
Approches Locales-Globales

Méthode Arlequin

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RÉSUMÉ. Des approches numériques permettant des analyses locales de modèles globaux sont discutées, la méthode Arlequin étant le point de focalisation. Par superposition d'états mécaniques se partageant les énergies, cette méthode crée des partitions de modèles, donnant de la "plasticité" à la (mono-)modélisation classique, et ce de manière consistante. Elle consiste en une famille de formulations des problèmes mécaniques, obtenues par combinaisons de briques élémentaires dont le choix est analysé. La capacité de la méthode Arlequin à permettre la réalisation de zooms numériques établissant des dialogues locaux de modèles et d'échelles est éclairée par des exemples.

ABSTRACT. Numerical approaches allowing the local analysis of global models are discussed, the Arlequin method being the topic of focus. By superposing mechanical states sharing the energies, this method generates a partition of models framework that gives a consistent "plasticity" to the classical (mono-)modeling. It consists in a family of formulations of mechanical problems, each of them being derived by combining basic bricks whose choices are analyzed. The effectiveness of this partition of models framework to allow concurrent multimodel and multiscale analysis is exemplified.

MOTS-CLÉS : multi-modèle, multi-échelle ; partition de modèles ; Local-GLobal ; méthode Arlequin

KEYWORDS: multimodel; multiscale; partition of models; Local-Global; Arlequin method

1. Introduction

Pour répondre à des demandes actuelles de conception et d'assurance d'intégrité des structures, il devient essentiel, sur le plan de la modélisation et du calcul numériques, de pouvoir

- introduire avec flexibilité des altérations locales (fissure, excavation, raidisseur, choc localisé, etc.) d'un modèle numérique global existant, en assurant le couplage Local-Global,

- enrichir localement le modèle global lorsque les hypothèses le sous-tendant ne sont plus en accord avec les objectifs visés.

Dans le cadre de la mécanique des milieux continus, la difficulté essentielle pour réaliser les tâches citées ci-dessus réside dans le manque de flexibilité des méthodes numériques classiques telles que la méthode des éléments finis (MEF). Dans la MEF et jusque récemment, le raffinement restait fondamentalement attaché, soit au maillage, soit au degré d'interpolation des champs recherchés. Ces deux mécanismes, à la base des méthodes puissantes de raffinements adaptatifs, peuvent toutefois se ressentir d'un manque de flexibilité et/ou occasionner des modèles lourds et des coûts prohibitifs pour des calculs de structures complexes.

Les années 90 ont vu le développement de nouvelles méthodes numériques visant à améliorer la flexibilité de la MEF et à en étendre ainsi le spectre d'usage. Citons particulièrement les méthodes sans maillages (meshless) [NAY 92, BEL 94] (et [ALT 04], pour des références plus étendues) qui proposent basiquement de remplacer la notion d'éléments finis par celle de particules finies (plus flexible), la méthode mixant la MEF à la précédente [BEL 95] et les méthodes de Partition de l'Unité (PUM) [MEL 96, BAB 97, BAB 00] qui introduisent localement une connaissance analytique de la solution recherchée pour éviter les raffinements EF classiques. S'appuyant sur la MEF et les PUM, la X-Fem [BEL 99, MOE 99] propose d'enrichir localement les bases des espaces d'approximation EF, utilisées dans le modèle global, par des modes de déformation relevant les altérations locales (fissure, par exemple) ou ses effets locaux (singularité, par exemple), atténuant ainsi les nécessités de remaillages locaux. Ces approches ont à l'évidence repoussé les frontières de la MEF. Elles se limitent cependant à travailler sur les seuls espaces d'approximation. Leur efficacité dans des cas généraux reste de ce fait dépendante de l'hypothèse que les modes relevant les altérations subies et/ou voulues soient (quasi) connus analytiquement. D'autres méthodes, de type micro-macro, travaillent à des niveaux d'approximation fine et grossière, à la fois ; les champs fins corrigeant ceux grossiers [HUG 95, HUG 98, ZOH 96, ODE 97, LAD 99, FEY 00]. Elles sont particulièrement dédiées au calcul des structures hétérogènes.

Les stratégies précédentes sont monomodèles et correctives. La s-méthode de type Locale-Globale de J. Fish [FIS 92] rompt avec elles par l'usage, en discret, d'un prin-

cipe de superposition d'un maillage local au maillage global avec le souci d'introduire des changements du modèle global. Observons ici que ce principe de superposition a été utilisé dans d'autres domaines pour répondre à des besoins aussi variés que l'établissement de l'existence de solutions pour problèmes d'équations aux dérivées partielles par Schwarz (1870), le développement de techniques de sous-structuration (e.g. [LET 94]), l'extension de l'usage de la méthode des différences finies pour la résolution de problèmes de mécanique des fluides dans des domaines non structurés donnant lieu à la méthode Chimère (e.g. [STE 87]), ou encore la dérivation de modèles limites de structures par des méthodes asymptotiques (e.g. [LED 91]).

Comme la s-Méthode, la méthode Arlequin [BEN 98, BEN 99] utilise la superposition de modèles dans une zone S , supposée connue. Mais, contrairement à cette dernière, le modèle local n'est pas rajouté au global (avec le risque de redondance), mais croisé et collé à lui sur une sous-zone S_g de S , pour générer une partition de modèles, moyennant l'usage d'une partition de l'unité sur S . De ce fait, la méthode est localement multi-modèle/multi-échelle.

La méthode Arlequin se présente en fait comme une famille de formulations d'un problème mécanique donné. Chaque formulation repose sur un choix de certains composants élémentaires. On donne dans la suite des éléments décrivant les composants de la méthode en continu et en discret, discutant, par l'analyse, le choix de ses composants et montrant par les exemples l'étendue de ses capacités.

2. Arlequin formulations

For the sake of the clarity, we consider a static linearized elasticity problem defined in a polyhedral domain Ω . We let Γ , \mathbf{f} , $\boldsymbol{\varepsilon}(\mathbf{v})$ and $\boldsymbol{\sigma}(\mathbf{v})$ respectively denote the clamped part of the boundary $\partial\Omega$, the applied density of body forces, the linearized strain and stress tensors associated to the displacement field \mathbf{v} . Without restriction, the complementary part of Γ in $\partial\Omega$ is assumed to be free. We also assume that the constitutive material follows a Hooke's law, which reads using usual convention of summation over repeated indices :

$$\boldsymbol{\sigma}_{ij}(\mathbf{v}) = \mathbf{R}_{ijkl} \boldsymbol{\varepsilon}_{kl}(\mathbf{v}) \quad (1)$$

with elasticity moduli \mathbf{R}_{ijkl} satisfying classical hypotheses.

The "monomodel" displacement problem of the considered mechanical system reads :

$$\text{Inf}_{\mathbf{v} \in \mathbf{W}} E(\mathbf{v}) \quad (2)$$

where, using classical notations,

$$\mathbf{W} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) ; \mathbf{v} = \mathbf{0} \text{ on } \Gamma\} \quad (3)$$

$$E(\mathbf{v}) = \frac{1}{2} \int_{\Omega} \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) d\Omega - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega \quad (4)$$

To rewrite (1) - (4) according to the Arlequin *vision*, we consider that Ω is partitioned into two overlapping polyhedral domains Ω_1 and Ω_2 . The clamped part Γ is assumed to be, say, in $\partial\Omega_1$. We let S_g denote the gluing zone supposed to be a non zero measured polyhedral subset of the superposition zone $S = \Omega_1 \cap \Omega_2$. Moreover, it is assumed that the boundary of S is contained in the boundary of S_g .

The Arlequin formulations of the model problem are obtained by

- 1) a multiplication of mechanical states in S ;
- 2) an energy distribution between the mechanical states in S , by using weight functions ;
- 3) a weak and compatible gluing of these states in S_g .

2.1. Mixed Arlequin formulations

In the mixed Arlequin approach, the gluing density of forces is a Lagrange multiplier field belonging to the dual of the space of the admissible displacement fields restricted to S_g . This leads to a coupling operator based on a duality bracket between $\mathbf{H}^1(S_g)$ and its dual space, denoted by $\langle \cdot, \cdot \rangle$. The first mixed continuous Arlequin problem is then the following : [BEN 99].

$$\text{Inf}_{(\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{W}_1 \times \mathbf{W}_2} \text{Sup}_{\boldsymbol{\lambda} \in \mathbf{W}'_g} \left\{ E_1(\mathbf{v}_1) + E_2(\mathbf{v}_2) + C_d(\boldsymbol{\lambda}, \mathbf{v}_1 - \mathbf{v}_2) \right\} \quad (5)$$

where

$$\mathbf{W}_1 = \{\mathbf{v}_1 \in \mathbf{H}^1(\Omega_1) ; \mathbf{v}_1 = \mathbf{0} \text{ on } \Gamma\} \quad (6)$$

$$\mathbf{W}_2 = \mathbf{H}^1(\Omega_2) \quad (7)$$

$$\mathbf{W}_g = \mathbf{H}^1(S_g) \quad (8)$$

$$E_i(\mathbf{v}_i) = \frac{1}{2} \int_{\Omega_i} \alpha_i \boldsymbol{\sigma}(\mathbf{v}_i) : \boldsymbol{\varepsilon}(\mathbf{v}_i) d\Omega - \int_{\Omega_i} \beta_i \mathbf{f} \cdot \mathbf{v}_i d\Omega \quad (9)$$

$$C_d(\boldsymbol{\lambda}, \mathbf{v}) = \langle \boldsymbol{\lambda}; \mathbf{v} \rangle \quad (10)$$

and where α_i and β_i denote two weight parameter functions that are assumed to be positive piecewise continuous functions in Ω_i , satisfying the following equalities :

$$\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1 \quad \text{in } S \quad (11)$$

$$\alpha_i = \beta_i = 1 \quad \text{in } \Omega_i \setminus S \quad (12)$$

Remark 1

In the superposition zone, distinct mechanical states concurrently exist. The stress tensor field satisfying the mechanical equilibrium is defined as the weighting of the stress tensor fields associated to both models through the α functions pair :

$$\boldsymbol{\sigma}^{arl} = \begin{cases} \boldsymbol{\sigma}(\mathbf{u}_1) & \text{in } \Omega_1 \setminus S \\ \boldsymbol{\sigma}(\mathbf{u}_2) & \text{in } \Omega_2 \setminus S \\ \alpha_1 \boldsymbol{\sigma}(\mathbf{u}_1) + \alpha_2 \boldsymbol{\sigma}(\mathbf{u}_2) & \text{in } S \end{cases} \quad (13)$$

The field defined by (13) is labelled as Arlequin stress tensor field [BEN 98].

The use of the duality bracket in the discrete level is not so clear. One can replace it, as for surface coupling, by an $\mathbf{L}^2(S_g)$ scalar product (for which a continuous Arlequin problem would be with no sense!). Another strategy consists in observing that, by using the Riesz representation theorem, a natural scalar product of $\mathbf{H}^1(S_g)$ can be substituted to the duality bracket. By the way, we notice that this last aspect stands for an advantage of the *volume* coupling operator (intimately related to the structure of the Arlequin method) when compared to the more usual surface coupling (which could of course be used in the Arlequin framework).

Baring these elements in mind, a second mixed Arlequin problem can be written as following :

$$\text{Inf}_{(\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{W}_1 \times \mathbf{W}_2} \text{Sup}_{\boldsymbol{\lambda} \in \mathbf{W}_g} \left\{ E_1(\mathbf{v}_1) + E_2(\mathbf{v}_2) + C(\boldsymbol{\lambda}, \mathbf{v}_1 - \mathbf{v}_2) \right\} \quad (14)$$

where

$$C(\boldsymbol{\lambda}, \mathbf{v}) = \int_{S_g} \{ \boldsymbol{\lambda} \cdot \mathbf{v} + \ell^2 \boldsymbol{\varepsilon}(\boldsymbol{\lambda}) : \boldsymbol{\varepsilon}(\mathbf{v}) \} d\Omega \quad (15)$$

and where ℓ denotes a strictly positive parameter homogeneous to a length.

Remark 2

The coupling operator $C(., .)$, defined by (15), can be replaced by any other scalar product that is equivalent to the $\mathbf{H}^1(S_g)$ natural scalar product.

2.2. Penalty-based Arlequin formulation

One can also use elastic springs (or more fuzzy ones) to activate gluing forces in the gluing zone. This basically leads to the following penalty-based Arlequin formulation of the elasticity problem :

$$\text{Inf}_{(\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{W}_1 \times \mathbf{W}_2} \left\{ E_1(\mathbf{v}_1) + E_2(\mathbf{v}_2) + C_p(\mathbf{v}_1 - \mathbf{v}_2) \right\} \quad (16)$$

where

$$C_p(\mathbf{v}) = \int_{S_g} \frac{1}{2} p \mathbf{v} \cdot \mathbf{v} d\Omega \quad (17)$$

and p is a strictly positive penalty parameter (that may be a function). One can notice that the mixed Arlequin formulations may be stabilized by a penalty term like the one defined by (17).

3. How to choose the Arlequin components

Before showing some examples of application of the Arlequin method, let us discuss first the Arlequin elements choices issue.

For the results given in this section, it will be assumed that :

$$\forall i \in \{1, 2\}, \exists \alpha_0 > 0 ; \alpha_i \geq \alpha_0, \text{ in } S \quad (18)$$

Notice that the condition (18) on the weight parameter functions α_i presents no practical difficulty.

For the stability result concerning the discrete mixed Arlequin problems, derived by using the finite element method, it will also be assumed that the space of rigid body motions over the gluing zone is contained in the spaces of approximation of the gluing forces.

3.1. Analysis of various gluing operators

We show here how the analysis can help us to choose the more suitable Arlequin gluing operator. Let us begin with a stability result for the penalty-based Arlequin problem.

Result 1- *Under the hypotheses (11), (12), (18) and other classical ones, the Arlequin penalized continuous problem, defined by (16), (17) and (6)-(9) and the associated*

discrete problems admit each a unique solution, for each strictly positive parameter p .

However, when the super-imposed mechanical models are significantly different then the penalty discrete solutions show very localized and unrealistic stresses behaviour in the gluing zones, unless an appropriate projection operator is used to modify the gluing penalty operator [BEN 98, BEN 99], which complicates significantly the otherwise simple to implement penalty gluing operator.

Concerning the Lagrange multiplier based gluing operators, we have the following results [BEN 99, BEN 01a], based on classical theories of mixed problems [BRE 74].

Result 2- *Under the hypotheses of result 1, the first and second mixed continuous Arlequin problems, defined by (5)-(10) and (6)-(9), (14), (15), respectively, admit each a unique solution.*

By a uniqueness argument, one can identify the volume gluing multiplier defined in the first mixed Arlequin problem with the classical surface coupling multiplier which suggests that the Lagrange multiplier defined in the first mixed Arlequin problem could be quite irregular (this fact will be exemplified numerically). This is one of the reasons for which the gluing operator we favour is the one leading to the second mixed Arlequin problem. As a matter of fact, let us mention that by adding the following hypothesis :

$$\mathbf{W}_{h_g} \subset \mathbf{W}_{h_1|S_g} \quad \text{or} \quad \mathbf{W}_{h_g} \subset \mathbf{W}_{h_2|S_g} \quad (19)$$

we can establish the following result [BEN 01a] for the discrete mixed Arlequin problems derived in a straightforward manner from the problem (14) :

Result 3- *Under the hypotheses (19) and those of result 1, the discrete mixed Arlequin problems derived from the second continuous mixed Arlequin problem by means of the finite element method are well-posed. Moreover, if a sufficient regularity is assumed for the continuous fields then we have the following optimal a priori error estimate :*

$$\exists C > 0, \text{ independent of } h_1, h_2 \text{ and } h_g ;$$

$$\|\mathbf{u}_1 - \mathbf{u}_{h_1}\|_{\mathbf{W}_1} + \|\mathbf{u}_2 - \mathbf{u}_{h_2}\|_{\mathbf{W}_2} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{h_g}\|_{\mathbf{W}_g} \leq C \max(h_1, h_2, h_g) \quad (20)$$

In the sequel, we only consider the second mixed Arlequin problems.

3.2. Choice of the weight functions

The weight functions, α_1 and α_2 , are assumed to be given. One can check easily that since the boundary of S is contained in the one of S_g , the Arlequin solution does not depend on these parameters when identical models are superposed to each other. This is a consistency argument for the approach. In the contrary, when different models are superposed, the Arlequin solutions do depend on the weight parameters. The question is then : how to choose these parameters in practice ?

Let us give here some answers that are basically oriented by one of the fundamental reasons which has motivated the development of the Arlequin method, namely the flexible zooming of a model.

3.2.1. General considerations

Though optimal choices (if ever necessary) seem to constitute a rather intricate issue in general, operational ones may be guided by the consideration of the relative local refinements of the superposed models. An absolute limit situation consists in super-imposing (locally) a rigid model to a deformable one. In this situation there is no need for the distribution of the internal energies since the one associated to a rigid model is identically equal to zero. Notice that in these very particular situations, one can establish a link between the fictitious domain method with a distributed Lagrange multiplier [GLO 00] and the second mixed Arlequin method.

3.2.2. A limit behaviour result

When considering deformable bodies, the stability analysis of the Arlequin problems requires that each α_i has to be strictly positive. But one can ask the question of existence of a limit behaviour of the Arlequin solutions whenever either α_1 or α_2 tends to the unity (the other tending to zero) in relevant situations where, in the unglued part of S , the two models are quite different (true multimodel or multiscale scenarios). For this, let us for instance assume that in the unglued part of S , one model is fractured and the other is not. Moreover, let us assume that the crack is strictly embedded in the interior of the unglued part of the fractured structure. Let us then define two global monomodel problems we denote by M_1 and M_2 , respectively. The first problem is associated to the fractured domain, while the second is associated to the “same” but sound domain. We denote by u_{M_1} and u_{M_2} the respective solutions. Now, if in the Arlequin framework, a partition of a local model LM_1 , part of M_1 containing the crack, and a global sound model GM_2 , here identical to M_2 , is made with α_1 and α_2 associated to LM_1 and GM_2 , respectively, then we can prove the following limit behaviour result :

Result 4- *Under ad hoc hypotheses, the Arlequin solutions tend to u_{M_i} when α_i tends to 1 and when β_i has the same order as α_i , $i = 1, 2$*

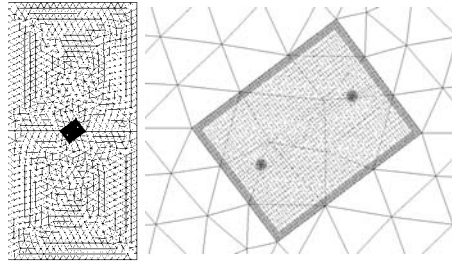


Figure 1. Meshes of the global and the local models

4. Examples of application

Two examples are given here. Others could be found in [BEN 01b, BEN 02, BEN 04, BEN 05] and some of them will be shown during the conference. Let us notice here that to obtain these results, it was necessary to tackle some geometrical and numerical issues. Indeed, by construction, the Arlequin framework allows the coexistence of incompatible models, sharing the energies of the system in the superposition regions and linked to each other in the gluing subregions. These heterogeneities require significant numerical and technical developments (see [BEN 05] and the references therein). A price to pay for the enhancement of the flexibility.

4.1. Slant cracked 2D plate under tension

This example aims at illustrating both the possibility of super-imposing with great flexibility a local slant cracked model on a sound plate and the effectiveness of the limit behaviour result 4, given in the previous section. The numerical energy release rate and the first and second stress intensity factors K_I and K_{II} are compared to their closed form expressions (for an infinite slant cracked plate). The used meshes for the global model and the local cracked one are depicted in (figure 1) where the tinted area stands for the chosen gluing zone.

For a weight function parameter associated to the local cracked model merely equal to unity, the resulting deformed numerical Arlequin model in local zones of interest, and the major principal stresses field, are given in figure 2.

The limit behaviours of energy release rates and stress intensity factors are shown by figure 3. Notice that in this figure α_{plate} refers to the weight function parameter associated to the globalplate model.

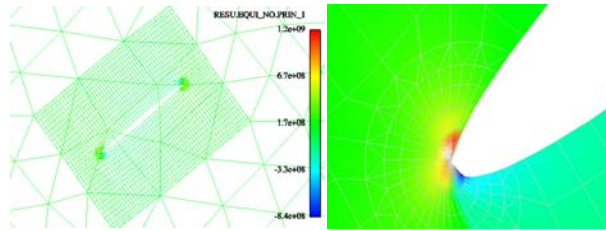


Figure 2. *Deformed meshes and major principal stress field*

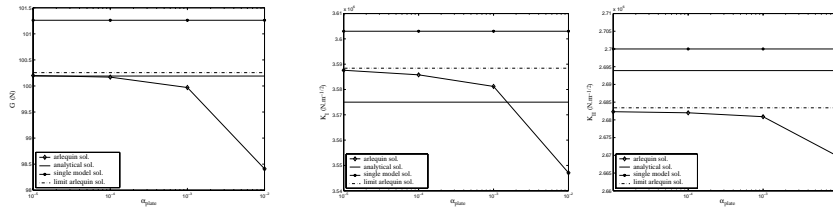


Figure 3. *Energy release rate and stress intensity factors K_I and K_{II}*

4.2. A cracked 3D/plate models partition

Our second example show how a local 3D partially cracked model can be easily superposed and glued to a global sound plate model by the Arlequin method. In figure 4, the iso-major principal stresses are represented on the global deformed configuration. A zoom around the defect shows more precisely the singularity of the stresses in the front of the crack. For the latter, only half of the local model is shown (see the electronic version for the same figure in colour).

5. Some concluding remarks

The Arlequin method has been presented. The practical choices of its components has been analyzed. The effectiveness of the approach to locally change a global mo-

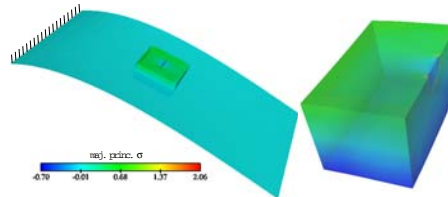


Figure 4. *Including a 3D local cracked model in a sound global plate model*

del with great flexibility has been exemplified. Definitely multimodel, this framework is also believed to be multiscale (see [XIA 04] for developments in this sense). Its capabilities to fit with contact or impact problems requirements has also been tested with success [BEN 01b, BEN 04]. An important aspect which will be shown during the conference is that in the dynamic regime, the Arlequin method can avoid spurious wave reflections by interfaces between fine and coarse scales.

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