0 to η_X, as desired. $\square$

To summarise this discussion, let us record:

7.4.3. **Corollary.** *We have the following implications for a smooth projective variety X over an infinite F (proper is enough *in characteristic 0): X is retract-rational $\Rightarrow$ X is strongly linearly connected $\Rightarrow$ X is separably rationally connected.*

Proof. In characteristic 0, use Theorem 7.3.1 and Proposition 7.4.2 to prove the first implication; if X is projective, replace Theorem 7.3.1 by Remark 7.3.2 b). The second implication follows from Kollár's work [27]. $\square$

7.4.4. **Remark.** In characteristic 0, if X is a smooth compactification of a torus, then it verifies Conditions (1) – (6) of Theorem 7.3.1 if and only if it is retract-rational, by [6, Prop. 7.4] (*i.e.* the first implication in the previous corollary is an equivalence for such X). This may also be true by replacing "torus" by "connected reductive group": at least it is so in many special cases, see [15, Th. 7.2 and Cor. 5.10].

Nevertheless, Proposition 7.4.2 suggests that in Corollary 7.4.3 the first implication may be strict, *i.e.* there are smooth proper varieties which satisfy the equivalent conditions (1)–(6) of Theorem 7.3.1 but are not retract-rational. To start with, do conditions (1)–(6) of Theorem 7.3.1 imply that F-rational points are dense in X? Here is another hint in this direction.

7.4.5. Lemma. *Let F be infinite, and suppose that the smooth proper variety X over F is retract rational. Then there exists a nonempty open subset $U \subseteq X$ such that, for every extension K/k and every nonempty open subset $V \subseteq U_K$, the set $V(K)/R$ has 1 element. In particular, any open subset of U is strongly linearly connected.*

Proof. This is obvious by taking for U a retract of an open subset of $\mathbf{A}^n$. □

7.4.6. Question. In Lemma 7.4.5, is it true that all open subsets of X are strongly linearly connected?

7.4.7. Remark (Colliot-Thélène)**.** If X is a separably rationally connected smooth projective variety, then any open subset of X is linearly connected, thanks to the analogue of property (7) of Theorem 7.3.1 (*cf.* Remark 7.3.2 c)).

7.5. S_r-local objects. In this rather disappointing subsection, we show that there are not enough of these objects. They are the exact opposite of rationally connected varieties.

7.5.1. Definition. A proper F-variety X is *nonrational* if it does not carry any nonconstant rational curve (over the algebraic closure of F), or equivalently if the map

$$X(\bar{F}) \to X(\bar{F}(t))$$

is bijective.

7.5.2. Lemma. *a) Nonrationality is stable by product and by passing to closed subvarieties.*
b) Curves of genus > 0 and torsors under abelian varieties are nonrational.
c) Nonrational smooth projective varieties are minimal in the sense that their canonical bundle is nef.

Proof. a) and b) are obvious; c) follows from the Miyaoka-Mori theorem ([32], see also [28, Th. 1.13] or [8, Th. 3.6]). □

On the other hand, an anisotropic conic is not a nonrational variety.
Smooth nonrational varieties are the local objects of $\mathbf{Sm}^{\mathrm{proj}}(F)$ with respect to S_r in the sense of Definition 1.1.1: the following lemma is valid in all characteristics.

7.5.3. Lemma. *a) A proper variety X is nonrational if and only if, for any morphism $f : Y \to Z$ between smooth varieties such that $f \in S_r$, the map*

$$f^* : Map(Z, X) \to Map(Y, X)$$

is bijective.
b) A smooth proper nonrational variety X is stably minimal in the
following sense: any morphism in S_r with source X is an isomorphism.

Proof. a) Necessity is clear (take $f : \mathbf{P}^1 \to \operatorname{Spec} F$). For sufficiency, f^* is clearly injective since f is dominant, and we have to show surjectivity. We may assume F algebraically closed. Let U be a common open subset to Y and $Z \times \mathbf{P}^n$ for suitable n. Let $\psi : Y \to X$. By [28, Cor. 1.5] or [8, Cor. 1.44], $\psi_{|U}$ extends to a morphism φ on $Z \times \mathbf{P}^n$. But for any closed point $z \in Z$, $\varphi(\{z\} \times \mathbf{P}^1)$ is a point, where $\mathbf{P}^1$ is any line of $\mathbf{P}^n$. Therefore $\varphi(\{z\} \times \mathbf{P}^n)$ is a point, which implies that φ factors through the first projection.

b) immediately follows from a).   □

7.5.4. Lemma. *If X is nonrational, it remains nonrational over any extension K/F.*

Proof. It is a variant of the previous one: we may assume that F is algebraically closed and that K/F is finitely generated. Let $f : \mathbf{P}^1_K \to X_K$. Spread f to a U-morphism $\widetilde{f} : U \times \mathbf{P}^1 \to U \times X$ and compose with the second projection. Any closed point $u \in U$ defines a map $f_u : \mathbf{P}^1 \to X$, which is constant, hence $p_2 \circ \widetilde{f}$ factors through the first projection, which implies that f is constant.   □

7.6. The function field of BG. Here F is any field. Let G be a linear algebraic group over F. Morel and Voevodsky have introduced in [34, §4.2] an object $B_{et}G \in \mathcal{H}(F)$, where $\mathcal{H}(F)$ is the $\mathbf{A}^1$-homotopy category of schemes defined in *loc. cit.* Here we associate to G an object $F(BG) \in S_r^{-1}\mathbf{field}(F)$: this notion is informally denoted by $F(G)$ in Saltman's paper [39]. It would take us too far to justify in what sense $F(BG)$ may be considered as the "function field" of $B_{et}G$, so we prefer to do like Totaro, who in [42] defined the Chow ring of BG without defining BG. We shall follow his method to achieve this:

Let V a faithful linear representation of G; we assume that V contains a proper closed subset S such that G acts freely on $V - S$, the quotient $(V - S)/G$ exists as a quasi-projective variety and the projection $V - S \to (V - S)/G$ is a G-torsor: this can be achieved as in [42, Rk. 1.4]. In the special case where G is reductive, this assumption is verified if the action of G is "almost free" in the sense of [7, Def. 2.13], *cf. loc. cit.* , Cor. 2.19.

The following is a version of the "no name lemma" [7, Cor. 3.9]:

7.6.1. Proposition. *a) The function field $F(BG) := F(V)^G$ is well-defined in $S_r^{-1}\,\mathbf{field}(F)$.*
b) The assignment $G \mapsto F(BG)$ is contravariant in G.

Proof. a) Let V' be another faithful representation of G verifying the same assumptions as V. We have inclusions of function fields

$$F(V \oplus V')^G$$

$$F(V)^G \qquad\qquad\qquad F(V')^G$$

Arguing as in [42, Proof of Th. 1.1], we get that both extensions are purely transcendental, hence an isomorphism $\varphi_{V,V'} : F(V)^G \xrightarrow{\sim} F(V')^G$ in $S_r^{-1}\,\mathbf{field}(F)$. Taking a third faithful representation and a larger diagram shows that the isomorphisms $\varphi_{V,V'}$ are transitive. We may then define $F(BG)$ independently of V as the direct limit of the $F(V)^G$ with respect to the transitive set of isomorphisms $\varphi_{V,V'}$.

b) Let $f : G \to H$ be a homomorphism. We want to define a morphism $f^* : F(BH) \to F(BG)$. Pick two faithful representations, V_G of G and V_H of H. We construct f^* from the diagram of extensions (where G acts on V_H via f):

$$F(BG) \xleftarrow{\sim} F(V_G)^G \xrightarrow{\sim} F(V_H \oplus V_G)^G$$

$$\uparrow$$

$$F(V_H)^G \longleftarrow F(V_H)^H \xrightarrow{\sim} F(BH)$$

Taking other representatives $F(V_G')^G$ and $F(V_H')^H$ and chasing in a bigger diagram shows that f^* commutes with the transitivity isomorphisms $\varphi_{V_G,V_G'}$ and $\varphi_{V_H,V_H'}$ of a). Similarly, one shows that $(g \circ f)^* = f^* \circ g^*$ if $g : H \to K$ is another homomorphism. $\square$

Here is one easy computation:

7.6.2. Proposition. *Let G be a split group of multiplicative type (by which we mean a closed subgroup of a split torus). Then, for any H, the map $F(B(G \times H)) \to F(BH)$ is an isomorphism in $S_r^{-1}\,\mathbf{field}(F)$. In particular, if F contains the m-th roots of unity, then $F(BA) \simeq F$ in $S_r^{-1}\,\mathbf{field}(F)$ for any abelian group A of exponent m.*

Proof. For two linear groups G, H we have $B(G \times H) \simeq BG \times BH$ in the sense that if V (*resp.* W) is a faithful linear representation of G (*resp.* of H), then $V \times W$ is a faithful representation of $G \times H$. By Cartier duality and the structure of finitely generated abelian groups,

this reduces us to the cases $G = \mu_m$ or $G = \mathbb{G}_m$. Then we may choose the 1-dimensional faithful representation $E = \mathbf{A}^1$ of G given by the action by homotheties; also, up to base-changing from F to $F(BH)$ (for a suitable choice of $F(BH)$), we may assume $H = 1$. In the first case, the map $x \mapsto x^m$ on $\mathbf{A}^1$ identifies the quotient $(E - \{0\})/\mu_m$ with $\mathbf{A}^1 - \{0\}$, hence $F(E)^{\mu_m} \simeq F(t)$. (One could also conclude more crudely by using Lüroth's theorem.) In the second case, $(E - \{0\})/\mathbb{G}_m \xrightarrow{\sim}$ Spec F, hence $F(E)^{\mathbb{G}_m} = F$.

For the last statement, note that $A(1) := A \otimes \mu_m$ is a split group of multiplicative type. $\qquad\square$

7.6.3. Remark. The proof shows more precisely that $F(V)^G$ is rational over F for a suitable V, which goes back to Fischer [9] when G is finite of exponent m and F contains a primitive m-th root of unity. On the other hand, it is well-known that $F(BA)$ is not stably rational in general if A is abelian (constant) and F does not contain enough roots of unity (Swan [41], $F = \mathbf{Q}$, $A = \mathbf{Z}/47$).

There is also the following important result of Saltman [39, Prop. 2.3]:

7.6.4. Proposition. *Suppose F is algebraically closed of characteristic zero and G is connected and reductive. Let T be a maximal torus of G and N its normaliser. Then the restriction map $F(BG) \to F(BN)$ is an isomorphism in $S_r^{-1}\mathbf{field}(F)$.*

7.6.5. Question. Suppose G is connected. Is it true that $F(BG) \simeq F(BG_{\mathrm{an}})$, where G_{an} is the anisotropic kernel of G?

To understand the scope of this question, note that if F is algebraically closed, it asks whether $F(BG) \simeq F$ for all (connected) G. This is a weakening of the open problem mentioned at the beginning of [7], which asks whether $F(BG)$ is always purely transcendental over F.

For a general field F of characteristic 0, a positive answer to Question 7.6.5 in the special case where $G = SL_n/\mu_p$, p prime and n a power of p, would be sufficient to imply at least surjectivity in the Merkurjev-Suslin theorem[4] by an extension of a well-known argument (see Le Bruyn [18, p. 101]). Namely, for any $K \in \mathbf{field}(F)$, let $N(K)$ (*resp.* $C(K)$) denote the kernel (*resp.* cokernel) of the norm residue homomorphism $K_2(K)/p \to H^2(K, \mu_p^{\otimes 2})$: these are functors on $\mathbf{field}(F)$ and by Bloch's lemma [2, Th. 3.1] they invert elements of S_r, hence

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[4]There exist arguments showing that surjectivity implies injectivity.

induce functors on $S_r^{-1}\mathbf{field}(F)$ (we keep the same notation). Assume now that $F = \mathbf{Q}(\mu_p)$: then $N(F) = C(F) = 0$ by Tate's theorem. Let $K = F(BSL_n/\mu_p) \in S_r^{-1}\mathbf{field}(F)$. If $K \simeq F$, then $C(K) = 0$. But by Saltman's theorem [39, Cor. 3.3], (a representative of) K is the centre of the generic division algebra A of degree n and exponent p:[5] hence $[A] \otimes \mu_p \in \operatorname{Im}(K_2(K)/p \to H^2(K, \mu_p^{\otimes 2}))$ and, by specialisation (Procesi [38]), the class of any algebra of degree n and exponent p over any $L \in \mathbf{field}(F)$ is in the image of the norm residue homomorphism. Letting n vary, we get $C(L) = 0$ for all $L \in \mathbf{field}(F)$, hence for all L of characteristic 0, any finally for any L of characteristic $\neq p$ by a classical henselian argument.

To attack Question 7.6.5, we would presumably need to consider G-torsors which do not necessarily arise from linear representations. Suppose that E is the total space of a G-torsor, and that E is a rational variety. Under what conditions is the function field $F(E/G)$ isomorphic to $F(BG)$ in $S_r^{-1}\mathbf{field}(F)$, or even in $S_r^{-1}\mathbf{place}(F)$? The answer to this question is far from clear, as the following example shows:

7.6.6. Example. Consider $G = \mathbf{Z}/2$. By Proposition 7.6.2, $F(BG) \simeq F$ in $S_r^{-1}\mathbf{field}(F)$. Now take an anisotropic conic C over F (we assume there is one). A point of degree 2 on C defines a G-torsor with base C and total space $\mathbf{P}^1$, but $F(C) \not\simeq F$ in $S_r^{-1}\mathbf{place}(F)$ (because $\operatorname{Ker}(Br(F) \to Br(F(C)))$ is non trivial, for example).

The only thing we are able to prove in the direction of Question 7.6.5 is the following lemma, probably well-known to experts:

7.6.7. Lemma. *Let G be a smooth F-algebraic group and let X, Y be two F-varieties with flat G-actions. Let $f : X \to Y$ be a G-equivariant morphism. Suppose that f is birational. Then the largest open subset $U \subseteq X$ such that $f_{|U}$ is an open immersion is G-stable.*

Proof. Let $\mu_X : G \times X \to X$ be the G-action on X and similarly μ_Y. If U is an open subset of X, then its G-saturation GU is open as the image of $G \times U$ under the flat map μ_X. We want to show that if $f_{|U}$ is an open immersion, then $f_{|GU}$ is an open immersion.

Suppose first that F is separably closed. Then $G(F)$ is Zariski-dense in G, hence $G(F)U = \bigcup_{g \in G(F)} gU$ is Zariski-dense in GU and therefore equal to GU. Given a point $x \in X$, f is an open immersion at x if and only if $f^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism; if this is true for x, it is clearly also true for gx for any $g \in G(F)$.

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[5]In his paper, Saltman assumes his ground field to be algebraically closed, but he kindly pointed out that the result only needs the presence of a primitive p-th root of unity.

In general, given $x \in GU$, f induces an isomorphism $f^* : \mathcal{O}_{Y_{F_s}, f(x)} \xrightarrow{\sim} \mathcal{O}_{X_{F_s}, x}$ where F_s is a separable closure of F, $\mathcal{O}_{X_{F_s}, x}$ is the semi-local ring of X_{F_s} at the inverse image of x and similarly for $\mathcal{O}_{Y_{F_s}, f(x)}$. As $\mathcal{O}_{X_{F_s}, x} = \mathcal{O}_{X,x} \otimes_F F_s$ and $\mathcal{O}_{Y_{F_s}, f(x)} = \mathcal{O}_{Y, f(x)} \otimes_F F_s$, we find that f is an open immersion at x. $\square$

7.7. **Open questions.** We finish by listing a few problems that are not answered in this paper.

(1) Compute Hom sets in $S_r^{-1} \mathbf{Var}(F)$. In [26, Rk. 8.11], it is shown that the functor $S_r^{-1} \mathbf{Sm}(F) \to S_r^{-1} \mathbf{Var}(F)$ is neither full nor faithful and that the Hom sets are in fact completely different.
(2) Give a categorical interpretation of rationally connected varieties.
(3) Can one make Lemma 6.2.1 more explicit in the special case $\operatorname{trdeg}(K/F) = 2$, $L = F = \bar{F}$, K not ruled, and understand things in terms of the minimal model of K? *cf.* [19, Ch. II, Ex. 4.12].
(4) Finally one should develop additional functoriality: products and internal Homs, change of base field.

APPENDIX A. INVARIANCE BIRATIONNELLE ET INVARIANCE HOMOTOPIQUE

par Jean-Louis Colliot-Thélène

Soit k un corps. Soit F un foncteur contravariant de la catégorie des k-schémas vers la catégorie des ensembles. Si sur les morphismes k-birationnels de surfaces projectives, lisses et géométriquement connexes ce foncteur induit des bijections, alors l'application $F(k) \to F(\mathbf{P}_k^1)$ est une bijection.

Démonstration. Toutes les variétés considérées sont des k-variétés. On écrit $F(k)$ pour $F(\operatorname{Spec}(k))$. Soit W l'éclaté de $\mathbf{P}^1 \times \mathbf{P}^1$ en un k-point M. Les transformés propres des deux génératrices L_1 et L_2 passant par M sont deux courbes exceptionnelles de première espèce $E_1 \simeq \mathbf{P}^1$ et $E_2 \simeq \mathbf{P}^1$ qui ne se rencontrent pas. On peut donc les contracter simultanément, la surface que l'on obtient est le plan projectif $\mathbf{P}^2$. Notons M_1 et M_2 les k-points de $\mathbf{P}^2$ sur lesquels les courbes E_1 et E_2 se contractent.

On réalise facilement cette construction de manière concrète. Dans $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ avec coordonnées multihomogènes $(u, v; w, z; X, Y, T)$ on prend pour W la surface définie par l'idéal $(uT - vX, wT - zY)$, et on considère les deux projections $W \to \mathbf{P}^1 \times \mathbf{P}^1$ et $W \to \mathbf{P}^2$.

On a un diagramme commutatif de morphismes

$$\begin{array}{ccc} E_1 & \longrightarrow & W \\ \wr\downarrow & & \downarrow \\ L_1 & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1. \end{array}$$

Le composé de l'inclusion $L_1 \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ et d'une des deux projections $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ est un isomorphisme. Par fonctorialité, la restriction $F(\mathbf{P}^1 \times \mathbf{P}^1) \to F(L_1)$ est donc surjective. Par fonctorialité, le diagramme ci-dessus implique alors que la restriction $F(W) \to F(E_1)$ est surjective.

Considérons maintenant la projection $W \to \mathbf{P}^2$. On a ici le diagramme commutatif de morphismes

$$\begin{array}{ccc} E_1 & \longrightarrow & W \\ \downarrow & & \downarrow \\ M_1 & \longrightarrow & \mathbf{P}^2. \end{array}$$

Par l'hypothèse d'invariance birationnelle, on a la bijection $F(\mathbf{P}^2) \xrightarrow{\sim} F(W)$. Donc la flèche composée $F(\mathbf{P}^2) \to F(W) \to F(E_1)$ est surjective. Mais par le diagramme commutatif ci-dessus la flèche composée se factorise aussi comme $F(\mathbf{P}^2) \to F(M_1) \to F(E_1)$. Ainsi $F(M_1) \to F(E_1)$, c'est-à-dire $F(k) \to F(\mathbf{P}^1)$, est surjectif. L'injectivité de $F(k) \to F(\mathbf{P}^1)$ résulte de la fonctorialité et de la considération d'un k-point sur $\mathbf{P}^1$.

Appendix B. A letter from O. Gabber

June 12, 2007

Dear Kahn,

I discuss a proof of

B.0.1. Theorem. *Let A be a regular local ring with residue field k, $X' \to X = Spec(A)$ a proper birational morphism, X'_{reg} the regular locus of X', X'_s the special fiber of X', $X'_{reg,s} = X'_s \cap X'_{reg}$, which is known to be open in X'_s, F a field extension of k, then any two points of $X'_{reg,s}(F)$ are R-equivalent in $X'_s(F)$.*

The proof I tried to sketch by joining centers of divisorial valuations has a gap in the imperfect residue field case. It is easier to adapt the proof by deformation of local arcs.

(1) If $Y' \to Y$ is a proper map between k-schemes whose fibers are projective spaces then for every F/k, $Y'(F)/R \to Y(F)/R$ is bijective.

In particular the theorem holds if X' is obtained from X by a sequence of blow-ups with regular centers.

(2) If A is a regular local ring of dimension > 1 with maximal ideal m, U an open non empty in $Spec(A)$, then there is $f \in m - m^2$ s.t. the generic point of $V(f)$ is in U.

This is because U omits only a finite number of height 1 primes and there are infinitely many possibilities for $V(f)$, e.g. $V(x - y^i)$ where x, y is a part of a regular system of parameters.

Inductively we get that there is $P \in U$ s.t. A/P is regular 1-dimensional.

(3) If A is a regular local ring and P, P' different prime ideals with A/P and A/P' regular one dimensional, then there is a prime ideal $Q \subset P \cap P'$ with A/Q regular 2-dimensional.

Indeed let $x_1, \ldots, x_n$ be a minimal system of generators of P; their images in A/P' generate a principal ideal; we may assume this ideal is generated by the image of x_1, and then we can substract some multiples of x_1 from $x_2, \ldots, x_n$ so that the images of $x_2, \ldots, x_n$ are 0; take $Q = (x_2, \ldots, x_n)$.

To prove the theorem we may assume F is a finitely generated extension of k, so F is a finite extension of a purely transcendental extension k' of k. We replace A by the local ring at the generic point of the special fiber of an affine space over A that has residue field k'. So we reduce to F/k finite. Let x, y be F-points of X'_s centered at closed points a, b at which X' is regular. Let U be dense open of X above which $X' \to X$ is an isomorphism. Let $X'(a), X'(b)$ be the local schemes (Spec of the local rings at a and b). There are regular one dimensional closed subschemes

$$C \subset X'(a), C' \subset X'(b)$$

whose generic points map to U.

By EGA 0_{III} 10.3 there are finite flat $D \to C$, $D' \to C'$ which are $Spec(F)$ over the closed points of C, C'. Then D, D' are Spec's of DVRs essentially of finite type over A (localization of finite type A-algebras). We form the pushout of $D \leftarrow Spec(F) \to D'$, which is Spec of a fibered product ring, which by some algebraic exercise is still an A-algebra essentially of finite type. The pushout can be embedded as a closed subscheme in Spec of a local ring of an affine space over A and then by (3) in some Y a 2-dimensional local regular A-scheme essentially of finite type. Now D, D' are subschemes of Y. We have a rational map $Y \to X'$ defined on the inverse image of U and in particular at the generic points of D and D'. By e.g. Theorem 26.1 in Lipman's paper on rational singularities (Publ. IHES 36) there is

$Y' \to Y$ obtained as a succession of blow-ups at closed points s.t. the rational map gives a morphism $Y' \to X'$. Then x, y are images of F-points of Y' (closed points of the proper transforms of D, D'),and by (1) any two F-points of the special fiber of $Y' \to Y$ are R-equivalent.

Sincerely,

Ofer Gabber

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