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1 **A collocated finite volume scheme to solve free**  
2 **convection for general non-conforming grids**

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8 **Abstract**

9 We present a new collocated numerical scheme for the approximation of the Navier-  
10 Stokes and energy equations under the Boussinesq assumption for general grids,  
11 using the velocity-pressure unknowns. This scheme is based on a recent scheme for  
12 the diffusion terms. Stability properties are drawn from particular choices for the  
13 pressure gradient and the non-linear terms. Numerical results show the accuracy of  
14 the scheme on irregular grids.

15 *Key words:* Collocated finite volume schemes, general non-conforming grids,  
16 Navier-Stokes Equations, Boussinesq assumption

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## 18 **1 Introduction**

19 Finite volume methods have been widely used in computational fluid dynamics  
20 for a long time; they are well adapted to the discretization of partial differential  
21 equations under conservative form, and one of their attractive features is that  
22 the resulting discretized equation has a clear physical interpretation [9]. In  
23 the framework of incompressible fluid flows, two strategies are often opposed,  
24 namely staggered and collocated schemes. The staggered strategy, which has  
25 become very popular since Patankar’s book [9], remains mainly restricted to  
26 geometrical domains with parallel and orthogonal boundary faces. Therefore,  
27 for computations on complex domains with general meshes, the collocated  
28 strategy which consists in approximating all unknowns on the same set of  
29 points (called collocation points but also cell-centers or simply centers), is  
30 often preferred, even though the pressure-velocity coupling demands some cure  
31 for the stabilization of the well-known checkerboard pressure modes; to this  
32 purpose, various pressure stabilization procedures, based on improvements of  
33 the *Momentum Interpolation Method* proposed by Rhie and Chow [10], are  
34 frequently used [8]).

35 In [3,11], a collocated finite volume scheme for incompressible flows is devel-  
36 oped on so called “admissible” unstructured meshes, that is meshes satisfying  
37 the two following conditions: the straight line joining the centers of two adja-  
38 cent control volumes is perpendicular to the common edge, and the neighbour-  
39 ing control volumes and the associated centers are arranged in the same order,  
40 with respect to the common edge. Rectangular or orthogonal parallelepipedic  
41 meshes, triangular (2D) or tetrahedral (3D) Delaunay meshes, and Voronoi  
42 meshes fulfill these requirements. Under this assumption, the isotropic diffu-

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43 sion fluxes can be consistently approximated by a two-point finite difference  
44 scheme. Using this approximation for a pure diffusion problem yields a sym-  
45 metric “M-matrix” (which ensures monotony); the stencil is limited to the  
46 control volume itself and its natural neighbours and it leads to the classical  
47 5- and 7-point schemes on rectangles and orthogonal parallelepipeds. Unfor-  
48 tunately, although the use of such grids considerably widens the variety of  
49 geometric shapes which can be gridded, it is far from solving all the critical  
50 needs resulting from actual problems:

- 51 • for complex 3D domains, to the large number of flat tetrahedra producing  
52 high discretization errors are often preferred generalized hexahedric meshes  
53 made of 3D elevations of quadrangular meshes, for which the faces of the  
54 control volumes are no longer planar;
- 55 • to our knowledge, there is yet no tool which is able to grid any geomet-  
56 rical shape in 3D using Voronoï or Delaunay tessellations, respecting the  
57 boundaries and the local refinement requirements;
- 58 • in compressible flows, the approximation of the full tensor by the usual  
59 two point scheme is no longer consistent even on admissible meshes, and  
60 multi-point approximations are therefore required;
- 61 • boundary layers are classically meshed with refined grids, so that the dis-  
62 cretization scheme should be able to deal with non-conforming meshes.

63 Whereas there is no real difficulty to discretize the convective terms for gen-  
64 eral and non-conforming grids, the writing of accurate approximations of the  
65 diffusion contributions, particularly relevant for low Reynolds (Péclet) flows,  
66 is still a challenge on such meshes.

67 In the early 80's, Kershaw [7] first proposed a nine-point scheme on structured  
68 quadrilateral grids by using the restrictive assumption of a smooth mapping  
69 between the logical mesh and the spatial coordinates. Since then, numerous  
70 works have been published to efficiently solve the diffusion equations in general  
71 geometry (see [1] for a review of recent papers). The drawbacks of the actual  
72 schemes for diffusion are often linked with one or several of these key points:

- 73 • a non-local stencil (quite dense matrices);
- 74 • cell-centered but also face-centered unknowns (large matrices);
- 75 • non-symmetric definite positive matrices (loss of the energy balance);
- 76 • loss of the convergence or of the accuracy on some particular grids;
- 77 • loss of monotony for solutions in purely diffusive problems (the resulting  
78 matrix is not a “M-matrix”).

79 We focus in this paper on the approximation of the Navier-Stokes and energy  
80 equations under the Boussinesq assumption, using a new scheme for diffu-  
81 sion terms. This scheme is shown to provide a cell-centered approximation  
82 with a quite reduced stencil, leading to symmetric definite positive matrices  
83 and to mathematical convergence proofs. Although the diffusion matrix is not  
84 known to satisfy the M-matrix property in general, the maximum principle  
85 was nevertheless preserved in our numerical three-dimensional simulations.

86 In this scheme, the discrete pressure gradient and the non-linear contributions  
87 are approximated so that the discrete kinetic and energy balances mimic their  
88 continuous counterparts. Indeed, the pressure gradient is chosen as the dual  
89 operator of the discrete divergence, and the discretization is such that there is  
90 no contribution of the non-linear velocity transport in the increase of kinetic  
91 energy. In order to suppress the pressure checkerboard modes, the mass bal-

92 ance is stabilized by a pressure expression which only redistributes the fluid  
 93 mass within subsets of control volumes, the characteristic size of which is two  
 94 or three times the local mesh size.

95 The remainder of this paper is divided into four sections. In section 2, the con-  
 96 tinuous formulation is presented in the framework of natural convection. The  
 97 following section presents the discrete scheme and pays particular attention  
 98 to the diffusive contribution. The fourth section is devoted to the numeri-  
 99 cal studies, first with analytical solutions and then for a natural convection  
 100 problem.

## 101 **2 Continuous formulation**

102 Let  $d$  be the dimension of the space ( $d = 2$  or  $3$ ) and let  $\Omega \subset \mathbb{R}^d$  be an  
 103 open polygonal connected domain. For  $\mathbf{x} \in \Omega$ , our aim is to compute an  
 104 approximation of the velocity  $\mathbf{u}(\mathbf{x}) = \sum_{i=1}^d u^{(i)}(\mathbf{x})\mathbf{e}_i$ , the pressure  $p(\mathbf{x})$  and  
 105 the temperature  $T(\mathbf{x})$ , solution of the steady and dimensionless Navier-Stokes  
 106 and energy equations under the Boussinesq approximation:

$$-\text{Pr}\Delta\mathbf{u} + \nabla p + (\mathbf{u} \cdot \nabla)\mathbf{u} - \text{Ra Pr } T\mathbf{e}_3 = \mathbf{f}(\mathbf{x}) \text{ in } \Omega \quad (1a)$$

$$-\Delta T + (\mathbf{u} \cdot \nabla)T = g(\mathbf{x}) \text{ in } \Omega \quad (1b)$$

$$\text{div}\mathbf{u} = 0 \text{ in } \Omega \quad (1c)$$

107 where  $\mathbf{e}_3$  indicates the vertical upward direction,  $\mathbf{f}(\mathbf{x}) = \sum_{i=1}^d f^{(i)}(\mathbf{x})\mathbf{e}_i$  and  $g(\mathbf{x})$   
 108 are dimensionless regular functions modeling source or sink in the momen-  
 109 tum or heat balances; Pr and Ra denote the Prandtl and Rayleigh numbers

110 respectively.

111 We consider the case of the homogeneous Dirichlet boundary conditions for  
 112 the velocity and of the mixed Dirichlet- Neumann boundary conditions for the  
 113 temperature. These boundary conditions read as follows:

$$\begin{cases} \mathbf{u}(\mathbf{x}) = \mathbf{0} & \mathbf{x} \in \Gamma, \\ T(\mathbf{x}) = T_b(\mathbf{x}) & \mathbf{x} \in \Gamma_1, \\ -\nabla T(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = q_b(\mathbf{x}) & \mathbf{x} \in \Gamma_2, \end{cases} \quad (2)$$

114 where  $\Gamma$  denotes the boundary of the domain,  $\Gamma_1, \Gamma_2 \subset \Gamma$  are such that  $\Gamma_1 \cap$   
 115  $\Gamma_2 = \emptyset$  and  $\Gamma_1 \cup \Gamma_2 = \Gamma$ , and  $\mathbf{n}(\mathbf{x})$  is the outward unit normal vector to the  
 116 boundary.

117 We assume that  $T_b$  is the trace on  $\Gamma_1$  of a function, again denoted  $T_b$ , such  
 118 that  $T_b \in H^1(\Omega)$ , and we define the functional space  $H_{\Gamma_1,0}^1(\Omega) = \{T \in$   
 119  $H^1(\Omega); T(\mathbf{x}) = 0 \text{ on } \Gamma_1\}$ . Then a weak formulation of equations (1a-1c) with  
 120 boundary conditions (2) reads: find  $\mathbf{u} \in H_0^1(\Omega)^d$ ,  $p \in L^2(\Omega)$  with  $\int_{\Omega} p(\mathbf{x})d\mathbf{x} =$   
 121 0, and  $T$  with  $T - T_b \in H_{\Gamma_1,0}^1(\Omega)$ , such that

$$\begin{aligned} \text{Pr} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} d\mathbf{x} - \int_{\Omega} p \text{div} \mathbf{v} d\mathbf{x} + \int_{\Omega} \text{div}(\mathbf{u} \otimes \mathbf{u}) \cdot \mathbf{v} d\mathbf{x} \\ - \text{Ra Pr} \int_{\Omega} T \mathbf{e}_3 \cdot \mathbf{v} d\mathbf{x} = \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{v} d\mathbf{x}, \quad \forall \mathbf{v} \in H_0^1(\Omega)^d, \end{aligned} \quad (3a)$$

$$\begin{aligned} \int_{\Omega} \nabla T \cdot \nabla \theta d\mathbf{x} + \int_{\Omega} \text{div}(\mathbf{u}T)\theta d\mathbf{x} \\ = \int_{\Omega} g(\mathbf{x})\theta d\mathbf{x} - \int_{\Gamma_2} q_b(\mathbf{x})\theta(\mathbf{x})d\mathbf{x}, \quad \forall \theta \in H_{\Gamma_1,0}^1(\Omega). \end{aligned} \quad (3b)$$

$$\operatorname{div} \mathbf{u}(\mathbf{x}) = 0 \text{ for a.e. } \mathbf{x} \in \Omega, \quad (3c)$$

122 Although we focus in this paper on finite volume schemes, the principles of  
 123 the discretization are also much inspired by the weak form (3a-3c).

### 124 3 Numerical scheme

125 We denote by  $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$  a space discretization, where (see Fig. 1):

- 126 •  $\mathcal{M}$  is a finite family of non empty connected open disjoint subsets of  $\Omega$   
 127 (the “control volumes”) such that  $\overline{\Omega} = \cup_{K \in \mathcal{M}} \overline{K}$ . For any  $K \in \mathcal{M}$ , let  
 128  $\partial K = \overline{K} \setminus K$ ,  $m_K > 0$  and  $h_K$  respectively denote the boundary, the  
 129 measure and the diameter of  $K$ .
- 130 •  $\mathcal{E}$  is a finite family of disjoint subsets of  $\overline{\Omega}$  (the “edges” of the mesh), such  
 131 that, for all  $\sigma \in \mathcal{E}$ ,  $\sigma$  is a non empty open subset of a hyperplane of  $\mathbb{R}^d$ ,  
 132 whose (d-1)-dimensional measure  $m_\sigma$  is positive. We assume that, for all  
 133  $K \in \mathcal{M}$ , there exists a subset  $\mathcal{E}_K$  of  $\mathcal{E}$  such that  $\partial K = \cup_{\sigma \in \mathcal{E}_K} \sigma$ . We then  
 134 denote by  $\mathcal{M}_\sigma = \{K \in \mathcal{M}, \sigma \in \mathcal{E}_K\}$ . The set  $\mathcal{E}$  is assumed to be partitioned  
 135 into  $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$ , such that, for all  $\sigma \in \mathcal{E}_{\text{ext}}$  (boundary edge),  $\mathcal{M}_\sigma$  has  
 136 exactly one element and  $\sigma \subset \partial\Omega$ , and for all  $\sigma \in \mathcal{E}_{\text{int}}$  (interior edge),  $\mathcal{M}_\sigma$   
 137 has exactly two elements. We also assume that, if  $\sigma \in \mathcal{E}_{\text{ext}}$ , then either  
 138  $\sigma \subset \Gamma_1$  or  $\sigma \subset \Gamma_2$ . For all  $\sigma \in \mathcal{E}$ , we denote by  $\mathbf{x}_\sigma$  the barycenter of  $\sigma$ . For  
 139 all  $K \in \mathcal{M}$  and  $\sigma \in \mathcal{E}_K$ , we denote by  $\mathbf{n}_{K,\sigma}$  the unit vector normal to  $\sigma$   
 140 outward to  $K$ .
- 141 •  $\mathcal{P}$  is a family of points of  $\Omega$  indexed by  $\mathcal{M}$  (the collocation points), denoted  
 142 by  $\mathcal{P} = (\mathbf{x}_K)_{K \in \mathcal{M}}$ , such that for all  $K \in \mathcal{M}$ ,  $K$  is assumed to be  $\mathbf{x}_K$ -star-  
 143 shaped, which means that for all  $\mathbf{x} \in K$ , the property  $[\mathbf{x}_K, \mathbf{x}] \subset K$  holds.



144 The Euclidean distance  $d_{K,\sigma}$  between  $\mathbf{x}_K$  and the hyperplane including  $\sigma$   
 145 is assumed positive. We then denote by  $C_{K,\sigma}$  the cone with vertex  $\mathbf{x}_K$  and  
 146 basis  $\sigma$ , and by  $m_{K,\sigma} = m_\sigma d_{K,\sigma}/d$  its measure.

147 For any edge  $\sigma \in \mathcal{E}_{\text{int}}$ , we define a linear mapping  $\Pi_\sigma : \mathbb{R}^{\mathcal{M}} \rightarrow \mathbb{R}$  (where an  
 148 element  $v \in \mathbb{R}^{\mathcal{M}}$  is defined by the family of real values  $(v_K)_{K \in \mathcal{M}}$ ) such that  
 149 for any regular function  $\psi$ , setting  $v_K = \psi(\mathbf{x}_K)$  for all  $K \in \mathcal{M}$ , then  $\Pi_\sigma(v)$  is  
 150 a second order approximation of  $\psi(\mathbf{x}_\sigma)$ . In such a case, there exist coefficients  
 151  $(\beta_\sigma^L)_{L \in \mathcal{M}}$  such that

$$\Pi_\sigma(u) = \sum_{L \in \mathcal{M}} \beta_\sigma^L u_L \text{ with } \sum_{L \in \mathcal{M}} \beta_\sigma^L = 1, \mathbf{x}_\sigma = \sum_{L \in \mathcal{M}} \beta_\sigma^L \mathbf{x}_L. \quad (4)$$

152 In three space dimensions, it is always possible to restrict the number of  
 153 nonzero  $\beta_\sigma^L$  to four (in practice, the scheme has been shown to be robust  
 154 with respect to the choice of these four control volumes, taken close enough  
 155 to the considered edge). We now define the finite dimensional space  $\mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{E}}$   
 156 (where an element  $v \in \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{E}}$  is defined by the family of real values  
 157  $((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}})$ ) and the following subspaces:

- 158 •  $X^{\mathcal{D}} = \{u \in \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{E}}, \forall \sigma \in \mathcal{E}_{\text{int}}, u_\sigma = \Pi_\sigma(u)\}$  (the dimension of  $X^{\mathcal{D}}$  is the  
 159 number of control volumes plus that of boundary edges),
- 160 •  $X_0^{\mathcal{D}} = \{u \in X^{\mathcal{D}}, \forall \sigma \in \mathcal{E}_{\text{ext}}, u_\sigma = 0\}$  (the dimension of  $X_0^{\mathcal{D}}$  is the number of  
 161 control volumes),
- 162 •  $X_{\Gamma_1,0}^{\mathcal{D}} = \{\theta \in X^{\mathcal{D}}, \forall \sigma \in \mathcal{E}_{\text{ext}} \cap \Gamma_1, \theta_\sigma = 0\}$  (the dimension of  $X_{\Gamma_1,0}^{\mathcal{D}}$  is the  
 163 number of control volumes plus that of boundary edges on  $\Gamma_2$ ).

165 Let us first define a discrete gradient for the elements of  $X^{\mathcal{D}}$  on cell  $K \in \mathcal{M}$ .

166 We set, for any  $u \in X^{\mathcal{D}}$  and  $K \in \mathcal{M}$ :

$$\nabla_K u = \frac{1}{\mathfrak{m}_K} \sum_{\sigma \in \mathcal{E}_K} \mathfrak{m}_\sigma (u_\sigma - u_K) \mathbf{n}_{K,\sigma}.$$

167 We could construct a discrete gradient with this formula but its use to approx-  
 168 imate the diffusive contributions (first terms of the left-hand sides of equations  
 169 (3a) and (3b)) is not adequate because we cannot construct a coercive form  
 170 from it, as we do below with a modified gradient in (5). Indeed, for all  $\sigma \in \mathcal{E}_K$ ,  
 171 we define  $R_{K,\sigma} u \in \mathbb{R}$  which may be seen as a consistency error on the normal  
 172 flux, by:

$$R_{K,\sigma} u = \frac{\sqrt{d}}{d_{K,\sigma}} (u_\sigma - u_K - \nabla_K u \cdot (\mathbf{x}_\sigma - \mathbf{x}_K)).$$

173 (note that  $R_{K,\sigma} u = 0$  if  $u_K$  and  $u_\sigma$  are the exact values of a linear function at  
 174 points  $x_K$  and  $x_\sigma$ , for all  $K$  and  $\sigma$ ). We then give the following expression for  
 175 a discrete gradient of  $u \in X^{\mathcal{D}}$  in each cone  $C_{K,\sigma}$ :

$$\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma} u \mathbf{n}_{K,\sigma},$$

176 and choose a global discrete gradient as the function  $\nabla_{\mathcal{D}} u$  which is such that

$$\nabla_{\mathcal{D}} u(\mathbf{x}) = \nabla_{K,\sigma} u, \text{ for a.e. } \mathbf{x} \in C_{K,\sigma}, \forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K.$$

177 We then get that

$$\int_{\Omega} \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{m_{\sigma} d_{K,\sigma}}{d} \nabla_{K,\sigma} u \cdot \nabla_{K,\sigma} v, \quad \forall u, v \in X^{\mathcal{D}}. \quad (5)$$

178 It may then be shown that  $\int_{\Omega} \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x}$  defines a symmetric inner  
 179 product on  $X^{\mathcal{D}}$  which provides a good approximation for  $\int_{\Omega} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}$   
 180 [4]. It is then possible, expressing  $u_{\sigma}$  and  $v_{\sigma}$  for all  $\sigma \in \mathcal{E}_{\text{int}}$  thanks to the  
 181 relations (4), to show that

$$\begin{aligned} \int_{\Omega} \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} = & \\ & \sum_{K \in \mathcal{M}} \left( \sum_{L \in \mathcal{N}_K} F_{K,L}(u) v_K + \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}} F_{K,\sigma}(u) (v_K - v_{\sigma}) \right), \end{aligned} \quad (6)$$

182 where for any  $K \in \mathcal{M}$ ,  $\mathcal{N}_K \subset \mathcal{M}$  is such that if  $L \in \mathcal{N}_K$ , then  $K \in \mathcal{N}_L$ , and  
 183 for any  $K \in \mathcal{M}$  and  $L \in \mathcal{N}_K$ ,  $F_{K,L}(u)$  is a linear mapping from  $X^{\mathcal{D}}$  to  $\mathbb{R}$  such  
 184 that  $F_{K,L}(u) = -F_{L,K}(u)$ .

185 The approximation of  $-\int_K \Delta u d\mathbf{x}$  is obtained by letting  $v_K = 1$ ,  $v_L = 0$  for  
 186  $L \neq K$  and  $v_{\sigma} = 0$  for  $\sigma \in \mathcal{E}_{\text{ext}}$  in (6):

$$-\int_K \Delta u d\mathbf{x} \simeq \sum_{L \in \mathcal{N}_K} F_{K,L}(u) + \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}} F_{K,\sigma}(u),$$

187 so that we may define an approximate Laplace operator  $\Delta_{\mathcal{M}}$  by the constant  
 188 values  $\Delta_K u$  on the cells  $K$ :

$$\Delta_K u = \frac{1}{m_K} \left( \sum_{L \in \mathcal{N}_K} F_{K,L}(u) + \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}} F_{K,\sigma}(u) \right). \quad (7)$$

189

190 The stencil of the discrete operator is then the set of control volumes  $M$  such  
 191 that there exists  $L \in \mathcal{N}_K$  with  $F_{K,L}(u)$  depending on  $u_M$ . In practice, the  
 192 equation for a given cell usually concerns the unknowns associated to itself,  
 193 its neighbours and its neighbours' neighbours. In general, the resulting matrix  
 194 is not an "M-matrix". Nevertheless, this property is fully recovered when using  
 195 particular meshes such as, for example, conforming orthogonal parallelepipeds.  
 196 Indeed, in such a case, locating  $\mathbf{x}_K$  at the center of gravity of the cell  $K$ , the  
 197 relation  $(\mathbf{x}_\sigma - \mathbf{x}_K)/d_{K,\sigma} = \mathbf{n}_{K,\sigma}$  holds. It is then possible to define the second  
 198 order linear interpolation  $\Pi_\sigma(v)$  by  $\Pi_\sigma(v) = (d_{L,\sigma}v_K + d_{K,\sigma}v_L)/(d_{L,\sigma} + d_{K,\sigma})$   
 199 for all  $\sigma$  such that  $\mathcal{M}_\sigma = \{K, L\}$ , and for all  $v \in X^{\mathcal{D}}$ . Using the identity

$$\sum_{\sigma \in \mathcal{E}_K} m_\sigma (\mathbf{x}_\sigma - \mathbf{x}_K) \mathbf{n}_{K,\sigma}^t = m_K \mathbb{I}$$

200 where  $^t$  designates the transposition and  $\mathbb{I}$  the identity matrix, we obtain,  
 201 thanks to simple computations:

$$\begin{aligned} \int_{\Omega} \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} &= \sum_{\sigma \in \mathcal{E}_{\text{int}}, \mathcal{M}_\sigma = \{K, L\}} \frac{m_\sigma}{d_{K,\sigma} + d_{L,\sigma}} (u_L - u_K)(v_L - v_K) \\ &+ \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \mathcal{M}_\sigma = \{K\}} \frac{m_\sigma}{d_{K,\sigma}} (u_\sigma - u_K)(v_\sigma - v_K). \end{aligned}$$

202 Then the previous relation leads to define  $\mathcal{N}_K$  as the set of the natural neigh-  
 203 bours of  $K$ , and to define the fluxes by the natural two-point difference scheme,  
 204 in the same manner as in [3,11]:

$$F_{K,L}(u) = \frac{m_\sigma}{d_{K,\sigma} + d_{L,\sigma}}(u_K - u_L) \quad \text{for } \sigma \in \mathcal{E}_{\text{int}}, \mathcal{M}_\sigma = \{K, L\}$$

$$F_{K,\sigma}(u) = \frac{m_\sigma}{d_{K,\sigma}}(u_K - u_\sigma) \quad \text{for } \sigma \in \mathcal{E}_{\text{ext}}, \mathcal{M}_\sigma = \{K\}.$$

205 The classical and cheap 5- and 7-point schemes on rectangular or orthogonal  
 206 parallelepipedic meshes is then recovered. An advantage can then be taken  
 207 from this property, by using meshes which consist in orthogonal parallelepi-  
 208 pedic control volumes in the main part of the interior of the domain, as we  
 209 show in the numerical examples.

### 210 3.2 Pressure-velocity coupling, mass balance and convective contributions

211 For all  $\mathbf{v} \in (X_0^{\mathcal{D}})^d$ , we define a discrete divergence operator by:

$$\operatorname{div}_K \mathbf{v} = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma \mathbf{v}_\sigma \cdot \mathbf{n}_{K,\sigma}, \quad \forall K \in \mathcal{M}.$$

212 Notice that

$$\operatorname{div}_K \mathbf{v} = \sum_{i=1}^d (\nabla_K v^{(i)})^{(i)}.$$

213 We then define the function  $\operatorname{div}_{\mathcal{D}} \mathbf{v}$  by the relation

$$\operatorname{div}_{\mathcal{D}} \mathbf{v}(\mathbf{x}) = \operatorname{div}_K \mathbf{v}, \quad \text{for a.e. } \mathbf{x} \in K, \quad \forall K \in \mathcal{M}.$$

214 As recalled in the introduction of this paper, a pressure stabilization method  
 215 is implemented in the mass conservation equation in order to prevent from  
 216 oscillations of the pressure, as for instance in [2] in the finite element setting,  
 217 [8], [10] in the finite volume setting. The originality of our approach is that  
 218 we directly include the stabilizing diffusive pressure flux in the approximated

219 mass flux, so that it will appear not only (as usual) in the mass equation, but  
 220 also in the momentum equation through the non-linear convective term. From  
 221 the mathematical point of view, this helps in obtaining simple estimates on the  
 222 velocity and pressure, but more importantly, it ensures that the contribution  
 223 of the discrete non-linear convective term to the kinetic (and thermal) energy  
 224 balance is zero, just as in the continuous case. Let us define the stabilized  
 225 mass flux across  $\sigma \in \mathcal{E}_{\text{int}}$  with  $\mathcal{M}_\sigma = \{K, L\}$ , by

$$\Phi_{K,\sigma}^\lambda(\mathbf{u}, p) = m_\sigma (\mathbf{u}_\sigma \cdot \mathbf{n}_{K,\sigma} + \lambda_\sigma (p_K - p_L)), \quad (8)$$

226 where  $(\lambda_\sigma)_{\sigma \in \mathcal{E}_{\text{int}}}$  is a given family of positive real numbers, the choice of which  
 227 is discussed below. Note that the quantity  $\lambda_\sigma (p_K - p_L)$  may be seen as a numer-  
 228 ical pressure diffusion flux, and that the overall numerical flux remains conser-  
 229 vative, that is, if  $\sigma \in \mathcal{E}_{\text{int}}$  with  $\mathcal{M}_\sigma = \{K, L\}$ , then  $\Phi_{K,\sigma}^\lambda(\mathbf{u}, p) + \Phi_{L,\sigma}^\lambda(\mathbf{u}, p) = 0$ .  
 230 We then use this modified flux, in order to define a stabilized centered trans-  
 231 port operator which is defined, for all  $\mathbf{v} \in (X_0^{\mathcal{D}})^d$ ,  $w \in X^{\mathcal{D}}$  and  $K \in \mathcal{M}$ ,  
 232 by

$$\text{div}_K^\lambda(w, \mathbf{u}, p) = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}, \mathcal{M}_\sigma = \{K, L\}} \Phi_{K,\sigma}^\lambda(\mathbf{u}, p) \frac{w_K + w_L}{2}.$$

233

234 An interesting remark is that, in the case where the mass balance equation in  
 235 the control volume  $K$  is satisfied, that is:

$$\text{div}_K^\lambda(1, \mathbf{u}, p) = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}} \Phi_{K,\sigma}^\lambda(\mathbf{u}, p) = 0,$$

236 then

$$\sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}} \Phi_{K,\sigma}^\lambda(\mathbf{u}, p) w_K = 0,$$

237 so that the following relation also holds:

$$m_K \text{div}_K^\lambda(w, \mathbf{u}, p) = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}, \mathcal{M}_\sigma = \{K, L\}} \Phi_{K,\sigma}^\lambda(\mathbf{u}, p) \frac{w_L - w_K}{2}.$$

238 We shall use this latter form in the practical implementation, in particular  
 239 in the discretization of the non-linear convection term. Indeed, it is more  
 240 efficient when computing the Jacobian matrix of the momentum equation,  
 241 since it avoids summing up values of the same amplitude. When the local grid  
 242 Reynolds (or Péclet) number is much larger than 1, an upwind scheme must  
 243 be applied that consists in substituting  $\text{div}_K^\lambda(w, \mathbf{v}, p)$  by

$$\begin{aligned} \text{div}_K^{\lambda, \text{up}}(w, \mathbf{u}, p) = \\ \frac{1}{m_K} \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}} \\ \mathcal{M}_\sigma = \{K, L\}}} \left( \max(\Phi_{K,\sigma}^\lambda(\mathbf{u}, p), 0) w_K + \min(\Phi_{K,\sigma}^\lambda(\mathbf{u}, p), 0) w_L \right). \end{aligned}$$

244 In both cases, the functions  $\text{div}_D^\lambda(w, \mathbf{u}, p)$  and  $\text{div}_D^{\lambda, \text{up}}(w, \mathbf{u}, p)$  are defined by  
 245 their constant values in each control volume.

246 For  $u, v \in (X_0^D)^d$ , we shall also define the centered vector valued divergence  
 247 operator  $\mathbf{div}_D^\lambda(\mathbf{w}, \mathbf{u}, p)$  and  $\mathbf{div}_D^{\lambda, \text{up}}(\mathbf{w}, \mathbf{u}, p)$  such that the  $i$ -th component of  
 248  $\mathbf{div}_D^\lambda(\mathbf{w}, \mathbf{u}, p)$  is equal to  $\text{div}_D^\lambda(w_i, \mathbf{u}, p)$ , for  $i = 1, \dots, d$ , and the same goes

249 for the upwind divergence.

### 250 3.3 Choice for the parameters $(\lambda_\sigma)_{\sigma \in \mathcal{E}_{\text{int}}}$

251 Different strategies can be applied to define the parameters  $(\lambda_\sigma)_{\sigma \in \mathcal{E}_{\text{int}}}$ . Amongst  
 252 all of them we applied the "cluster stabilization method" [3,11] that consists  
 253 in constructing a partition of  $\mathcal{M}$ , denoted  $\mathcal{G}$ , and set  $\lambda_\sigma = \lambda$  if there exists  
 254  $G \in \mathcal{G}$  (such  $G \subset \mathcal{M}$  is called a cluster) with  $\mathcal{M}_\sigma \subset G$ , otherwise we set  
 255  $\lambda_\sigma = 0$ . Here is an example of such an algorithm: first for all cells  $K$ , initialize  
 256 a new cluster if  $K$  and its neighbouring cells do not already belong to a cluster  
 257 (Figs. 2a,b); then for any remaining isolated cell  $L$ , connect it to the closest  
 258 cluster having the largest number of common edges with  $L$  (Fig. 2c).

### 259 3.4 Resulting discrete equations

260 We denote by  $T_{b,\mathcal{D}}$  the element  $T \in X^{\mathcal{D}}$  such that  $T_K = 0$  for all  $K \in \mathcal{M}$ ,  
 261  $T_\sigma = 0$  for all  $\sigma \in \mathcal{E}_{\text{int}}$  and all  $\sigma \in \mathcal{E}_{\text{ext}}$  with  $\sigma \subset \Gamma_2$ , and, for all  $\sigma \in \mathcal{E}_{\text{ext}}$  with  
 262  $\sigma \subset \Gamma_1$ ,

$$T_\sigma = \frac{1}{m_\sigma} \int_\sigma T_b(\mathbf{x}) ds(\mathbf{x}). \quad (9)$$

263 Let  $\mathcal{H}_{\mathcal{M}}(\Omega) \subset L^2(\Omega)$  denote the set of functions which are constant in each  
 264  $K \in \mathcal{M}$ ; for any function  $q \in \mathcal{H}_{\mathcal{M}}(\Omega)$ , we shall denote by  $q_K$  its constant  
 265 value on  $K \in \mathcal{M}$ . We then define the mapping  $P_{\mathcal{M}} : X^{\mathcal{D}} \rightarrow \mathcal{H}_{\mathcal{M}}(\Omega)$  by  
 266  $v \in X^{\mathcal{D}} \mapsto P_{\mathcal{M}}v$  with  $P_{\mathcal{M}}v(\mathbf{x}) = v_K$  for a.e.  $\mathbf{x} \in K$  and all  $K \in \mathcal{M}$ . We also  
 267 define the mapping  $P_{\mathcal{E}} : X^{\mathcal{D}} \rightarrow L^2(\Gamma)$  by  $v \in X^{\mathcal{D}} \mapsto P_{\mathcal{E}}v$  with  $P_{\mathcal{E}}v(\mathbf{x}) = v_\sigma$   
 268 for a.e.  $\mathbf{x} \in \sigma$  and all  $\sigma \in \mathcal{E}_{\text{ext}}$ .



269 Let us then use the previously defined discrete operators to formulate a discrete  
 270 approximation to problem (3a-3b) :

271 find  $\mathbf{u} = (u^{(i)})_{i=1,d} \in (X_0^{\mathcal{D}})^d$ ,  $p \in \mathcal{H}_{\mathcal{M}}(\Omega)$  with  $\int_{\Omega} p(\mathbf{x}) d\mathbf{x} = \sum_{K \in \mathcal{M}} m_K p_K = 0$   
 272 and  $T - T_{b,\mathcal{D}} \in X_{\Gamma_1,0}^{\mathcal{D}}$  such that:

273 the discrete momentum balance holds:

$$\begin{aligned} & \Pr \int_{\Omega} \nabla_{\mathcal{D}} \mathbf{u} : \nabla_{\mathcal{D}} \mathbf{v} \, d\mathbf{x} - \int_{\Omega} p \operatorname{div}_{\mathcal{D}} \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \mathbf{div}_{\mathcal{D}}^{\lambda}(\mathbf{u}, \mathbf{u}, p) \cdot P_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} \\ & - \operatorname{Ra} \Pr \int_{\Omega} P_{\mathcal{M}} T \, \mathbf{e}_3 \cdot P_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot P_{\mathcal{M}} \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in (X_0^{\mathcal{D}})^d, \end{aligned} \quad (10)$$

274 the energy balance holds:

$$\begin{aligned} & \int_{\Omega} \nabla_{\mathcal{D}} T \cdot \nabla_{\mathcal{D}} \theta \, d\mathbf{x} + \int_{\Omega} \operatorname{div}_{\mathcal{D}}^{\lambda}(T, \mathbf{u}, p) P_{\mathcal{M}} \theta \, d\mathbf{x} \\ & = \int_{\Omega} g P_{\mathcal{M}} \theta \, d\mathbf{x} - \int_{\Gamma_2} q_b P_{\mathcal{E}} \theta \, ds, \quad \forall \theta \in X_{\Gamma_1,0}^{\mathcal{D}}, \end{aligned} \quad (11)$$

275 the mass balance holds:

$$\operatorname{div}_{\mathcal{D}}^{\lambda}(1, \mathbf{u}, p) = 0 \text{ a.e. in } \Omega. \quad (12)$$

276 We then deduce from (10) the  $d$  discrete equations of the conservation of  
 277 momentum in the control volume  $K$ , letting  $v^{(i)} = 1$  in  $K$ , and 0 otherwise.

278 These equations read, in vector form:

$$\begin{aligned}
& m_K \Delta_K \mathbf{u} + \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \mathcal{M}_\sigma = \{M, L\}}} m_\sigma \beta_\sigma^K (p_M - p_L) \mathbf{n}_{L, \sigma} \\
& + \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}} \\ \mathcal{M}_\sigma = \{K, L\}}} \Phi_{K, \sigma}^\lambda(\mathbf{u}, p) \frac{\mathbf{u}_K + \mathbf{u}_L}{2} - \text{Ra Pr } m_K T_K \mathbf{e}_3 = \int_K \mathbf{f} \, d\mathbf{x}
\end{aligned} \tag{13}$$

279 (where  $\Delta_K \mathbf{u}$  is the vector valued discrete Laplace operator defined by (7) for  
280 each of its component). Similarly, we deduce from (11) the discrete equation  
281 of the conservation of energy in the control volume  $K$ , letting  $\theta = 1$  in  $K$ , and  
282 0 otherwise. This equation reads:

$$m_K \Delta_K T + \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}} \\ \mathcal{M}_\sigma = \{K, L\}}} \Phi_{K, \sigma}^\lambda(\mathbf{u}, p) \frac{T_K + T_L}{2} = \int_K g \, d\mathbf{x}. \tag{14}$$

283 Recall that, for all  $K \in \mathcal{M}$ , and all  $\sigma \in \mathcal{E}_K$  such that  $\sigma \subset \Gamma_1$ , the following  
284 Dirichlet boundary condition (9) is imposed. We deduce from (11) the relation  
285 imposed by the Neumann boundary condition for the thermal flux, letting  
286  $\theta_\sigma = 1$  and 0 otherwise, for some  $\sigma \in \mathcal{E}_K$  with  $\sigma \subset \Gamma_2$ :

$$F_{K, \sigma}(T) = \int_\sigma q_b(\mathbf{x}) \, ds(\mathbf{x}). \tag{15}$$

287 Note that the above relation is natural, accounting for the fact that  $F_{K, \sigma}(T)$   
288 approximates the heat flux at the edge  $\sigma$ . Finally, we write (12) in a given  
289 control volume  $K$ :

$$\sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}} \Phi_{K, \sigma}^\lambda(\mathbf{u}, p) = 0. \tag{16}$$

291 *3.5 Some mathematical properties*

292 The system of discrete equations (13-16) appears as a system of non-linear  
 293 equations. The mathematical proof of the existence of at least a solution can  
 294 be shown in the particular case  $T_b = 0$  and  $q_b = 0$ , which we consider in this  
 295 section. Indeed, in this case, we can show some a priori bounds on  $T$  and  $\mathbf{u}$ .  
 296 We first let  $\theta = T$  in (11). Using the relation

$$\int_{\Omega} \operatorname{div}_{\mathcal{D}}^{\lambda}(T, \mathbf{u}, p) P_{\mathcal{M}}T \, d\mathbf{x} = 0,$$

297 which results from (12), we get

$$\|\nabla_{\mathcal{D}}T\|_{L^2(\Omega)^d}^2 = \int_{\Omega} g P_{\mathcal{M}}T \, d\mathbf{x}.$$

298 Thanks to a discrete Poincaré inequality which follows from [5, Lemma 5.3],  
 299 we get that there exists  $C_T$ , only depending on the regularity of the mesh and  
 300 on  $g$ , but not on the size of the mesh, such that

$$\|\nabla_{\mathcal{D}}T\|_{L^2(\Omega)^d} \leq C_T.$$

301 We then let  $\mathbf{v} = \mathbf{u}$  in (10). We get, thanks to (8) and (12),

$$\begin{aligned} & \Pr \|\nabla_{\mathcal{D}}\mathbf{u}\|_{(L^2(\Omega)^d)^d}^2 + \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \mathcal{M}_{\sigma} = \{K, L\}}} m_{\sigma} \lambda_{\sigma} (p_L - p_K)^2 \\ &= \int_{\Omega} (\mathbf{f} + \operatorname{Ra} \Pr P_{\mathcal{M}}T \mathbf{e}_3) \cdot P_{\mathcal{M}}\mathbf{u} \, d\mathbf{x}. \end{aligned}$$

302 Again using the Poincaré inequality, we conclude that there exists  $C_{\mathbf{u}}$ , only  
 303 depending on the regularity of the mesh, on  $\text{Ra}$ ,  $\text{Pr}$ ,  $f$  and  $g$ , but not on the  
 304 size of the mesh, such that

$$\|\nabla_{\mathcal{D}}\mathbf{u}\|_{(L^2(\Omega)^d)^d} \leq C_{\mathbf{u}}.$$

305 Hence, using the topological degree method, we can prove the existence of  
 306 at least one solution. Moreover, these inequalities are then sufficient to get  
 307 compactness properties, which show that, from a sequence of discrete solutions  
 308 with the space step tending to zero, we can extract a converging subsequence,  
 309 for suitable norms. Then we can prove that the limit of this subsequence has a  
 310 sufficient regularity, in relation with the weak sense provided by (3). It is then  
 311 possible to pass to the limit on (11), (10) and (12), using test functions which  
 312 are interpolation of regular ones. We then get that the limit of the converging  
 313 subsequence satisfies (3).

#### 314 4 Numerical validation

315 The set of non-linear equations (13-16) is solved by an under-relaxed Newton  
 316 method where the unknowns are the velocity  $\mathbf{u}_K$ , the pressure  $p_K$  and the  
 317 temperature  $T_K$  and  $T_\sigma$  for all  $K \in \mathcal{M}$  and  $\sigma \in \mathcal{E}_{\text{ext}} \cap \Gamma_2$ . The solutions of the  
 318 linear systems are computed with a parallel Generalized Minimal RESidual  
 319 method provided by the scalable linear solvers package *HYPRE* with a pre-  
 320 conditioning based on the block Jacobi iLU factorization carried out by *Euclid*

321 [6].

#### 322 4.1 Analytical solutions

323 We consider two closed cavities, cubic or cone-shaped, in which the fluid flow  
 324 and the heat transfer are known *a priori*. Let  $p_{\text{ref}}(\mathbf{x})$ ,  $\mathbf{u}_{\text{ref}}(\mathbf{x})$  and  $T_{\text{ref}}(\mathbf{x})$  be the  
 325 pressure, the velocity satisfying the mass equation (1c) and the temperature  
 326 field. Since the exact solutions are known, the right-hand sides  $\mathbf{f}(\mathbf{x})$  and  $g(\mathbf{x})$   
 327 in equations (13) and (14) are chosen as the residuals of the Navier-Stokes  
 328 (Eq. 1a) and energy equations (Eq. 1b) with  $\mathbf{u}(\mathbf{x}) \equiv \mathbf{u}_{\text{ref}}(\mathbf{x})$ ,  $p(\mathbf{x}) \equiv p_{\text{ref}}(\mathbf{x})$   
 329 and  $T(\mathbf{x}) \equiv T_{\text{ref}}(\mathbf{x})$ . We denote by  $(n_i)_{i=1,\dots,3}$  the number of cells in directions  
 330  $(\mathbf{e}_i)_{i=1,\dots,3}$ . For any regular function  $\psi(\mathbf{x}) \in \{(u^{(i)}(\mathbf{x}))_{i=1,\dots,3}, p(\mathbf{x}), T(\mathbf{x})\}$ , the  
 331 relative accuracy of the scheme is measured by  $\|\psi - \psi_{\text{ref}}\|_{\infty}/\|\psi_{\text{ref}}\|_{\infty}$ ,  $\|\psi -$   
 332  $\psi_{\text{ref}}\|_2/\|\psi\|_2$  and  $\|\psi - \psi_{\text{ref}}\|_{H_1}/\|\psi_{\text{ref}}\|_{H_1}$ , with  $\|\psi - \psi_{\text{ref}}\|_{\infty} = \max_{K \in \mathcal{M}} |\psi_K -$   
 333  $\psi_{\text{ref}}(\mathbf{x}_K)|$ ,  $\|\psi - \psi_{\text{ref}}\|_2 = \sqrt{\sum_{K \in \mathcal{M}} m_K (\psi_K - \psi_{\text{ref}}(\mathbf{x}_K))^2}$  and  $\|\psi - \psi_{\text{ref}}\|_{H_1} =$   
 334  $\sqrt{\sum_{i=1}^d (\|\nabla_{\mathcal{D}, \mathcal{M}}^{(i)} \psi - \nabla^{(i)} \psi_{\text{ref}}\|_2)^2}$

335 Three kinds of meshes are studied for the unit cubic enclosure. The first one  
 336 is the simplest mesh consisting of regular parallelepipeds where  $\mathbf{x}_K$  is located  
 337 at the gravity center of the cell. The second one (Fig. 3a) is constructed by a  
 338 smooth mapping between the logical mesh and the spatial coordinates [1]. The  
 339 vertices  $\mathbf{x}_s(i, j, k) = \left( \mathbf{x}_s^{(l)}(i, j, k) \right)_{l=1,\dots,3}$  of the elementary distorted cubes are  
 340 defined by:  $\forall (i, j, k) \in N([1, n_1 + 1]) \times N([1, n_2 + 1]) \times N([1, n_3 + 1])$ ,

$$\begin{aligned} \mathbf{x}_s^{(1)}(i, j, k) &= 1 - \cos\left(\frac{\pi(i-1)}{2n_1}\right) \\ \mathbf{x}_s^{(2)}(i, j, k) &= \frac{(j-1)}{n_2} + 0.1 \sin\left(\frac{2\pi(j-1)}{n_2}\right) \sin\left(\frac{2\pi(k-1)}{n_3}\right) \\ \mathbf{x}_s^{(3)}(i, j, k) &= \frac{(k-1)}{n_3} + 0.1 \sin\left(\frac{2\pi(j-1)}{n_2}\right) \sin\left(\frac{2\pi(k-1)}{n_3}\right) \end{aligned}$$

341 and  $(\mathbf{x}_K)_{K \in \mathcal{M}}$  are located at the gravity centers of the cells. The third and last  
342 mesh for the cubic cavity (Fig. 3b) is based on parallelepipeds where  $\mathbf{x}_K$  is  
343 again at the gravity center of  $K$ . Then, each vertex  $\mathbf{x}_s$  of the cells is randomly  
344 displaced in the  $l = 1, \dots, 3$  space directions of a magnitude at most equal  
345 to 90% of the quantity  $\min_{K \in \mathcal{M}} |\mathbf{x}_s^{(l)} - \mathbf{x}_K^{(l)}|$ . Unlike the previous mesh, which  
346 consisted of hexahedra with plane faces, the four edges of a face are now not  
347 included into a same plane, with the exception of edges which belong to the  
348 boundaries of the cubic domain.

349 The conic-shaped cavity is bounded by the lateral surface  $((x^{(1)} - 0.5)^2 +$   
350  $(x^{(2)} - 0.5)^2 = ((6 - 5x^{(3)})/12)^2$  for  $x^{(3)} \in [0, 1]$  and by two plane discs  
351  $(x^{(1)} - 0.5)^2 + (x^{(2)} - 0.5)^2 \leq 1/4$  for  $x^{(3)} = 0$  and  $(x^{(1)} - 0.5)^2 + (x^{(2)} - 0.5)^2 \leq$   
352  $1/12^2$  for  $x^{(3)} = 1$ . The mesh of this enclosure is based on cubes which were  
353 only cut to match the lateral curved boundary. Thus, the mesh error tends to  
354 zero quadratically with respect to the mesh size. Remark that the resulting  
355 boundary cells having a volume less than  $0.1 \prod_{i=1}^d 1/n_i$  are merged into adja-  
356 cent cells in order to avoid too large differences of volumes between adjacent  
357 cells that may deteriorate the numerical accuracy. To illustrate the mesh at  
358 the boundaries (Fig. 3c), the polyhedra  $K$  have to be divided into tetrahe-  
359 drons by using the nodes  $\mathbf{x}_K$ ,  $\mathbf{x}_\sigma$ , and two successive vertices which define the  
360 edge  $\sigma \in \mathcal{E}_K$ .

361

362 We are first interested in the pure diffusive problem of a scalar variable (Eq.  
 363 (14) with  $\mathbf{u} = 0$ ) where  $\Gamma_1 = \Gamma$  and  $\Gamma_2 = \{\emptyset\}$ . It was first checked that  
 364 the errors obtained with a linear analytical solution on the different meshes  
 365 and cavities are of the order of the computer accuracy, even for the coars-  
 366 est grids. The next analytical test consists in choosing the reference solution  
 367  $T_{\text{ref}}(x^{(1)}, x^{(2)}, x^{(3)}) = \sin(\pi x^{(1)}) \cos(\pi x^{(2)}) \cos(\pi x^{(3)})$  with appropriate Dirichlet  
 368 boundary conditions (Fig. 4a-c). The orders of convergence are evaluated by  
 369 the values of the mean slopes of the curves representing the relative errors as  
 370 a function of the mesh size (Tab. 1). The accuracy of the scheme is therefore  
 371 close to 2 when considering the  $L^2$ -norm and it slightly decreases with the  
 372  $L^\infty$ -norm but always remains larger than 1.5. As expected, the order of con-  
 373 vergence for the gradients ( $H^1$ -norm) is 1.

374

375 We now examine the convergence behavior of the isothermal Navier-Stokes  
 376 equations by setting  $\mathbf{u}_{\text{ref}}(\mathbf{x}) = \nabla \wedge \sum_{i=1}^d (4x^{(1)}(x^{(1)}-1))^3 (4x^{(2)}(x^{(2)}-1))^4 (4x^{(3)}(x^{(3)}-1))^5 \mathbf{e}_i$   
 377 and  $p_{\text{ref}}(\mathbf{x}) = \cos(\pi x^{(1)}) \cos(\pi x^{(2)}) \cos(\pi x^{(3)})$  in (Eq. 3a) with  $\text{Pr} = 1$  and  
 378  $\text{Ra} = 0$  (note that the dimensionless writing of the equations is meaningless be-  
 379 cause the current reference velocity is related to the thermal diffusivity which  
 380 never appears for isothermal problems. Another velocity reference should be  
 381 used, based on the viscous diffusivity so that the Péclet number was replaced  
 382 by the Reynolds number,  $\text{Re} = 1$ ). The table (2) indicates that the convergence  
 383 rates of the velocity components are larger than 1.90 on the three finer meshes  
 384 when the relative error is based on the  $L^2$ -norm and first order accurate for the  
 385 pressure for distorted meshes. In accordance with the diffusion problem when  
 386 the  $L^\infty$ -norm is used, the orders of convergence slightly decrease for the veloc-  
 387 ity but a convergence rate larger than 1.6 is still observed. The convergence

388 rates of the gradients are better than the expected first order. Unsurprisingly,  
389 the  $L^\infty$  and  $H^1$ -norms of the pressure do not tend to zero with the mesh size  
390 because it simply appears in the momentum equation as lagrangian multiplier  
391 of the mass equation. Thus the only guaranteed convergence for the pressure  
392 is based on the  $L^2$ -norm.

#### 393 4.2 Natural convection problem

394 We consider an air filled unit-cubic enclosure with isolated walls except the  
395 two face to face vertical isothermal surfaces at  $x^{(1)} = 0$  and 1. The govern-  
396 ing fluid flow equations are solution of system (1) and (2) with  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ ,  
397  $g(\mathbf{x}) = 0$ ,  $q_b(\mathbf{x}) = 0$ ,  $T(0, x^{(2)}, x^{(3)}) = -0.5$  and  $T(1, x^{(2)}, x^{(3)}) = 0.5$ . The  
398 Prandtl and Rayleigh numbers are fixed to  $\text{Pr} = 0.71$  and  $\text{Ra} = 10^7$  and the  
399 stabilization parameter is chosen equal to  $\lambda_\sigma = 10^{-8}$  in the mass equation  
400 (see relation (8)). Because very small boundary layers take place along the  
401 walls, the mesh size is non-uniformly distributed in each direction  $l \in [1, d]$ :  
402  $\forall (i, j, k) \in N([1, n_1 + 1]) \times N([1, n_2 + 1]) \times N([1, n_3 + 1])$ ,  $\mathbf{x}_s^{(l)}(i, j, k) =$   
403  $\left(1 - \cos \frac{\pi(i\delta_{l1} + j\delta_{l2} + k\delta_{l3} - 1)}{n_l}\right) / 2$ . To also study the effect of non-cubic  
404 meshes, the coordinates of the previous defined vertices are randomly dis-  
405 placed of a magnitude at most of  $0.9 \min_{K \in \mathcal{M}} |\mathbf{x}_s^{(l)} - \mathbf{x}_K^{(l)}|$ , for  $l \in [1, d]$ . Ta-  
406 ble (5) presents the maxima of the velocity components, the average Nusselt  
407 number on the isothermal walls and their relative differences with respect to  
408 reference data [12]. For cubic meshes and the  $\|\cdot\|_\infty$ -norm, the scheme seems  
409 to be second order at least between  $n_i = 40$  and 60 where the relative gap is  
410 then divided by about 2 or more. Remark that the results are quite accurate  
411 and depart from less than 1% from the reference values when  $n_i = 60$ . The  
412 second order accuracy of numerical solutions is also obtained for randomly



413 perturbed meshes except for the third velocity component where the con-  
414 vergence order is rather difficult to define. Although  $e(u^{(2)})$  is large, it can be  
415 noticed firstly that its value is divided by 4 between  $n_i = 20$  and 40 and sec-  
416 ondly that the corresponding reference value is smaller than the other velocity  
417 components, about 10 times smaller in comparison with  $u^{(3)}$ .

## 418 5 Conclusion

419 In this paper we presented a new scheme which is well suited for the simu-  
420 lation of incompressible viscous flows on irregular and non-conforming grids.  
421 This possibility seems to open a large field of new applications (grid refinement  
422 as a function of an *a posteriori* error computation, free boundaries, ...). We  
423 emphasize that the convergence of the scheme may be proven mathematically,  
424 and that the obtained numerical results are accurate. Although we presented  
425 this scheme in the steady case, its extension to transient regimes is straight-  
426 forward. In this latter case, one should consider optimizing the linear solving  
427 step by using suitable projection algorithms.

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461 **List of Tables**

|     |   |   |    |
|-----|---|---|----|
| 462 | 1 | Slopes of the linear approximations of convergence curves for   |    |
| 463 |   | $10 \leq n_i \leq 100$ ( $50 \leq n_i \leq 100$ ).  | 28 |
| 464 | 2 | Slopes of the linear approximations of convergence curves for   |    |
| 465 |   | $10 \leq n_i \leq 60$ ( $40 \leq n_i \leq 60$ ).  | 29 |
| 466 | 3 | Maxima of the velocity components $\ u^{(i)}\ _\infty$ ( $i = 1, \dots, 3$ ),                                 |    |
| 467 |   | average Nusselt number $\bar{\text{Nu}} = \int_0^1 \int_0^1 (\nabla T \cdot \mathbf{n})_{x=0} dydz$ and their |    |
| 468 |   | relative differences to reference values [12], $e = \ \psi\ _\infty / \psi_{\text{ref}} - 1$ ,                |    |
| 469 |   | $\psi \in \{u^{(i)}, \bar{\text{Nu}}, i = 1, \dots, 3\}$ .  | 30 |

470 **List of Figures**

|     |   |   |    |
|-----|---|---|----|
| 471 | 1 | Main mesh notations   | 31 |
| 472 | 2 | Example of cluster building: a. First cluster nucleus, b. All the   |    |
| 473 |   | cluster nuclei are constructed but isolated cells (unnumbered   |    |
| 474 |   | cells) still remain, c. The isolated cells are connected to a   |    |
| 475 |   | neighbouring cluster.   | 32 |
| 476 | 3 | Example of meshes applied for convergence analysis, $n_i = 20$  |    |
| 477 |   | with $i = 1, \dots, 3$ and $x \equiv x^{(1)}$ , $y \equiv x^{(2)}$ , $z \equiv x^{(3)}$ : a. Smooth                     |    |
| 478 |   | mapping, b. Random meshes, c. Truncated cone mesh.  | 33 |
| 479 | 4 | Solution accuracy of the diffusive problem for different norms,   |    |
| 480 |   | $n_i = n_1 = n_2 = n_3$ : a. $e_2 = \ T - T_{\text{ref}}\ _2$ , b. $e_{\text{max}} = \ T - T_{\text{ref}}\ _{\infty}$ , |    |
| 481 |   | c. $e_{H_1} = \ T - T_{\text{ref}}\ _{H_1}$ .   | 34 |

|                  | Cubic enclosure |               |               | Truncated conic |
|------------------|-----------------|---------------|---------------|-----------------|
|                  | cubic meshes    | smooth meshes | random meshes | enclosure       |
| $L^2$ -norm      | 2 (2)           | 1.95 (1.97)   | 1.79 (1.93)   | 1.97 (1.99)     |
| $L^\infty$ -norm | 1.99 (2)        | 1.80 (1.68)   | 1.66 (1.88)   | 1.81 (1.55)     |
| $H^1$ -norm      | 2 (2)           | 1.49 (1.28)   | 1.11 (1.07)   | 1.46 (1.46)     |

Table 1

|            |           | Cubic meshes | Smooth meshes | Random meshes |
|------------|-----------|--------------|---------------|---------------|
| $L^2$      | $u^{(1)}$ | 2 (2)        | 1.91 (1.98)   | 1.71 (1.91)   |
|            | $u^{(2)}$ | 1.99 (2)     | 1.93 (1.98)   | 1.73 (1.90)   |
|            | $u^{(3)}$ | 2 (2)        | 1.92 (1.98)   | 1.76 (1.90)   |
|            | $p$       | 2 (2)        | 1.08 (0.76)   | 0.80 (0.91)   |
| $L^\infty$ | $u^{(1)}$ | 2 (2)        | 1.64 (2)      | 1.52 (1.80)   |
|            | $u^{(2)}$ | 1.75 (1.78)  | 1.37 (1.64)   | 1.56 (1.69)   |
|            | $u^{(3)}$ | 1.88 (1.74)  | 1.53 (1.59)   | 1.62 (1.82)   |
|            | $p$       | 1.72 (1.93)  | --            | --            |
| $H^1$      | $u^{(1)}$ | 1.98 (2)     | 1.77 (1.85)   | 1.34 (1.29)   |
|            | $u^{(2)}$ | 2 (2)        | 1.78 (1.84)   | 1.33 (1.25)   |
|            | $u^{(3)}$ | 1.95 (1.99)  | 1.74 (1.83)   | 1.31 (1.23)   |
|            | $p$       | 1.91 (1.97)  | --            | --            |

Table 2

| Mesh types | $n_i$ | $\ u^{(1)}\ _\infty$ | $e(u^{(1)})$ | $\ u^{(2)}\ _\infty$ | $e(u^{(2)})$ | $\ u^{(3)}\ _\infty$ | $e(u^{(3)})$ | Nu       | $e(\text{Nu})$ |
|------------|-------|----------------------|--------------|----------------------|--------------|----------------------|--------------|----------|----------------|
| cubic      | 20    | 333, 23              | -13%         | 70, 959              | -15%         | 767, 01              | -0.15%       | 16, 380  | 0.23%          |
|            | 30    | 371, 89              | -3.1%        | 79, 105              | -5.1%        | 761, 11              | -0.91%       | 16, 366  | 0.14%          |
|            | 40    | 377, 71              | -1.6%        | 81, 097              | -2.7%        | 761, 15              | -0.91%       | 16, 361  | 0.11%          |
|            | 50    | 380, 19              | -0.95%       | 82, 234              | -1.4%        | 767, 25              | -0.11%       | 16, 357  | 0.086%         |
|            | 60    | 380, 47              | -0.88%       | 82, 615              | -0.93%       | 767, 90              | -0.031%      | 16, 353  | 0.065%         |
| Random     | 20    | 497, 85              | 30%          | 363, 48              | 340%         | 869, 95              | 13%          | 16, 023  | -2.0%          |
|            | 30    | 419, 43              | 9.3%         | 276, 63              | 230%         | 777, 25              | 1.2%         | 16, 198  | -0.88%         |
|            | 40    | 400, 18              | 4.3%         | 151, 27              | 81%          | 779, 93              | 1.5%         | 16, 259  | -0.51%         |
| [12]       |       | 383, 8357            | 0%           | 83, 3885             | 0%           | 768, 1393            | 0%           | 16, 3427 | 0%             |

Table 3

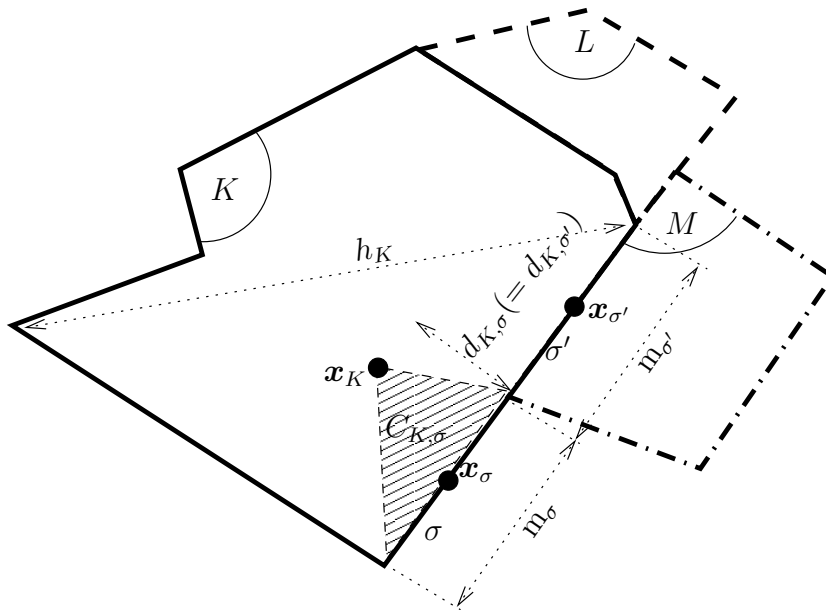
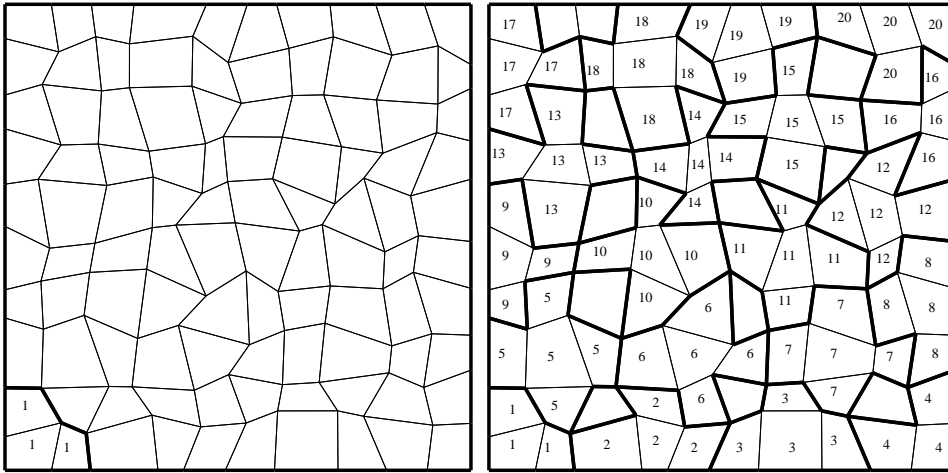
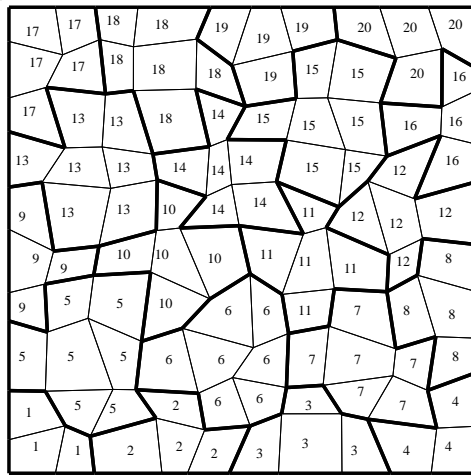


Fig. 1.



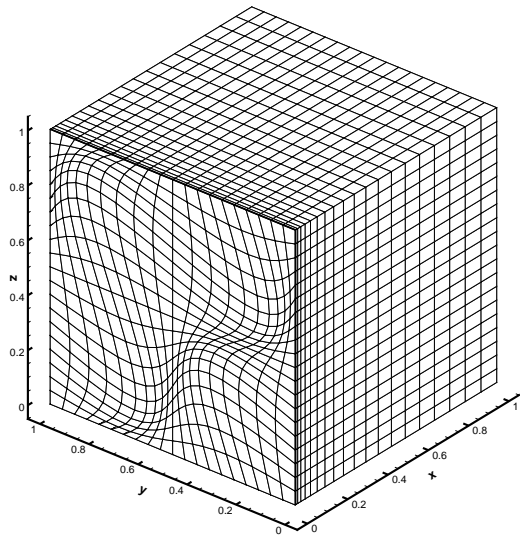


a. b.

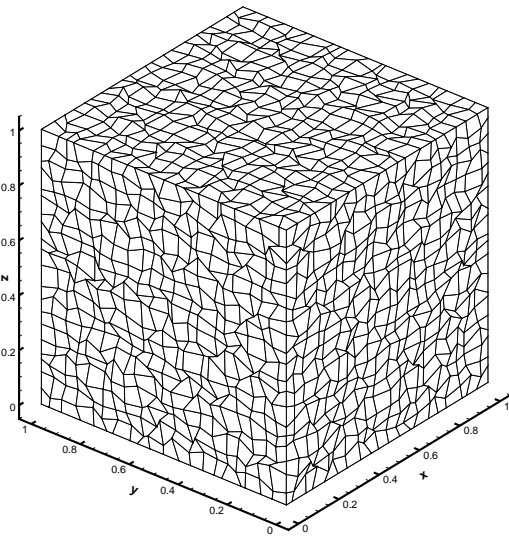


c.

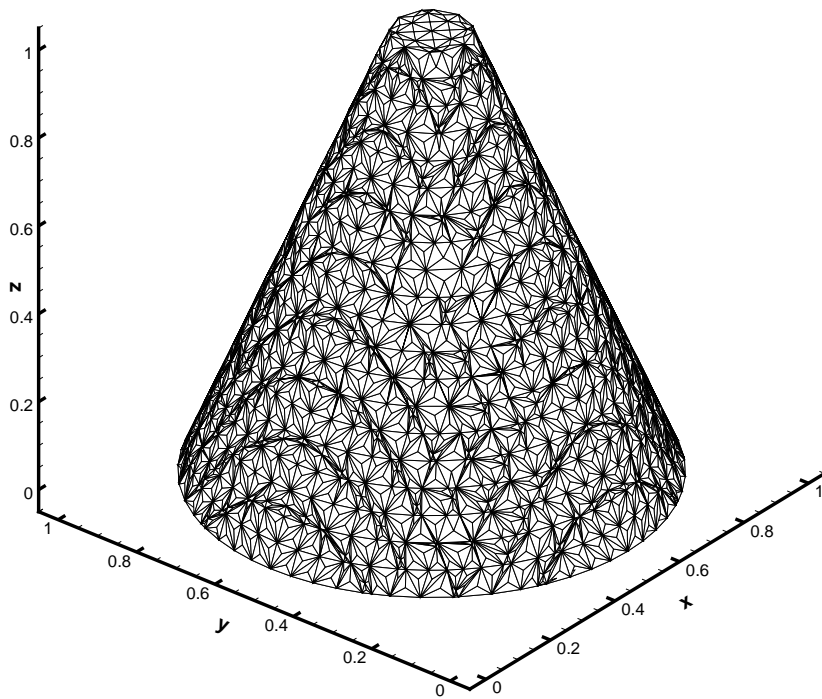
Fig. 2.



a.



b.



c.

Fig. 3.

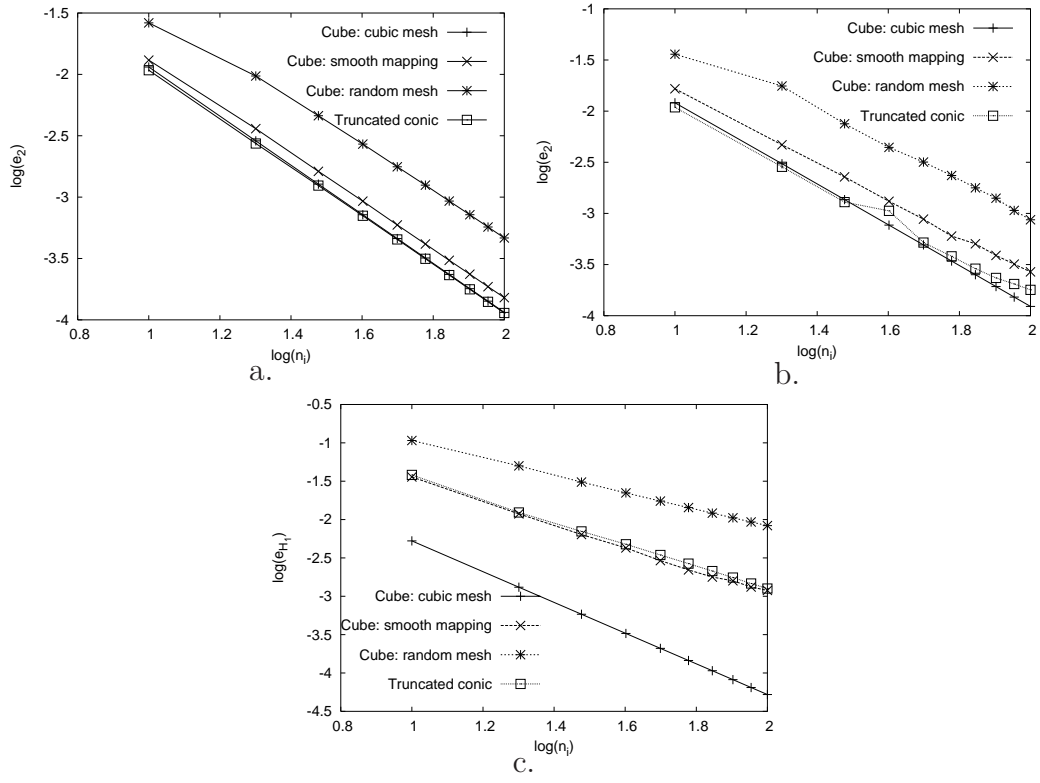


Fig. 4.