# Sharp estimates for the convergence of the density of the Euler scheme in small time

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**Abstract**

In this work, we approximate a diffusion process by its Euler scheme and we study the convergence of the density of the marginal laws. We improve previous estimates especially for small time.

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# 1 Introduction

Let us consider a $d$-dimensional diffusion process $(X_s)_{0\leq s\leq T}$ and a $q$-dimensional Brownian motion $(W_s)_{0\leq s\leq T}$. $X$ satisfies the following SDE

$$dX_s^i = b_i(s, X_s)ds + \sum_{j=1}^{q} \sigma_{ij}(s, X_s)dW_s^j, \quad X_0^i = x^i, \forall i \in \{1, \cdots, d\}. \tag{1.1}$$

We approximate $X$ by its Euler scheme with $N$ ($N \geq 1$) time steps, say $X^N$, defined as follows. We consider the regular grid $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ of the interval $[0, T]$, i.e. $t_k = k\frac{T}{N}$. We put $X_0^N = x$ and for all $i \in \{1, \cdots, d\}$ we define

$$X_u^{N,i} = X_{t_k}^{N,i} + b_i(t_k, X_{t_k}^N)(u - t_k^N) + \sum_{j=1}^{q} \sigma_{ij}(t_k, X_{t_k}^N)(W_u^j - W_{t_k}^j), \text{ for } u \in [t_k, t_{k+1}[. \tag{1.2}$$

The continuous Euler scheme is an Itô process verifying

$$X_u^N = x + \int_0^u b(\varphi(s), X_{\varphi(s)}^N)ds + \int_0^u \sigma(\varphi(s), X_{\varphi(s)}^N)dW_s$$

where $\varphi(u) := \sup\{t_k : t_k \leq u\}$. If $\sigma$ is uniformly elliptic, the Markov process $X$ admits a transition probability density $p(0, x; s, y)$. Concerning $X^N$ (which is not Markovian except at times $(t_k)_k$), $X_s^N$ has a probability density $p^N(0, x; s, y)$, for any $s > 0$. We aim at proving sharp estimates of the difference $p(0, x; s, y) - p^N(0, x; s, y)$.

It is well known (see Bally and Talay (1996), Konakov and Mammen (2002), Guyon (2006)) that this difference is of order $\frac{1}{N}$. However, the known upper bounds of this difference are too rough for small values of $s$. In this work, we provide tight upper bounds of $|p(0, x; s, y) - p^N(0, x; s, y)|$ in $s$ (see Theorem 3), so that we can estimate quantities like

$$\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] \text{ or } \mathbb{E}\left[\int_0^T f(X_{\varphi(s)}^N)ds\right] - \mathbb{E}\left[\int_0^T f(X_s)ds\right] \tag{1.3}$$

(without any regularity assumptions on $f$) more accurately than before (see Theorem 5). For other applications, see Labart (2007). Unlike previous references, we allow $b$ and $\sigma$ to be time-dependent and assume they are only $C^3$ in space. Besides, we use Malliavin's calculus tools.

**Background results**

The difference $p(0, x; s, y) - p^N(0, x; s, y)$ has been studied a lot. We can found several results in the literature on expansions w.r.t. $N$. First, we mention a result from Bally and Talay (1996) (Corollary 2.7). The authors assume

**Hypothesis 1** *$\sigma$ is elliptic (with $\sigma$ only depending on $x$) and $b, \sigma$ are $C^\infty(\mathbb{R}^d)$ functions whose derivatives of any order greater or equal to 1 are bounded.*

By using Malliavin's calculus, they show that

$$p(0, x; T, y) - p^N(0, x; T, y) = \frac{1}{N}\pi_T(x, y) + \frac{1}{N^2}R_T^N(x, y), \tag{1.4}$$

with $|\pi_T(x,y)| + |R_T^N(x,y)| \leq \frac{K(T)}{T^q}\exp(-c\frac{|x-y|^2}{T})$, where $c > 0$, $q > 0$ and $K(\cdot)$ is a non decreasing function. We point out that $q$ is unknown, which doesn't enable to deduce the behavior of $p - p^N$ when $T \to 0$.

Besides that, Konakov and Mammen (2002) have proposed an analytical approach based on the so-called parametrix method to bound $p(0,x;1,y) - p^N(0,x;1,y)$ from above. They assume

**Hypothesis 2** *$\sigma$ is elliptic and $b, \sigma$ are $C^\infty(\mathbb{R}^d)$ functions whose derivatives of any order are bounded.*

For each pair $(x,y)$ they get an expansion of arbitrary order $j$ of $p^N(0,x;1,y)$. The coefficients of the expansion depend on $N$

$$p(0,x;1,y) - p^N(0,x;1,y) = \sum_{i=1}^{j-1} \frac{1}{N^i}\pi_{N,i}(0,x;1,y) + O(\frac{1}{N^j}). \tag{1.5}$$

The coefficients have Gaussian tails : for each $i$ they find constants $c_1 > 0$, $c_2 > 0$ s.t. for all $N \geq 1$ and all $x, y \in \mathbb{R}^d$, $|\pi_{N,i}(0,x;1,y)| \leq c_1 \exp(-c_2|x-y|^2)$. To do so, they use upper bounds for the partial derivatives of $p$ (coming from Friedman (1964)) and prove analogous results on the derivatives of $p^N$. Strong though this result may be, nothing is said when replacing 1 by $t$, for $t \to 0$. That's why we present now the work of Guyon (2006).

Guyon (2006) improves (1.4) and (1.5) in the following way.

**Definition 1.** *Let $\mathcal{G}_l(\mathbb{R}^d), l \in \mathbb{Z}$ be the set of all measurable functions $\pi : \mathbb{R}^d \times (0,1] \times \mathbb{R}^d \to \mathbb{R}$ s.t.*

- *for all $t \in (0,1], \pi(\cdot;t,\cdot)$ is infinitely differentiable,*
- *for all $\alpha, \beta \in \mathbb{N}^d$, there exist two constants $c_1 \geq 0$ and $c_2 > 0$ s.t. for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$|\partial_x^\alpha \partial_y^\beta \pi(x;t,y)| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2}\exp(-c_2|x-y|^2/t).$$

Under Hypothesis 2 and for $T = 1$, the author has proved the following expansions

$$p^N - p = \frac{\pi}{N} + \frac{\pi_N}{N^2}, \tag{1.6}$$

$$p^N - p = \sum_{i=1}^{j-1} \frac{\pi_{N,i}}{N^i} + \sum_{i=2}^{j} \left(t - \frac{\lfloor Nt \rfloor}{N}\right)^i \pi'_{N,i} + \frac{\pi''_{N,j}}{N^j}, \tag{1.7}$$

where $\pi \in \mathcal{G}_1(\mathbb{R}^d)$ and $(\pi_N, N \geq 1)$ is a bounded sequence in $\mathcal{G}_4(\mathbb{R}^d)$. For each $i \geq 1$, $(\pi_{N,i}, N \geq 1)$ is a bounded family in $\mathcal{G}_{2i-2}(\mathbb{R}^d)$, and $(\pi'_{N,i}, N \geq 1), (\pi''_{N,i}, N \geq 1)$ are two bounded families in $\mathcal{G}_{2i}(\mathbb{R}^d)$. These expansions can be seen as improvements of (1.4) and (1.5) : it also allows infinite differentiations w.r.t. $x$ and $y$ and makes precise the way the coefficients explode when $t$ tends to 0.

As a consequence (see Guyon (2006, Corollary 22)), one gets

$$|p(0,x;s,y) - p^N(0,x;s,y)| \leq \frac{c_1}{N s^{\frac{d+2}{2}}} e^{-c_2 \frac{|x-y|^2}{s}}, \tag{1.8}$$

for two positive constants $c_1$ and $c_2$, and for any $x, y$ and $s \leq 1$. This result should be compared with the one of Theorem 3 (when $T = 1$), in which the upper bound is tighter ($s$ has a smaller power).

# 2 Main Results

Before stating the main result of the paper, we introduce the following notation

**Definition 2.** *$C_b^{k,l}$ denotes the set of continuously differentiable bounded functions $\phi : (t,x) \in [0,T] \times \mathbb{R}^d$ with uniformly bounded derivatives w.r.t. $t$ (resp. w.r.t. $x$) up to order $k$ (resp. up to order $l$).*

The main result of the paper, whose proof is postponed to Section 4, is established under the following Hypothesis

**Hypothesis 3** *$\sigma$ is uniformly elliptic, $b$ and $\sigma$ are in $C_b^{1,3}$ and $\partial_t \sigma$ is in $C_b^{0,1}$.*

**Theorem 3.** *Assume Hypothesis 3. Then, there exist a constant $c > 0$ and a non decreasing function $K$, depending on the dimension $d$ and on the upper bounds of $\sigma, b$ and their derivatives s.t. $\forall (s,x,y) \in ]0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, one has*

$$|p(0,x;s,y) - p^N(0,x;s,y)| \leq \frac{K(T)T}{N s^{\frac{d+1}{2}}} \exp\left(-\frac{c|x-y|^2}{s}\right).$$

**Corollary 4.** *Assume Hypothesis 3. From the last inequality and Aronson's inequality (A.1), we deduce*

$$\left|\frac{p(0,x;T,x) - p^N(0,x;T,x)}{p(0,x;T,x)}\right| \leq \frac{K(T)}{N}\sqrt{T}. \tag{2.1}$$

*This inequality yields $p(0,x;T,x) \sim p^N(0,x;T,x)$ when $T \to 0$.*

Theorem 3 enables to bound quantities like in (1.3) in the following way

**Theorem 5.** *Assume Hypothesis 3. For any function $f$ such that $|f(x)| \leq c_1 e^{c_2|x|}$, it holds*

$$\begin{aligned}
&\left|\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)]\right| \leq c_1 e^{c_2|x|} K(T) \frac{\sqrt{T}}{N}, \\
&\left|\mathbb{E}\left[\int_0^T f(X^N_{\varphi(s)})ds\right] - \mathbb{E}\left[\int_0^T f(X_s)ds\right]\right| \leq c_1 e^{c_2|x|} K(T) \frac{T}{N}.
\end{aligned}$$

Had we used the results stated by Guyon (2006) (and more precisely the one recalled in (1.8)), we would have obtained $\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] = O(\frac{1}{N})$. Intuitively, this result is not optimal: the right hand side doesn't tend to 0 when $T$ goes to 0 while it should. Analogously, regarding $\mathbb{E}\left[\int_0^T f(X^N_{\varphi(s)})ds\right] - \mathbb{E}\left[\int_0^T f(X_{\varphi(s)})ds\right]$, we would obtain $O(\frac{T \ln N}{N})$ instead of $O(\frac{T}{N})$.

*Proof of Theorem 5.* Writing $\mathbb{E}[f(X_T^N)]-\mathbb{E}[f(X_T)]$ as $\int_{\mathbb{R}^d} f(y)(p^N(0,x;T,y)-p(0,x;T,y))dy$ and using Theorem 3 yield the first result. Concerning the second result, we split $\mathbb{E}\left[\int_0^T (f(X^N_{\varphi(s)})-f(X_s))ds\right]$ in two terms $\mathbb{E}\left[\int_0^T (f(X^N_{\varphi(s)})-f(X_{\varphi(s)}))ds\right]$ and $\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)})-f(X_s))ds\right]$. First, using Theorem 3 leads to

$$\left|\mathbb{E}\left[\int_0^T (f(X^N_{\varphi(s)})-f(X_{\varphi(s)}))ds\right]\right| = \left|\int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds f(y)(p^N(0,x;\varphi(s),y)-p(0,x;\varphi(s),y))\right|,$$
$$\leq \frac{K(T)T}{N} c_1 e^{c_2|x|} \int_{\frac{T}{N}}^T \frac{ds}{\sqrt{\varphi(s)}},$$

where we use the easy inequality $\int_{\mathbb{R}^d} e^{c_2|y|} \frac{e^{\frac{-c|x-y|^2}{s}}}{s^{d/2}} dy \leq K(T)e^{c_2|x|}$. Since $\varphi(s) \geq s - \frac{T}{N}$, we get $\left|\mathbb{E}\left[\int_0^T (f(X^N_{\varphi(s)})-f(X_{\varphi(s)}))ds\right]\right| \leq \frac{K(T)T^{3/2}}{N} c_1 e^{c_2|x|}$. Second, we write

$$\left|\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)})-f(X_s))ds\right]\right| \leq c_1 e^{c_2|x|}\frac{T}{N} + \int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds c_1 e^{c_2|y|} \int_{\varphi(s)}^s du |\partial_u p(0,x;u,y)|.$$

Then, Proposition 13 yields $\left|\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)})-f(X_s))ds\right]\right| \leq c_1 e^{c_2|x|}\left(\frac{T}{N} + C\int_{\frac{T}{N}}^T \ln(\frac{s}{\varphi(s)})ds\right)$. Moreover, $\int_{\frac{T}{N}}^T \ln(\frac{s}{\varphi(s)})ds = \sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}} \ln(\frac{s}{t_k})ds = \frac{T}{N}\sum_{k=1}^{N-1}((k+1)\ln(\frac{k+1}{k})-1) \leq C\frac{T}{N}$, using a second order Taylor expansion. This gives $\left|\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)})-f(X_s))ds\right]\right| \leq c_1 e^{c_2|x|}K(T)\frac{T}{N}$. ■

In the next section, we give results related to Malliavin's calculus, that will be useful for the proof of Theorem 3.

# 3 Basic results on Malliavin's calculus

We refer the reader to Nualart (2006), for more details. Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and let $(W_t)_{t\geq 0}$ be a $q$-dimensional Brownian motion. For $h(\cdot) \in H = \mathbb{L}^2([0,T],\mathbb{R}^q)$, $W(h)$ is the Wiener stochastic integral $\int_0^T h(t)dW_t$. Let $\mathcal{S}$ denote the class of random variables of the form $F = f(W(h_1),\cdots,W(h_n))$ where $f$ is a $C^\infty$ function with derivatives having a polynomial growth, $(h_1,\cdots,h_n) \in H^n$ and $n \geq 1$. For $F \in \mathcal{S}$, we define its derivative $\mathcal{D}F = (\mathcal{D}_t F := (\mathcal{D}^1_t F,\cdots,\mathcal{D}^q_t F))_{t\in[0,T]}$ as the $H$ valued random variable given by

$$\mathcal{D}_t F = \sum_{i=1}^n \partial_{x_i} f(W(h_1),\cdots,W(h_n))h_i(t).$$

The operator $\mathcal{D}$ is closable as an operator from $\mathbb{L}^p(\Omega)$ to $\mathbb{L}^p(\Omega;H)$, for $p \geq 1$. Its domain is denoted by $\mathbb{D}^{1,p}$ w.r.t. the norm $\|F\|_{1,p} = [\mathbb{E}|F|^p + \mathbb{E}(\|\mathcal{D}F\|^p_H)]^{1/p}$. We can define the iteration of the operator $\mathcal{D}$, in such a way that for a smooth random variable $F$, the derivative $\mathcal{D}^k F$ is a random variable with values on $H^{\otimes k}$. As in the case $k=1$, the operator $\mathcal{D}^k$ is closable from $\mathcal{S} \subset \mathbb{L}^p(\Omega)$ into $\mathbb{L}^p(\Omega;H^{\otimes k}), p \geq 1$. If we define the norm

$$\|F\|_{k,p} = [\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}(\left\|\mathcal{D}^j F\right\|^p_{H^{\otimes j}})]^{1/p},$$

we denote its domain by $\mathbb{D}^{k,p}$. Finally, set $\mathbb{D}^{k,\infty} = \cap_{p\geq 1}\mathbb{D}^{k,p}$, and $\mathbb{D}^{\infty} = \cap_{k,p\geq 1}\mathbb{D}^{k,p}$. One has the following chain rule property

**Proposition 6.** *Fix $p \geq 1$. For $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$, and $F = (F_1, \cdots, F_d)^*$ a random vector whose components belong to $\mathbb{D}^{1,p}$, $f(F) \in \mathbb{D}^{1,p}$ and for $t \geq 0$, one has $\mathcal{D}_t(f(F)) = f'(F)\mathcal{D}_t F$, with the notation*

$$\mathcal{D}_t F = \begin{pmatrix} \mathcal{D}_t F_1 \\ \vdots \\ \mathcal{D}_t F_d \end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{R}^q.$$

We now introduce the Skorohod integral $\delta$, defined as the adjoint operator of $\mathcal{D}$.

**Proposition 7.** *$\delta$ is a linear operator on $\mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^q)$ with values in $\mathbb{L}^2(\Omega)$ s.t.*

- *the domain of $\delta$ (denoted by $Dom(\delta)$) is the set of processes $u \in \mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^q)$ s.t. $|\mathbb{E}(\int_0^T \mathcal{D}_t F \cdot u_t dt)| \leq c(u)|F|_{\mathbb{L}^2}$ for any $F \in \mathbb{D}^{1,2}$.*
- *If $u$ belongs to $Dom(\delta)$, then $\delta(u)$ is the one element of $\mathbb{L}^2(\Omega)$ characterized by the integration by parts formula*

$$\forall F \in \mathbb{D}^{1,2}, \;\; \mathbb{E}(F\delta(u)) = \mathbb{E}\left(\int_0^T \mathcal{D}_t F \cdot u_t dt\right).$$

**Remark 8.** *If $u$ is an adapted process belonging to $\mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^q)$, then the Skorohod integral and the Itô integral coincide : $\delta(u) = \int_0^T u_t dW_t$, and the preceding integration by parts formula becomes*

$$\forall F \in \mathbb{D}^{1,2}, \;\; \mathbb{E}\left(F\int_0^T u_t dW_t\right) = \mathbb{E}\left(\int_0^T \mathcal{D}_t F \cdot u_t dt\right). \tag{3.1}$$

*This equality is also called the duality formula.*

This duality formula is the corner stone to establish general integration by parts formula of the form

$$\mathbb{E}[\partial^\alpha g(F) G] = \mathbb{E}[g(F) H_\alpha(F, G)]$$

for any non degenerate random variables $F$. We only give the formulation in the case of interest $F = X_t^N$.

**Proposition 9.** *We assume that $\sigma$ is uniformly elliptic and $b$ and $\sigma$ are in $C_b^{0,3}$. For all $p > 1$, for all multi-index $\alpha$ s.t. $|\alpha| \leq 2$, for all $t \in ]0,T]$, all $u, r, s \in [0,T]$ and for any functions $f$ and $g$ in $C_b^{|\alpha|}$, there exist a random variable $H_\alpha \in \mathbb{L}^p$ and a function $K(T)$ (uniform in $N, x, s, u, r, t, f$ and $g$) s.t.*

$$\mathbb{E}[\partial_x^\alpha f(X_t^N) g(X_u^N, X_r^N, X_s^N)] = \mathbb{E}[f(X_t^N) H_\alpha], \tag{3.2}$$

*with*

$$|H_\alpha|_{\mathbb{L}_p} \leq \frac{K(T)}{t^{\frac{|\alpha|}{2}}} \|g\|_{C_b^{|\alpha|}}. \tag{3.3}$$

These results are given in the article of Kusuoka and Stroock (1984): (3.3) is owed to Theorem 1.20 and Corollary 3.7.
Another consequence of the duality formula is the derivation of an upper bound for $p^N$.

**Proposition 10.** *Assume $\sigma$ is uniformly elliptic and $b$ and $\sigma$ are in $C_b^{0,2}$. Then, for any $x, y \in \mathbb{R}^d$, $s \in ]0,T]$, one has*

$$p^N(0,x;s,y) \leq \frac{K(T)}{s^{d/2}} e^{-c\frac{|x-y|^2}{s}}, \tag{3.4}$$

*for a positive constant $c$ and a non decreasing function $K$, both depending on $d$ and on the upper bounds for $b, \sigma$ and their derivatives.*

Although this upper bound seems to be quite standard, to our knowledge such a result has not appeared in the literature before, except in the case of time homogeneous coefficients (see Konakov and Mammen (2002), proof of Theorem 1.1).

*Proof.* The inequality (1.32) of Kusuoka and Stroock (1984, Theorem 1.31) gives $p^N(0,x;s,y) \leq \frac{K(T)}{s^{d/2}}$ for any $x$ and $y$. This implies the required upper bound when $|x-y| \leq \sqrt{s}$. Let us now consider the case $|x-y| > \sqrt{s}$. Using the same notations as in Kusuoka and Stroock (1984), we denote $\psi(y) = \rho(\frac{|y-x|}{r})$ where $r > 0$ and $\rho$ is a $C_b^\infty$ function such that $\mathbf{1}_{\{[3/4,\infty[\}} \leq \rho \leq \mathbf{1}_{\{[1/2,\infty[\}}$. Then, combining inequality (1.33) of Kusuoka and Stroock (1984, Theorem 1.31) and Corollary 3.7 leads to

$$\sup_{|y-x|\geq r} p^N(0,x;s,y) \leq K(T) \frac{e^{-c\frac{r^2}{s}}}{s^{d/2}} \left(1 + \sqrt{\frac{s}{r^2}}\right),$$

where we use $|\psi(X_s^N)|_{1,q} \leq K(T) e^{-c\frac{r^2}{s}} \left(1 + \sqrt{\frac{s}{r^2}}\right)$. This easily completes the proof in the case $|x-y| \geq \sqrt{s}$. ■

# 4 Proof of Theorem 3

In the following, $K(\cdot)$ denotes a generic non decreasing function (which may depend on $d$, $b$ and $\sigma$). To prove Theorem 3, we take advantage of Propositions 9 and 10. The scheme of the proof is the following

- Use a PDE and Itô's calculus to write the difference $p^N(0,x;s,y) - p(0,x;s,y)$

$$= \int_0^s \mathbb{E}\left[\sum_{i=1}^d (b_i(\varphi(r), X^N_{\varphi(r)}) - b_i(r, X_r^N))\partial_{x_i} p(r, X_r^N; s, y)\right.$$
$$\left. + \frac{1}{2}\sum_{i,j=1}^d (a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r, X_r^N))\partial^2_{x_i x_j} p(r, X_r^N; s, y)\right] dr := E_1 + E_2. \tag{4.1}$$

- Prove the intermediate result $\forall (r,x,y) \in [0,s[\times\mathbb{R}^d \times \mathbb{R}^d$ and $c > 0$

$$\mathbb{E}\left[\exp\left(-c\frac{|y - X_r^N|^2}{s-r}\right)\right] \leq K(T)\left(\frac{s-r}{s}\right)^{\frac{d}{2}} \exp\left(-c'\frac{|x-y|^2}{s}\right), \tag{4.2}$$

where $c' > 0$.

- Use Malliavin's calculus, Proposition 10 and the intermediate result, to show that each term $E_1$ and $E_2$ (see (4.1)) is bounded by $\frac{K(T)T}{N}\frac{1}{s^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{s})$.

**Definition 11.** *We say that a term $E(x,s,y)$ satisfies property $\mathcal{P}$ if* $\forall (x,s,y)\in\mathbb{R}^d\times]0,T]\times\mathbb{R}^d$

$$|E(x,s,y)|\leq \frac{K(T)T}{N}\frac{1}{s^{\frac{d+1}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right). \quad (\mathcal{P})$$

## 4.1 Proof of equality (4.1)

First, the transition density function $(r,x)\longmapsto p(r,x;s,y)$ satisfies the PDE

$$(\partial_r+\mathcal{L}_{(r,x)})p(r,x;s,y)=0,\ \ \forall r\in[0,s[,\forall x\in\mathbb{R}^d,$$

where $\mathcal{L}_{(r,x)}$ is defined by $\mathcal{L}_{(r,x)}=\sum_{i,j}a_{ij}(r,x)\partial^2_{x_ix_j}+\sum_i b_i(r,x)\partial_{x_i}$, and $a_{ij}(r,x)=\frac{1}{2}[\sigma\sigma^*]_{ij}(r,x)$. The function, as well as its first derivatives, are uniformly bounded by a constant depending on $\epsilon$ for $|s-r|\geq\epsilon$ (see Appendix A).
Second, since $p^N(0,x;s,y)$ is a continuous function in $s$ and $y$ (convolution of Gaussian densities), we observe that

$$p^N(0,x;s,y)-p(0,x;s,y)=\lim_{\epsilon\longrightarrow 0}\mathbb{E}[p(s-\epsilon,X^N_{s-\epsilon};s,y)-p(0,x;s,y)].$$

Then, for any $\epsilon>0$, Itô's formula leads to

$$\mathbb{E}[p(s-\epsilon,X^N_{s-\epsilon};s,y)-p(0,x;s,y)]=\mathbb{E}\left[\int_0^{s-\epsilon}\partial_r p(r,X^N_r;s,y)dr\right]$$
$$+\mathbb{E}\left[\int_0^{s-\epsilon}\sum_{i=1}^d b_i(\varphi(r),X^N_{\varphi(r)})\partial_{x_i}p(r,X^N_r;s,y)dr+\frac{1}{2}\int_0^{s-\epsilon}\sum_{i,j=1}^d a_{ij}(\varphi(r),X^N_{\varphi(r)})\partial^2_{x_ix_j}p(r,X^N_r;s,y)dr\right].$$

From the PDE, the above equality becomes

$$\begin{aligned}\mathbb{E}[p(s-\epsilon,X^N_{s-\epsilon};s,y)-p(0,x;s,y)]&=\mathbb{E}\left[\int_0^{s-\epsilon}\sum_{i=1}^d(b_i(\varphi(r),X^N_{\varphi(r)})-b_i(r,X^N_r))\partial_{x_i}p(r,X^N_r;s,y)dr\right]\\&+\frac{1}{2}\mathbb{E}\left[\int_0^{s-\epsilon}\sum_{i,j=1}^d(a_{ij}(\varphi(r),X^N_{\varphi(r)})-a_{ij}(r,X^N_r))\partial^2_{x_ix_j}p(r,X^N_r;s,y)dr\right],\\&:=\int_0^{s-\epsilon}\mathbb{E}[\phi(r)]dr,\end{aligned}$$

where $\phi(r)=\sum_{i=1}^d(b_i(\varphi(r),X^N_{\varphi(r)})-b_i(r,X^N_r))\partial_{x_i}p(r,X^N_r;s,y)+\frac{1}{2}\sum_{i,j=1}^d(a_{ij}(\varphi(r),X^N_{\varphi(r)})-a_{ij}(r,X^N_r))\partial^2_{x_ix_j}p(r,X^N_r;s,y)$. To get (4.1), it remains to prove that $\mathbb{E}(\phi(r))$ is integrable over $[0,s]$. We check it by looking at the rest of the proof.

### 4.2 Proof of the intermediate result (4.2)

We prove inequality (4.2). $\mathbb{E}[\exp(-c\frac{|y-X_r^N|^2}{s-r})] = \int_{\mathbb{R}^d} \exp(-c\frac{|y-z|^2}{s-r}) p^N(0,x;r,z)dz$. Using Proposition 10, we get

$$\mathbb{E}\left[\exp\left(-c\frac{|y-X_r^N|^2}{s-r}\right)\right] \leq \frac{K(T)}{r^{\frac{d}{2}}}\int_{\mathbb{R}^d}\exp\left(-c\frac{|y-z|^2}{s-r}\right)\exp\left(-c'\frac{|x-z|^2}{r}\right)dz$$
$$\leq K(T)\Pi_{i=1}^d\int_{\mathbb{R}}\frac{1}{\sqrt{r}}\exp\left(-c\frac{|y_i-z_i|^2}{s-r}\right)\exp\left(-c'\frac{|x_i-z_i|^2}{r}\right)dz_i,$$

and $\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\frac{(s-r)}{2c}}}\exp(-c\frac{|y_i-z_i|^2}{s-r})\frac{1}{\sqrt{2\pi\frac{r}{2c'}}}\exp(-c'\frac{|x_i-z_i|^2}{r})dz_i$ is the convolution product of the density of two independant Gaussian random variables $\mathcal{N}(-x_i,\frac{r}{2c'})$ and $\mathcal{N}(y_i,\frac{s-r}{2c})$ computed at 0. Hence, the integral is equal to $\frac{1}{\sqrt{2\pi(\frac{r}{2c'}+\frac{s-r}{2c})}}\exp\left(-\frac{|x_i-y_i|^2}{\frac{r}{c'}+\frac{s-r}{c}}\right)$. Then,

$$\int_{\mathbb{R}}\frac{1}{\sqrt{r}}\exp\left(-c\frac{|y_i-z_i|^2}{s-r}\right)\exp\left(-c'\frac{|x_i-z_i|^2}{r}\right)dz_i \leq C\left(\frac{s-r}{s}\right)^{\frac{1}{2}}\exp\left(-c''\frac{|x_i-y_i|^2}{s}\right)$$

and (4.2) follows.

### 4.3 Upper bound for $E_1$

We recall that $E_1 = \int_0^s \mathbb{E}\left[\sum_{i=1}^d (b_i(\varphi(r),X^N_{\varphi(r)}) - b_i(r,X_r^N))\partial_{x_i}p(r,X_r^N;s,y)\right]dr$. For each $i$, we apply Itô's formula to $b_i(u,X_u^N)$ between $u=\varphi(r)$ and $u=r$. We get

$$b_i(\varphi(r),X^N_{\varphi(r)}) - b_i(r,X_r^N) = \int_{\varphi(r)}^r \alpha_u^i du + \int_{\varphi(r)}^r \sum_{k=1}^q \beta_u^{i,k} dW_u^k, \tag{4.3}$$

where $\alpha_u^i$ depends on $\partial_t b, \partial_x b, \partial_x^2 b, \sigma$, and $\beta_u^i = -\nabla_x b_i(u,X_u^N)\sigma(\varphi(r),X^N_{\varphi(r)})$. Since $b,\sigma$ belong to $C_b^{1,3}$, $\alpha^i$ and $(\beta^{i,k})_{1\leq k\leq q}$ are uniformly bounded. Using (4.3) and the duality formula (3.1) yield

$$E_1 = \sum_{i=1}^d \int_0^s \{\mathbb{E}[\int_{\varphi(r)}^r \partial_{x_i}p(r,X_r^N;s,y)\alpha_u^i du + \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i}p(r,X_r^N;s,y))\cdot\beta_u^i du]\}dr$$
$$:= E_{11} + E_{12}, \tag{4.4}$$

where $\beta_u^i$ is a row vector of $q$ components. We upper bound $E_{11}$ and $E_{12}$.

**Bound for $E_{11} = \sum_{i=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \partial_{x_i}p(r,X_r^N;s,y)\alpha_u^i du]dr$.**
Since $|\sum_{i=1}^d \partial_{x_i}p(r,X_r^N;s,y)\alpha_u^i| \leq |\alpha_u||\partial_x p(r,X_r^N;s,y)|$ and $\alpha_u$ is uniformly bounded in $u$, we have

$$|E_{11}| \leq C\frac{T}{N}\int_0^s \mathbb{E}|\partial_x p(r,X_r^N;s,y)|dr.$$

Besides that, from Proposition 13, $|\partial_x p(r, X_r^N; s, y)| \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \exp\big(-c\frac{|y-X_r^N|^2}{s-r}\big)$. Then,

$$|E_{11}| \leq K(T)\frac{T}{N}\int_0^s \frac{1}{(s-r)^{\frac{d+1}{2}}}\mathbb{E}\left[\exp\left(-c\frac{|y-X_r^N|^2}{s-r}\right)\right]dr.$$

Using the intermediate result (4.2) yields

$$|E_{11}| \leq K(T)\frac{T}{N}\int_0^s \frac{1}{\sqrt{s-r}}\frac{1}{s^{\frac{d}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right)dr \leq K(T)\frac{T}{N}\frac{1}{s^{\frac{d-1}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right)$$

and thus, $E_{11}$ satisfies property $\mathcal{P}$ (see Definition 11).

**Bound for** $E_{12} = \sum_{i=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) \cdot \beta_u^i du]dr$.

To rewrite $E_{12}$, we use the expression of $\beta_u^i$ and Proposition 6, which gives $\mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) = \nabla_x(\partial_{x_i} p(r, X_r^N; s, y))\sigma(\varphi(r), X_{\varphi(r)}^N)$. Then,

$$E_{12} = -\int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial^2_{x_i x_k} p(r, X_r^N; s, y)[(\sigma\sigma^*)(\varphi(r), X_{\varphi(r)}^N)(\nabla_x b_i(u, X_u^N))^*]_k]du. \quad (4.5)$$

Using the integration by parts formula (3.2), we get that

$$E_{12} = -\int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial_{x_i} p(r, X_r^N; s, y) H^i_{e_k}(r, u)]du$$

where $e_k$ is a vector whose $k$-th component is 1 and other components are 0. From (3.3), we deduce $\mathbb{E}[|H^i_{e_k}(r,u)|^p]^{1/p} \leq C\frac{K(T)}{r^{1/2}}$, where $C$ only depends on $|\sigma|_\infty$, $|\partial_x\sigma|_\infty$, $|\partial_x b|_\infty$, $|\partial^2_{xx} b|_\infty$. By the Hölder inequality, it follows that

$$|E_{12}| \leq K(T)\int_0^s dr \int_{\varphi(r)}^r \frac{1}{r^{1/2}}\mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}}du.$$

Using Proposition 13 leads to $|\partial_x p(r, X_r^N; s, y)| \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}}\exp(-c\frac{|y-X_r^N|^2}{s-r})$, and combining this inequality with the intermediate result (4.2) yields

$$\mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{d/(d+1)} \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}}\left(\frac{s-r}{s}\right)^{\frac{d^2}{2(d+1)}}\exp\left(-c\frac{|y-x|^2}{s}\right). \quad (4.6)$$

Hence, $E_{12}$ is bounded by

$$\frac{K(T)}{s^{\frac{d^2}{2(d+1)}}}\frac{T}{N}\exp\left(-c\frac{|y-x|^2}{s}\right)\int_0^s \frac{1}{r^{1/2}}\frac{1}{(s-r)^{\frac{d+1}{2}-\frac{d^2}{2(d+1)}}}dr.$$

The above integral equals $s^{\frac{1}{2}-\frac{d+1}{2}+\frac{d^2}{2(d+1)}}B(\frac{1}{2}, \frac{1}{2(d+1)})$ where $B$ is the function Beta. Thus $|E_{12}| \leq \frac{K(T)}{s^{d/2}}\frac{T}{N}\exp(-c\frac{|y-x|^2}{s})$, and $E_{12}$ satisfies property $\mathcal{P}$.

### 4.4 Upper bound for $E_2$

We recall $E_2 = \frac{1}{2}\int_0^s \mathbb{E}[\sum_{i,j=1}^d (a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r, X^N_r))\partial^2_{x_ix_j} p(r, X^N_r; s, y)]dr$. As we did for $E_1$, we apply Itô's formula to $a_{ij}(u, X^N_u)$ between $\varphi(r)$ and $r$. We get $a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r, X^N_r) = \int_{\varphi(r)}^r \gamma^{ij}_u du + \int_{\varphi(r)}^r \delta^{ij}_u dW_u$, where $\gamma^{ij}_u$ depends on $\sigma, \partial_t\sigma, \partial_x\sigma, b, \partial^2_{xx}\sigma$ and $\delta^{ij}_u$ is a row vector of size $q$, with $l$-th component $(\delta^{ij}_u)_l = -\sum_{k=1}^d \partial_{x_k} a_{ij}(u, X^N_u)\sigma_{kl}(\varphi(r), X^N_{\varphi(r)})$. Then, the duality formula (3.1) leads to

$$\begin{aligned} E_2 &= \sum_{i,j=1}^d \int_0^s \{\mathbb{E}[\int_{\varphi(r)}^r \partial^2_{x_ix_j} p(r, X^N_r; s, y)\gamma^{ij}_u du + \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial^2_{x_ix_j} p(r, X^N_r; s, y)) \cdot \delta^{ij}_u du]\}dr \\ &:= E_{21} + E_{22}. \end{aligned}$$

**Bound for** $E_{21} = \sum_{ij=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \partial^2_{x_ix_j} p(r, X^N_r; s, y)\gamma^{ij}_u du]dr$**.**
As $\sigma, b, \partial_t\sigma, \partial_x\sigma, \partial^2_x\sigma$ are $C^1_b$ in space, $\gamma^{ij}_u$ has the same smoothness properties as the term $[(\sigma\sigma^*)(\varphi(r), X^N_{\varphi(r)})(\nabla_x b_i(u, X^N_u))^*]_k$ appearing in (4.5). Thus, $E_{21}$ can be treated as $E_{12}$ and satisfies to the same estimate.

**Bound for** $E_{22} = \sum_{i,j=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial^2_{x_ix_j} p(r, X^N_r; s, y)) \cdot \delta^{ij}_u du]dr$.

To rewrite $E_{22}$, we use the expression of $\delta^{ij}_u$ and Proposition 6, which asserts $\mathcal{D}_u(\partial^2_{x_ix_j} p(r, X^N_r; s, y)) = \nabla_x(\partial^2_{x_ix_j} p(r, X^N_r; s, y))\sigma(\varphi(r), X^N_{\varphi(r)})$. Thus,

$$E_{22} = -\sum_{i,j,k=1}^d \int_0^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial^3_{x_ix_jx_k} p(r, X^N_r; s, y)[(\sigma\sigma^*)(\varphi(r), X^N_{\varphi(r)})(\nabla_x a_{ij}(u, X^N_u))^*]_k]du.$$

To complete this proof, we split $E_{22}$ in two terms : $E^1_{22}$ (resp $E^2_{22}$) corresponds to the integral in $r$ from 0 to $\frac{s}{2}$ (resp. from $\frac{s}{2}$ to $s$).

- On $[0, \frac{s}{2}]$, $E^1_{22}$ is bounded by $C\frac{T}{N}\int_0^{\frac{s}{2}} \mathbb{E}[|\partial^3_{x_ix_jx_k} p(r, X^N_r; s, y)|]dr$. Using Proposition 13 and (4.2), it gives

  $$|E^1_{22}| \leq \frac{K(T)T}{N}\frac{1}{s^{d/2}} \exp\left(-c\frac{|x-y|^2}{s}\right)\int_0^{\frac{s}{2}} \frac{1}{(s-r)^{3/2}}dr.$$

  Hence, $E_{22}$ satisfies $\mathcal{P}$.

- On $[\frac{s}{2}, s]$, we use the integration by parts formula (3.2) of Proposition 9, with $|\alpha| = 2$.

  $$E^2_{22} = -\sum_{i,j,k=1}^d \int_{\frac{s}{2}}^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial_{x_i} p(r, X^N_r; s, y)H^i_{e_{jk}}]du,$$

  where $e_{jk}$ is a vector full of zeros except the $j$-th and the $k$-th components. Using Hölder's inequality and (3.3) (remember that $\sigma \in C^{1,3}_b$), we obtain

  $$|E^2_{22}| \leq K(T)\frac{T}{N}\int_{\frac{s}{2}}^s \frac{1}{r}\mathbb{E}[|\partial_x p(r, X^N_r; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}}dr. \tag{4.7}$$

By applying (4.6), we get

$$|E_{22}^2| \leq K(T)\frac{T}{N}\frac{1}{s^{1+\frac{d^2}{2(d+1)}}}\exp\left(-c\frac{|x-y|^2}{s}\right)\int_{\frac{s}{2}}^{s}\frac{1}{(s-r)^{\frac{2d+1}{2d+2}}}dr,$$

and the result follows.

# A Bounds for the transition density function and its derivatives

We bring together classical results related to bounds for the transition probability density of $X$ defined by (1.1).

**Proposition 12** ( Aronson (1967)). *Assume that the coefficients $\sigma$ and $b$ are bounded measurable functions and that $\sigma$ is uniformly elliptic. There exist positive constants $K, \alpha_0, \alpha_1$ s.t. for any $x, y$ in $\mathbb{R}^d$ and any $0 \leq t < s \leq T$, one has*

$$\frac{K^{-1}}{(2\pi\alpha_1(s-t))^{\frac{d}{2}}}e^{-\frac{|x-y|^2}{2\alpha_1(s-t)}} \leq p(t,x;s,y) \leq K\frac{1}{(2\pi\alpha_2(s-t))^{\frac{d}{2}}}e^{-\frac{|x-y|^2}{2\alpha_2(s-t)}}. \tag{A.1}$$

**Proposition 13** ( Friedman (1964)). *Assume that the coefficients $b$ and $\sigma$ are Hölder continuous in time, $C_b^2$ in space and that $\sigma$ is uniformly elliptic. Then, $\partial_x^{m+a}\partial_y^b p(t,x;s,y)$ exist and are continuous functions for all $0 \leq |a| + |b| \leq 2, |m| = 0, 1$. Moreover, there exist two positive constants $c$ and $K$ s.t. for any $x, y$ in $\mathbb{R}^d$ and any $0 \leq t < s \leq T$, one has*

$$|\partial_x^{m+a}\partial_y^b p(t,x;s,y)| \leq \frac{K}{(s-t)^{(|m|+|a|+|b|+d)/2}}\exp\left(-c\frac{|y-x|^2}{s-t}\right).$$

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