Electre Tri-C: A Multiple Criteria Sorting Method Based on Central Reference Actions

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Résumé

On propose dans cet article une nouvelle méthode qui s'insère dans la famille Electre. Elle permet de traiter les problèmes de tri ordinaux lorsque les catégories sont définies par des actions de référence centrales et non pas des profils limites. Nous avons appelé cette nouvelle méthode Electre Tri-C. La méthode appelée jusqu'à présent Electre Tri qui prend appui sur des profils limites qui servent à borner les catégories sera ci-après appelée Electre Tri-B. Après avoir mis en évidence l'intérêt de cette méthode, on introduit les hypothèses fondamentales sur lesquelles elle repose ainsi que les exigences structurelles auxquelles il nous paraît naturel qu'elle satisfasse. Electre Tri-C est constituée de deux procédures dites descendante et ascendante. Celles-ci sont similaires aux procédures pseudo-conjonctive (initialement appelée pessimiste) et pseudo-disjonctive (initialement appelée optimiste) qui constituent Electre Tri-B. Elles conduisent toutefois à des affectations différentes comme on le montre dans cet article.

Mots Clés: Aide multicritère à la décision, Tri ordinal, Méthodes Electre, Actions de référence centrales

Electre Tri-C: A Multiple Criteria Sorting Method
Based on Central Reference Actions

Abstract

In this paper, we propose a new method within the Electre framework. This method deals with sorting problems where the pre-defined and ordered categories are based on central reference actions instead of boundary actions (boundary profiles). We will call this method Electre Tri-C. Therefore, the well-known method called up to now Electre Tri based on boundary actions will be designated here by Electre Tri-B. After setting the interest of this new sorting method, we introduce the assumptions and structural requirements which seem natural to be fulfilled by the method. Electre Tri-C provides two assignment rules: a descending rule and an ascending rule. These rules are quite similar to the pseudo-conjunctive rule (formerly called pessimistic) and the pseudo-disjunctive rule (formerly called optimistic) belonging to Electre Tri-B. Therefore, there exist differences on the assignment results which will be outlined in this paper.

Keywords: Multiple Criteria Decision Aiding, Sorting, Electre methods, Central Reference Actions

1 Introduction

In Multiple Criteria Decision Aiding (MCDA), the analyst can envisage the decisional analysis according to several perspectives, or problematics, which provide an idea of what is expected to be done with the object of a decision (i.e., the possible output) (Roy, 1996; Bouyssou et al., 2006). One of these problematics, the sorting problem, considers a set of categories $C_1, \ldots, C_h, \ldots, C_q$, which are defined *a priori* by those to whom the decision aiding is offered (i.e., the decision-makers).

The categories are defined in order to assign the objects of a decision (i.e., the actions). These actions can be credit demand files, patients waiting for treatment, risk zones, or R&D projects, among the many possibilities, and are subject to treatment or analysis depending on the category to which they are assigned. A credit demand file can be accepted without additional information, accepted subject to additional information, sent to a particular department for further analysis, rejected under certain conditions, or rejected with no conditions at all. Or, based on a set of exams, a patient can be subject to a certain type of medical treatment among those defined *a priori* for the set of pathologies studied. Thus, the set of categories emerges naturally from the decision-aiding context through a process of interaction with the decision-makers.

Many outranking based sorting procedures have been proposed, such as the trichotomic segmentation procedure (Moscarola and Roy, 1977), N-Tomic (Massaglia and Ostanello, 1991), filtering based procedures (Perny, 1998), Proaftn (Belacel, 2000), Trinomfc (Léger and Martel, 2002), multi-profile trichotomic procedure (Norese and Viale, 2002), Iris (Dias et al., 2002b; Dias and Mousseau, 2003), PairClas (Doumpos and Zopounidis, 2004), Smaa-Tri (Tervonen et al., 2007), and a variant of Electre Tri based on "central" profiles (Nemery, 2008) which is significantly different from our approach. However, Electre Tri (Yu, 1992; Roy and Bouyssou, 1993) is still currently one of the most used procedures in MCDA for dealing with the sorting problem, with many applications having been analyzed with Electre Tri over the last fifteen years (see Dimitras et al., 1995; Arondel and Girardin, 2000; Raju et al., 2000; Joerin et al., 2001; Georgopoulou et al., 2003; Merad et al., 2004; André and Roy, 2007).

In Electre Tri the categories are assumed to be ordered from the "worst", C_1 (e.g., the one that contains the worst actions, the lowest priority actions, the most risky actions) to the "best", C_q, where $q \geq 2$ (e.g., the one that contains the best actions, the highest priority actions, the least risky actions). Boundary actions are defined to mark the frontiers between two consecutive ordered categories. Each category is, therefore, delimited by a lower and an upper boundary action. These boundary actions are defined through reference actions that can be realistic or unrealistic. They can be defined either by direct interaction with the decision-maker or by using an aggregation/disaggregation procedure in order to elicit the boundary actions that allow a correct assignment of some training actions previously assigned by the decision-maker to the categories (see Mousseau and Słowiński, 1998; Ngo The and Mousseau, 2002; Dias et al., 2002a; Doumpos and

Zopounidis, 2002; Mousseau and Dias, 2004; Doumpos and Zopounidis, 2007).

Defining reference actions is often a very hard task. This is particularly the case when the decision-maker has a fuzzy idea of the frontier between two consecutive categories. In order to improve the interaction with the decision-maker, the difficulties that may occur of setting boundary actions led us to apprehend the categories by defining central reference actions, where the frontiers between two consecutive categories are not explicitly defined. The procedure proposed in this paper, designated Electre Tri-C, is intended to achieve this goal. Therefore, the well-known method based on boundary actions called up to now Electre Tri will be designated here by Electre Tri-B.

Electre Tri-C is, therefore, a new assignment procedure. Like Electre Tri-B, it assumes that each action to be considered for an assignment to a certain category is evaluated on a coherent family of criteria $g_1, \ldots, g_j, \ldots, g_n$. The assignment of an action only takes into account the intrinsic evaluation of this action on all the criteria and does not depend on nor influence the category to which another action should be assigned. The actions to be assigned are not compared to reference actions that define a lower and an upper bound of the category, but instead are compared to reference actions that we call "central". To perform this new kind of actions comparison, the same outranking credibility indices used in Electre Tri-B are used as they were originally defined (i.e., the same as in Electre III; see, Roy and Bouyssou, 1993).

The rest of this paper is organized as follows. Section 2 introduces and reviews the concepts, definitions, and notation related to the outranking credibility indices. Section 3 is devoted to the proposed Electre Tri-C method which contains the basic assumptions and the two assignment rules. Section 4 presents the properties of the Electre Tri-C assignment rules. Section 5 presents an overview of Electre Tri-B and a comparison between Electre Tri-C and Electre Tri-B. Section 6 provides a numerical example of Electre Tri-C. Section 7 presents some additional results such as the comparison of the two Electre Tri-C assignment rules. Finally, the last section offers our concluding remarks and some avenues for future research.

2 The credibility index: Definitions and notation

Let $a_1, a_2, \ldots$ denote the *potential actions*. The set of such actions, A, can be partially known *a priori*, and the actions can appear progressively during the decision aiding process. The objective is to assign the actions to a set of *ordered categories* C_1, ..., C_h, ..., C_q, with $q \geq 2$, the nature of which was provided in the previous section. Suppose that a coherent family F of n *criteria* $g_1, \ldots, g_j, \ldots, g_n$, with $n \geq 2$, has been defined in order to evaluate any action considered for assignment to a certain category.

Let us consider the criterion $g_j \in F$ and two actions a and a'. Taking into account the preference direction of this criterion, the *advantage* of action a over action a' is defined as follows:

$$\Omega_j(a, a') = \begin{cases} g_j(a) - g_j(a') & \text{if } g_j \text{ is to be maximized} \\ g_j(a') - g_j(a) & \text{if } g_j \text{ is to be minimized} \end{cases} \tag{2.1}$$

Below, each criterion g_j will be considered as a *pseudo-criterion*, which means that two thresholds are associated to g_j: an *indifference threshold*, q_j, and a *preference threshold*, p_j, such that $p_j \geq q_j \geq 0$. These thresholds are introduced in order to take into account the imperfect character of the data from the computation of the performances $g_j(a)$ as well as the arbitrariness that affects the definition of the criteria. Based on the definition of such thresholds, their values should be interpreted as follows:

1) $|\Omega_j(a, a')| \leq q_j$ represents a non-significant advantage of a over a', meaning that a is *indifferent* to a' according to g_j, denoted aI_ja'.

2) $\Omega_j(a, a') > p_j$ represents a significant advantage of a over a', meaning that a is *strictly preferred* to a' according to g_j, denoted aP_ja'.

3) $q_j < \Omega_j(a, a') \leq p_j$ represents an ambiguity zone. The advantage of a over a' is a little large to conclude about an indifference between a and a', but this advantage is not enough to conclude about a strict preference in favour of a. This means that there is a hesitation between indifference and strict preference. In such a case, a is *weakly preferred* to a', denoted aQ_ja'.

Let us notice that q_j can be null and/or equal to p_j.

The indifference and preference have been presented as constants. However, in practice, they can vary according to the performances $g_j(a)$ or $g_j(a')$. In order to simplify the Electre Tri-C method the basic formulae (2.2) and (2.6) have been written with constants thresholds. The way to generalized these formulae for taking into account variable thresholds, which is often absolutely required in several Case Studies, is analyzed in the Appendix A.1 (for more details, see Roy and Vincke, 1984 and Roy, 1996, p. 184-194).

When using the outranking concept, the main idea is that an action a *outranks* an action a' according to the criterion g_j (denoted aS_ja') if a is judged *at least as good as* a' on the criterion g_j. This is true without ambiguity when $\Omega_j(a, a') \geq -q_j$. But, when $-p_j \leq \Omega_j(a, a') < -q_j$, the possibility of indifference between a and a' cannot be excluded. This indifference is less and less credible when $\Omega_j(a, a')$ moves closer to $-p_j$. From this point of view, the *credibility indices* $c_j(a, a')$, or the *partial concordance indices*, of an outranking of a over a' are defined as follows:

$$c_j(a, a') = \begin{cases} 0 & \text{if } \Omega_j(a, a') < -p_j \\ \frac{\Omega_j(a,a') + p_j}{p_j - q_j} & \text{if } -p_j \leq \Omega_j(a, a') < -q_j \\ 1 & \text{if } \Omega_j(a, a') \geq -q_j \end{cases} \tag{2.2}$$

Let us notice that despite the way the value of $c_j(a, a')$ is modeled within the "small range" $[-p_j, -q_j[$, this value is only related to the ordinal definition of the criterion g_j.

Finally, the credibility of the comprehensive outranking of a over a', meaning that a may be judged at least as good as a' when taking all the criteria from F into account, is defined as follows. Let $\sigma(a,a')$ denote such a credibility index.

$$\sigma(a,a') = c(a,a') \prod_{j=1}^{n} T_j(a,a') \tag{2.3}$$

where

$$T_j(a,a') = \begin{cases} \frac{1 - d_j(a,a')}{1 - c(a,a')} & \text{if } d_j(a,a') > c(a,a') \\ 1 & \text{otherwise} \end{cases} \tag{2.4}$$

This credibility index aggregates a *comprehensive concordance index*, $c(a,a')$, and the *partial discordance indices*, $d_j(a,a')$, $j = 1,\dots,n$. These two types of indices make use of two more parameters associated with each criterion g_j, $j = 1,\dots,n$: the *weights*, denoted w_j, where $w_j > 0$, and the *veto thresholds*, denoted v_j, such that $v_j \geq p_j$. When using variable thresholds, see the Appendix A.1 for more details. In the following, assume without loss of generality, that $\sum_{j=1}^{n} w_j = 1$.

$$c(a,a') = \sum_{j=1}^{n} w_j \; c_j(a,a') \tag{2.5}$$

$$d_j(a,a') = \begin{cases} 0 & \text{if } \Omega_j(a,a') \geq -p_j \\ \frac{\Omega_j(a,a') + p_j}{p_j - v_j} & \text{if } -v_j \leq \Omega_j(a,a') < -p_j \\ 1 & \text{if } \Omega_j(a,a') < -v_j \end{cases} \tag{2.6}$$

See, for example, (Roy, 1991; Yu, 1992; Roy and Bouyssou, 1993) for more details about the original formulae as well as their interpretations. Different variants, or extensions, for such indices have been proposed without changing the fundamental properties (see, for example, Mousseau and Dias, 2004; Figueira et al., 2006; Roy and Słowiński, 2008). The definitions and results presented in the next sections remain valid with these variants, or extensions.

3 Problem statement and assignment rules

The aim of this section is to present the Electre Tri-C method, including the basic assumptions, the structural requirements, and the assignment rules.

3.1 Basic assumptions and structural requirements

Let b_h denote a *central reference action* introduced to characterize category C_h. Assume that the actions b_h, $h = 1,\dots,q$, have been defined through an interaction procedure with the decision-maker. Notice that C_1 is the worst category and C_q the best one, with $q \geq 2$. Therefore, let $B = \{b_0, b_1, \dots, b_h, \dots, b_q, b_{q+1}\}$ denote the set of $(q+2)$ reference actions, where b_0 and b_{q+1} are two particular reference actions defined as follows: $g_j(b_0)$ is the worst possible performance on criterion g_j, and $g_j(b_{q+1})$ is the best possible performance on the same criterion g_j, for all $g_j \in F$.

Consider two reference actions, b_h and b_{h+1}. According to the ordered character of the categories, it does not seem restrictive to assume that b_{h+1} strictly dominates b_h. Let $b_{h+1} \Delta_F b_h$ denote the *(strict) dominance relation*, such that

$$\forall j,\ \Omega_j(b_{h+1}, b_h) \geq 0 \text{ and } \exists j,\ \Omega_j(b_{h+1}, b_h) > 0; h = 0, \dots, q \tag{3.7}$$

However, the intuition lead us to think that if $\Omega_j(b_{h+1}, b_h)$ is too small with respect to the indifference and preference thresholds, there is the possibility of having a certain ambiguity on the assignment of some actions to categories C_h and C_{h+1}. It will be proved in Section 4 that this is true for Electre Tri-C when the following condition is not verified.

Condition 1 (Strict separability)
The set of reference actions, B, fulfills the strict separability condition if and only if

$$\Omega_j(b_{h+1}, b_h) > p_j,\ j = 1, \dots, n;\ h = 0, \dots, q \tag{3.8}$$

When using variable thresholds, see the Appendix A.1 for more details. If the strict separability condition holds, then $c_j(b_{h+1}, b_h) = 1$ and $c_j(b_h, b_{h+1}) = 0$, for all $g_j \in F$. Since there is no discordance on all criteria, $\sigma(b_{h+1}, b_h) = 1$ and $\sigma(b_h, b_{h+1}) = 0$.

For further analysis of the impact of the (strict) dominance relation and the strict separability condition on the results as well as the consistency of the assignment rules based on central reference actions, it seems natural to introduce the following structural requirements.

Definition 1 (Structural requirements)

1) *Conformity: Each central reference action, b_h, must be assigned to the category, C_h, $h = 1, \dots, q$.*

2) *Monotonicity: If an action a strictly dominates a', then a is assigned to a category at least as good as the category a' is assigned to.*

3) *Homogeneity: Two actions must be assigned to the same category when they compare themselves in an identical manner with respect to the reference actions.*

4) *Stability: After a modification of the set B by applying either a merging or a splitting procedure (see Definition 2), the non-adjacent categories to the modified ones will remain with the same actions as before the modification. More precisely:*

a) After merging two consecutive categories, any action a previously assigned to the non-modified categories will remain in the same category. Moreover, the actions previously assigned to the merged categories will be assigned either to the new category or to one of the two adjacent categories.

b) After splitting the category C_h into two new consecutive categories, any action that was not previously assigned to C_{h+1} nor to C_{h-1} will remain in the same category. Furthermore, the actions previously assigned to the former category C_h will be assigned to one of the two new categories. Moreover, an action previously assigned to C_{h+1} will be assigned either to the same category or to the best of the two new categories. Similarly, an action previously assigned to C_{h-1} will be assigned either to the same category or to the worst of the two new categories.

Definition 2 (Basic modification procedures)

1) Merging procedure: The distinction between two consecutive categories, C_{h-1} and C_h, will be ignored by introducing a new central reference action, b'_h, such that $\Omega_j(b'_h, b_{h-1}) \geq 0$ and $\Omega_j(b_h, b'_h) \geq 0$, for all $g_j \in F$.

2) Splitting procedure: The category C_h will be split into two new consecutive categories by introducing two new central reference actions, b'_h and b''_h, such that b_{h+1} strictly dominates b''_h, b''_h strictly dominates b'_h, b'_h strictly dominates b_{h-1}, $\Omega_j(b''_h, b_h) \geq 0$, and $\Omega_j(b_h, b'_h) \geq 0$, for all $g_j \in F$.

It should be noticed that adding or removing a category are particular cases of these two basic procedures.

3.2 Electre Tri-C assignment rules

As for the definition of the assignment rules, it is useful to introduce the concept of *slackness functions* as well as the related properties as follows.

Definition 3 (Slackness functions)
Let $\lambda \in [0.5, 1]$ denote the chosen majority level:

1) Direct slackness function: $\xi_h^+(a, \lambda) = \sigma(a, b_h) - \lambda$, $h = (q+1), \ldots, 0$.

2) Reverse slackness function: $\xi_h^-(a, \lambda) = \sigma(b_h, a) - \lambda$, $h = 0, \ldots, (q+1)$.

Proposition 1

a) The direct slackness function does not decrease when moving from a given category to a worst one.

b) The reverse slackness function does not decrease when moving from a given category to a best one.

Proof

a) When moving from a given category to a worst one, the direct slackness function $\xi_h^+(a,\lambda)$ does not decrease because the credibility indices $\sigma(a,b_h)$ are non-increasing functions on the set B.

b) When moving from a given category to a best one, the reverse slackness function $\xi_h^-(a,\lambda)$ does not decrease because the credibility indices $\sigma(b_h,a)$ are non-decreasing functions on the set B. □

Therefore, two assignment rules for Electre Tri-C are defined as follows.

Definition 4 (Descending assignment rule)
Choose a majority level λ ($0.5 \leq \lambda \leq 1$). Decrease h from $(q+1)$ until the first value t such that $\xi_t^+(a,\lambda) \geq 0$. If $t > 0$ and $\xi_t^+(a,\lambda) \leq |\xi_{t+1}^+(a,\lambda)|$, then assign action a to category C_t. Otherwise, assign a to C_{t+1}.

Taking into account that for all a, $\xi_{q+1}^+(a,\lambda) < 0$ and $\xi_0^+(a,\lambda) \geq 0$, according to Proposition 1.a there exists necessarily a value t such that $\xi_t^+(a,\lambda) \geq 0$ and $\xi_{t+1}^+(a,\lambda) < 0$. Thus, any action a is assigned to a *unique category* by the descending rule.

Definition 5 (Ascending assignment rule)
Choose a majority level λ ($0.5 \leq \lambda \leq 1$). Increase h from 0 until the first value t such that $\xi_t^-(a,\lambda) \geq 0$. If $t < (q+1)$ and $\xi_t^-(a,\lambda) \leq |\xi_{t-1}^-(a,\lambda)|$, then assign action a to category C_t. Otherwise, assign a to C_{t-1}.

Taking into account that for all a, $\xi_0^-(a,\lambda) < 0$ and $\xi_{q+1}^-(a,\lambda) \geq 0$, according to Proposition 1.b there exists necessarily a value t such that $\xi_t^-(a,\lambda) \geq 0$ and $\xi_{t-1}^-(a,\lambda) < 0$. Thus, any action a is assigned to a *unique category* by the ascending rule.

Notice that the assignment of a potential action a is *independent* from any others.

Remark 1 (A mirror equivalence)
Let F^* denote the coherent family of criteria g_j^* obtained from g_j through the inversion of the preference direction, for $j = 1, \ldots, n$. Let $\sigma^*(a,b_h)$ and $\xi_h^{*+}(a,\lambda)$ denote the credibility indices and the direct slackness functions, respectively. It is trivial to verify that, for all a and b_h:

1) $\sigma^*(a,b_h) = \sigma(b_h,a)$

2) $\xi_h^{*+}(a,\lambda) = \xi_h^-(a,\lambda)$

When using the new family of criteria, F^*, the category C_1 becomes the best category and C_q the worst one. Let $C_h^* = C_{q+1-h}$, for $h = 1, \ldots, q$, and $b_h^* = b_{q+1-h}$, for $h = 0, \ldots, (q+1)$. When the Electre Tri-C *descending rule is applied to these new categories, the direct slackness function $\xi_h^{*+}(a, \lambda)$ is used as the reverse slackness function $\xi_h^-(a, \lambda)$ in the* Electre Tri-C *ascending rule when this method is applied to the initial categories.*

This equivalence shows a way to replace the descending rule by the ascending rule. It will be referred in what follows as a "transposition in the mirror".

4 Properties of the assignment rules

The aim of this section is to analyze the properties of the Electre Tri-C assignment rules according to the conditions imposed to the set of reference actions, B.

Theorem 1

a) The monotonicity, homogeneity, and stability requirements hold.

b) If the strict separability condition is fulfilled, then the conformity requirement holds.

The following proof is done for the descending rule. It remains valid for the ascending rule by the transposition in the mirror (see Remark 1).

Proof

a.1) Monotonicity: $a\Delta_F a' \Rightarrow \xi_k^+(a, \lambda) \geq \xi_k^+(a', \lambda)$, $k = q, \ldots, 1$. Therefore, according to the descending rule if h is the first value of k such that $\xi_k^+(a, \lambda) \geq 0$ and h' the first value of k such that $\xi_k^+(a', \lambda) \geq 0$, then one necessarily has $h \geq h'$. If $h > h'$, then a is assigned either to the same category as a' is assigned to or to a better category. If $h = h'$, then the monotonicity would not be verified only when a is assigned to C_h and a' is assigned to C_{h+1}. Let us prove that this is impossible. Action a' is assigned to C_{h+1} if and only if $\xi_h^+(a', \lambda) > |\xi_{h+1}^+(a', \lambda)|$. Since $\xi_h^+(a', \lambda) = \sigma(a', b_h) - \lambda$ and $|\xi_{h+1}^+(a', \lambda)| = \lambda - \sigma(a', b_{h+1})$, then $\sigma(a', b_h) + \sigma(a', b_{h+1}) > 2\lambda$. Similarly, a is assigned to C_h if and only if $\sigma(a, b_h) + \sigma(a, b_{h+1}) \leq 2\lambda$. Therefore, $\sigma(a, b_h) + \sigma(a, b_{h+1}) \leq 2\lambda < \sigma(a', b_h) + \sigma(a', b_{h+1})$. This is impossible because $a\Delta_F a'$ means that $\sigma(a, b_h) \geq \sigma(a', b_h)$ and $\sigma(a, b_{h+1}) \geq \sigma(a', b_{h+1})$.

a.2) Homogeneity: By definition two different actions, a and a', are compared themselves in an identical manner with the reference actions if and only if the following conditions are verified: $\sigma(a, b_h) = \sigma(a', b_h)$ and $\sigma(b_h, a) = \sigma(b_h, a')$, for all $h = 1, \ldots, q$. Therefore, the homogeneity condition is verified because the assignment of an action a to a category C_h by the descending rule only depends on $\sigma(a, b_h)$.

a.3) Stability under a merging procedure: Assume that the consecutive categories C_{h-1} and C_h are merged into only one category, denoted C'_h. Let b'_h denote the central reference action introduced to characterize the new category C'_h. From the conditions imposed to b'_h according to the merging procedure (Definition 2.1), the new set B' obtained from B by replacing b_h and b_{h-1} to b'_h leads to $b_{h+1}\Delta_F b'_h$ and $b'_h\Delta_F b_{h-2}$. According to the descending rule, we will prove successively that:

1) *If a was previously assigned to category C_k, $k \geq (h+1)$, then a will be assigned to the same category, after modification.*
2) *If a was previously assigned to category C_k, $k \leq (h-2)$, then a will be assigned to the same category, after modification.*
3) *If a was previously assigned either to C_h or C_{h-1}, then a will be assigned either to the new category C'_h or to one of the two adjacent categories to the modified ones, i.e., C_{h-2} and C_{h+1}, after modification.*

Let us prove these three cases:

1) *Any action a such that $\xi^+_{h+1}(a,\lambda) \geq 0$ is assigned to one category C_k, $k \geq (h+1)$. It is clear that a merging procedure does not change the assignment of a. Thus, on the one hand, a was previously assigned to C_k, $k > (h+1)$ if and only if $\xi^+_{h+1}(a,\lambda) \geq 0$. On the other hand, a could have been assigned to C_{h+1} with $\xi^+_{h+1}(a,\lambda) < 0$ when $\xi^+_h(a,\lambda) > |\xi^+_{h+1}(a,\lambda)|$. Let us prove that according to these conditions a remains assigned to C_{h+1} after modification. Let $\xi'^+_h(a,\lambda) = \sigma(a,b'_h) - \lambda$. After modification a will be assigned to C_{h+1} if and only if $\xi'^+_h(a,\lambda) > |\xi^+_{h+1}(a,\lambda)|$. This inequality is necessarily verified since $\xi'^+_h(a,\lambda) \geq \xi^+_h(a,\lambda)$ because $\Omega_j(b_h,b'_h) \geq 0 \Rightarrow \sigma(a,b'_h) \geq \sigma(a,b_h)$.*
2) *Any action a such that $\xi^+_{h-2}(a,\lambda) < 0$ is assigned to one category C_k, $k \leq (h-2)$. It is clear that a merging procedure does not change the assignment of a. Thus, on the one hand, a was previously assigned to C_k, $k < (h-2)$ if and only if $\xi^+_{h-2}(a,\lambda) < 0$. On the other hand, a could have been assigned to C_{h-2} with $\xi^+_{h-2}(a,\lambda) \geq 0$ when $\xi^+_{h-1}(a,\lambda) < 0$ and $\xi^+_{h-2}(a,\lambda) \leq |\xi^+_{h-1}(a,\lambda)|$. Let us prove that according to these conditions a remains assigned to C_{h-2} after modification. Since $\Omega_j(b'_h,b_{h-1}) \geq 0$ one obtains $\sigma(a,b'_h) \leq \sigma(a,b_{h-1})$. Thus, $\xi'^+_h(a,\lambda) \leq \xi^+_{h-1}(a,\lambda)$. This latter quantity being negative, then $|\xi'^+_h(a,\lambda)| \geq |\xi^+_{h-1}(a,\lambda)|$. Therefore, $\xi^+_{h-2}(a,\lambda) \leq |\xi'^+_h(a,\lambda)|$, which proves that a will remain assigned to category C_{h-2}, after modification.*
3) *From the two above paragraphs, only the actions previously assigned to the former categories C_{h-1} and C_h could be assigned to the new category C'_h after modification. But, nothing proves that all of those actions are assigned to the new category. Some of them can be assigned to one of the two adjacent categories (if they exist), C_{h-2} and C_{h+1}, after modification.*

i) *Consider an action a previously assigned to* C_{h-1} *with* $\xi^+_{h-1}(a,\lambda) < 0$ *and* $\xi^+_{h-2}(a,\lambda) > |\xi^+_{h-1}(a,\lambda)|$*. From 2), according to these conditions,* $|\xi'^+_h(a,\lambda)| \geq |\xi^+_{h-1}(a,\lambda)|$*. Thus, an action a can verify the following condition:* $|\xi'^+_h(a,\lambda)| > \xi^+_{h-2}(a,\lambda)$*. In such a case, a will be assigned to* C_{h-2} *after modification.*

ii) *Consider an action a previously assigned to* C_h *with* $\xi^+_h(a,\lambda) \geq 0$ *and* $\xi^+_h(a,\lambda) \leq |\xi^+_{h+1}(a,\lambda)|$*. Since* $\Omega_j(b_h, b'_h) \geq 0$ *one obtains* $\sigma(a, b'_h) \geq \sigma(a, b_h)$*. Thus,* $\xi'^+_h(a,\lambda) \geq \xi^+_h(a,\lambda)$*. This latter quantity being positive, the following condition can hold:* $\xi'^+_h(a,\lambda) > |\xi^+_{h+1}(a,\lambda)|$*. In such a case, a will be assigned to* C_{h+1} *after modification.*

a.4) *Stability under a splitting procedure: Assume that the category* C_h *is split into two new consecutive categories, denoted* C'_h *and* C''_h*. Let* b'_h *denote the central reference action introduced to characterize the worst of the two new categories,* C'_h*, and* b''_h *the central reference action introduced to characterize the best of the two new categories,* C''_h*. According to the descending rule, we will prove successively that:*

1) *If a was previously assigned to* C_k*,* $k \neq \{h-1, h, h+1\}$*, then a will be assigned to the same category, after modification.*
2) *If a was previously assigned to the former category* C_h*, then a will be assigned to one of the new categories,* C'_h *or* C''_h*, after modification.*
3) *If a was previously assigned to* C_{h+1}*, then a will be assigned either to the same category or to the best of the two new categories,* C''_h*, after modification.*
4) *If a was previously assigned to* C_{h-1}*, then a will be assigned either to the same category or to the worst of the two new categories,* C'_h*, after modification.*

Let us prove these four cases:

1) *The proof is similar to the first two cases of the merging procedure, when a was previously assigned either to category* C_k*,* $k \geq (h+2)$ *or to* C_k*,* $k \leq (h-2)$*.*
2) *Consider an action previously assigned to* C_h*. The following two cases must be analyzed:*

   i) $\xi^+_h(a,\lambda) \geq 0$ *and* $\xi^+_{h+1}(a,\lambda) < 0$*, when* $\xi^+_h(a,\lambda) \leq |\xi^+_{h+1}(a,\lambda)|$*. Since* $\Omega_j(b''_h, b_h) \geq 0$ *one obtains* $\xi''^+_h(a,\lambda) \leq \xi^+_h(a,\lambda)$*. Since* $\Omega_j(b_{h+1}, b''_h) \geq 0$ *one obtains* $\xi^+_{h+1}(a,\lambda) \leq \xi''^+_h(a,\lambda)$*. Therefore,* $\xi^+_{h+1}(a,\lambda) \leq \xi''^+_h(a,\lambda) \leq \xi^+_h(a,\lambda)$*. When* $\xi''^+_h(a,\lambda) \geq 0$*, if* $\xi''^+_h(a,\lambda) \leq |\xi^+_{h+1}(a,\lambda)|$*, then a will be assigned to* C''_h*, after modification. Otherwise, if* $\xi''^+_h(a,\lambda) < 0$*, then a will be assigned at most to* C''_h*.*

   ii) $\xi^+_h(a,\lambda) < 0$ *and* $\xi^+_{h-1}(a,\lambda) \geq 0$*, when* $\xi^+_{h-1}(a,\lambda) > |\xi^+_h(a,\lambda)|$*. Since* $\Omega_j(b_h, b'_h) \geq 0$ *one obtains* $\xi'^+_h(a,\lambda) \geq \xi^+_h(a,\lambda)$*. Since* $\Omega_j(b'_h, b_{h-1}) \geq 0$ *one obtains* $\xi'^+_h(a,\lambda) \leq \xi^+_{h-1}(a,\lambda)$*. Therefore,* $\xi^+_h(a,\lambda) \leq \xi'^+_h(a,\lambda) \leq \xi^+_{h-1}(a,\lambda)$*.*

When $\xi^+_{h-1}(a,\lambda) < 0$*, if* $\xi^+_{h-1}(a,\lambda) > |\xi'^+_h(a,\lambda)|$*, then a will be assigned to* C'_h*, after modification. Otherwise, if* $\xi'^+_h(a,\lambda) \geq 0$*, then a will be assigned at least to* C'_h*.*

3) *Consider an action previously assigned to* C_{h+1}*. The following two cases must be analyzed:*

   i) $\xi^+_{h+1}(a,\lambda) \geq 0$ *and* $\xi^+_{h+2}(a,\lambda) < 0$*, when* $\xi^+_{h+1}(a,\lambda) \leq |\xi^+_{h+2}(a,\lambda)|$*. In such a case, it is trivial to verify that a will remain in the same category, after modification.*

   ii) $\xi^+_{h+1}(a,\lambda) < 0$ *and* $\xi^+_h(a,\lambda) \geq 0$*, when* $\xi^+_h(a,\lambda) > |\xi^+_{h+1}(a,\lambda)|$*. From 2.i) with the same conditions it was proved that* $\xi^+_{h+1}(a,\lambda) \leq \xi''^+_h(a,\lambda) \leq \xi^+_h(a,\lambda)$*. According to these inequalities the following condition holds:* $\xi''^+_h(a,\lambda) \leq |\xi^+_{h+1}(a,\lambda)|$*. In such a case, a will be assigned to category* C''_h*.*

4) *Consider an action previously assigned to* C_{h-1}*. The following two cases must be analyzed:*

   i) $\xi^+_{h-1}(a,\lambda) \geq 0$ *and* $\xi^+_h(a,\lambda) < 0$*, when* $\xi^+_{h-1}(a,\lambda) \leq |\xi^+_h(a,\lambda)|$ *From 2.ii) with the same conditions it was proved that* $\xi^+_h(a,\lambda) \leq \xi'^+_h(a,\lambda) \leq \xi^+_{h-1}(a,\lambda)$*. According to these inequalities, the following condition holds:* $\xi^+_{h-1}(a,\lambda) > |\xi'^+_h(a,\lambda)|$*. In such a case, a will be assigned to category* C'_h*.*

   ii) $\xi^+_{h-1}(a,\lambda) < 0$ *and* $\xi^+_{h-2}(a,\lambda) \geq 0$*, when* $\xi^+_{h-2}(a,\lambda) > |\xi^+_{h-1}(a,\lambda)|$*. In such a case, it is trivial to verify that a will remain in the same category, after modification.*

b) *Conformity: Assume that the strict separability condition holds. By construction of the credibility indices:* $\sigma(b_h,b_h) = 1$*,* $\sigma(b_{h+1},b_h) = 1$*, and* $\sigma(b_h,b_{h+1}) = 0$*, for all* $h = 0,\ldots,q$*. Therefore, from the direct slackness function (Definition 3.1)* $\xi^+_h(b_h,\lambda) = 1-\lambda \geq 0$ *and* $\xi^+_{h+1}(b_h,\lambda) = -\lambda < 0$*. When applying the descending rule, each central reference action* b_h *is assigned to* C_h *if and only if* $\xi^+_h(b_h,\lambda) \leq |\xi^+_{h+1}(b_h,\lambda)|$ *which is equivalent to* $1-\lambda \leq \lambda$*. This is true because* $\lambda \geq \frac{1}{2}$*. Thus, the conformity of the reference actions is always verified.* □

The decision-maker can have some good reasons to introduce a set of reference actions, B, that does not fulfill the strict separability condition, for certain ordered pairs (b_{h+1}, b_h). As noticed in Section 3.1, when considering the possible minimum differences in the performances, the strict dominance condition is not enough to clearly separate the categories. Indeed, the weak separability condition below defines the minimum differences in the performances which seem to us adequate to impose between two consecutive central reference actions.

Condition 2 (Weak separability)
The set of reference actions, B, fulfills the weak separability condition if and only if

$$\forall j,\ \Omega_j(b_{h+1}, b_h) \geq 0 \text{ and } \exists j,\ \Omega_j(b_{h+1}, b_h) > p_j; h = 0, \ldots, q. \tag{4.9}$$

When using variable thresholds, see the appendix A.1 for more details. The weak separability condition allows the existence of some criteria g_j such that $0 \leq \Omega_j(b_{h+1}, b_h) \leq p_j$. In such criteria, one obtains $c_j(b_{h+1}, b_h) = 1$ and $c_j(b_h, b_{h+1}) \leq 1$. Therefore, as there is no discordance, if the weak separability condition is fulfilled, then $\sigma(b_{h+1}, b_h) = 1$ and $\sigma(b_h, b_{h+1}) \leq 1$. As noticed in Section 3.1, when the strict separability condition is fulfilled, then $\sigma(b_{h+1}, b_h) = 1$ and $\sigma(b_h, b_{h+1}) = 0$.

According to Theorem 1, the conformity requirement holds if the strict separability condition is fulfilled. Let us analyze in which conditions such a requirement holds when the strict separability condition is not fulfilled. Indeed, when the conformity requirement does not hold, the assignment model becomes inconsistent and, therefore, it is not able to support the assignment of the potential actions.

Theorem 2
If the weak separability condition is fulfilled, then there exists a compatible majority level, λ^c, for which the conformity requirement holds, whenever the chosen majority level $\lambda \geq \lambda^c$, such that

$$\lambda^c = \frac{1}{2} + \frac{1}{2} \max_{h\,=\,0,\,\ldots,\,q} \left\{ \sigma(b_h, b_{h+1}) \right\} \tag{4.10}$$

The proof is done for the descending rule. It remains valid for the ascending rule by the transposition in the mirror (see Remark 1).

Proof *According to the descending rule $\sigma(b_h, b_h) = 1 \Rightarrow \xi_h^+(b_h, \lambda) = 1 - \lambda \geq 0$ and b_h is assigned to a category at least as good as C_h. The central reference action b_h is assigned to C_h if and only if $\xi_{h+1}^+(b_h, \lambda) = \sigma(b_h, b_{h+1}) - \lambda < 0$ and $\xi_h^+(b_h, \lambda) \leq |\xi_{h+1}^+(b_h, \lambda)|$ which are equivalent to the following two inequalities:*

$$\lambda > \sigma(b_h, b_{h+1}) \text{ and } 1 - \lambda \leq \lambda - \sigma(b_h, b_{h+1}) \tag{4.11}$$

We have shown above that with the weak separability condition one obtains $\sigma(b_h, b_{h+1}) < 1$. It follows that

$$\frac{1}{2} + \frac{1}{2}\sigma(b_h, b_{h+1}) \geq \sigma(b_h, b_{h+1}) \tag{4.12}$$

Therefore, if the second inequality of 4.11 is true, then the first one is also true, and consequently b_h is assigned to C_h. Taking all central reference actions b_h, $h = 1, \ldots, q$, into account, the conformity requirement holds if and only if

$$\lambda \geq \frac{1}{2} + \frac{1}{2} \max_{h\,=\,0,\,\ldots,\,q} \left\{ \sigma(b_h, b_{h+1}) \right\} \tag{4.13}$$

□

Let us analyze the impact of the two separability conditions on the two basic modification procedures defined for the stability requirement (Definition 2):

1) Let B denote a set of reference actions emerged from a practical situation with a compatible conformity majority level λ^c, and λ denote the chosen majority level such that $\lambda \geq \lambda^c$. The value of λ will remain at least as good as λ'^c, where λ'^c is the conformity majority level associated with B' obtained after applying a merging procedure (see $a.3$ in the Proof of Theorem 1).

2) The splitting procedure can provide different conclusions in comparison with the merging procedure (see $a.4$ in the Proof of Theorem 1). Indeed, after applying a splitting procedure, if $\lambda < \lambda'^c$, then the conformity requirement does not hold. Therefore, one can easily prove that:

   - according to the descending procedure, b'_h can be assigned to C''_h instead of C'_h.
   - according to the ascending procedure, b''_h can be assigned to C'_h instead of C''_h.

Remark 2
When comparing an action a to the reference actions b_h, $h = 0, 1, \ldots, (q+1)$, it is also useful to analyze the set of reference actions, B, as follows:

1) *If $\xi^+_q(a, \lambda) \geq 0$, then the action a is assigned to C_q by the* Electre Tri-C *descending rule. This means that the action a is judged "very good" in comparison to all central reference actions in B. Moreover, if the actions are systematically assigned to C_q, then a deeply analysis must be done in order to conclude about the under-evaluation of the set B, or to assume that all potential actions are really very good.*

2) *If $\xi^-_1(a, \lambda) \geq 0$, then the action a is assigned to C_1 by the* Electre Tri-C *ascending rule. This means that the action a is judged "very poor" in comparison to all central reference actions in B. Moreover, if the actions are systematically assigned to C_1, then a deeply analysis must be done in order to conclude about the over-evaluation of the set B, or to assume that all potential actions are really very poor.*

3) *From the two above cases, when a deeply analysis is required we can start to choose a different majority level, if possible, since the previous one has been chosen too high.*

5 Comparison with Electre Tri-B

This section presents an overview of Electre Tri-B and a comparison between the Electre Tri-C and Electre Tri-B assignment results.

5.1 An overview of Electre Tri-B

According to (Yu, 1992) and (Roy and Bouyssou, 1993, p. 389-401), the assignment of an action a by Electre Tri-B is based on pairwise comparisons between the action a and the boundary actions which characterize the pre-defined and ordered categories. The set of such categories is denoted here $\widehat{C} = \{\widehat{C}_1, \ldots, \widehat{C}_h, \ldots, \widehat{C}_q\}$, where $\widehat{C}_1$ is the worst category and $\widehat{C}_q$ is the best one, with $q \geq 2$. Each category $\widehat{C}_h$ is defined by a lower boundary action, $\hat{b}_{h-1}$, and an upper boundary action, $\hat{b}_h$, such that $\hat{b}_h \Delta_F \hat{b}_{h-1}$, $h = 1, \ldots, q$. Let $\widehat{B} = \{\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_h, \ldots, \hat{b}_q\}$ denote the set of the $(q+1)$ boundary actions. Furthermore, the role played by $\hat{b}_0$ and $\hat{b}_q$ when using boundary actions is the same as b_0 and b_{q+1} with central reference actions.

When using the slackness functions (Definition 3), the most well-known assignment rules of Electre Tri-B (formerly called pessimistic and optimistic, respectively) are rewritten as follows. (Roy, 2002) showed that it would be suitable to replace pessimistic by pseudo-conjunctive and optimistic by pseudo-disjunctive.

Definition 6 (Pseudo-conjunctive assignment rule)
Choose a majority level λ ($0.5 \leq \lambda \leq 1$). Decrease h from q until the first value such that $\xi^+_{h-1}(a, \lambda) \geq 0$. Assign action a to category $\widehat{C}_h$.

Definition 7 (Pseudo-disjunctive assignment rule)
Choose a majority level λ ($0.5 \leq \lambda \leq 1$). Increase h from 0 until the first value such that $\xi^-_h(a, \lambda) \geq 0$ and $\xi^+_h(a, \lambda) < 0$. Assign action a to category $\widehat{C}_h$.

Based on the above Electre Tri-B assignment rules, if an action a is assigned to category $\widehat{C}_k$ by the pseudo-conjunctive rule and to $\widehat{C}_h$ by the pseudo-disjunctive rule, then it was proved that $k \leq h$ (see Roy and Bouyssou, 1993, p. 395). Furthermore, the two assignment rules provide the same results if and only if there is no t such that $\xi^+_t(a, \lambda) < 0$ and $\xi^-_t(a, \lambda) < 0$ or there is at most one t such that $\xi^+_t(a, \lambda) \geq 0$ and $\xi^-_t(a, \lambda) \geq 0$.

5.2 Comparing the assignment results

When applying Electre Tri-C, each category is characterized by a central reference action. In Electre Tri-B, each category is delimited by a lower and an upper boundary action. Theorem 3 allows to compare the assignment results of Electre Tri-C and Electre Tri-B taking into account either the descending rule and the pseudo-conjunctive rule, respectively, or the ascending rule and the pseudo-disjunctive rule, respectively.

Theorem 3

1) *Consider* $(q+2)$ *reference actions defined to apply* Electre Tri-C *with* q *categories. When such reference actions are used as the boundary actions of the* $(q+1)$ *categories in* Electre Tri-B,

   a) *if an action* a *is assigned to* C_h *by the* Electre Tri-C *descending rule, then* a *is assigned to* $\widehat{C}_h$ *or* $\widehat{C}_{h+1}$ *by the* Electre Tri-B *pseudo-conjunctive rule.*

   b) *if an action* a *is assigned to* C_t *by the* Electre Tri-C *ascending rule, then* a *is assigned to* $\widehat{C}_k$*, with* $k \geq t$*, by the* Electre Tri-B *pseudo-disjunctive rule.*

2) *Consider* $(q+1)$ *boundary actions defined to apply* Electre Tri-B *with* q *categories. When such boundary actions are used as the reference actions of the* $(q-1)$ *categories in* Electre Tri-C,

   a) *if an action* a *is assigned to* $\widehat{C}_h$ *by the* Electre Tri-B *pseudo-conjunctive rule, then* a *is assigned to* C_h *or* C_{h-1} *by the* Electre Tri-C *descending rule.*

   b) *if an action* a *is assigned to* $\widehat{C}_t$ *by the* Electre Tri-B *pseudo-disjunctive rule, then* a *is assigned to* C_k*, with* $k \leq t$*, by the* Electre Tri-C *ascending rule.*

Proof

1) *Assume that the* $(q+2)$ *reference actions were defined.*

   a) *When applying the* Electre Tri-C *descending rule (Definition 4), an action* a *is assigned to* C_h *if one of the two following cases holds. First,* $\xi_h^+(a,\lambda) \leq |\xi_{h+1}^+(a,\lambda)|$*, with* $\xi_h^+(a,\lambda) \geq 0$ *and* $\xi_{h+1}^+(a,\lambda) < 0$*. In such a case,* a *is assigned to* $\widehat{C}_{h+1}$ *according to the* Electre Tri-B *pseudo-conjunctive rule (Definition 6). Second,* $\xi_{h-1}^+(a,\lambda) > |\xi_h^+(a,\lambda)|$*, with* $\xi_{h-1}^+(a,\lambda) \geq 0$ *and* $\xi_h^+(a,\lambda) < 0$*. In such a case,* a *is assigned to* $\widehat{C}_h$ *according to the* Electre Tri-B *pseudo-conjunctive rule.*

   b) *When applying the* Electre Tri-C *ascending rule (Definition 5), an action* a *is assigned to* C_t *if one of the following two cases holds. First,* $\xi_t^-(a,\lambda) \leq |\xi_{t-1}^-(a,\lambda)|$*, with* $\xi_t^-(a,\lambda) \geq 0$ *and* $\xi_{t-1}^-(a,\lambda) < 0$*. In such a case, according to the* Electre Tri-B *pseudo-disjunctive rule (Definition 7), on the one hand if* $\xi_t^+(a,\lambda) < 0$*, then* a *is assigned to* $\widehat{C}_t$*, but on the other hand if there exist* p *central reference actions such that* $\xi_{t+s}^+(a,\lambda) \geq 0$*,* $s = 0, \ldots, (p-1)$*, then* a *is assigned to* $\widehat{C}_{t+p}$*. Second,* $\xi_{t+1}^-(a,\lambda) > |\xi_t^-(a,\lambda)|$*, with* $\xi_{t+1}^-(a,\lambda) \geq 0$ *and* $\xi_t^-(a,\lambda) < 0$*. In such a case, according to the* Electre Tri-B *pseudo-disjunctive rule, on the one hand if* $\xi_{t+1}^+(a,\lambda) < 0$*, then* a *is assigned to* $\widehat{C}_{t+1}$*, but on the other hand if there exist* p *central reference actions such that* $\xi_{t+s+1}^+(a,\lambda) \geq 0$*,* $s = 0, \ldots, (p-1)$*, then* a *is assigned to* $\widehat{C}_{t+p+1}$*.*

2) Assume that the $(q+1)$ *boundary actions were defined.*

a) According to the Electre Tri-B *pseudo-conjunctive rule (Definition 6), an action* a *is assigned to* $\widehat{C}_h$ *if, for the highest* h, $\xi^+_{h-1}(a,\lambda) \geq 0$. *Furthermore,* $\xi^+_h(a,\lambda) < 0$. *Thus, from the* Electre Tri-C *descending rule (Definition 4), if* $\xi^+_{h-1}(a,\lambda) \leq |\xi^+_h(a,\lambda)|$, *then* a *is assigned to category* C_{h-1}. *Otherwise,* a *is assigned to* C_h.

b) According to the Electre Tri-B *pseudo-disjunctive rule (Definition 7), an action* a *is assigned to* $\widehat{C}_t$ *if, for the lowest* t, $\xi^-_t(a,\lambda) \geq 0$ *and* $\xi^+_t(a,\lambda) < 0$. *Thus, if* $\xi^-_{t-1}(a,\lambda) < 0$, *then, according to the* Electre Tri-C *ascending rule (Definition 5), if* $\xi^-_t(a,\lambda) \leq |\xi^-_{t-1}(a,\lambda)|$, *then* a *is assigned to category* C_t, *or to* C_{t-1}, *otherwise. When there exist* p *boundary actions such that* $\xi^-_{t-s-1}(a,\lambda) \geq 0$ *and* $\xi^+_{t-s-1}(a,\lambda) \geq 0$, $s = 0,\ldots,(p-1)$, *according to the* Electre Tri-C *ascending rule, if* $\xi^-_{t-s-1}(a,\lambda) \leq |\xi^-_{t-s-2}(a,\lambda)|$, *then* a *is assigned to category* C_{t-s-1}. *Otherwise,* a *is assigned to* C_{t-s-2}. □

6 A numerical example

This section presents the assignment results provided by the Electre Tri-C and Electre Tri-B methods. This numerical example is based on a Case Study which concentrates on France's Lorraine region, where iron has been mined for more than a century. The underground mining tunnels have caused land subsidence, which led buildings to collapse. The object of this study was to make a partition of a piece of land into zones and assign such zones to pre-defined risk categories for decision concerning permanent surveillance (Merad et al., 2004).

Four categories have been defined to apply Electre Tri-B according to a surveillance system that should be applied to the zones assigned to each category. The zones assigned to category $\widehat{C}_4$ will be subject to a permanent monitoring system, the zones assigned to $\widehat{C}_3$ will require a deeply investigation, the zones assigned to $\widehat{C}_2$ will be subject to annually topographic surveys, and the zones assigned to $\widehat{C}_1$ will need only topographic surveys. These risk categories are ordered and separated by three boundary actions: $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$. Since $\widehat{C}_4$ is the highest risk category, $\hat{b}_3$ "displays a risk at least as high as" $\hat{b}_2$ for all criteria. Similarly, $\hat{b}_2$ "displays a risk at least as high as" $\hat{b}_1$ for all criteria.

The data for the numerical example are composed of 10 homogeneous zones (actions), $a_1,\ldots,a_{10}$, which are evaluated on 10 criteria, $g_1,\ldots,g_{10}$, and 3 boundary actions which allow to define the 4 categories (see Tables 1, 2, and 3).

Table 1: Definition of the criteria

	Description and preference direction		Parameters		
Criteria	Short description	Direction	q_j	p_j	w_j
g_1	Corrected mean stress applied on pillars	increasing	0.05	0.1	5
g_2	Existence of fault	increasing	0	0	1
g_3	Superimposition of pillars	increasing	0	0	1
g_4	Size and regularity of pillars	increasing	0	0	1
g_5	Sensitivity of rock to flooding	increasing	0	0	5
g_6	Depth of the top mined layer	decreasing	10	20	1
g_7	Maximum expected subsidence	increasing	0.10	0.20	1
g_8	Expected surface deformation	increasing	0.05	0.09	20
g_9	Zone extent	increasing	0.5	1.0	1
g_{10}	Vulnerability of building	increasing	0	0	10

Source: Adapted from Merad et al., 2004.

Table 2: Performance matrix (potential actions)

Actions	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
a_1	5.8	10	0	20	0	35	2.37	6.80	3.6	20
a_2	4.8	0	40	0	0	70	1.28	1.83	0.2	10
a_3	9.7	10	10	0	30	200	1.67	0.84	7.4	30
a_4	10.4	10	10	10	30	203	1.68	0.83	9.0	20
a_5	9.7	0	10	0	10	222	1.20	0.54	1.8	20
a_6	9.8	10	0	20	0	50	1.27	2.54	6.7	20
a_7	12.3	0	0	0	30	155	0.96	0.61	14.1	10
a_8	11.2	10	0	0	30	180	0.71	0.39	6.4	20
a_9	11.3	0	40	20	0	115	2.18	1.89	2.5	10
a_{10}	11.0	10	0	10	30	170	0.31	0.18	2.6	20

Source: Adapted from Merad et al., 2004.

Table 3: Boundary actions

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
$\hat{b}_1$	8.0	0	10	10	10	190	1.00	0.63	6.0	20
$\hat{b}_2$	10.0	10	10	10	10	150	1.40	0.82	20.0	20
$\hat{b}_3$	14.0	10	40	20	20	110	1.80	1.00	35.0	30

Source: Adapted from Merad et al., 2004.

In order to illustrate the two Electre Tri-C assignment rules, assume that it is possible to obtain four central reference actions, b_h, one for each category, C_h, $h = 1, \ldots, 4$. This is an alternative way to define the categories. Table 4 presents the proposed central reference actions. These actions were defined taking into account their position between two consecutive boundary actions as well as the scale associated with each criterion as in Merad et al., 2004.

Table 4: Central reference actions

Actions	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
b_1	7.0	0	10	10	10	210	0.80	0.53	3.0	20
b_2	9.0	10	10	10	10	170	1.20	0.725	13.0	20
b_3	12.0	10	40	20	20	130	1.60	0.91	27.5	30
b_4	16.0	10	40	20	30	90	2.00	1.10	37.5	40

Source: Adapted from Merad et al., 2004.

The credibility indices of the comprehensive outranking of the potential actions over the central reference actions, and *vice-versa*, are presented in Table 5. These indices and the chosen majority level are used to compute both assignment results of Electre Tri-C and Electre Tri-B methods.

Table 5: Outranking credibility (potential actions)

	$\sigma(a, b_h)$				$\sigma(b_h, a)$			
Actions	b_1	b_2	b_3	b_4	b_1	b_2	b_3	b_4
a_1	0.7609	0.7391	0.5217	0.5217	0.4739	0.5000	0.5217	0.5217
a_2	0.5217	0.5000	0.4783	0.4783	0.5000	0.5217	0.5435	0.5435
a_3	0.9783	0.9348	0.5870	0.1304	0.0652	0.1087	0.8913	1.0000
a_4	1.0000	0.9565	0.2609	0.1304	0.2826	0.3261	0.8913	1.0000
a_5	0.9522	0.4783	0.0000	0.0000	0.8696	0.8913	1.0000	1.0000
a_6	0.8696	0.8478	0.5000	0.5000	0.3478	0.4130	0.5435	0.5435
a_7	0.7391	0.2609	0.2174	0.1087	0.4000	0.7500	0.7826	1.0000
a_8	0.5217	0.4783	0.1304	0.1304	0.7174	0.7826	0.8913	1.0000
a_9	0.6739	0.6304	0.5217	0.5000	0.3696	0.3696	0.5326	0.5478
a_{10}	0.5217	0.5000	0.1304	0.1304	0.7391	0.7826	0.8913	1.0000

The set of central reference actions (Table 4) does not fulfil the strict separability condition, but the weak separability condition is fulfilled since, for instance, $\Omega_j(b_2, b_1) = 0$, for all $j \in \{3, 4, 5, 10\}$. Moreover, there are some criteria in which the difference in the performances between two consecutive central reference actions is at least as good as the preference thresholds. The outranking credibility indices of the central reference actions over the same reference actions are presented in Table 6.

Table 6: Outranking credibility (central reference actions)

$\sigma(b_h, b_t)$	b_1	b_2	b_3	b_4
b_1	1.0000	0.3696	0.0000	0.0000
b_2	1.0000	1.0000	0.0217	0.0217
b_3	1.0000	1.0000	1.0000	0.0652
b_4	1.0000	1.0000	1.0000	1.0000

According to Theorem 2, the compatible majority level, λ^c, is computed from the credibility indices of the central reference actions (Table 6). For this numerical example, $\lambda^c = 0.69$. Consequently, we must choose a majority level within the range $[0.69, 1]$ in order to obtain a consistent assignment model. Let $\lambda = 0.70$ be the chosen majority level for this numerical example.

When using the four central reference actions presented in Table 4 as boundary actions defining five categories according to Theorem 3, the Electre Tri-C and Electre Tri-B assignment results are the ones presented in Table 7. On the one hand the descending assignment results must be compared with the pseudo-conjunctive assignment results, and on the other hand the ascending assignment results must be compared with the pseudo-disjunctive assignment results.

Table 7: Comparing the assignment results

	Electre Tri-C		Electre Tri-B	
Actions	Descending	Ascending	Pseudo-conjunctive	Pseudo-disjunctive
a_1	C_2	C_4	$\widehat{C}_3$	$\widehat{C}_5$
a_2	C_1	C_4	$\widehat{C}_1$	$\widehat{C}_5$
a_3	C_3	C_3	$\widehat{C}_3$	$\widehat{C}_3$
a_4	C_2	C_3	$\widehat{C}_3$	$\widehat{C}_3$
a_5	C_2	C_1	$\widehat{C}_2$	$\widehat{C}_2$
a_6	C_2	C_4	$\widehat{C}_3$	$\widehat{C}_5$
a_7	C_1	C_2	$\widehat{C}_2$	$\widehat{C}_2$
a_8	C_1	C_2	$\widehat{C}_1$	$\widehat{C}_1$
a_9	C_1	C_4	$\widehat{C}_1$	$\widehat{C}_5$
a_{10}	C_1	C_1	$\widehat{C}_1$	$\widehat{C}_1$

Note: $\lambda = 0.70$

Observe that when b_2 plays the role of a central reference action. The Electre Tri-C descending rule leads to the assignment of actions a_1, a_4, a_5, and a_6 to category C_2, i.e. $C_2 = \{a_1, a_4, a_5, a_6\}$. If we now decide that b_2 plays the role of the upper boundary action of category $\widehat{C}_2$ in Electre Tri-B, $\hat{b}_2 = b_2$, then the pseudo-conjunctive rule assigns the same four actions to two different but consecutive categories, i.e. a_5 is assigned to $\widehat{C}_2$ and a_1, a_4, and a_6 are assigned to $\widehat{C}_3$. The actions initially assigned to C_2 are now shared between $\widehat{C}_2$ and $\widehat{C}_3$, which clearly shows that b_2 plays the role of a central reference action in Electre Tri-C.

7 Additional results

This section presents a comparison between the two Electre Tri-C assignment rules and some particular results. Both results are based on the following definition.

Definition 8 (λ-binary relations)
Let λ be the chosen majority level and consider two actions a and a'.

1) *λ-outranking:* $aS^{\lambda}a' \Leftrightarrow \sigma(a,a') - \lambda \geq 0$

2) *λ-indifference:* $aI^{\lambda}a' \Leftrightarrow \sigma(a,a') - \lambda \geq 0 \ \wedge \ \sigma(a',a) - \lambda \geq 0$

3) *λ-incomparability:* $aR^{\lambda}a' \Leftrightarrow \sigma(a,a') - \lambda < 0 \ \wedge \ \sigma(a',a) - \lambda < 0$

4) *λ-preference:* $a \succ a' \Leftrightarrow \sigma(a,a') - \lambda \geq 0 \ \wedge \ \sigma(a',a) - \lambda < 0$

The credibility indices obtained when comparing an action a to the reference actions b_h are then compared to the chosen majority level as in the Electre Tri-B method. Therefore, the result provided for Electre Tri-B (Roy and Bouyssou, 1993, Rés. 6.3.1, p. 392) is still valid for Electre Tri-C. According to this result, the comparison of an action a to the reference actions b_h provides one and only one of the three following cases:

1) There is no b_t such that $aR^{\lambda}b_t$ and there is a b_h such that $aI^{\lambda}b_h$. If b_h is not unique, then the reference actions which are λ-indifferent to the action a are consecutive.

2) There is no b_t such that $aI^{\lambda}b_t$ and there is a b_h such that $aR^{\lambda}b_h$. If b_h is not unique, then the reference actions which are λ-incomparable to the action a are consecutive.

3) There is no b_h such that $aI^{\lambda}b_h$ or $aR^{\lambda}b_h$.

Theorem 4 establishes a comparison between the two Electre Tri-C assignment rules.

Theorem 4

a) *If an action a is λ-indifferent to at least one reference action, then a is assigned by the descending rule to a category at least as good as the one a is assigned to when using the ascending rule.*

b) *If an action a is λ-incomparable to at least one reference action, then a is assigned by the descending rule to a category at most as good as the one a is assigned to when using the ascending rule.*

c) *Otherwise, both rules assign the action a to the same category or to two different but consecutive categories.*

Proof

a) *If an action a is λ-indifferent to at least one reference action, then the following case occurs:* $a \succ b_0$, $a \succ b_1$, …, $a \succ b_t$, $aI^\lambda b_{t+1}$, …, $aI^\lambda b_s$, $b_{s+1} \succ a$, …, $b_{q+1} \succ a$, *with* $0 \le t \le (q-1)$ *and* $(t+1) \le s \le q$. *According to the descending rule (Definition 4), the highest index h such that an action a is λ-indifferent to* b_h *is* $h = s$. *Thus, if* $\xi^+_s(a,\lambda) \le |\xi^+_{s+1}(a,\lambda)|$, *then the action a is assigned to category* C_s. *Otherwise, a is assigned to* C_{s+1}. *According to the ascending rule (Definition 5), the lowest index h, such that an action a is λ-indifferent to* b_h *is* $h = (t+1)$. *Thus, if* $\xi^-_{t+1}(a,\lambda) \le |\xi^-_t(a,\lambda)|$, *then the action a is assigned to category* C_{t+1}. *Otherwise, a is assigned to* C_t. *Consequently, the descending rule provides always a category at least as good as the one provided by the ascending rule because* $t < (t+1) \le s < (s+1)$.

b) *If an action a is λ-incomparable to at least one reference action, then the following case occurs:* $a \succ b_0$, $a \succ b_1$, …, $a \succ b_t$, $aR^\lambda b_{t+1}$, …, $aR^\lambda b_s$, $b_{s+1} \succ a$, …, $b_{q+1} \succ a$, *with* $0 \le t \le (q-1)$ *and* $(t+1) \le s \le q$. *According to the descending rule (Definition 4), the lowest index h, such that an action a is λ-incomparable to* b_h *is* $h = (t+1)$. *Thus, if* $\xi^+_t(a,\lambda) \le |\xi^+_{t+1}(a,\lambda)|$, *then the action a is assigned to category* C_t. *Otherwise, a is assigned to* C_{t+1}. *According to the ascending rule (Definition 5), the highest index h, such that an action a is λ-incomparable to* b_h *is* $h = s$. *Thus, if* $\xi^-_{s+1}(a,\lambda) \le |\xi^-_s(a,\lambda)|$, *then the action a is assigned to category* C_{s+1}. *Otherwise, a is assigned to* C_s. *Consequently, the descending rule provides always a category at most as good as the one provided by the ascending rule because* $t < (t+1) \le s < (s+1)$.

c) *If there is only λ-preference relations between an action a and all reference actions* b_h, *then the following case occurs:* $a \succ b_0$, $a \succ b_1$, …, $a \succ b_t$, $b_{t+1} \succ a$, …, $b_{q+1} \succ a$, *with* $0 \le t \le q$. *According to the descending rule (Definition 4), the highest index h, such that an action a is λ-preferred to* b_h *is* $h = t$. *Thus, if* $\xi^+_t(a,\lambda) \le |\xi^+_{t+1}(a,\lambda)|$, *then the action a is assigned to category* C_t. *Otherwise, a is assigned to* C_{t+1}. *According to the ascending rule (Definition 5), the lowest index h, such that a reference action* b_h *is λ-preferred to an action a is* $h = (t+1)$. *Thus, if* $\xi^-_{t+1}(a,\lambda) \le |\xi^-_t(a,\lambda)|$, *then the action a is assigned to category* C_{t+1}. *Otherwise, a is assigned to* C_t. *Consequently, both descending and ascending rules can provide either the same category* (C_t *or* C_{t+1}) *or the descending rule provides the category* C_t *and the ascending rule the category* C_{t+1} *or vice-versa.* □

The results expressed in this theorem are illustrated in Table 8 from the numerical example studied in Section 6.

Table 8: Electre Tri-C assignment results

Actions	b_1	b_2	b_3	b_4	Descending	Ascending
a_1	$\succ$	$\succ$	R^λ	R^λ	C_2	C_4
a_2	R^λ	R^λ	R^λ	R^λ	C_1	C_4
a_3	$\succ$	$\succ$	$\prec$	$\prec$	C_3	C_3
a_4	$\succ$	$\succ$	$\prec$	$\prec$	C_2	C_3
a_5	I^λ	$\prec$	$\prec$	$\prec$	C_2	C_1
a_6	$\succ$	$\succ$	R^λ	R^λ	C_2	C_4
a_7	$\succ$	$\prec$	$\prec$	$\prec$	C_1	C_2
a_8	$\succ$	$\prec$	$\prec$	$\prec$	C_1	C_1
a_9	R^λ	R^λ	R^λ	R^λ	C_1	C_4
a_{10}	$\succ$	$\prec$	$\prec$	$\prec$	C_1	C_1

Note: $\lambda = 0.70$

Observe that the action a_5 is λ-indifferent to b_1. In such a case, a_5 is assigned to C_2 by the descending rule and to C_1 by the ascending rule. It clearly shows that it is possible that an action can be assigned to a better category by the descending rule than by the ascending rule. We can also observe that there is a strong λ-incomparability between the actions a_2 and a_9 with respect to all central reference actions. In such cases, the descending rule assigns these two actions to C_1, while the ascending rule assigns the same actions to C_4.

Consider an action a and a central reference action b_h. The following proposition establishes the links between two λ-binary relations and Electre Tri-C assignment results according to both the descending and the ascending rules.

Proposition 2

a) If a λ-outranks b_h, then a is assigned at least to C_h by the descending rule.

b) If b_h λ-outranks a, then a is assigned at most to C_h by the ascending rule.

c) If a is λ-preferred to b_h, then a is assigned at least to C_h by both rules.

d) If b_h is λ-preferred to a, then a is assigned at most to C_h by both rules.

Proof

a) Assume that $aS^\lambda b_h$. Based on Definition 8.1 and Proposition 1.a, $aS^\lambda b_h \Rightarrow \xi^+_h(a,\lambda) \geq 0 \Rightarrow \xi^+_k(a,\lambda) \geq 0,\ k \leq h$. Thus, a is assigned to category C_t, such that $t \geq h$ because of the two following situations. First, when $\xi^+_{h+1}(a,\lambda) < 0$ if $\xi^+_h(a,\lambda) \leq |\xi^+_{h+1}(a,\lambda)|$, then a is assigned to C_h by the descending rule. Otherwise, a is assigned to C_{h+1}. Second, if $\xi^+_{h+1}(a,\lambda) \geq 0$, then a is necessarily assigned to a category at least as good as C_{h+1}.

b) *Assume that* $b_h S^\lambda a$. *Based on Definition 8.1 and Proposition 1.b,* $b_h S^\lambda a \Rightarrow \xi_h^-(a,\lambda) \geq 0 \Rightarrow \xi_k^-(a,\lambda) \geq 0,\ k \geq h$. *Thus, a is assigned to a category* C_t*, such that* $t \leq h$ *because of the two following situations. First, when* $\xi_{h-1}^-(a,\lambda) < 0$ *if* $\xi_h^-(a,\lambda) \leq |\xi_{h-1}^-(a,\lambda)|$*, then a is assigned to* C_h *by the ascending rule. Otherwise, a is assigned to* C_{h-1}*. Second, if* $\xi_{h-1}^-(a,\lambda) \geq 0$*, then a is necessarily assigned to a category at most as good as* C_{h-1}*.*

c) *As for the descending rule, the proof is similar to* a) *by using Definition 8.4 and Proposition 1.a. As for the ascending rule, the proof is similar to* b) *by using Definition 8.4 and Proposition 1.b.*

d) *As for the descending rule, the proof is similar to* a) *by using Definition 8.4 and Proposition 1.a. As for the ascending rule, the proof is similar to* b) *by using Definition 8.4 and Proposition 1.b.* □

8 Conclusions

This paper dealt with a new sorting method, called Electre Tri-C, which categories are defined through central reference actions instead of boundary actions. A comparison with Electre Tri-B shows the main similarities of the two methods. Defining categories through central reference actions is, in our opinion, of the uttermost importance for modelling a wide variety of practical decision aiding situations dealing with the assignment of actions to pre-defined and ordered categories.

Central reference actions and boundary actions are two alternative ways for defining ordered categories. These reference actions must be defined *a priori* to play an appropriate role. The procedures must preserve this role when assigning the potential actions to the categories. In several situations, it is more adequate to define the categories through an interaction process with the decision-maker by using central reference actions. Defining the categories through boundary actions can be difficult when the frontier between the criteria used to delimit them is rather fuzzy in the mind of the decision-maker.

The two proposed assignment rules fulfill the structural properties of uniqueness, independence, conformity, monotonicity, homogeneity, and stability. When the set of reference actions does not fulfill the strict separability condition, but only a weak separability condition, a compatible majority level must be computed in order to obtain a consistent assignment model.

As for future research avenues, we intend to analyse a possible extension to multiple typical actions. Such an extension will allow us modelling a larger number of decision aiding situations in the field of sorting problems. When using the concept of central reference actions, we intend to study assignment procedures to deal with partially ordered categories and completely non-ordered categories. Furthermore, it is more difficult to introduce boundary actions than central reference actions for the definition of partially ordered categories. Currently there is no method incorporating the notion of category size in the assignment procedures to limit the number of actions that can be assigned to each category. Indeed, we also intend to study this particular issue by introducing a notion of relative independence for characterizing a new sorting problematic. At the same time, we should focus our attention on the inference of some parameters through an aggregation-disaggregation procedure using central reference actions.

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A Appendix

A.1 Generalization of the formulae (2.2), (2.6), (3.8), and (4.9) to take into account variable thresholds

Let g_j be a criterion such that the difference between the best and the worst possible performances allows the definition of several performance levels. It is often the case when the performance is defined by a physical or a monetary measure. In such a case, the same values for the indifference and the preference thresholds can be judged inappropriate. For instance, a variable threshold defined as a percentage of one of the two performances can be considered more appropriate. It can also be the case with a verbal scale including an enough number of levels. The way of defining these verbal levels can produce a clear distinction between two consecutive levels in certain zones of the scale and an unclear distinction in other zones. In practice, it is often necessary to take into account variable thresholds instead of constant ones.

As noticed in Section 2, the indifference and the preference thresholds have been introduced to discriminate situations of indifference, weak preference, and strict preference when comparing two actions a and a' according to the criterion g_j. With the constant thresholds this discrimination only takes into account the value of the advantage $\Omega_j(a, a')$. In other words, the discrimination is based on the amplitude of the interval $[g_j(a), g_j(a')]$ independently of the position that this interval occupies along the scale of criterion g_j. The introduction of variable thresholds allows to take into account the position of this interval along the scale. This is possible by defining threshold functions, which are non-negative functions, in the two following ways:

1) The thresholds called **direct**, denoted q_j and p_j, conceived to characterize the borderline of the indifference zone and the borderline of the preference zone, respectively, while progressing in the scale of the criterion g_j in the sense of the increasing preferences. For instance, with a direct preference threshold, a is strictly preferred to a' if and only if:

    i) $\Omega_j(a, a') = g_j(a) - g_j(a') \geq p_j[g_j(a')]$ if the preferences increase when the performances increase too.

    ii) $\Omega_j(a, a') = g_j(a') - g_j(a) \geq p_j[g_j(a)]$ if the preferences increase when the performances decrease.

2) The thresholds called **inverse**, denoted q'_j and p'_j, conceived to characterize the borderline of the indifference zone and the borderline of the preference zone, respectively, while progressing in the scale of the criterion g_j in the sense of the decreasing preferences. For instance, with a inverse preference threshold, a is strictly preferred to a' if and only if:

    i) $\Omega_j(a, a') = g_j(a) - g_j(a') \geq p'_j[g_j(a)]$ if the preferences increase when the performances increase too.

    ii) $\Omega_j(a, a') = g_j(a') - g_j(a) \geq p'_j[g_j(a')]$ if the preferences increase when the performances decrease.

Let us notice that direct and inverse thresholds are particular functions subject to certain coherence conditions. Furthermore, if the analyst chose to define only direct thresholds it is possible to obtain automatically the inverse thresholds, and *vice-versa*, because these two types of variable thresholds are functionally linked (for more details on these questions see, for example, Roy, 1996, § 9.3.2, p. 188).

When taking into account the notation and definitions introduced in this Appendix, the formula (2.2) is generalized in case of variable thresholds as follows:

1) With **direct thresholds** q_j and p_j must be replaced respectively by:

    i) $q_j[g_j(a)]$ and $p_j[g_j(a)]$ if the preferences increase when the performances increase too.

    ii) $q_j[g_j(a')]$ and $p_j[g_j(a')]$ if the preferences increase when the performances decrease.

2) With **inverse thresholds** q_j and p_j must be replaced respectively by:

    i) $q'_j[g_j(a')]$ and $p'_j[gj(a')]$ if the preferences increase when the performances increase too.

ii) $q'_j[g_j(a)]$ and $p'_j[g_j(a)]$ if the preferences increase when the performances decrease.

The reasons that led us to introduce variable indifference and preference thresholds (rather than constant ones) are the same that led us to introduce variable veto thresholds, in the case where a veto threshold is associated to criterion g_j. The analyst can again choose between direct veto thresholds, denoted v_j, and inverse veto thresholds, denoted v'_j.

The formula (2.6) is generalized as follows:

1) With **direct thresholds**, p_j and v_j must be replaced respectively by:

   i) $p_j[g_j(a)]$ and $v_j[g_j(a)]$ if the preferences increase when the performances increase too.

   ii) $p_j[g_j(a')]$ and $v_j[g_j(a')]$ if the preferences increase when the performances decrease.

2) With **inverse thresholds**, p_j and v_j must be replaced respectively by:

   i) $p'_j[g_j(a')]$ and $v'_j[g_j(a')]$ if the preferences increase when the performances increase too.

   ii) $p'_j[g_j(a)]$ and $v'_j[g_j(a)]$ if the preferences increase when the performances decrease.

When using variable preference thresholds the formulae (3.8) and (4.9) which characterize the strict and the weak separability conditions, respectively, are generalized as follows:

i) $\Omega_j(b_{h+1}, b_h) > p_j[g_j(b_h)]$ if the preferences increase when the performances increase too.

ii) $\Omega_j(b_{h+1}, b_h) > p'_j[g_j(b_{h+1})]$ if the preferences increase when the performances decrease.

Finally, let us notice that when using variable thresholds in the sorting problematic it is not necessary to compute the threshold values for each scale levels of criterion g_j. Indeed, all the Electre Tri-C formulae (the same is applied to Electre Tri-B) can be written by making use of the values that are assigned to the direct and inverse thresholds only for the performances of the reference actions, $g_j(b_h)$, $h = 0, \ldots, (q+1)$. It comes from the fact that in a given formula, concerning the way in which two actions a and a' are compared, that makes use of a direct threshold computed for a it is possible to substitute it by an inverse threshold computed for a', and *vice-versa*.

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