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WIGNER FUNCTIONS AND STOCHASTICALLY PERTURBED LATTICE DYNAMICS

GIADA BASILE, STEFANO OLLA, AND HERBERT SPOHN

ABSTRACT. We consider lattice dynamics with a small stochastic perturbation of order ε and prove that for a space-time scale of order ε^{-1} the Wigner function evolves according to a linear transport equation describing inelastic collisions. For an energy and momentum conserving chain the transport equation predicts a slow decay, as $1/\sqrt{t}$, for the energy current correlation in equilibrium. This is in agreement with previous studies using a different method.

1. INTRODUCTION

Wigner functions are a very convenient and versatile tool in the analysis of wave equations. For multi-component linear wave equations the semiclassical part of the solution is covered by the time evolved Wigner function, see [9] with refinements in [5]. If the coefficients of the wave equation are weakly random, then in the semiclassical limit the Wigner function is governed by a transport equation, which accounts for the finite life-time of the modes. We refer to the very informative survey [11] and to [1], [8] for two completely worked out benchmarks. A similar, but more involved scheme works for weakly nonlinear wave equations, see [13, 12].

In our contribution we develop a novel application for Wigner functions. Rather than stochastically perturbing the coefficients of the wave equation we add stochastic terms to the equation. They can be written down most easily for a discrete wave equation (lattice dynamics) which is the only case considered here. As a Hamiltonian system the lattice dynamics conserves energy and, depending on the couplings, also momentum. The basic idea is to have the added stochastic terms respect locally such conservation laws. In the context of interacting mechanical particles related models have been studied, e.g., in [10]. But in the context of wave equations such an approach is very recent [4, 2, 3].

To have a closed equation for the evolution of the Wigner function the stochastic part of the generator has to be of order ε with ε the semiclassical parameter, $0 < \varepsilon \ll 1$. We will prove that in the limit $\varepsilon \rightarrow 0$ the Wigner function is governed by a linear transport equation. In the cases mentioned above the collision operator of the transport equation describes elastic collisions, while in our case the collisions are inelastic with energy conserved only on average.

The Wigner function evolution is very efficient for the understanding of the long-time properties of the stochastic wave dynamics. We will provide only one

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such illustration for which we choose the one-dimensional chain with momentum conservation. From its transport equation it follows that the energy transport is anomalous. Phonons with wave vector k travel, for small k , with unit speed but have a life-time which diverges as $1/k^2$.

For the most general (linear) lattice dynamics the lattice is d -dimensional, the displacement vector is n -dimensional, there are possibly several particles per unit cell, and the interaction potential is quadratic. While our methods would apply, at such generality our text would be hard to read. Therefore we work out the simplest case ($d = 1, n = 1$, one particle per unit cell) in complete detail (sections 2 to 4) and explain the generalization to $d = n$ arbitrary and one particle per unit cell in section 5. In these cases the Wigner function is assumed to have a spatial decay at infinity. To discuss energy transport and the energy current correlations one has to consider initial probability measures which are translation invariant. This requires some modifications which are studied in section 6. Anomalous transport will be discussed in section 6.4.

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2. THE MODEL (ONE-DIMENSIONAL CASE)

To develop the necessary techniques, we consider first the case of a one-dimensional chain. It consists of an infinite system of harmonic oscillators where particles are labelled by $y \in \mathbb{Z}$. The phase space is $(\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$ and a configuration at time t is denoted by $\{q_y(t), p_y(t)\}_{y \in \mathbb{Z}}$, where p_y and q_y , respectively, are the momentum and the displacement from the equilibrium position of the y -th particle. The Hamiltonian of the system is given by

$$H(p, q) = \frac{1}{2} \sum_{y \in \mathbb{Z}} p_y^2 + \frac{1}{2} \sum_{y, y' \in \mathbb{Z}} \alpha(y - y') q_y q_{y'}. \quad (1)$$

We denote with $\hat{v}(k)$, $k \in \mathbb{T} = [0, 1]$, the Fourier transform of a function v on \mathbb{Z} ,

$$\hat{v}(k) = \sum_{z \in \mathbb{Z}} e^{-2\pi i k z} v(z), \quad (2)$$

and with $\tilde{f}(z)$, $z \in \mathbb{Z}$, the inverse Fourier transform of a function f on \mathbb{T} ,

$$\tilde{f}(z) = \int_{\mathbb{T}} dk e^{2\pi i k z} f(k). \quad (3)$$

The function $\omega(k) = \sqrt{\hat{\alpha}(k)}$ is called *dispersion relation*.

We assume $\alpha(\cdot)$ to satisfy the following properties:

Assumption 1.

- (a1) $\alpha(y) \neq 0$ for some $y \neq 0$.
- (a2) $\alpha(y) = \alpha(-y)$ for all $y \in \mathbb{Z}$.
- (a3) There are constants $C_1, C_2 > 0$ such that for all y

$$|\alpha(y)| \leq C_1 e^{-C_2 |y|}.$$

- (a4) We require either $\hat{\alpha}(k) > 0$ for all $k \in \mathbb{T}$ (pinned case), or $\hat{\alpha}(k) > 0 \forall k \neq 0$, and $\hat{\alpha}(0) = 0$ (unpinned case). Moreover, in the unpinned case we assume $\hat{\alpha}''(0) > 0$.

Assumptions (a2), (a3) ensure that $\hat{\alpha}$ is a real analytic function on \mathbb{T} . Notice that in the pinned case ω is strictly positive and also analytic on \mathbb{T} . In the unpinned case, the condition (a4) says that $\omega(k) = c|k|$ with $c > 0$ for small k , to say ω is a *regular acoustic* dispersion relation.

We consider the Hamiltonian dynamics weakly perturbed by a stochastic noise acting only on momenta and locally preserving momentum and kinetic energy. The generator of the dynamics is

$$L = A + \varepsilon\gamma S \quad (4)$$

with $\varepsilon > 0$, where A is the usual Hamiltonian vector field

$$A = \sum_{y \in \mathbb{Z}} p_y \partial_{q_y} - \sum_{y, y' \in \mathbb{Z}} \alpha(y - y') q_{y'} \partial_{p_y}, \quad (5)$$

while S is the generator of the stochastic perturbation. The operator S acts only on the momenta $\{p_y\}$ and generates a diffusion on the surface of constant kinetic energy and constant momentum. S is defined as

$$S = \frac{1}{6} \sum_{z \in \mathbb{Z}} (Y_z)^2, \quad (6)$$

where

$$Y_z = (p_z - p_{z+1}) \partial_{p_{z-1}} + (p_{z+1} - p_{z-1}) \partial_{p_z} + (p_{z-1} - p_z) \partial_{p_{z+1}}$$

which is a vector field tangent to the surface of constant kinetic energy and of constant momentum for three neighbouring particles. As a consequence energy and momentum are locally conserved which, of course, implies also the conservation of total momentum and total energy of the system,

$$S \sum_{y \in \mathbb{Z}} p_y = 0, \quad SH = 0.$$

The evolution of $\{p(t), q(t)\}$ is given by the following stochastic differential equations

$$\begin{aligned} dq_y &= p_y dt, \\ dp_y &= -(\alpha * q)_y dt + \frac{\varepsilon\gamma}{6} \Delta(4p_y + p_{y-1} + p_{y+1}) dt \\ &\quad + \sqrt{\frac{\varepsilon\gamma}{3}} \sum_{k=-1,0,1} (Y_{y+k} p_y) dw_{y+k}(t). \end{aligned} \quad (7)$$

Here $\{w_y(t)\}_{y \in \mathbb{Z}}$ are independent standard Wiener processes and Δ is the discrete laplacian on \mathbb{Z} ,

$$\Delta f(z) = f(z+1) + f(z-1) - 2f(z).$$

Let us introduce a complex valued field $\psi : \mathbb{Z} \rightarrow \mathbb{C}$ defined as

$$\psi(y, t) = \frac{1}{\sqrt{2}} ((\tilde{\omega} * q)_y(t) + ip_y(t)). \quad (8)$$

Observe that $|\psi(y)|^2 = \frac{1}{2}p_y^2 + \frac{1}{2}\sum_{y' \in \mathbb{Z}} \alpha(y-y')q_y q_{y'} = e_y$ is the energy of particle y and conservation of total energy is equivalent to the conservation of the ℓ_2 -norm. For every $t \geq 0$ the evolution of ψ is given by the following SDE,

$$\begin{aligned} d\psi(y, t) = & -i(\tilde{\omega} * \psi)(y, t)dt + \frac{1}{2}\varepsilon\gamma\beta * (\psi - \psi^*)(y, t)dt \\ & + \sqrt{\frac{\varepsilon\gamma}{3}} \sum_{k=-1,0,1} (Y_{y+k} \frac{1}{2}(\psi - \psi^*)(y, t))dw_{y+k}(t), \end{aligned} \quad (9)$$

where β is defined through

$$(\beta * f)(z) = \frac{1}{6}\Delta(4f(z) + f(z-1) + f(z+1)). \quad (10)$$

3. WIGNER DISTRIBUTION AND THE BOLTZMANN PHONON EQUATION

Given a function J on $\mathbb{R} \times \mathbb{T}$, we define on $\mathbb{R} \times \mathbb{Z}$

$$\tilde{J}(x, z) = \int_{\mathbb{T}} dk e^{2\pi ikz} J(x, k). \quad (11)$$

We also define on $\mathbb{R} \times \mathbb{T}$

$$\hat{J}(p, k) = \int_{\mathbb{R}} dx e^{-2\pi ipx} J(x, k). \quad (12)$$

We choose a class of test-functions J on $\mathbb{R} \times \mathbb{T}$ such that $J(\cdot, k) \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ for any $k \in \mathbb{T}$.

Let us fix $\varepsilon > 0$. We denote by $\langle \cdot \rangle_\varepsilon$ the expectation value with respect to a family of probability measures on phase space which satisfies the following properties:

- (b1) $\langle \psi(y) \rangle_\varepsilon = 0, \quad \forall y \in \mathbb{Z};$
- (b2) $\langle \psi(y')\psi(y) \rangle_\varepsilon = 0, \quad \forall y, y' \in \mathbb{Z};$
- (b3) $\sup_{\varepsilon > 0} \varepsilon \langle \|\psi\|_{\ell_2}^2 \rangle_\varepsilon \leq K$ for some $K > 0$.

Observe that, since $\langle \|\psi\|^2 \rangle_\varepsilon = \langle H \rangle_\varepsilon$ is the expectation value of the energy, we are considering states with an energy of order ε^{-1} . We define the Wigner distribution W^ε through

$$\begin{aligned} \langle J, W^\varepsilon \rangle &= \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle \psi(y')^* \psi(y) \rangle_\varepsilon \int_{\mathbb{T}} dk e^{2\pi ik(y'-y)} J(\varepsilon(y'+y)/2, k)^* \\ &= \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle \psi(y')^* \psi(y) \rangle_\varepsilon \tilde{J}(\varepsilon(y'+y)/2, y-y')^*, \end{aligned} \quad (13)$$

where $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$. By condition (b3) this distribution is well defined, see proposition 13 in the appendix.

More traditionally, the Wigner distribution can be written as

$$\langle J, W^\varepsilon \rangle = \frac{\varepsilon}{2} \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \hat{J}(p, k)^*, \quad (14)$$

where $\hat{\psi}$ is the Fourier transform of ψ , periodically extended to the whole of \mathbb{R} . It is easy to prove the following proposition.

Proposition 2. *Under the assumption (b3) for every test function $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$, there exist constants K_1, K_2 such that*

$$\begin{aligned} |\langle J, W^\varepsilon \rangle| &\leq K_1 \int_{\mathbb{R}} dp \sup_{k \in \mathbb{T}} |\hat{J}(p, k)| < \infty, \\ |\langle J, W^\varepsilon \rangle| &\leq K_2 \sum_{z \in \mathbb{Z}} \sup_{x \in \mathbb{Z}} |\tilde{J}(x, z)| < \infty \end{aligned} \quad (15)$$

for every $\varepsilon > 0$.

Remark 3. *Notice that W^ε is well defined on a wider class of test functions than $\mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d)$. In particular we can take $J(x, k) = J(k)$, a bounded real valued function on \mathbb{T} , and by (14) we have*

$$\langle J, W^\varepsilon \rangle = \frac{\varepsilon}{2} \int_{\mathbb{T}} dk \langle |\hat{\psi}(k)|^2 \rangle_\varepsilon J(k),$$

while choosing $J(x, k) = J(x)$, a bounded real valued function on \mathbb{R} , we have

$$\langle J, W^\varepsilon \rangle = \frac{\varepsilon}{2} \sum_{y \in \mathbb{Z}} \langle e_y \rangle_\varepsilon J(\varepsilon y).$$

Let us start our dynamics with an initial measure satisfying conditions (b1), (b2), (b3). We want to study the evolution of the Wigner distribution W^ε on the time scale $\varepsilon^{-1}t$, i.e. we define for $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$,

$$\begin{aligned} &\langle J, W^\varepsilon(t) \rangle \\ &= \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle \psi(y', t/\varepsilon)^* \psi(y, t/\varepsilon) \rangle_\varepsilon \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} J(\varepsilon(y' + y)/2, k)^*. \end{aligned} \quad (16)$$

Observe that since the dynamics preserves the total energy, the condition $\varepsilon \langle \|\psi\| \rangle \leq K$ holds at any time and, by proposition 13, the Wigner distribution is well defined at any time.

According to remark 3, if we choose test functions depending only on k , then we obtain the distribution of energy in k -space. It turns out that in the limit as $\varepsilon \rightarrow 0$, this distribution converges to the solution of the homogeneous Boltzmann equation, and this will be our first result. Define the distribution $\mathcal{E}^\varepsilon(t)$ on \mathbb{T} by

$$\langle J, \mathcal{E}^\varepsilon(t) \rangle = \langle J, W^\varepsilon(t) \rangle = \frac{\varepsilon}{2} \int_{\mathbb{T}} dk \langle |\hat{\psi}(k, t/\varepsilon)|^2 \rangle_\varepsilon J(k) \quad (17)$$

for any bounded function J .

Theorem 4. *Assume that the initial measure satisfies conditions (b1), (b2), (b3), and furthermore $\mathcal{E}^\varepsilon(0)$ converges to a positive measure $\mathcal{E}_0(dk)$ on \mathbb{T} . Then $\mathcal{E}^\varepsilon(t)$ converges to $\mathcal{E}(t, dk)$, the solution of*

$$\partial_t \langle J, \mathcal{E}(t) \rangle = \gamma \langle CJ, \mathcal{E}(t) \rangle \quad (18)$$

for every bounded function $J : \mathbb{T} \rightarrow \mathbb{R}$. The collision operator, C , is defined by

$$CJ(x, k) = \int_{\mathbb{T}} dk' R(k, k') (J(x, k') - J(x, k)) \quad (19)$$

with

$$R(k, k') = \frac{4}{3} (2 \sin^2(2\pi k) \sin^2(\pi k') + 2 \sin^2(2\pi k') \sin^2(\pi k) - \sin^2(2\pi k) \sin^2(2\pi k')). \quad (20)$$

In order to prove the next theorem, the full inhomogeneous equation, we need an additional condition on the initial distribution in the unpinned case ($\hat{\alpha}(0) = 0$):

(b4) In the unpinned case we require

$$\lim_{R \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\varepsilon}{2} \int_{|k| < R} dk \langle |\hat{\psi}(k)|^2 \rangle_\varepsilon = 0.$$

Condition (b4) ensures that there is no initial concentration of energy at wave number $k = 0$. This condition can be omitted if we consider a dispersion relation ω which is analytic on \mathbb{T} (as in the pinned case).

Theorem 5. *Let Assumptions (b1-b4) hold and assume that $W^\varepsilon(0)$ converges to a positive measure $\mu_0(dx, dk)$. Then, for all $t \in [0, T]$, $W^\varepsilon(t)$ converges to a positive measure $\mu_0(t, dx, dk)$, which is the unique solution of the Boltzmann equation*

$$\partial_t \langle J, \mu(t) \rangle = \frac{1}{2\pi} \langle \omega'(k) \partial_x J, \mu(t) \rangle + \gamma \langle CJ, \mu(t) \rangle \quad (21)$$

with initial condition $\mu_0(dx, dk)$.

In (21) $\langle J, \mu(t) \rangle$ denotes the linear functional $\int_{\mathbb{R} \times \mathbb{T}} J(x, k) \mu(dx, dk)$.

Observe that the kernel R of (20) is non-negative, symmetric, and is equal to zero only if $k = 0$ or $k' = 0$. Moreover, it is easy to see that

$$\int_{\mathbb{T}} dk' R(k, k') = -\hat{\beta}(k), \quad (22)$$

where $\hat{\beta}(k)$ is the Fourier transform of the function β defined in (10). Thus the Boltzmann equation (21) can be interpreted as the forward equation of a Markov process $(X(t), K(t))$ on $\mathbb{R} \times \mathbb{T}$ for the dynamics of a particle, which in the context of lattice dynamics is called *phonon*. The phonon with momentum k travels with velocity $\omega'(k)$ and suffers random collisions. More precisely $K(t)$

is an autonomous reversible jump Markov process with jump rate R , while the position $X(t)$ is determined through

$$X(t) = X(0) + \int_0^t \omega'(K(s)) ds.$$

4. PROOF OF THEOREMS 4 AND 5.

4.1. Relative Compactness of the Wigner distribution. Existence of the limit of the Wigner distributions $W^\varepsilon(t)$ will be established as in [1]. The limit distribution $W(t)$ is non-negative, as it is proved in [7], [8], i.e. for every $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}, \mathbb{C})$

$$\langle |J|^2, W^\varepsilon \rangle = \frac{\varepsilon}{2} \left\langle \int_{\mathbb{T}} dk \left| \sum_{y \in \mathbb{Z}} J(\varepsilon y, k) e^{-2\pi i k y} \psi(y) \right|^2 \right\rangle_\varepsilon + \mathcal{O}(\varepsilon). \quad (23)$$

Let us introduce the space \mathcal{A} of functions J on $\mathbb{R}^d \times \mathbb{T}^d$ such that

$$\|J\|_{\mathcal{A}} = \sum_{z \in \mathbb{Z}} \sup_{x \in \mathbb{R}} |\tilde{J}(x, z)| < \infty, \quad (24)$$

where \tilde{J} is defined in (11). The following lemma shows that if the distributions W^ε are uniformly bounded in \mathcal{A}' , the dual space to \mathcal{A} , then at every time t one can choose a sequence $\varepsilon_j \rightarrow 0$ such that W^{ε_j} converge in the *-weak topology in \mathcal{A}' to a limit distribution $W(t)$.

Lemma 6. *There exists a constant $C > 0$ independent of t such that $\forall \varepsilon > 0$*

$$\|W^\varepsilon(t)\|_{\mathcal{A}'} \leq C.$$

Proof. For every $J \in \mathcal{A}$

$$\langle J, W^\varepsilon(t) \rangle = \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle \psi(y') \psi(y) \rangle_\varepsilon \tilde{J}(\varepsilon(y + y')/2, y - y')^*.$$

Then, using Schwarz inequality and Assumption (b3),

$$|\langle J, W^\varepsilon(t) \rangle_\varepsilon| \leq \frac{\varepsilon}{2} \sum_{y \in \mathbb{Z}} \langle |\psi(y)|^2 \rangle_\varepsilon \|J\|_{\mathcal{A}} \leq K \|J\|_{\mathcal{A}}.$$

□

4.2. Proof of theorem 4. We consider a class of test functions J depending only on $k \in \mathbb{T}$. In particular we choose J real valued and bounded. Recall the definition

$$\langle J, \mathcal{E}^\varepsilon \rangle = \frac{\varepsilon}{2} \int_{\mathbb{T}} dk \langle |\hat{\psi}(k)|^2 \rangle_\varepsilon J(k),$$

which is well defined since $|\langle J, \mathcal{E}^\varepsilon \rangle| \leq \frac{1}{2} K \sup_{k \in \mathbb{T}} |J(k)|$.

The evolution of the distribution $\mathcal{E}^\varepsilon(t)$ is determined by Ito's formula, namely

$$\partial_t \langle J, \mathcal{E}^\varepsilon(t) \rangle = \frac{\varepsilon}{2} \int_{\mathbb{T}} dk \varepsilon^{-1} \langle L |\hat{\psi}(k, t/\varepsilon)|^2 \rangle_\varepsilon J(k),$$

where

$$L|\hat{\psi}(k)|^2 = A|\hat{\psi}(k)|^2 + \varepsilon\gamma S|\hat{\psi}(k)|^2$$

and A, S are respectively defined in (5), (6). We have

$$A|\hat{\psi}(k)|^2 = [A\hat{\psi}(k)^*]\hat{\psi}(k) + \hat{\psi}(k)^*[A\hat{\psi}(k)],$$

where by direct computation

$$A\hat{\psi}(k) = \sum_{y \in \mathbb{Z}} e^{-2\pi iky} A\psi(y) = -i \sum_{y \in \mathbb{Z}} e^{-2\pi iky} (\tilde{\omega} * \psi)(y) = -i\omega(k)\hat{\psi}(k) \quad (25)$$

and thus $A|\hat{\psi}(k)|^2 = 0$. For the stochastic part, since S is a second order operator, we have

$$S|\hat{\psi}(k)|^2 = [S\hat{\psi}(k)^*]\hat{\psi}(k) + \hat{\psi}(k)^*[S\hat{\psi}(k)] + \frac{1}{3} \sum_{z \in \mathbb{Z}} [Y_z \hat{\psi}(k)^*][Y_z \hat{\psi}(k)],$$

where by direct computation

$$\begin{aligned} S\hat{\psi}(k) &= \sum_{y \in \mathbb{Z}} e^{-2\pi iky} S\psi(y) = \frac{1}{2} \sum_{y \in \mathbb{Z}} e^{-2\pi iky} \beta * (\psi - \psi^*)(y) \\ &= \frac{1}{2} \hat{\beta}(k)(\hat{\psi}(k) - \hat{\psi}(-k)^*) \end{aligned} \quad (26)$$

with β defined in (10). Thus

$$\begin{aligned} &(S\hat{\psi}(k)^*)\hat{\psi}(k) + \hat{\psi}(k)^*(S\hat{\psi}(k)) \\ &= \hat{\beta}|\hat{\psi}(k)|^2 - \frac{1}{2} \hat{\beta}(\hat{\psi}(k)\hat{\psi}(-k) + \hat{\psi}(k)^*\hat{\psi}(-k)^*), \end{aligned}$$

where

$$\hat{\beta}(k) = -\frac{4}{3} \sin^2(\pi k)(1 + 2 \cos^2(\pi k)). \quad (27)$$

Finally we have to compute $\frac{1}{3} \sum_{z \in \mathbb{Z}} [Y_z \hat{\psi}(k)^*][Y_z \hat{\psi}(k)]$. It holds

$$\sum_{z \in \mathbb{Z}} [Y_z \hat{\psi}(k)^*][Y_z \hat{\psi}(k)] = \sum_{y, y' \in \mathbb{Z}} e^{2\pi ik(y'-y)} \sum_{z \in \mathbb{Z}} [Y_z \psi(y')^*][Y_z \psi(y)], \quad (28)$$

where

$$\begin{aligned} &\sum_{z \in \mathbb{Z}} [Y_z \psi(y')^*][Y_z \psi(y)] \\ &= [Y_{y+1} \psi(y')^*][Y_{y+1} \psi(y)] + [Y_y \psi(y')^*][Y_y \psi(y)] + [Y_{y-1} \psi(y')^*][Y_{y-1} \psi(y)]. \end{aligned}$$

This expression is explicitly computed in the appendix, see eq. (77). By inserting it in (28) we get

$$\begin{aligned} & \sum_{y, y' \in \mathbb{Z}} e^{2\pi i k(y' - y)} \sum_{z \in \mathbb{Z}} [Y_z \psi(y')^*] [Y_z \psi(y)] \\ &= \cos(4\pi k) \sum_{y \in \mathbb{Z}} (2p_y p_{y+1} - p_y p_{y+2} - p_y^2) \\ & \quad + \cos(2\pi k) \sum_{y \in \mathbb{Z}} (2p_y p_{y+2} - 2p_y^2) + \sum_{y \in \mathbb{Z}} (-2p_y p_{y+1} - p_y p_{y+2} + 3p_y^2), \end{aligned}$$

which is equal to

$$\begin{aligned} & \int_{\mathbb{T}} d\xi |\hat{p}(\xi)|^2 (\cos(4\pi k) [2 \cos(2\pi \xi) - \cos(4\pi \xi) - 1] \\ & \quad + 2 \cos(2\pi k) [\cos(4\pi \xi) - 1] + [3 - 2 \cos(2\pi \xi) - \cos(4\pi \xi)]). \end{aligned}$$

Finally, after some trigonometric identities and using the relation

$$|\hat{p}(k)|^2 = \frac{1}{2} (|\hat{\psi}(k)|^2 + |\hat{\psi}(-k)|^2 - \hat{\psi}(k)\hat{\psi}(-k) - \hat{\psi}(k)^*\hat{\psi}(-k)^*),$$

we get

$$\begin{aligned} & \frac{1}{3} \sum_{z \in \mathbb{Z}} [Y_z \hat{\psi}(k)^*] [Y_z \hat{\psi}(k)] \\ &= \int_{\mathbb{T}} d\xi R(k, \xi) (|\hat{\psi}(\xi)|^2 - \frac{1}{2} [\hat{\psi}(\xi)\hat{\psi}(-\xi) + \hat{\psi}(\xi)^*\hat{\psi}(-\xi)^*]), \end{aligned}$$

where $R(k, \xi)$ is given by (20).

Since $\int_{\mathbb{T}} d\xi R(k, \xi) = -\hat{\beta}(k)$, we can write

$$S|\hat{\psi}(k)|^2 = C|\hat{\psi}(k)|^2 - \frac{1}{2} C(\hat{\psi}(k)\hat{\psi}(-k) + \hat{\psi}(k)^*\hat{\psi}(-k)^*),$$

where C is the operator defined in (19), i.e.

$$Cf(k) = \int_{\mathbb{T}} d\xi R(k, \xi) (f(\xi) - f(k)).$$

The evolution of $\mathcal{E}^\varepsilon(t)$ is given by

$$\begin{aligned} \partial_t \langle J, \mathcal{E}^\varepsilon(t) \rangle &= \gamma \frac{\varepsilon}{2} \int_{\mathbb{T}} dk \langle |\hat{\psi}(k, t/\varepsilon)|^2 \rangle_\varepsilon C J(k) \\ & \quad - \gamma \frac{\varepsilon}{2} \int_{\mathbb{T}} dk \frac{1}{2} [\langle (\hat{\psi}(k)\hat{\psi}(-k))(t/\varepsilon) \rangle_\varepsilon + \langle (\hat{\psi}(k)^*\hat{\psi}(-k)^*)(t/\varepsilon) \rangle_\varepsilon] (C J)(k). \end{aligned}$$

Defining the distribution $Y^\varepsilon(t)$ on \mathbb{T} through

$$\begin{aligned} \langle J, Y^\varepsilon(t) \rangle &= \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle \psi(y', t/\varepsilon) \psi(y, t/\varepsilon) \rangle_\varepsilon \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} J(k) \\ &= \frac{\varepsilon}{2} \int_{\mathbb{T}} dk \langle [\hat{\psi}(k)\hat{\psi}(-k)](t/\varepsilon) \rangle_\varepsilon J(k), \end{aligned}$$

we can rewrite the evolution equation as

$$\partial_t \langle J, \mathcal{E}^\varepsilon(t) \rangle = \gamma \langle CJ, \mathcal{E}^\varepsilon(t) \rangle - \frac{\gamma}{2} (\langle CJ, Y^\varepsilon(t) \rangle + \langle CJ, Y^\varepsilon(t)^* \rangle). \quad (29)$$

This is not a closed equation for $\mathcal{E}^\varepsilon(t)$. However we expect that in the limit $\varepsilon \rightarrow 0$ the terms containing the distributions $Y^\varepsilon(t)$, $Y^\varepsilon(t)^*$ disappear. In order to prove it, let us consider the evolution of the distribution $Y^\varepsilon(t)$ on the kinetic time scale. Calculations are similar to the previous ones, but with the difference that now $A[\hat{\psi}(k)\hat{\psi}(-k)] \neq 0$, and precisely

$$A[\hat{\psi}(k)\hat{\psi}(-k)] = -2i\omega(k)\hat{\psi}(k)\hat{\psi}(-k).$$

We arrive at

$$\begin{aligned} \partial_t \langle J, Y^\varepsilon(t) \rangle &= -\frac{2i}{\varepsilon} \langle \omega J, Y^\varepsilon(t) \rangle + \frac{\gamma}{2} \langle \hat{\beta} J, Y^\varepsilon(t) \rangle \\ &+ \frac{\gamma}{2} (\langle CJ, Y^\varepsilon(t) \rangle + \langle CJ, Y^\varepsilon(t)^* \rangle) - \frac{\gamma}{2} \langle \hat{\beta} J, Y^\varepsilon(t)^* \rangle - \gamma \langle CJ, \mathcal{E}^\varepsilon(t) \rangle. \end{aligned} \quad (30)$$

Observe that by integrating eq. (30) in time, we obtain

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t dt \langle \omega J, Y^\varepsilon(t) \rangle \right| = 0$$

for every bounded function J . In particular, since by item (i) of lemma 14

$$\sup_{k \in \mathbb{T}} \frac{R(k, k')}{\omega(k)} < \infty,$$

we can choose a function $\omega^{-1}CJ$ with J bounded and obtain

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t dt \langle CJ, Y^\varepsilon(t) \rangle \right| = 0.$$

In the same way we have $\lim_{\varepsilon \rightarrow 0} \left| \int_0^t dt \langle CJ, Y^\varepsilon(t)^* \rangle \right| = 0$ and any limit distribution $\mathcal{E}(t)$ of $\mathcal{E}^\varepsilon(t)$ solves the equation

$$\langle J, \mathcal{E}(t) \rangle = \langle J, \mathcal{E}(0) \rangle + \gamma \int_0^t ds \langle CJ, \mathcal{E}(s) \rangle$$

for every bounded real valued function J . \square

4.3. Proof of theorem 5. Now we will give the proof of (21) for the class test function $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}, \mathbb{C})$. The main difference to the previous case is that the Hamiltonian part of the generator contributes to the evolution of $W^\varepsilon(t)$, resulting in a ballistic transport term. In order to control this term, we need to ensure that there is no mass concentration at $k = 0$ for every macroscopic time $t \in [0, T]$ with $T > 0$. This is stated in the following lemma.

Lemma 7. *Let assumption (b4) hold. Then for every $t \in [0, T]$*

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\varepsilon}{2} \int_{|k| < \rho} dk \langle |\hat{\psi}(k, t/\varepsilon)|^2 \rangle_\varepsilon = 0.$$

Proof. We use the evolution equation (29) for $J_\rho(k) = 1_{[-\rho, \rho]}(k)$. Since $|CJ_\rho(k)| \leq c_1(2\rho + J_\rho(k))$ and $\langle |CJ_\rho(k)|, |Y^\varepsilon(t) + Y^\varepsilon(t)^*| \rangle \leq \langle |CJ_\rho(k)|, \mathcal{E}^\varepsilon(t) \rangle$, we obtain the bound

$$\begin{aligned} \langle J_\rho, \mathcal{E}^\varepsilon(t) \rangle &\leq \langle J_\rho, \mathcal{E}^\varepsilon(0) \rangle + c_2\gamma \int_0^t ds \langle |CJ_\rho|, \mathcal{E}^\varepsilon(s) \rangle \\ &\leq \langle J_\rho, \mathcal{E}^\varepsilon(0) \rangle + c_3\gamma \left(2\rho Kt + \int_0^t ds \langle J_\rho, \mathcal{E}^\varepsilon(s) \rangle \right), \end{aligned}$$

where K is the bound on the total energy from condition (b3). Then by Gronwall's inequality

$$\langle J_\rho, \mathcal{E}^\varepsilon(t) \rangle \leq (2\rho K + \langle J_\rho, \mathcal{E}^\varepsilon(0) \rangle) e^{c_3\gamma t},$$

where, by assumption (b4), $\overline{\lim}_{\varepsilon \rightarrow 0} \langle J_\rho, \mathcal{E}^\varepsilon(0) \rangle \rightarrow 0$ for $\rho \rightarrow 0$. \square

4.3.1. *Proof of theorem 5.* For every $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}, \mathbb{C})$ the evolution of the distribution $W^\varepsilon(t)$ on the kinetic time-scale is given by

$$\begin{aligned} \partial_t \langle J, W^\varepsilon(t) \rangle &= (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \partial_t \langle \psi(y', t/\varepsilon)^* \psi(y, t/\varepsilon) \rangle_\varepsilon \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} J(\varepsilon(y + y')/2, k)^* \\ &= (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \varepsilon^{-1} \langle L[\psi(y')^* \psi(y)] \rangle_\varepsilon \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} J(\varepsilon(y + y')/2, k)^*, \end{aligned}$$

where

$$L[\psi(y')^* \psi(y)] = A[\psi(y')^* \psi(y)] + \varepsilon\gamma S[\psi(y')^* \psi(y)]$$

and A, S are defined in (5), (6), respectively. We start by computing the evolution determined by A , the Hamiltonian part of the generator. Using the representation of the Wigner distribution in Fourier space we get

$$\begin{aligned} \frac{\varepsilon}{2} \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \varepsilon^{-1} A[\langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon] \widehat{J}(p, k)^* \\ = -i \frac{\varepsilon}{2} \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \\ \varepsilon^{-1} [\omega(k + \varepsilon p/2) - \omega(k - \varepsilon p/2)] \widehat{J}(p, k)^*. \end{aligned}$$

Now we prove that one can replace $\varepsilon^{-1}[\omega(k + \varepsilon p/2) - \omega(k - \varepsilon p/2)]$ with $\omega'(k)p$ in the last expression. For every $0 < \rho < 1/2$

$$\begin{aligned} \frac{\varepsilon}{2} \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \\ (\varepsilon^{-1}[\omega(k + \varepsilon p/2) - \omega(k - \varepsilon p/2)] - \omega'(k)p) \widehat{J}(p, k)^* = I_{>}^\varepsilon(\rho) + I_{<}^\varepsilon(\rho), \end{aligned}$$

where

$$\begin{aligned} I_{>}^\varepsilon(\rho) &= \frac{\varepsilon}{2} \int_{\mathbb{R}} dp \int_{|k|>\rho} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \\ &\quad (\varepsilon^{-1}[\omega(k + \varepsilon p/2) - \omega(k - \varepsilon p/2)] - \omega'(k)p) \widehat{\mathcal{J}}(p, k)^* \\ I_{<}^\varepsilon(\rho) &= \frac{\varepsilon}{2} \int_{\mathbb{R}} dp \int_{|k|<\rho} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \\ &\quad (\varepsilon^{-1}[\omega(k + \varepsilon p/2) - \omega(k - \varepsilon p/2)] - \omega'(k)p) \widehat{\mathcal{J}}(p, k)^*. \end{aligned}$$

Using Schwarz inequality and points (i), (ii) of lemma 14 in the appendix

$$\begin{aligned} |I_{<}^\varepsilon(\rho)| &\leq \int_{\mathbb{R}} dp (C|p| + \|\nabla\omega\|_\infty) \sup_{k \in \mathbb{T}} |\widehat{\mathcal{J}}(p, k)| \left(\frac{\varepsilon}{2} \int_{|k| \leq \rho} dk \langle |\hat{\psi}(k)|^2 \rangle_\varepsilon \right) \\ &\leq C_0 \frac{\varepsilon}{2} \int_{|k| \leq \rho} dk \langle |\hat{\psi}(k)|^2 \rangle_\varepsilon \end{aligned}$$

and, by lemma 7,

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} |I_{<}^\varepsilon(\rho)| = 0.$$

To compute $I_{>}^\varepsilon(\rho)$ we split it into two parts,

$$\begin{aligned} I_{>}^\varepsilon(\rho) &= \frac{\varepsilon}{2} \int_{\mathbb{R}, \varepsilon|p| \geq \rho} dp \int_{|k|>\rho} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \\ &\quad (\varepsilon^{-1}[\omega(k + \varepsilon p/2) - \omega(k - \varepsilon p/2)] - \omega'(k)p) \widehat{\mathcal{J}}(p, k)^* \\ &\quad + \frac{\varepsilon}{2} \int_{\mathbb{R}, \varepsilon|p| < \rho} dp \int_{|k|>\rho} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \\ &\quad (\varepsilon^{-1}[\omega(k + \varepsilon p/2) - \omega(k - \varepsilon p/2)] - \omega'(k)p) \widehat{\mathcal{J}}(p, k)^*. \end{aligned}$$

Again we use Schwarz inequality and points (i), (ii) of lemma 14 to show that the first term on the RHS is negligible, since for all $\rho > 0$ it is bounded by

$$K \int_{|p| \geq \rho/\varepsilon} dp (c|p| + \|\nabla\omega\|_\infty) \sup_{k \in \mathbb{T}} |\widehat{\mathcal{J}}(p, k)|,$$

which tends to 0 as $\varepsilon \rightarrow 0$.

For the second term on the RHS we use the point (iii) of lemma 14, since $|k| > \rho$, $\varepsilon|p| < \rho$ implies $|k| > \varepsilon|p|$, and for all $\rho > 0$ we get

$$\begin{aligned} &\frac{\varepsilon}{2} \int_{|p| < \rho/\varepsilon} dp \int_{|k|>\rho} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \\ &\quad (\varepsilon^{-1}[\omega(k + \varepsilon p/2) - \omega(k - \varepsilon p/2)] - \omega'(k)p) \widehat{\mathcal{J}}(p, k)^* \\ &\leq K \int_{\mathbb{R}} dp \varepsilon \frac{C_4}{\rho} |p|^2 \sup_{k \in \mathbb{T}} |\widehat{\mathcal{J}}(p, k)|, \end{aligned}$$

which tends to 0 as $\varepsilon \rightarrow 0$. Then we have

$$\begin{aligned}
 & \frac{\varepsilon}{2} \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \varepsilon^{-1} A [\langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_{\varepsilon}] \widehat{J}(p, k)^* \\
 &= \frac{\varepsilon}{2} \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \langle \hat{\psi}(k - \varepsilon p/2)^* \hat{\psi}(k + \varepsilon p/2) \rangle_{\varepsilon} (-i p) \omega'(k) \widehat{J}(p, k)^* + \mathcal{O}(\varepsilon) \quad (31) \\
 &= \frac{1}{2\pi} \left\langle \nabla \omega \nabla_r J, W^{\varepsilon}(t) \right\rangle_{\varepsilon} + \mathcal{O}(\varepsilon).
 \end{aligned}$$

We have to compute

$$\begin{aligned}
 & (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \langle S[\psi(y')^* \psi(y)] \rangle_{\varepsilon} \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} J(\varepsilon(y + y')/2, k)^* \\
 &= \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle S[\psi(y')^* \psi(y)] \rangle_{\varepsilon} \tilde{J}(\varepsilon(y + y')/2, y - y')^*.
 \end{aligned}$$

Since S is a second order operator, we have

$$S[\psi(y')^* \psi(y)] = \psi(y')^* S\psi(y) + [S\psi(y')^*] \psi(y) + \frac{1}{3} \sum_{z \in \mathbb{Z}} [Y_z \psi(y')^*] [Y_z \psi(y)],$$

where by direct computation

$$S\psi(y) = \frac{i}{\sqrt{2}} S p_y = \frac{1}{2} \beta * (\psi^* - \psi)(y) \quad (32)$$

and

$$\psi(y')^* S\psi(y) + [S\psi(y')^*] \psi(y) = \frac{i}{\sqrt{2}} (\psi(y')^* (\beta * p)_y - \psi(y) (\beta * p)_{y'}).$$

Integrating by parts and using the symmetry of β , we can rewrite

$$\begin{aligned}
 & \gamma(\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} [\langle \psi(y')^* S\psi(y) \rangle_{\varepsilon} + \langle (S\psi(y')^*) \psi(y) \rangle_{\varepsilon}] \tilde{J}(\varepsilon(y + y')/2, y - y')^* \\
 &= \gamma(\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \frac{i}{\sqrt{2}} \langle \psi(y')^* p_y \rangle_{\varepsilon} \sum_{z \in \mathbb{Z}} \beta(z) \tilde{J}(\varepsilon(y + y' + z)/2, y - y' + z)^* \\
 &\quad - \gamma(\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \frac{i}{\sqrt{2}} \langle \psi(y) p_{y'} \rangle_{\varepsilon} \sum_{z \in \mathbb{Z}} \beta(z) \tilde{J}(\varepsilon(y + y' - z)/2, y - y' + z)^*.
 \end{aligned}$$

Using the energy bound and the properties of the test functions J , one can write the first term on right hand side as

$$\begin{aligned}
 & \gamma(\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \frac{i}{\sqrt{2}} \langle \psi(y')^* p_y \rangle_{\varepsilon} \sum_{z \in \mathbb{Z}} \beta(z) \tilde{J}(\varepsilon(y + y')/2, y - y' + z)^* + \mathcal{O}(\varepsilon) \\
 &= \gamma(\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \frac{i}{\sqrt{2}} \langle \psi(y')^* p_y \rangle_{\varepsilon} \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} \hat{\beta}(k) J(\varepsilon(y + y')/2, k)^* \\
 &\quad + \mathcal{O}(\varepsilon)
 \end{aligned}$$

and the same can be done for the other term. Finally we obtain

$$\begin{aligned}
& (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} [\langle \psi(y')^* S \psi(y) \rangle_\varepsilon + \langle (S \psi(y')^*) \psi(y) \rangle_\varepsilon] \tilde{J}(\varepsilon(y + y')/2, y - y')^* \\
&= (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \frac{i}{\sqrt{2}} \langle \psi(y')^* p_y - \psi(y) p_{y'} \rangle_\varepsilon \\
& \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} \hat{\beta}(k) J(\varepsilon(y + y')/2, k)^* + \mathcal{O}(\varepsilon).
\end{aligned} \tag{33}$$

About the other term in (32) first observe that $\sum_{z \in \mathbb{Z}} [Y_z \psi(y')^*] [Y_z \psi(y)]$ is just a finite sum for any y, y' fixed and, computing it explicitly and identifying terms that differ by translations, see details of the computation in appendix 7.1, one obtains

$$\begin{aligned}
& \frac{1}{3} (\varepsilon/2) \sum_{y, y', z \in \mathbb{Z}} \langle [Y_z \psi(y')^*] [Y_z \psi(y)] \rangle \tilde{J}(\varepsilon(y' + y)/2, y - y')^* \\
&= (\varepsilon/2) \sum_{y \in \mathbb{Z}} \sum_{z, u = -2}^2 \alpha(z, u) \langle p_y p_{y+z} \rangle_\varepsilon \tilde{J}(\varepsilon y, u)^* + \mathcal{O}(\varepsilon),
\end{aligned} \tag{34}$$

where $\alpha(z, u) = \alpha(-z, u) = \alpha(z, -u) = \alpha(u, z)$ and is given by (78). We can rewrite it as

$$\begin{aligned}
& (\varepsilon/2) \sum_{y \in \mathbb{Z}} \sum_{z, u = -2}^2 \alpha(z, u) \langle p_y p_{y+z} \rangle_\varepsilon \tilde{J}(\varepsilon(y + z/2), u)^* + \mathcal{O}(\varepsilon) \\
&= (\varepsilon/2) \sum_{y \in \mathbb{Z}} \sum_{z, u \in \mathbb{Z}} \alpha(z, u) \langle p_y p_{y+z} \rangle_\varepsilon \tilde{J}(\varepsilon(y + z/2), u)^* + \mathcal{O}(\varepsilon),
\end{aligned}$$

where we put $\alpha(z, u) = 0$ if $|z| > 2$ or $|u| > 2$ and by changing variables we obtain

$$(\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \langle p_y p_{y'} \rangle_\varepsilon \sum_{u \in \mathbb{Z}} \alpha(y' - y, u) \tilde{J}(\varepsilon(y' + y)/2, u)^* + \mathcal{O}(\varepsilon). \tag{35}$$

Defining

$$R(k, k') = \sum_{z \in \mathbb{Z}} \sum_{u \in \mathbb{Z}} e^{-2\pi i k z} e^{-2\pi i k' u} \alpha(z, u),$$

we can rewrite (35) as

$$(\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \langle p_y p_{y'} \rangle_\varepsilon \int dk e^{2\pi i k(y' - y)} \int dk' R(k, k') J(\varepsilon(y' + y)/2, k')^* + \mathcal{O}(\varepsilon),$$

where direct computation gives

$$\begin{aligned}
R(k, k') &= \frac{2}{3} (3 - 2 \cos(2\pi k) - \cos(4\pi k) - 2 \cos(2\pi k') + 2 \cos(2\pi(k' + 2k)) \\
& \quad - \cos(4\pi k') + 2 \cos(2\pi(2k' + k)) - \cos(2\pi(2k' + 2k))) \\
&= \frac{4}{3} (2 \sin^2(2\pi k) \sin^2(\pi k') + 2 \sin^2(2\pi k') \sin^2(\pi k) - \sin^2(2\pi k) \sin^2(2\pi k')),
\end{aligned}$$

which is the kernel defined in (20). Using the relation $\int_{\mathbb{T}} dk' R(k, k') = -\hat{\beta}(k)$ and

$$\begin{aligned} & \langle p_y p_{y'} \rangle_\varepsilon \\ &= \frac{1}{2} [\langle \psi(y')^* \psi(y) \rangle_\varepsilon + \langle \psi(y') \psi(y)^* \rangle_\varepsilon] - \frac{1}{2} [\langle \psi(y') \psi(y) \rangle_\varepsilon + \langle \psi(y')^* \psi(y)^* \rangle_\varepsilon] \\ & \frac{i}{\sqrt{2}} \langle \psi(y')^* p_y - \psi(y) p_{y'} \rangle_\varepsilon \\ &= \langle \psi(y')^* \psi(y) \rangle_\varepsilon - \frac{1}{2} [\langle \psi(y') \psi(y) \rangle_\varepsilon + \langle \psi(y')^* \psi(y)^* \rangle_\varepsilon], \end{aligned}$$

we obtain

$$\begin{aligned} & \gamma \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle S(\psi(y')^* \psi(y)) \rangle_\varepsilon J(\varepsilon(y' + y)/2, y' - y) \\ &= \gamma \langle CJ, W^\varepsilon \rangle - \frac{\gamma}{2} (\langle CJ, Y^\varepsilon \rangle + \langle CJ, Y^{\varepsilon*} \rangle) + \mathcal{O}(\varepsilon), \end{aligned} \quad (36)$$

where the *collision operator* C is defined in (19) and the distributions $Y^\varepsilon(t)$, $Y^\varepsilon(t)^*$ are defined as

$$\begin{aligned} \langle J, Y^\varepsilon(t) \rangle &= (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \langle \psi(y') \psi(y) \rangle_\varepsilon \tilde{J}(\varepsilon(y + y')/2, y - y')^*, \\ \langle J, Y^\varepsilon(t)^* \rangle &= (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \langle \psi(y')^* \psi(y)^* \rangle_\varepsilon \tilde{J}(\varepsilon(y + y')/2, y - y')^* \end{aligned} \quad (37)$$

for every $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T}, \mathbb{C})$.

The evolution of $W^\varepsilon(t)$ is not a closed equation,

$$\begin{aligned} \partial_t \langle J, W^\varepsilon(t) \rangle &= \langle (\nabla \omega \nabla_x J), W^\varepsilon(t) \rangle + \gamma \langle (CJ), W^\varepsilon(t) \rangle \\ & \quad - \frac{\gamma}{2} \langle (CJ), Y^\varepsilon(t) \rangle - \frac{\gamma}{2} \langle (CJ), Y^\varepsilon(t)^* \rangle + \mathcal{O}(\varepsilon). \end{aligned} \quad (38)$$

However we expect that in the kinetic limit $\varepsilon \rightarrow 0$ the terms containing the distributions $Y^\varepsilon(t)$, $Y^\varepsilon(t)^*$ to disappear. To prove this, we consider the evolution of $Y^\varepsilon(t)$. Again by Ito's formula

$$\begin{aligned} \partial_t \langle J, Y^\varepsilon(t) \rangle &= (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \partial_t \langle \psi(y') \psi(y) \rangle_\varepsilon \tilde{J}(\varepsilon(y + y')/2, y - y')^* \\ &= (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \varepsilon^{-1} \langle L[\psi(y') \psi(y)] \rangle_\varepsilon \tilde{J}(\varepsilon(y + y')/2, y - y')^* \\ &= (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \varepsilon^{-1} \langle A[\psi(y') \psi(y)] \rangle_\varepsilon \tilde{J}(\varepsilon(y + y')/2, y - y')^* \\ & \quad + \gamma (\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \langle S[\psi(y') \psi(y)] \rangle_\varepsilon \tilde{J}(\varepsilon(y + y')/2, y - y')^*. \end{aligned}$$

For the stochastic part of the generator, by a similar computation as above, we obtain

$$\begin{aligned} & \gamma(\varepsilon/2) \sum_{y, y' \in \mathbb{Z}} \langle S[\psi(y')\psi(y)] \rangle_\varepsilon \tilde{J}(\varepsilon(y + y')/2, y' - y) \\ &= \frac{\gamma}{2} \langle (CJ), Y^\varepsilon \rangle + \frac{\gamma}{2} \langle (CJ), Y^{\varepsilon*} \rangle + \frac{\gamma}{2} \langle (\hat{\beta}J), Y^\varepsilon \rangle - \frac{\gamma}{2} \langle (\hat{\beta}J), Y^{\varepsilon*} \rangle \\ & \quad - \frac{\gamma}{2} [\langle (CJ), W^\varepsilon \rangle + \langle (CJ), W^{\varepsilon*} \rangle] + \mathcal{O}(\varepsilon). \end{aligned} \quad (39)$$

To compute the Hamiltonian contribution to the evolution of $Y^\varepsilon(t)$, we use the representation of $Y^\varepsilon(t)$ in the Fourier space and get

$$\begin{aligned} & (\varepsilon/2) \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \varepsilon^{-1} A[\langle \hat{\psi}(k - \varepsilon p/2) \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon] \\ &= -2i\varepsilon^{-1} (\varepsilon/2) \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \varepsilon^{-1} \langle \hat{\psi}(k - \varepsilon p/2) \hat{\psi}(k + \varepsilon p/2) \rangle_\varepsilon \\ & \quad (\omega(k + \varepsilon p/2) + \omega(k - \varepsilon p/2)) \hat{J}(p, k)^*, \end{aligned}$$

where with similar arguments as above one can replace $\omega(k + \varepsilon p/2) + \omega(k - \varepsilon p/2)$ by $2\omega(k)$, with an error of order ε^2 . Then we arrive at the following equation for the evolution of $Y^\varepsilon(t)$,

$$\begin{aligned} \partial_t \langle J, Y^\varepsilon(t) \rangle &= -\frac{2i}{\varepsilon} \langle (\omega J), Y^\varepsilon(t) \rangle + \frac{\gamma}{2} \langle (CJ), Y^\varepsilon \rangle + \frac{\gamma}{2} \langle (\hat{\beta}J), Y^\varepsilon \rangle \\ & \quad + \frac{\gamma}{2} \langle (CJ), Y^{\varepsilon*} \rangle - \frac{\gamma}{2} \langle (\hat{\beta}J), Y^{\varepsilon*} \rangle - \frac{\gamma}{2} [\langle (CJ), W^\varepsilon \rangle + \langle (CJ), W^{\varepsilon*} \rangle] + \mathcal{O}(\varepsilon). \end{aligned} \quad (40)$$

After time integration, we obtain, for any $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$,

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t dt \langle (\omega J), Y^\varepsilon(t) \rangle \right| = 0.$$

Observe that $R(k, k')/\omega(k) \in C^\infty(\mathbb{T}/\{0\})$ and, using item (i) of lemma 14 in the appendix,

$$\sup_{k \in \mathbb{T}} \frac{R(k, k')}{\omega(k)} < \infty.$$

Then equation (40) holds for any function

$$\omega(k)^{-1} CJ(y, k) = \int_{\mathbb{T}} dq \frac{R(k, q)}{\omega(k)} (J(y, q) - J(y, k))$$

with $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ and consequently

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t dt \langle (CJ), Y^\varepsilon(t) \rangle \right| = 0.$$

5. EXTENSION TO DIMENSIONS $d \geq 2$

We consider a particular generalisation of our model to d dimensions, $d \geq 2$. The perfect lattice is \mathbb{Z}^d . Deviations from the equilibrium position $\mathbf{y} \in \mathbb{Z}^d$ is $\mathbf{q}_{\mathbf{y}} \in \mathbb{R}^d$ and $\mathbf{p}_{\mathbf{y}}$ denotes the corresponding momentum. Thus the phase space is $(\mathbb{R}^d \times \mathbb{R}^d)^{\mathbb{Z}^d}$. The Hamiltonian of the system is given by

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \sum_{\mathbf{y} \in \mathbb{Z}^d} \mathbf{p}_{\mathbf{y}}^2 + \frac{1}{2} \sum_{\mathbf{y}, \mathbf{y}' \in \mathbb{Z}^d} \alpha(\mathbf{y} - \mathbf{y}') \mathbf{q}_{\mathbf{y}} \cdot \mathbf{q}_{\mathbf{y}'}. \quad (41)$$

For simplicity the couplings α are taken to be scalar. In general, α would be a $d \times d$ matrix. We denote

$$\hat{v}(\mathbf{k}) = \sum_{\mathbf{z} \in \mathbb{Z}^d} e^{-2\pi i \mathbf{k} \cdot \mathbf{z}} v(\mathbf{z}), \quad \tilde{f}(\mathbf{z}) = \int_{\mathbb{T}^d} d\mathbf{k} e^{2\pi i \mathbf{k} \cdot \mathbf{z}} f(\mathbf{k}). \quad (42)$$

We assume $\alpha(\cdot)$ to satisfy the following properties:

Assumption 8.

- (a1) $\alpha(\mathbf{y}) \neq 0$ for some $\mathbf{y} \neq 0$.
- (a2) $\alpha(\mathbf{y}) = \alpha(-\mathbf{y})$ for all $\mathbf{y} \in \mathbb{Z}^d$.
- (a3) There are constants $C_1, C_2 > 0$ such that for all \mathbf{y}

$$|\alpha(\mathbf{y})| \leq C_1 e^{-C_2 |\mathbf{y}|}.$$

- (a4) We require $\hat{\alpha} > 0$ on \mathbb{T}^d (pinned case) or $\hat{\alpha}(\mathbf{k}) > 0, \forall \mathbf{k} \neq 0, \hat{\alpha}(0) = 0$ (unpinned case). Moreover, in the unpinned case we require that the Hessian of $\hat{\alpha}$ at $\mathbf{k} = 0$ is invertible.

The dynamics is determined by the generator $L = A + \varepsilon \gamma S$ with

$$A = \sum_{\mathbf{y} \in \mathbb{Z}^d} \mathbf{p}_{\mathbf{y}} \cdot \partial_{\mathbf{q}_{\mathbf{y}}} - \sum_{\mathbf{y}, \mathbf{y}' \in \mathbb{Z}^d} \alpha(\mathbf{y} - \mathbf{y}') \mathbf{q}_{\mathbf{y}'} \cdot \partial_{\mathbf{p}_{\mathbf{y}}}. \quad (43)$$

S is defined through the vector fields

$$X_{\mathbf{x}, \mathbf{z}}^{i,j} = (p_{\mathbf{z}}^j - p_{\mathbf{y}}^j)(\partial_{p_{\mathbf{z}}^i} - \partial_{p_{\mathbf{y}}^i}) - (p_{\mathbf{z}}^i - p_{\mathbf{y}}^i)(\partial_{p_{\mathbf{z}}^j} - \partial_{p_{\mathbf{y}}^j}),$$

according to

$$S = \frac{1}{2(d-1)} \sum_{\mathbf{y} \in \mathbb{Z}^d} \sum_{i,j,k=1}^d (X_{\mathbf{y}, \mathbf{y} + \mathbf{e}_k}^{i,j})^2 = \frac{1}{4(d-1)} \sum_{\substack{\mathbf{y}, \mathbf{z} \in \mathbb{Z}^d \\ \|\mathbf{y} - \mathbf{z}\| = 1}} \sum_{i,j=1}^d (X_{\mathbf{x}, \mathbf{z}}^{i,j})^2, \quad (44)$$

where $\mathbf{e}_1, \dots, \mathbf{e}_d$ is the canonical basis of \mathbb{Z}^d . As in the one-dimensional case

$$S \sum_{\mathbf{y} \in \mathbb{Z}^d} \mathbf{p}_{\mathbf{y}} = 0, \quad SH = 0.$$

Note that now it suffices to couple nearest neighbors.

The evolution of $\{\mathbf{p}(t), \mathbf{q}(t)\}$ is given by the following stochastic differential equations

$$\begin{aligned} d\mathbf{q}_{\mathbf{y}} &= \mathbf{p}_{\mathbf{y}} dt, \\ d\mathbf{p}_{\mathbf{y}} &= -(\alpha * \mathbf{q})_{\mathbf{y}} dt + 2\varepsilon\gamma\Delta\mathbf{p}_{\mathbf{y}} dt \\ &\quad + \frac{\sqrt{\varepsilon\gamma}}{2\sqrt{d-1}} \sum_{\substack{\mathbf{z} \in \mathbb{Z}^d, \\ \|\mathbf{z}-\mathbf{y}\|=1}} \sum_{i,j=1}^d (X_{\mathbf{y},\mathbf{z}}^{i,j} \mathbf{p}_{\mathbf{y}}) dw_{\mathbf{y},\mathbf{z}}^{i,j}(t) \end{aligned} \quad (45)$$

for all $\mathbf{y} \in \mathbb{Z}^d$. Here $\{w_{\mathbf{z},\mathbf{y}}^{i,j} = w_{\mathbf{y},\mathbf{z}}^{i,j}; \mathbf{z}, \mathbf{y} \in \mathbb{Z}^d; i, j = 1, \dots, d; \|\mathbf{y} - \mathbf{z}\| = 1\}$ are independent standard Wiener processes.

Define the complex valued vector field $\psi : \mathbb{Z}^d \rightarrow \mathbb{C}^d$ as

$$\psi(\mathbf{y}, t) = \frac{1}{\sqrt{2}}((\tilde{\omega} * \mathbf{q})_{\mathbf{y}}(t) + i\mathbf{p}_{\mathbf{y}}(t)) \quad (46)$$

with the inverse relation

$$\mathbf{p}_{\mathbf{y}}(t) = \frac{i}{\sqrt{2}}(\psi^* - \psi)(\mathbf{y}, t). \quad (47)$$

Observe that $|\psi(\mathbf{y})|^2 = e_{\mathbf{y}}$, the local energy at \mathbf{y} . For every $t \geq 0$, the evolution of ψ is given by the following SDE,

$$\begin{aligned} d\psi(\mathbf{y}, t) &= -i(\tilde{\omega} * \psi)(\mathbf{y}, t)dt + \frac{1}{2}\varepsilon\gamma\beta * (\psi - \psi^*)(\mathbf{y}, t)dt \\ &\quad + \frac{\sqrt{\varepsilon\gamma}}{4\sqrt{d-1}} \sum_{\substack{\mathbf{y}' \in \mathbb{Z}^d, \\ \|\mathbf{y}'-\mathbf{y}\|=1}} \sum_{i,j=1}^d (X_{\mathbf{y},\mathbf{y}'}^{i,j} (\psi - \psi^*)(\mathbf{y}, t)) dw_{\mathbf{y},\mathbf{y}'}^{i,j}(t), \end{aligned} \quad (48)$$

where β is determined through $(\beta * f)(\mathbf{z}) = \Delta f(\mathbf{z})$.

Given a function J on $\mathbb{R}^d \times \mathbb{T}^d$, we define

$$\tilde{J}(\mathbf{x}, \mathbf{z}) = \int_{\mathbb{T}^d} d\mathbf{k} e^{i2\pi\mathbf{k}\cdot\mathbf{z}} J(\mathbf{x}, \mathbf{k}) \quad (49)$$

on $\mathbb{R}^d \times \mathbb{Z}^d$. We also define

$$\hat{J}(\mathbf{p}, \mathbf{k}) = \int_{\mathbb{R}^d} d\mathbf{x} e^{-i2\pi\mathbf{p}\cdot\mathbf{x}} J(\mathbf{x}, \mathbf{k}). \quad (50)$$

We choose a class of test-functions J on $\mathbb{R}^d \times \mathbb{T}^d$ such that $J \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d, \mathbb{M}_d)$, where \mathbb{M}_d is the space of complex $d \times d$ matrices.

Fix $\varepsilon > 0$. We introduce the complex valued correlation matrices

$$\langle \psi(\mathbf{y}')^* \otimes \psi(\mathbf{y}) \rangle_{\varepsilon}, \quad \langle \psi(\mathbf{y}') \otimes \psi(\mathbf{y}) \rangle_{\varepsilon}, \quad (51)$$

where $\langle \cdot \rangle_{\varepsilon}$ denotes the expectation value with respect to a probability measure on phase space which satisfies the following properties:

- (c1) $\langle \psi(\mathbf{y}) \rangle_{\varepsilon} = 0, \quad \forall \mathbf{y} \in \mathbb{Z}^d,$
- (c2) $\langle \psi(\mathbf{y}') \otimes \psi(\mathbf{y}) \rangle_{\varepsilon} = 0, \quad \forall \mathbf{y}, \mathbf{y}' \in \mathbb{Z}^d,$
- (c3) $\langle \|\psi\|^2 \rangle_{\varepsilon} = \langle \sum_{\mathbf{z} \in \mathbb{Z}^d} |\psi(\mathbf{z})|^2 \rangle_{\varepsilon} \leq K\varepsilon^{-d}.$

Observe that, since $\langle \|\boldsymbol{\psi}\|^2 \rangle_\varepsilon = \langle H \rangle_\varepsilon$, we are considering states with an energy of order ε^{-d} . We define the matrix-valued Wigner distribution W^ε as

$$\begin{aligned} & \langle J, W^\varepsilon \rangle \\ &= (\varepsilon/2)^d \sum_{\mathbf{y}, \mathbf{y}' \in \mathbb{Z}^d} \sum_{i, j=1}^d \langle \psi_j(\mathbf{y}')^* \psi_i(\mathbf{y}) \rangle_\varepsilon \int_{\mathbb{T}^d} d\mathbf{k} e^{i2\pi\mathbf{k} \cdot (\mathbf{y}' - \mathbf{y})} J_{j,i}(\varepsilon(\mathbf{y}' + \mathbf{y})/2, k)^* \\ &= (\varepsilon/2)^d \sum_{\mathbf{y}, \mathbf{y}' \in \mathbb{Z}^d} \sum_{i, j=1}^d \langle \psi_j(\mathbf{y}')^* \psi_i(\mathbf{y}) \rangle_\varepsilon \tilde{J}_{j,i}(\varepsilon(\mathbf{y}' + \mathbf{y})/2, \mathbf{y} - \mathbf{y}')^* \end{aligned} \quad (52)$$

with $J \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d, \mathbb{M}_d)$. The evolution of the diagonal terms of the distribution W^ε on time scale $\varepsilon^{-1}t$ is determined through

$$\begin{aligned} \langle J, W^\varepsilon(t) \rangle &= (\varepsilon/2)^d \sum_{i=1}^d \sum_{\mathbf{y}, \mathbf{y}' \in \mathbb{Z}^d} \langle \psi_i(\mathbf{y}', t/\varepsilon)^* \psi_i(\mathbf{y}, t/\varepsilon) \rangle_\varepsilon \\ & \int_{\mathbb{T}^d} d\mathbf{k} e^{i2\pi\mathbf{k} \cdot (\mathbf{y}' - \mathbf{y})} J_i(\varepsilon(\mathbf{y}' + \mathbf{y})/2, k)^* \end{aligned} \quad (53)$$

for $J_i \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d)$, $i = 1 \dots, d$.

Observe that since the dynamics preserves the total energy, the condition $\varepsilon^d \langle \|\boldsymbol{\psi}\| \rangle \leq K$ holds at any time and, by proposition 13, the Wigner distribution is well defined at any time. On this time scale the diagonal terms of the distribution $W^\varepsilon(t)$ converge in a weak sense to a (vector valued) measure $\mu = \{\mu_i(t), i = 1, \dots, d\}$ on $\mathbb{R}^d \times \mathbb{T}^d$ which satisfies the following Boltzmann equation. For any vector valued function $J \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d, \mathbb{C}^d)$,

$$\langle J, \mu(t) \rangle - \langle J, \mu(0) \rangle = \frac{1}{2\pi} \int_0^t ds \left(\langle \nabla \omega \cdot \nabla_{\mathbf{x}} J, \mu(s) \rangle + \gamma \langle C J, \mu(t) \rangle \right), \quad (54)$$

where $\langle J, \mu(t) \rangle$ denotes the scalar product $\sum_{i=1}^d \int_{\mathbb{R}^d \times \mathbb{T}^d} J_i(\mathbf{x}, \mathbf{k})^* \mu_i(d\mathbf{x}, d\mathbf{k})$. The collision operator is given by

$$(CJ)_i(\mathbf{x}, \mathbf{k}) = \frac{1}{d-1} \sum_{\substack{1 \leq j \leq d, \\ j \neq i}} \int_{\mathbb{T}^d} d\mathbf{k}' R(\mathbf{k}, \mathbf{k}') (J_j(\mathbf{x}, \mathbf{k}') - J_i(\mathbf{x}, \mathbf{k})), \quad (55)$$

where the kernel $R : \mathbb{T}^d \times \mathbb{T}^d \rightarrow \mathbb{R}$ has the following expression,

$$R(\mathbf{k}, \mathbf{k}') = 16 \sum_{\ell=1}^d \sin^2(\pi k_\ell) \sin^2(\pi k'_\ell). \quad (56)$$

As in the one-dimensional case, in order to prove the inhomogeneous Boltzmann equation (54), we need an additional condition on the initial distribution in the unpinned case ($\hat{\alpha}(0) = 0$), which ensures that there is no initial concentration of energy at $\mathbf{k} = 0$:

(c4) In the unpinned case we require

$$\lim_{R \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} (\varepsilon/2)^d \int_{|\mathbf{k}| < R} d\mathbf{k} \langle |\hat{\psi}(\mathbf{k})|^2 \rangle_\varepsilon = 0.$$

Now we state the precise theorem. The proof is analogous to the one-dimensional case.

Theorem 9. *Let Assumptions (c1-c4) hold and assume that $W^\varepsilon(0)$ converges to a positive vector valued measure μ_0 . Then, for all $t \in [0, T]$, $W^\varepsilon(t)$ converges to a positive (vector-valued) measure $\mu_0(t)$ which is the unique solution of the Boltzmann equation*

$$\partial_t \langle J, \mu(t) \rangle = \frac{1}{2\pi} \langle \nabla \omega \cdot \nabla_{\mathbf{x}} J, \mu(t) \rangle + \gamma \langle C J, \mu(t) \rangle \quad (57)$$

with initial condition $\mu_0(t)$.

As in the one-dimensional case, the Boltzmann equation has a probabilistic interpretation as the forward equation of a Markov process. We consider the Markov process

$$(\mathbf{X}(t), \mathbf{K}(t), i(t)).$$

By (55), the jump rate from (i, \mathbf{k}) to $(j, d\mathbf{k}')$ is given by

$$\nu_{\mathbf{k}, i}(j, d\mathbf{k}') = \frac{1}{d-1} (1 - \delta_{i,j}) R(\mathbf{k}, \mathbf{k}') d\mathbf{k}', \quad \forall i, j = 1, \dots, d.$$

Transitions between states with the same index i are forbidden. The total collision rate is

$$\phi_i(\mathbf{k}) = \sum_{j=1}^d \int_{\mathbb{T}^d} \nu_{\mathbf{k}, i}(j, d\mathbf{k}') = \int_{\mathbb{T}^d} d\mathbf{k}' R(\mathbf{k}, \mathbf{k}'),$$

$i = 1, \dots, d$, which does not depend on i . Explicitly

$$\phi_i(\mathbf{k}) = \phi(\mathbf{k}) = 8 \sum_{\ell=1}^d \sin^2(\pi k_\ell). \quad (58)$$

Given a state (\mathbf{k}, i) at $t = 0$, it jumps at time τ to the state $(d\mathbf{k}', j)$ with a probability $\nu_{\mathbf{k}, i}(d\mathbf{k}', j)/\phi(\mathbf{k})$, where τ is an exponentially distributed random variable of mean $\phi(\mathbf{k})^{-1}$. As before the position process $\mathbf{X}(t)$ is defined through

$$\mathbf{X}(t) = \mathbf{X}(0) + \frac{1}{2\pi} \int_0^t ds \nabla \omega(\mathbf{K}(s)). \quad (59)$$

6. HOMOGENEOUS CASE: CORRELATIONS, ENERGY CURRENT AND CONDUCTIVITY

6.1. Translation invariant measures. We consider a situation where the initial measure on phase space is invariant under space translations. For simplicity we work in the one-dimensional setting. Using the methods from section 5, the generalization to $d \geq 2$ is straightforward.

Since energy will be now a.s. infinite, the results of section 3 do not apply. Let us denote with $\langle \cdot \rangle$ the expectation value with respect to this initial translation invariant measure, and assume that it has the following properties:

- (d1) $\langle \psi(y) \rangle = 0, \quad \forall y \in \mathbb{Z},$
- (d2) $\langle \psi(y)\psi(0) \rangle = 0, \quad \forall y \in \mathbb{Z},$
- (d3) $\sum_{z \in \mathbb{Z}} |\langle \psi(0)^* \psi(z) \rangle| < \infty.$

The Wigner distribution is still well defined for every function $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$. Using translation invariance

$$\begin{aligned} \langle J, W^\varepsilon \rangle &= \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle \psi(y')^* \psi(y) \rangle \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} J(\varepsilon(y' + y)/2, k)^* \\ &= \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \langle \psi(0)^* \psi(y - y') \rangle \int_{\mathbb{T}} dk e^{2\pi i k(y' - y)} J(\varepsilon(y' + y)/2, k)^* \\ &= \sum_{z \in \mathbb{Z}} \langle \psi(0)^* \psi(z) \rangle \int_{\mathbb{T}} dk e^{-2\pi i k z} \left[\frac{\varepsilon}{2} \sum_{y \in \mathbb{Z}} J(\varepsilon(2y + z)/2, k)^* \right], \end{aligned}$$

which is finite by condition (d3) and by the fast decay of J . In Fourier space the previous expression becomes

$$\begin{aligned} \langle J, W^\varepsilon \rangle &= \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk e^{2\pi i(k - \varepsilon p/2)y'} \langle \psi(y')^* \psi(y) \rangle e^{-2\pi i(k + \varepsilon p/2)y} \widehat{J}(p, k)^* \end{aligned}$$

and using translation invariance

$$\langle J, W^\varepsilon \rangle = \frac{1}{2} \int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \delta(p) \mathcal{W}(k) \widehat{J}(p, k)^* = \frac{1}{2} \int_{\mathbb{T}} dk \mathcal{W}(k) \widehat{J}(0, k)^*,$$

where $\mathcal{W}(k)$ is the Fourier transform of the correlation function $\langle \psi(0)^* \psi(z) \rangle$,

$$\mathcal{W}(k) = \sum_{z \in \mathbb{Z}} e^{-2\pi i k z} \langle \psi(0)^* \psi(z) \rangle. \quad (60)$$

By condition (d3), $\mathcal{W}(k)$ is well defined and in $L_1(\mathbb{T})$. Moreover, by translation invariance, $\mathcal{W}(k)$ is a real positive function.

If we consider the deterministic dynamics only, then \mathcal{W} is preserved by the dynamics, i.e. $\partial_t \mathcal{W}(t) = 0$. This follows from eq. (9) for $\gamma = 0$. Such property is no longer true if the system evolves according to the full dynamics, defined through the generator $L = A + \varepsilon \gamma S$. In order to observe an effective change of the covariance, hence of the function \mathcal{W} , we have to consider the time scale of order ε^{-1} . Denoting by $\mathcal{W}^\varepsilon(t) = \mathcal{W}(t/\varepsilon)$, we obtain the following evolution equation,

$$\partial_t \mathcal{W}^\varepsilon(k, t) = \gamma (C \mathcal{W}^\varepsilon)(k, t) - \frac{\gamma}{2} [C(\mathcal{Y}^\varepsilon + \mathcal{Y}^{\varepsilon*})](k, t), \quad (61)$$

where C is the collision operator defined in (19), while $\mathcal{Y}^\varepsilon(k, t)$ is the Fourier transform of the correlation function $\langle \psi(0)\psi(z) \rangle$ at the rescaled time t/ε ,

$$\mathcal{Y}^\varepsilon(k, t) = \sum_{z \in \mathbb{Z}} e^{-2\pi i k z} \langle \psi(t/\varepsilon, 0) \psi(t/\varepsilon, z) \rangle.$$

As before, we prove that in the limit $\varepsilon \rightarrow 0$ one obtains a closed equation for $\mathcal{W}^\varepsilon(t)$, as stated in the next theorem.

Theorem 10. *Assume that the initial state satisfies the above conditions and that $\mathcal{W}^\varepsilon(k, 0) = \mathcal{W}_0(k)$ is continuous on \mathbb{T} . Then, $\forall k \in \mathbb{T}, t \in [0, T]$,*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{W}^\varepsilon(k, t) = \mathcal{W}(k, t),$$

where $\mathcal{W}(k, t)$ satisfies the homogeneous Boltzmann equation

$$\begin{aligned} \partial_t \mathcal{W}(k, t) &= \gamma(C\mathcal{W})(k, t), \\ \mathcal{W}(k, 0) &= \mathcal{W}_0(k) \end{aligned} \tag{62}$$

with C defined in (19).

The proof of this theorem will be given in section 6.5.

6.2. Equilibrium time correlations. We consider the system in equilibrium and we denote by $\langle \cdot \rangle_T$ the average at respect to the equilibrium measure with temperature T . This is a translation invariant Gaussian centered measure with zero mean, uniquely characterised through its covariance

$$\mathcal{W}(k) = T, \quad \langle \psi(y)\psi(0) \rangle_T = 0, \quad \forall y \in \mathbb{Z}. \tag{63}$$

$\langle \cdot \rangle_T$ is a stationary measure for the SDE (9).

Consider a function $g \in \ell_1(\mathbb{Z})$ antisymmetric, $g(z) = -g(-z)$, and such that $\|\hat{g}/\omega\|_\infty < \infty$ and define the function

$$\Phi = \sum_{x \in \mathbb{Z}} g(x) p_x q_0.$$

The total time covariance is defined as

$$\mathcal{F}^\varepsilon(t) = \sum_{z \in \mathbb{Z}} \langle \Phi(t/\varepsilon) \tau_z \Phi(0) \rangle_T. \tag{64}$$

We want to compute $\mathcal{F}^\varepsilon(t)$ in the kinetic limit $\varepsilon \rightarrow 0$.

Consider the centered translation invariant Gaussian measure defined by the following covariance,

$$\mathcal{W}^{(T, \tau)}(k) = \frac{\omega(k)}{T^{-1}\omega(k) + i\tau\hat{g}(k)}, \quad \mathcal{Y}^{(T, \tau)}(k) = 0 \tag{65}$$

with $\tau > 0$. Observe that $\mathcal{W}^{(\beta, \tau)}$ is a real, continuous function which is positive for τ small enough. Formally (65) corresponds to the perturbed measure

$$Z^{-1} \exp \left[-T^{-1}H + \tau \sum_{z \in \mathbb{Z}} \tau_z \Phi \right].$$

We denote by $\langle \cdot \rangle_{(T,\tau)}$ its expectation.

Lemma 11. *For every $\varepsilon > 0$,*

$$\sum_{z \in \mathbb{Z}} \langle \Phi(t/\varepsilon) \tau_z \Phi(0) \rangle_T = \lim_{\tau \rightarrow 0} \langle \Phi(t/\varepsilon) \rangle_{(T,\tau)}. \quad (66)$$

The proof of this lemma is given in section 6.6 below.

By direct computation, at the rescaled time t/ε ,

$$\begin{aligned} \langle \Phi(t/\varepsilon) \rangle_{(T,\tau)} &= -\frac{1}{2} \sum_{z \in \mathbb{Z}} g(z) \langle q_z p_0 - q_0 p_z \rangle_{(T,\tau)} \\ &= -i \int_{\mathbb{T}} dk \frac{\hat{g}(k)}{\omega(k)} \mathcal{W}^{(T,\tau)}(k, t/\varepsilon). \end{aligned}$$

By Theorem 10, $\mathcal{W}^{(T,\tau)}(k, t/\varepsilon) \rightarrow \mathcal{W}(k, t)$ for $\varepsilon \rightarrow 0$, where $\mathcal{W}(k, t)$ satisfies the homogeneous Boltzmann equation (62) with initial condition $\mathcal{W}(k, 0) = \omega(k)(T^{-1}\omega(k) + i\tau\hat{g}(k))^{-1}$. It is easy to verify that for any bounded antisymmetric function f on \mathbb{T}

$$\int_{\mathbb{T}} dk (Cf)(k) \mathcal{W}(k, t) = - \int_{\mathbb{T}} dk \phi(k) \mathcal{W}(k, t),$$

with $\phi = -\hat{\beta}(k) = \frac{4}{3} \sin^2(\pi k)[1 + 2 \cos^2(\pi k)]$, see (27). Then

$$-i \int_{\mathbb{T}} dk \frac{\hat{g}(k)}{\omega(k)} \mathcal{W}^{(T,\tau)}(k, t) = -i \int_{\mathbb{T}} dk \frac{\hat{g}(k)}{\omega(k)} \mathcal{W}^{(T,\tau)}(k, 0) e^{-\gamma\phi(k)t}$$

and finally, for $t \geq 0$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathcal{F}^\varepsilon(t) &= \lim_{\tau \rightarrow 0} -\frac{i}{\tau} \int_{\mathbb{T}} dk \frac{\hat{g}(k)}{\omega(k)} \mathcal{W}^{(T,\tau)}(k, 0) e^{-\gamma\phi(k)t} \\ &= T^2 \int_{\mathbb{T}} dk \frac{|\hat{g}(k)|^2}{\omega(k)^2} e^{-\gamma\phi(k)t}. \end{aligned} \quad (67)$$

6.3. Energy current time correlation. The Hamiltonian energy current J is implicitly defined through the conservation law

$$Ae_x = \tau_{x-1}J - \tau_x J.$$

By direct computation $J = \sum_{z>0} j_{0,z}$ with

$$j_{0,z} = -\frac{1}{2} \alpha(z) \sum_{y=0}^{z-1} (q_{z-y} p_{-y} - q_{-y} p_{z-y}).$$

Denoting by $\langle \cdot \rangle$ the expectation value with respect to a translation invariant centered Gaussian measure, it is easy to see that

$$\begin{aligned} \langle J \rangle &= -\frac{1}{2} \sum_{z>0} z \alpha(z) \langle q_z p_0 - q_0 p_z \rangle \\ &= \frac{1}{4\pi} \int_{\mathbb{T}} dk \frac{\hat{\alpha}'(k)}{\omega(k)} \mathcal{W}(k) \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} dk \omega'(k) \mathcal{W}(k). \end{aligned}$$

Let us denote by $\mathcal{C}^\varepsilon(t)$ the energy time correlation function on the kinetic time scale t/ε at temperature T ,

$$\mathcal{C}^\varepsilon(t) = \sum_{x \in \mathbb{Z}} \langle J(t/\varepsilon) \tau_x J(0) \rangle_T.$$

Using the translation invariance of the Gaussian measure $\langle \cdot \rangle_T$ we have

$$\begin{aligned} \mathcal{C}^\varepsilon(t) &= \sum_{x \in \mathbb{Z}} \sum_{z \in \mathbb{Z}} \frac{z}{4} \alpha(z) \sum_{z' \in \mathbb{Z}} \frac{z'}{4} \alpha(z') \langle (q_z p_0 - q_0 p_z)(t/\varepsilon) \\ &\quad (q_{x+z'} p_x - q_x p_{x+z'}) (0) \rangle_T = \sum_{x \in \mathbb{Z}} \langle \tilde{J}(t/\varepsilon) \tau_x \tilde{J}(0) \rangle_T, \end{aligned}$$

where

$$\tilde{J} = -\frac{1}{4} \sum_{z \in \mathbb{Z}} z \alpha(z) (q_z p_0 - q_0 p_z).$$

Using the results of the previous subsection we arrive at

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathcal{C}^\varepsilon(t) &= \frac{T^2}{(4\pi)^2} \int_{\mathbb{T}} dk \frac{|\hat{\alpha}'(k)|^2}{\omega(k)^2} e^{-\gamma\phi(k)|t|} \\ &= \frac{T^2}{4\pi^2} \int_{\mathbb{T}} dk |\omega'(k)|^2 e^{-\gamma\phi(k)|t|} \end{aligned} \tag{68}$$

for all $t \in \mathbb{R}$.

6.4. Thermal Conductivity. A standard definition of the thermal conductivity, $\kappa^{(\varepsilon)}$, is by means of the Green-Kubo formula as

$$\kappa^{(\varepsilon)} = \frac{1}{T^2} \int_0^\infty dt \mathcal{C}^\varepsilon(\varepsilon t). \tag{69}$$

We refer to [2, 4] for details. In general, the kinetic limit provides the lowest order approximation in ε as

$$\kappa^{(\varepsilon)} = \varepsilon^{-1} \kappa^{(0)} + \mathcal{O}(1).$$

Inserting in (69) the limit (68) thus yields

$$\kappa^{(0)} = \frac{1}{4\pi^2} \int_{\pi} dk \frac{|\omega'(k)|^2}{\gamma\phi(k)}. \tag{70}$$

For the pinned case $\kappa^{(0)} < \infty$. $\kappa^{(\varepsilon)}$ has been computed in [2] with the result

$$\kappa^{(\varepsilon)} = \varepsilon^{-1}\kappa^{(0)} + \varepsilon\gamma.$$

Thus, somewhat unexpectedly, the kinetic theory captures already the main details of the conductivity.

For the unpinned case $\kappa^{(0)} = \infty$, hence $\kappa^{(\varepsilon)} = \infty$, for $d = 1, 2$. The Boltzmann equation (21) provides a simple explanation. For small k , $\omega(k) = 2\pi c|k|$. Thus small k phonons travel with speed c . On the other hand, the collision rate vanishes as k^2 for small k , see (27). Thus at small k there are only very few collisions which, together with $c > 0$, is responsible for the divergent conductivity. The positional part $X(t)$ of the process consists mostly of very long stretches of uniform motion. In fact on a large scale $X(t)$ is governed by a symmetric Levy process of index $\alpha = 3/2$, see [6] for details.

6.5. Proof of theorem 10. The proof of the theorem 10 is analogous to the proof above. We only have to control that $\mathcal{W}^\varepsilon(t)$ and $\mathcal{Y}^\varepsilon(t)$ are well defined for every $t \in [0, T]$. This is stated in the next lemma.

Lemma 12. *Let the conditions (d1-d3) hold. Then $\mathcal{W}^\varepsilon(t), \mathcal{Y}^\varepsilon(t) \in L_1(\mathbb{T})$ for every $t \in [0, T]$.*

Proof. By similar computations as above we find that $\mathcal{W}^\varepsilon(t), \mathcal{Y}^\varepsilon(t)$ satisfy the following evolution equations,

$$\begin{aligned} \partial_t \mathcal{W}^\varepsilon(k, t) &= \gamma(C\mathcal{W}^\varepsilon)(k, t) - \frac{\gamma}{2}(C(\mathcal{Y}^\varepsilon + \mathcal{Y}^{\varepsilon*}))(k, t), \\ \partial_t \mathcal{Y}^\varepsilon(k, t) &= -\frac{2i\omega(k)}{\varepsilon}\mathcal{Y}^\varepsilon(k, t) + \frac{\gamma}{2}\hat{\beta}(k)(\mathcal{Y}^\varepsilon + \mathcal{Y}^{\varepsilon*})(k, t) \\ &\quad + \frac{\gamma}{2}(C(\mathcal{Y}^\varepsilon + \mathcal{Y}^{\varepsilon*}))(k, t) - \frac{\gamma}{2}C(\mathcal{W}^\varepsilon(k, t) + \mathcal{W}^\varepsilon(-k, t)). \end{aligned}$$

In particular by Duhamel's formula we can rewrite the second equation as

$$\begin{aligned} \mathcal{Y}^\varepsilon(k, t) &= \gamma \int_0^t ds e^{-2i\omega(k)(t-s)/\varepsilon} \left(\frac{1}{2}\hat{\beta}(k)(\mathcal{Y}^\varepsilon + \mathcal{Y}^{\varepsilon*})(k, s) \right. \\ &\quad \left. + \frac{1}{2}(C(\mathcal{Y}^\varepsilon + \mathcal{Y}^{\varepsilon*}))(k, s) - \frac{1}{2}C(\mathcal{W}^\varepsilon(k, s) + \mathcal{W}^\varepsilon(-k, s)) \right). \end{aligned}$$

Then we get the following bounds

$$\begin{aligned} |\mathcal{W}^\varepsilon(k, t)| &\leq \mathcal{W}(k) + \gamma c_1 \int_0^t ds [|\mathcal{W}^\varepsilon(k, s)| + |\mathcal{Y}^\varepsilon(k, s)|] \\ &\quad + \gamma c_2 \int_0^t ds \int_{\mathbb{T}} dk (|\mathcal{W}^\varepsilon(k, s)| + |\mathcal{Y}^\varepsilon(k, s)|) \\ |\mathcal{Y}^\varepsilon(k, t)| &\leq +\gamma c_3 \int_0^t ds [|\mathcal{W}^\varepsilon(k, s)| + |\mathcal{W}^\varepsilon(-k, s)| + |\mathcal{Y}^\varepsilon(k, s)|] \\ &\quad + \gamma c_4 \int_0^t ds \int_{\mathbb{T}} dk (|\mathcal{W}^\varepsilon(k, s)| + |\mathcal{Y}^\varepsilon(k, s)|) \end{aligned}$$

and finally

$$\begin{aligned} \int_{\mathbb{T}} dk [|\mathcal{W}^\varepsilon(k, t)| + |\mathcal{Y}^\varepsilon(k, t)|] &\leq \int_{\mathbb{T}} dk \mathcal{W}(k) + \\ &\quad \gamma c_5 \int_0^t ds \int_{\mathbb{T}} dk (|\mathcal{W}^\varepsilon(k, s)| + |\mathcal{Y}^\varepsilon(k, s)|). \end{aligned}$$

By Gronwall's lemma

$$\int_{\mathbb{T}} dk [|\mathcal{W}^\varepsilon(k, t)| + |\mathcal{Y}^\varepsilon(k, t)|] \leq e^{\gamma c_5 t} \int_{\mathbb{T}} dk \mathcal{W}(k).$$

□

6.6. Proof of (66). We define the generator of the speeded up process as

$$L_\varepsilon = \varepsilon^{-1}L = \varepsilon^{-1}A + S.$$

Let us denote by $\langle \cdot \rangle_T$ the average with respect to the Gaussian measure with zero mean and covariance (63), and by $\langle \cdot \rangle_{(\beta, \tau)}$ the Gaussian measure with zero mean and covariance (65). We consider the Laplace transforms of $\langle \Phi(t/\varepsilon) \rangle_{(T, \tau)}$ and $\sum_{x \in \mathbb{Z}} \langle \Phi(t/\varepsilon) \tau_x \Phi(0) \rangle_T$,

$$\begin{aligned} \int_0^\infty dt e^{-\lambda t} \langle \Phi(t/\varepsilon) \rangle_{(T, \tau)} &= \langle (\lambda - L_\varepsilon)^{-1} \Phi \rangle_{(T, \tau)} = \langle u_\lambda \rangle_{(T, \tau)}, \\ \int_0^\infty dt e^{-\lambda t} \sum_x \langle \Phi(t/\varepsilon) \tau_x \Phi(0) \rangle_T &= \sum_{x \in \mathbb{Z}} \langle [(\lambda - L_\varepsilon)^{-1} \Phi] \tau_x \Phi \rangle_T \\ &= \sum_{x \in \mathbb{Z}} \langle u_\lambda \tau_x \Phi \rangle_T, \end{aligned} \quad (71)$$

where $u_\lambda = \sum_{z \in \mathbb{Z}} f_\lambda(z) q_0 p_z$ with f_λ the solution of the equation

$$\lambda f_\lambda(z) - \frac{\gamma}{6} \Delta(4f_\lambda(z) + f_\lambda(z+1) + f_\lambda(z-1)) = g(z).$$

Observe that u_λ does not depend on ε , because $L_\varepsilon u_\lambda = \varepsilon^{-1} \nabla F + \gamma S u_\lambda$ with F some non-local function. Since f_λ is antisymmetric, and by translation invariance of the measure, the gradient term does not contribute. We have

$$\begin{aligned} \langle u_\lambda \rangle_{T, \tau} &= \sum_{z \in \mathbb{Z}} f_\lambda(z) \langle q_0 p_z \rangle_{T, \tau} = -\frac{1}{2} \sum_{z \in \mathbb{Z}} f_\lambda(z) \langle q_z p_0 - q_0 p_z \rangle_{T, \tau} \\ &= -i \int_{\mathbb{T}} dk \hat{f}_\lambda(k) \frac{1}{\omega(k)} \mathcal{W}^{(T, \tau)}(k) \\ &= -i \int_{\mathbb{T}} dk \frac{\hat{g}(k)}{\lambda + \gamma \phi(k)} \frac{1}{T^{-1} \omega(k) + i \tau \hat{g}(k)} \end{aligned}$$

with $\phi(k) = -\hat{\beta}(k)$. For every positive λ , the right hand side is finite for every τ . In particular

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} \langle u_\lambda \rangle_{T,\tau} = T^2 \int_{\mathbb{T}} dk \frac{|\hat{g}(k)|^2}{\omega(k)^2(\lambda + \gamma\phi(k))}.$$

In the same way

$$\begin{aligned} \sum_{x \in \mathbb{Z}} \langle u_\lambda \tau_x \Phi \rangle_T &= T \sum_{y \in \mathbb{Z}} \sum_{z \in \mathbb{Z}} f_\lambda(y) g(z) \langle q_0 q_{y-z} \rangle_T \\ &= T \int_{\mathbb{T}} dk \hat{f}_\lambda(k)^* \frac{\hat{g}(k)}{\omega(k)^2} \mathcal{W}^{(\beta)}(k) = T^2 \int_{\mathbb{T}} dk \frac{|\hat{g}(k)|^2}{\omega(k)^2(\lambda + \phi(k))}, \end{aligned}$$

where Parseval's identity is used. \square

7. APPENDIX

Proposition 13. *Under the assumption (b3),*

$$|\langle J, W^\varepsilon \rangle| \leq C_J \quad (72)$$

for every test function $J \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d)$ and for every $\varepsilon > 0$.

Proof. For any test function $J \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d)$ we denote by $\|J\|_{n,\infty}$ the following norm,

$$\|J\|_{n,\infty} = \sup_{i,j \in \{1,\dots,d\}} \sup_{\mathbf{r} \in \mathbb{R}^d, \mathbf{k} \in \mathbb{T}^d} \left| \sum_{i_1=1}^d \dots \sum_{i_n=1}^d \partial_{k_{i_1}} \dots \partial_{k_{i_n}} J_{i,j}(\mathbf{r}, \mathbf{k}) \right|.$$

Let us define

$$\mathcal{W}^\varepsilon[J](\mathbf{y}, \mathbf{y}', i, j) = \int_{\mathbb{T}^d} d\mathbf{k} e^{2\pi i \mathbf{k} \cdot (\mathbf{y}' - \mathbf{y})} J_{j,i\varepsilon}(\mathbf{y}' + \mathbf{y}/2, \mathbf{k})^*,$$

for every $\mathbf{y}, \mathbf{y}' \in \mathbb{Z}^d$, $i, j = 1, \dots, d$. Integrating by part in \mathbf{k} for $(d+1)$ times, we get

$$|\mathcal{W}^\varepsilon[J](\mathbf{y}, \mathbf{y}', i, j)| \leq \frac{1}{(2\pi)^{d+1}} \frac{1}{|(y'_1 - y_1) + \dots + (y'_d - y_d)|^{d+1}} \|J\|_{d+1,\infty},$$

where y_i denotes the i -th component of the vector \mathbf{y} . We denote by

$$\langle \mathbf{y}' - \mathbf{y} \rangle^{d+1} = |(y'_1 - y_1) + \dots + (y'_d - y_d)|^{d+1}.$$

By Schwarz inequality

$$\begin{aligned} &|\langle J, W^\varepsilon \rangle| \\ &\leq (\varepsilon/2)^d \sum_{\mathbf{y}, \mathbf{y}' \in \mathbb{Z}^d} \sum_{i,j=1}^d [\langle |\psi_i(\mathbf{y})|^2 \rangle_\varepsilon]^{1/2} [\langle |\psi_j(\mathbf{y}')|^2 \rangle_\varepsilon]^{1/2} |\mathcal{W}^\varepsilon[J](\mathbf{y}, \mathbf{y}', i, j)| \\ &\leq (\varepsilon/2)^d \langle \|\boldsymbol{\psi}\|^2 \rangle_\varepsilon \sum_{\mathbf{z} \in \mathbb{Z}^d} \frac{1}{\langle \mathbf{z} \rangle^{d+1}} c_0 \|J\|_{d+1,\infty} \leq c \|J\|_{d+1,\infty}, \end{aligned}$$

where in the last inequality we used $\varepsilon^d \langle \|\boldsymbol{\psi}\|^2 \rangle \leq K$ with K positive. \square

Lemma 14. *Let Assumption 1 hold with $\hat{\alpha}(0) = 0$ (unpinned case). The following assertions hold.*

(i) *There are constants C_1, C_2, C_3 such that $\forall \mathbf{k} \in \mathbb{T}^d$*

$$|\nabla \hat{\alpha}(\mathbf{k})| \leq C_1 |\mathbf{k}|, \quad C_2 |\mathbf{k}| \leq \omega(\mathbf{k}) \leq C_3 |\mathbf{k}|. \quad (73)$$

In addition, $\|\nabla \omega\|_\infty < \infty$.

(ii) *For all $\mathbf{k} \in \mathbb{T}^d$ and $\mathbf{p} \in \mathbb{R}^d$, there is a positive constant C such that*

$$\varepsilon^{-1} |\omega(\mathbf{k} + \varepsilon \mathbf{p}/2) - \omega(\mathbf{k} - \varepsilon \mathbf{p}/2)| \leq C |\mathbf{p}| \quad (74)$$

for every $\varepsilon > 0$.

(iii) *For $\mathbf{p} \in \mathbb{R}^d$, there is a positive constant C_4 such that for all $\mathbf{k} \in \mathbb{T}^d$, $|\mathbf{k}| > \varepsilon |\mathbf{p}|$, with $\varepsilon > 0$ one has*

$$|\varepsilon^{-1} [\omega(\mathbf{k} + \varepsilon \mathbf{p}/2) - \omega(\mathbf{k} - \varepsilon \mathbf{p}/2)] - \mathbf{p} \cdot \nabla \omega(\mathbf{k})| \leq \varepsilon C_4 \frac{|\mathbf{p}|^2}{|\mathbf{k}|}. \quad (75)$$

Proof. The first inequality of item (i) follows from a Taylor expansion of $\hat{\alpha}$ around zero, using the fact that $\nabla \hat{\alpha}(0) = 0$ and $\|D^2 \hat{\alpha}\|_\infty < \infty$. Using the same argument we have $\omega(\mathbf{k}) \leq C_3 |\mathbf{k}|$, since $\omega(\mathbf{k}) = \hat{\alpha}(\mathbf{k})^{1/2}$ and $\hat{\alpha}(\mathbf{k}) \leq C |\mathbf{k}|^2$, with $C > 0$. Let us denote with A_0 the Hessian of $\hat{\alpha}$ at $\mathbf{k} = 0$. By assumption (a4), $\mathbf{k} \cdot A_0 \mathbf{k} > c |\mathbf{k}|^2$. Moreover, there is a $\delta > 0$ such that for every $|\mathbf{k}| < \delta$, $(|\hat{\alpha}(\mathbf{k}) - \frac{1}{2} \mathbf{k} \cdot A_0 \mathbf{k}|)^2 < \frac{1}{4} \mathbf{k} \cdot A_0 \mathbf{k}$. Then $\omega(\mathbf{k}) = ((\hat{\alpha}(\mathbf{k}) - \frac{1}{2} \mathbf{k} \cdot A_0 \mathbf{k}) + \frac{1}{2} \mathbf{k} \cdot A_0 \mathbf{k})^{1/2} \geq \frac{c}{2} |\mathbf{k}|^2$ if $|\mathbf{k}| < \delta$. For $|\mathbf{k}| \geq \delta$, ω is strictly positive and there is a constant c' such that $\omega(\mathbf{k}) \geq c' |\mathbf{k}|$ if $|\mathbf{k}| \geq \delta$, which proves the second inequality. To prove the last one it is enough to observe that for all $\mathbf{k} \neq 0$,

$$|\nabla \omega(\mathbf{k})| = \frac{1}{2} \frac{|\nabla \hat{\alpha}(\mathbf{k})|}{\omega(\mathbf{k})} \leq \frac{1}{2} \frac{C_1}{C_2} < \infty.$$

Hence $\|\omega\|_\infty < \infty$.

Let us prove item (iii). We observe that the function on the left hand side of (75) is zero if $|\mathbf{p}| = 0$ and we have to consider only the case $|\mathbf{p}| > 0$. Since we are assuming $|\mathbf{k}| > \varepsilon |\mathbf{p}|$, it follows that we need to discuss the case $|\mathbf{k}| > 0$. Observe that for any $s \in (0, \varepsilon]$, $|\mathbf{k} \pm \frac{1}{2} s \mathbf{p}| \geq \frac{1}{2} |\mathbf{k}| > 0$ if $|\mathbf{k}| > \varepsilon |\mathbf{p}|$ and the function $\omega(\mathbf{k} + \frac{1}{2} \varepsilon \mathbf{p}) - \omega(\mathbf{k} - \frac{1}{2} \varepsilon \mathbf{p})$ is C^∞ in this range. In particular

$$\begin{aligned} & \varepsilon^{-1} \omega(\mathbf{k} + \varepsilon \mathbf{p}/2) - \omega(\mathbf{k} - \varepsilon \mathbf{p}/2) - \mathbf{p} \cdot \nabla \omega(\mathbf{k}) \\ &= \varepsilon \left(\frac{\mathbf{p}}{2} \cdot \nabla \right)^2 \omega(\mathbf{k} + s \mathbf{p}/2) - \varepsilon \left(\frac{\mathbf{p}}{2} \cdot \nabla \right)^2 \omega(\mathbf{k} - \tilde{s} \mathbf{p}/2) \end{aligned}$$

with $s, \tilde{s} \in (0, \varepsilon)$, where, denoting $\mathbf{k}_\pm = \mathbf{k} \pm \frac{1}{2} s \mathbf{p}$,

$$(\mathbf{p} \cdot \nabla)^2 \omega(\mathbf{k}_\pm) = \frac{1}{2} \frac{1}{\omega(\mathbf{k}_\pm)} (\mathbf{p} \cdot \nabla)^2 \hat{\alpha}(\mathbf{k}_\pm) - \frac{1}{4} \frac{1}{\omega(\mathbf{k}_\pm)^3} (\mathbf{p} \cdot \nabla \hat{\alpha}(\mathbf{k}_\pm))^2$$

and, using item (i),

$$|(\mathbf{p} \cdot \nabla)^2 \omega(\mathbf{k}_\pm)| \leq 4C_4 \frac{|\mathbf{p}|^2}{|\mathbf{k}|}.$$

This prove item (iii).

Item (ii) follows from (iii) if $|\mathbf{k}| > \varepsilon|\mathbf{p}|$. If $|\mathbf{k}| \leq \varepsilon|\mathbf{p}|$ we use the bound

$$\varepsilon^{-1}|\omega(\mathbf{k} + \varepsilon\mathbf{p}/2) - \omega(\mathbf{k} - \varepsilon\mathbf{p}/2)| \leq C_3\varepsilon^{-1}(|\mathbf{k} + \varepsilon\mathbf{p}/2| + |\mathbf{k} - \varepsilon\mathbf{p}/2|) \leq C|\mathbf{p}|.$$

7.1. Proof of (34). First of all we observe that

$$\begin{aligned} \sum_{z \in \mathbb{Z}} [Y_z \psi(y')]^* [Y_z \psi(y)] &= [Y_{y+1} \psi(y')]^* [Y_{y+1} \psi(y)] \\ &\quad + [Y_y \psi(y')]^* [Y_y \psi(y)] + [Y_{y-1} \psi(y')]^* [Y_{y-1} \psi(y)], \end{aligned} \tag{76}$$

where

$$\begin{aligned} Y_{y+1} \psi(y) &= \frac{i}{\sqrt{2}}(p_{y+1} - p_{y+2}), & Y_y \psi(y) &= \frac{i}{\sqrt{2}}(p_{y+1} - p_{y-1}), \\ Y_{y-1} \psi(y) &= \frac{i}{\sqrt{2}}(p_{y-2} - p_{y-1}). \end{aligned}$$

Then (76) is equal to

$$\begin{aligned} &(p_{y+1} - p_{y+2}) [(p_{y+1} - p_{y+2}) \delta_{y',y} + (p_{y+2} - p_y) \delta_{y',y+1} \\ &\quad + (p_y - p_{y+1}) \delta_{y',y+2}] + (p_{y+1} - p_{y-1}) [(p_y - p_{y+1}) \delta_{y',y-1} \\ &\quad + (p_{y+1} - p_{y-1}) \delta_{y',y} + (p_{y-1} - p_y) \delta_{y',y+1}] + (p_{y-2} - p_{y-1}) \\ &\quad \times [(p_{y-1} - p_y) \delta_{y',y-2} + (p_y - p_{y-2}) \delta_{y',y-1} + (p_{y-2} - p_{y-1}) \delta_{y',y}] \\ &= (p_{y+1} - p_{y+2})(p_y - p_{y+1}) \delta_{y',y+2} + (p_{y-2} - p_{y-1})(p_{y-1} - p_y) \delta_{y',y-2} \\ &\quad + [(p_{y+1} - p_{y+2})(p_{y+2} - p_y) + (p_{y+1} - p_{y-1})(p_{y-1} - p_y)] \delta_{y',y+1} \\ &\quad + [(p_{y+1} - p_{y-1})(p_y - p_{y+1}) + (p_{y-2} - p_{y-1})(p_y - p_{y-2})] \delta_{y',y-1} \\ &\quad + [(p_{y+1} - p_{y+2})^2 + (p_{y+1} - p_{y-1})^2 + (p_{y-2} - p_{y-1})^2] \delta_{y',y} \\ &= 2 \sum_{r=-2}^2 A_{y,r} \delta_{y',y+r} \end{aligned} \tag{77}$$

and we can write

$$\begin{aligned} &\frac{1}{3}(\varepsilon/2) \sum_{y, y', z \in \mathbb{Z}} \langle [Y_z \psi(y')]^* [Y_z \psi(y)] \rangle_\varepsilon \tilde{J}(\varepsilon(y' + y)/2, y - y')^* \\ &= \frac{1}{3}(\varepsilon/2) \sum_{y \in \mathbb{Z}} \sum_{r=-2}^2 A_{y,r} \tilde{J}(\varepsilon(y + r/2), -r)^*. \end{aligned}$$

Expanding the $A_{y,r}$ and identifying terms that differ by translations, we arrive at

$$\begin{aligned} & \frac{1}{3}(\varepsilon/2) \sum_{y,y',z \in \mathbb{Z}} \langle [Y_z \psi(y')]^* [Y_z \psi(y)] \rangle_\varepsilon \tilde{J}(\varepsilon(y' + y)/2, y - y')^* \\ &= \frac{\varepsilon}{6} \sum_{y \in \mathbb{Z}} \left[\langle 2p_y p_{y+1} - p_y p_{y+2} - p_y^2 \rangle_\varepsilon (\tilde{J}(\varepsilon y, 2)^* + \tilde{J}(\varepsilon y, -2)^*) \right. \\ & \quad + \langle 2p_y p_{y+2} - 2p_y^2 \rangle_\varepsilon (\tilde{J}(\varepsilon y, 1)^* + \tilde{J}(\varepsilon y, -1)^*) \\ & \quad \left. + \langle -4p_y p_{y+1} - 2p_y p_{y+2} + 6p_y^2 \rangle_\varepsilon \tilde{J}(\varepsilon y, 0)^* \right] + \mathcal{O}(\varepsilon), \end{aligned}$$

where we have used the smoothness of \tilde{J} in $x \in \mathbb{R}$. We can rewrite the last expression as

$$(\varepsilon/2) \sum_{y \in \mathbb{Z}} \sum_{z, u = -2}^2 \alpha(z, u) \langle p_y p_{y+z} \rangle_\varepsilon \tilde{J}(\varepsilon y, u)^* + \mathcal{O}(\varepsilon),$$

where $\alpha(z, u) = \alpha(z, -u) = \alpha(u, z) = \alpha(-u, z)$ and is given by

$$\begin{aligned} \alpha(0, 0) &= 1, & \alpha(0, 1) &= -1/3, & \alpha(0, 2) &= -1/6 \\ \alpha(1, 1) &= 0, & \alpha(1, 2) &= 1/6 \\ \alpha(2, 2) &= -1/12. \end{aligned} \tag{78}$$

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