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# Semiclassical quantization of electrons in magnetic fields: the generalized Peierls substitution

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A generalized Peierls substitution which takes into account a Berry phase term must be considered for the semiclassical treatment of electrons in a magnetic field. This substitution turns out to be an essential element for the correct determination of the semiclassical equations of motion as well as for the semiclassical Bohr-Sommerfeld quantization condition for energy levels. A general expression for the cross-sectional area is derived and used as an illustration for the calculation of energy levels of Bloch and Dirac electrons.

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Semiclassical approaches are very important in many areas of physics for the study of the short wave length behavior of quantum systems, including Bloch electrons in crystals or Dirac particles in external fields. An essential ingredient of these approaches is the Bohr-Sommerfeld quantization condition, whose generalization from scalar to vector wave fields has revealed new gauge structures related to Berry's phases [1]. This paper presents a detailed study of the semiclassical quantization for a single quantum particle in a magnetic field, exemplified by electrons in a crystal and by Dirac electrons. This unified description of a particle in a magnetic field is based on a method of semiclassical diagonalization for an arbitrary matrix valued Hamiltonian developed previously [2] (for a generalization to higher order in  $\hbar$  see [3]). This method results in an effective diagonal Hamiltonian in terms of gauge-covariant but noncanonical, actually non-commutative, coordinates. It will be shown that a generalized Berry's phase dependent Peierls substitution is necessary for the establishment of the full equations of motion including Berry's phase terms. This substitution turns out to be also an essential ingredient for the Bohr-Sommerfeld quantization condition of an electron in a magnetic field. Indeed, when reformulated in terms of the generalized Peierls substitution, this condition leads to a modification of the semiclassical quantization rules as well as to a generalization of the cross-sectional area derived independently by Roth [4] and Fal'kovskii [5] in the context of Bloch electrons.

*Semiclassical diagonalization.* Let us consider a system of a quantum particle in an uniform external magnetic field  $\mathbf{B} = \nabla \times \tilde{\mathbf{A}}$  described by an arbitrary matrix valued Hamiltonian  $H(\mathbf{\Pi}, \mathbf{R})$ , where  $\mathbf{\Pi} = \mathbf{P} + e\tilde{\mathbf{A}}(\mathbf{R})$  is the covariant momentum and  $e > 0$  is the electric charge. We assume that the system can be separated into two contributions such that  $H(\mathbf{\Pi}, \mathbf{R}) = H_m(\mathbf{\Pi}) + \varphi(\mathbf{R})$  where  $H_m(\mathbf{\Pi})$  is the pure magnetic part and  $\varphi(\mathbf{R})$  is the external electric potential. In this paper we will be mainly

interested by the magnetic contribution. The exact diagonalization of this matrix valued operator through an unitary matrix  $U(\mathbf{\Pi})$  is in general not known, and in this paper we apply a recursive diagonalization procedure developed previously by two of the authors. This procedure is based on a series expansion in the Planck constant of the required diagonal Hamiltonian [2][3]. By diagonal Hamiltonian it is meant a matrix representation with block-diagonal matrix elements associated with energy band subspaces. The method is based on the knowledge of the zero-order diagonal representation  $\varepsilon = U_0 H_0 U_0^+$  where  $U_0$  is the zero-order transformation matrix,  $H_0$  the Hamiltonian  $H_m$ , in which the components  $\Pi^i$  are considered formally as classical, and therefore commuting operators. Quantum corrections are then re-introduced to yield the expression for the diagonal Hamiltonian  $H_d = U(\mathbf{\Pi}) H U^+(\mathbf{\Pi})$  which, if we limit ourselves to the semiclassical order (the semiclassical condition being that the radius of curvature of the orbit is large in comparison with wavelength), has diagonal operator elements labelled by the energy index  $n$  which reads :

$$(H_d)_{nn} = \varepsilon_n(\boldsymbol{\pi}_n) + \varphi(\mathbf{r}_n) - e\hbar\mathbf{M}\cdot\mathbf{B} \quad (1)$$

Here  $\varepsilon_n(\boldsymbol{\pi})$  is the zero-order matrix element of  $\varepsilon$  (it can itself be a matrix as for a Dirac Hamiltonian, in which case a block-diagonalization is considered) in which classical variables are now replaced by the quantum operators  $\boldsymbol{\pi}_n = \mathbf{\Pi} + \hbar A_\pi$  and  $\mathbf{r}_n = \mathbf{R} + \hbar A_r$ , where we have defined the Berry connection as being the projection on the  $n$ th energy band  $A_{r/\pi} = \mathcal{P}_n [A_{r/\pi}]$  of the matrix  $A_{r/\pi} = \pm i [U \nabla_{\pi/r} U^+]$ . It turns out that in the magnetic case the Berry connection in momentum has the following expression  $A_\pi = -e A_r \times \mathbf{B}$ . Looking at Eq. (1) we have proven that instead of the Peierls substitution [6], which amounts to replace the canonical momentum  $\mathbf{P}$  by the covariant one  $\mathbf{\Pi}$  in the energy band  $\varepsilon_n$ , one has to consider a generalization of the Peierls substitution via

the noncanonical covariant momentum

$$\boldsymbol{\pi}_n = \boldsymbol{\Pi} - e\hbar\mathbf{A} \times \mathbf{B} \quad (2)$$

(where we now use the notation  $A \equiv A_r$ ). The last term in Eq. (1) is the coupling between the uniform magnetic field and the magnetic moment defined as  $\mathbf{M}(\pi) = \frac{i}{2}\mathcal{P}_n([\varepsilon, \mathcal{A}] \times \mathcal{A}) = \frac{\hbar}{2}\mathcal{P}_n(\dot{\mathcal{A}} \times \mathcal{A})$ . Note that it is common (especially in solid state physics [7]) to write the matrix elements of the components of  $\mathbf{M}$  as  $M^i = \frac{i\hbar^2\varepsilon^{ijk}}{2} \sum_{m \neq n} \frac{(\mathcal{A}_j)_{nm}(\dot{\mathcal{A}}_k)_{mn}}{\varepsilon_n - \varepsilon_m}$  (where we used  $\dot{\mathcal{A}}_{nm} = \frac{i}{\hbar}(\varepsilon_n - \varepsilon_m)\mathcal{A}_{nm}$ ), which thus depend on the band-to-band matrix element of  $\mathcal{A}$ .

The appearance of the Berry connection allows us to define naturally non-Abelian (in general) Berry curvatures  $\Theta_{ij}(\boldsymbol{\pi}) = \partial_{r^i}A_j - \partial_{r^j}A_i + [A_i, A_j]$  where for simplicity we omit now band indices. Position operators then satisfy an unusual non-commutative algebra  $[r^i, r^j] = i\hbar^2\Theta^{ij}$ . The generalized covariant momentum satisfy an algebra  $[\pi^i, \pi^j] = -ie\hbar\varepsilon^{ijk}B_k + ie^2\hbar^2\varepsilon^{ipk}\varepsilon^{jqk}\Theta^{pq}B_kB_l$  slightly corrected with respect to the usual one  $[\Pi^i, \Pi^j] = -ie\hbar\varepsilon^{ijk}B_k$  by a term of order  $\hbar^2B^2$  which can in general be neglected. The Heisenberg relations between the coordinate and the momentum  $[r^i, \pi^j] = i\hbar\delta^{ij} + ie\hbar^2\varepsilon^{jlk}\Theta^{il}B_k$  is also slightly changed but by a term of order  $\hbar^2B$ . This contribution which is a direct consequence of introducing the generalized covariant momentum was overlooked in previous works, with the exception of Bliokh's work on the specific case of the Dirac equation [8]. It turns out that this term is essential for the determination of the genuine semiclassical equations of motion which are

$$\begin{aligned} \dot{\mathbf{r}} &= \partial\mathcal{E}/\partial\boldsymbol{\pi} - \hbar\dot{\boldsymbol{\pi}} \times \boldsymbol{\Theta}(\boldsymbol{\pi}) \\ \dot{\boldsymbol{\pi}} &= -e\mathbf{E} - e\dot{\mathbf{r}} \times \mathbf{B} \end{aligned} \quad (3)$$

where we defined  $\mathcal{E} \equiv \varepsilon - e\hbar\mathbf{M}\cdot\mathbf{B}$ . As consequence of the non-commutative algebra, the velocity equation is corrected by an anomalous velocity term  $\dot{\boldsymbol{\pi}} \times \boldsymbol{\Theta}$ , where the vector  $\boldsymbol{\Theta}$  defined as  $\Theta_i = \varepsilon_{ijk}\Theta^{jk}/2$  is the Berry curvature of an electronic state in the given  $n$ th band, associated to the electron motion in the  $n$ th energy band. These equations of motion, where first derived in solid state physics context in [9] (see also [10]) by considering the evolution of the wave packet of a Bloch electron in an electromagnetic field. In this picture, it is the mean over wave packets of the operator  $\mathbf{r}$  corresponding thus to the wave-packet center  $r_c$  and the mean of  $\boldsymbol{\pi}$  giving the mean wave vector  $\pi_c$  that are the variables in Eq (3). The operatorial approach reveals first that the operator  $\boldsymbol{\pi}$  is in fact a generalized covariant momentum operator which replaces the Peierls substitution, and second, that the operatorial equations of motion are not restricted to Bloch electrons in a magnetic field but are valid for any physical system described by an arbitrary matrix valued

Hamiltonian of the kind  $H(\boldsymbol{\Pi}, \mathbf{R}) = H_m(\boldsymbol{\Pi}) + \varphi(\mathbf{R})$ . In particular they are also valid for Dirac particles moving in an electromagnetic field.

Note that in solids, for crystals with simultaneous time-reversal and spatial inversion symmetry, the Berry curvature and the magnetic moment vanish identically throughout the Brillouin zone [9]. This is the case for most applications in solid state physics, but there are situations where these symmetries are not simultaneously present as in GaAs, where inversion symmetry is broken, or in ferromagnets, which break time reversal symmetries. In the same way, the presence of a strong magnetic field, the magnetic Bloch bands corresponding to the unperturbed system breaks the time inversion symmetries. In all these cases the dynamical and transport properties must be described by the full equations of motion given by Eq. (3). In the case of Dirac particles, both the Berry curvature and the magnetic moment are non zero and the full equations of motion have to be considered.

*Bohr-Sommerfeld quantization.* Having shown the necessity of the generalized Peierls substitution for the determination of the semiclassical equations of motion, we now investigate the relevance of this new concept at the level of the semiclassical quantization of the energy levels for an electron motion in an external uniform magnetic field only ( $\varphi = 0$ ), so that Eq. (3) becomes  $\dot{\mathbf{r}} = D(\frac{\partial\mathcal{E}}{\partial\boldsymbol{\pi}})$  and  $\dot{\boldsymbol{\pi}} = -eD(\frac{\partial\mathcal{E}}{\partial\boldsymbol{\pi}} \times \mathbf{B})$  with  $D^{-1} = 1 - e\hbar\mathbf{B}\boldsymbol{\Theta}$ . For convenience,  $\mathbf{B}$  is chosen to point in the  $z$ -direction  $\mathbf{B} = B\mathbf{k}$ . Consequently the orbits satisfies the conditions  $\mathcal{E} = \text{const}$  and  $\pi_z = \text{const}$ . The semiclassical quantization of energy levels can be done according to the Bohr-Sommerfeld quantization rule

$$\oint P_{\perp} dR_{\perp} = 2\pi\hbar(n + 1/2) \quad (4)$$

where  $P_{\perp}$  is the canonical momentum in the plane perpendicular to the axis  $\pi_z = cte$ . The integration is taken over a period of the motion and  $n$  is a large integer. Now, it turns out to be convenient to choose the gauge  $\dot{A}_y = BX$ ,  $\dot{A}_x = \dot{A}_z = 0$ . In this gauge, one has  $\pi_z = P_z = cte$ , and the usual covariant momentum  $\Pi_y = P_y + eBX$ . As  $BX = B(x - \hbar A_x)$  the generalized covariant momentum defined as  $\pi_y = \Pi_y + e\hbar BA_x$  becomes

$$\pi_y = P_y + eBx \quad (5)$$

which is formally the same relation as the one between the canonical variables, but now relating the new covariant generalized dynamical operators. This relation with the help of the equations of motion gives  $\dot{P}_y = \dot{\pi}_y - eB\dot{x} = 0$  thus  $P_y$  is a constant of motion so that  $\oint P_y dY = P_y \oint dY = 0$  and Eq.(4) becomes simply  $\oint P_x dX = 2\pi\hbar(n + 1/2)$ . Now using the definition of

the generalized momentum  $P_x = \pi_x + e\hbar A_y B$  and the differential of the canonical position  $dX = dx - \hbar dA_x = \frac{d\pi_y}{eB} - \hbar dA_x$ , the Bohr-Sommerfeld condition Eq. (4) becomes

$$\oint \pi_x d\pi_y = 2\pi\hbar eB \left( n + \frac{1}{2} - \frac{1}{2\pi} \oint A_{\perp} d\boldsymbol{\pi}_{\perp} \right) \quad (6)$$

where the integral is now taken along a closed trajectory  $\Gamma$  in the  $\pi$  space and  $\frac{1}{2\pi} \oint A_{\perp} d\boldsymbol{\pi}_{\perp} = \phi_B$  is the Berry phase for the orbit  $\Gamma$ . It is interesting to note that in terms of the usual covariant momentum (Peierls substitution) we have instead of Eq. (6) the condition  $\oint \Pi_x d\Pi_y = 2\pi\hbar eB (n + 1/2)$ . The integration in Eq. (6) defines the cross-sectional area  $S_0(\varepsilon, \pi_z)$  of the orbit  $\Gamma$  which is the intersection of the constant energy surface  $\varepsilon(\boldsymbol{\pi}) = \text{const}$  and the plane  $\pi_z = \text{const}$ . Therefore the condition Eq. (6) implicitly determines the energy levels  $\varepsilon_n(\pi_z)$ . Computing now the cross-sectional area  $S_0(\mathcal{E}, \pi_z) = S_0(\varepsilon - e\hbar M_z B, \pi_z) \approx S_0(\varepsilon, \pi_z) + dS$ , with  $dS = \oint d\kappa d\boldsymbol{\pi}_{\perp}$  the area of the annulus between the energy surface  $\varepsilon = \text{const}$  and the surface  $\varepsilon + d\varepsilon$  with  $d\varepsilon = -e\hbar M_z B$ , and where  $d\kappa = \sqrt{d\pi_x^2 + d\pi_y^2}$  is an elementary length of the  $\pi$  orbit. Then, as  $dS$  can be written  $dS = \oint \frac{d\varepsilon d\kappa}{|\partial\varepsilon/\partial\boldsymbol{\pi}_{\perp}|} = -e\hbar B \oint \frac{M_z d\kappa}{|\partial\varepsilon/\partial\boldsymbol{\pi}_{\perp}|}$  where the integral is taken over the orbit  $\Gamma$ , one has finally

$$S_0(\mathcal{E}, \pi_z) = 2\pi\hbar eB \left( n + \frac{1}{2} - \phi_B - \frac{1}{2\pi} \oint \frac{M_z(\boldsymbol{\pi}) d\kappa}{|\partial\varepsilon/\partial\boldsymbol{\pi}_{\perp}|} \right) \quad (7)$$

It is common to write  $S_0(\mathcal{E}, \pi_z) = 2\pi\hbar eB (n + \gamma)$  defining thus the coefficient  $\gamma - \frac{1}{2} = -\phi_B - \frac{1}{2\pi} \oint \frac{M_z d\kappa}{|\partial\varepsilon/\partial\boldsymbol{\pi}_{\perp}|}$ . This coefficient can also be written in a different form

$$\gamma - \frac{1}{2} = -\frac{1}{2\pi} \oint \frac{[\tilde{\mathbf{v}} \times A + \mathbf{M}]_z d\kappa}{|\partial\varepsilon/\partial\boldsymbol{\pi}_{\perp}|} \quad (8)$$

with  $\tilde{\mathbf{v}} \equiv \partial\varepsilon/\partial\boldsymbol{\pi}$ . Eq. (8) is a generalization of a previous result found by Roth [4] and Fal'kovskii [5], in the specific context of Bloch electrons in a magnetic field. The connection with Berry's phase was seen later by Mikitik and Sharlai [11]. In both [4] and [11], the term  $[\tilde{\mathbf{v}} \times A + \mathbf{M}]$  was written as  $\frac{1}{2}\mathcal{P}_n [(\frac{\boldsymbol{\Pi}}{m} + \mathbf{v}) \times \mathcal{A}]$  where  $\mathbf{v} = \frac{\boldsymbol{\Pi}}{m} + \hbar\mathcal{A}$  is the velocity operator before projection on a band, and  $\boldsymbol{\Pi} = m\mathbf{R}$ , a relation valid only for a Hamiltonian whose kinetic energy is  $\boldsymbol{\Pi}^2/2m$ . Therefore Eq. (7) is more general and has a broader field of application, as it is a general result which applies for any kind of single quantum particle system in a magnetic field, including Bloch and Dirac electrons. Importantly the derivation provided here is new, and it turns out to be the result of the generalized Peierls substitution in the Bohr-Sommerfeld condition.

*Bloch electron.* In a crystal, the Berry gauge  $A(\mathbf{k})$  is Abelian (a scalar operator), written in terms of the periodic part of the Bloch wave  $|u_n(\mathbf{k})\rangle$  as

$A(\mathbf{k}) = i \langle u_n(\mathbf{k}) | \partial_{\mathbf{k}} | u_n(\mathbf{k}) \rangle$ , where  $\mathbf{k}$  is the generalized covariant pseudo momentum ( $\mathbf{k} = \boldsymbol{\pi}/\hbar$ ). Application of Eq. (7) for electron trajectories in a crystal with time reversal and spatial inversion symmetry, where it is expected that, both  $\boldsymbol{\Theta}$  and  $\mathbf{M}$  vanish in the Brillouin zone, has been studied by Mikitik and Sharlai [11]. But these authors also pointed out the fact that the Berry's phase is non zero when the electron orbit surrounds the band-contact line of a metal, actually  $\phi_B = \pm 1/2$ . Consequently,  $\gamma = 0$  in this case, instead of the previously supposed constant value  $\gamma = 1/2$  which is commonly used in describing oscillation effect in metals. As these authors mentioned, measurements of  $\gamma$  can allow the detection of band contact lines.

As a simple application of Eq. (7) consider a crystal with time reversal and spatial inversion symmetry, and where the Fermi surface is an ellipsoid of revolution characterized by two effective masses, a transverse  $m_{\perp}$  and a longitudinal  $m_l$  one. The energy levels can easily be deduced. Indeed  $\mathcal{E} = \hbar^2 \left( \frac{\mathbf{k}_{\perp}^2}{2m_{\perp}} + \frac{K_z^2}{2m_l} \right)$  and the cross-sectional area  $S_0(\mathcal{E}, K_z)$  is a disc of radius square  $\mathbf{k}_{\perp}^2 = 2m_{\perp} \left( \mathcal{E}/\hbar^2 - \frac{K_z^2}{2m_l} \right)$  so that the energy levels are  $\mathcal{E}_n = \frac{eB\hbar}{m_{\perp}} \left( n + \frac{1}{2} \right) + \frac{\hbar^2 K_z^2}{2m_l}$  which actually coincide with the exact ones because the energy levels of an harmonic oscillator keep their form at large  $n$ .

*Dirac electron.* Let us consider the Dirac Hamiltonian  $H = \boldsymbol{\alpha} \cdot \boldsymbol{\Pi} + \beta m$  in the presence of an uniform magnetic field, with  $\boldsymbol{\alpha}$  and  $\beta$  the usual  $(4 \times 4)$  Dirac matrices. The semiclassical block-diagonalization followed by a projection on, say, the positive energy subspace, leads to the  $(2 \times 2)$  matrix valued energy operator  $\mathcal{E} = \varepsilon - e\hbar \mathbf{M} \cdot \mathbf{B}$  where  $\varepsilon = \sqrt{\boldsymbol{\pi}^2 + m^2}$  ( $c = 1$ ) and the magnetic moment is given by  $\mathbf{M} = \frac{\boldsymbol{\sigma}}{2\varepsilon} - \frac{\mathbf{L}}{\varepsilon}$ , with  $\mathbf{L} = \boldsymbol{\pi} \times A$  representing the intrinsic orbital angular momentum [2]. It turns out that for Dirac, the magnetic moment can also be expressed as  $\mathbf{M} = \varepsilon \boldsymbol{\Theta}$ , with the curvature vector given by the matrix [2]

$$\boldsymbol{\Theta}(\boldsymbol{\pi}) = -\frac{1}{2\varepsilon^3} \left[ m\boldsymbol{\sigma} + \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) \boldsymbol{\pi}}{\varepsilon + m} \right]$$

with  $\boldsymbol{\sigma}$  the Pauli matrices. Berry's connection is defined as  $A = i \langle +, \boldsymbol{\pi} | \partial_{\boldsymbol{\pi}} | +, \boldsymbol{\pi} \rangle$  where  $|+, \boldsymbol{\pi}\rangle$  is two components spinor of the positive energy subspace. Consider  $\mathbf{B}$  pointing in the  $z$ -direction so that  $\pi_z = P_z = \text{const}$ , with the goal to compute the Landau energy levels (LEL) as an application of Eq. (7). As the cross-sectional area  $S_0(\varepsilon, P_z)$  is a disc of radius square  $\boldsymbol{\pi}_{\perp}^2 = \varepsilon^2 - m^2 - P_z^2$ , the application of Eq. (7) consists in replacing  $\varepsilon$  by  $\mathcal{E}$  in  $\boldsymbol{\pi}_{\perp}^2$  so that we have  $S_0(\mathcal{E}, P_z) = \pi (\mathcal{E}_n^2 - m^2 - P_z^2)$ , which yields the semiclassical quantized LEL through the relation

$$\mathcal{E}_n^2 - m^2 - P_z^2 = 2\hbar eB \left( n + \frac{1}{2} - \phi_B - \frac{1}{2\pi} \oint \frac{M_z d\kappa}{|\partial\varepsilon/\partial\boldsymbol{\pi}_{\perp}|} \right)$$

Now from the Berry connection  $A = \frac{\boldsymbol{\pi} \times \boldsymbol{\sigma}}{2\varepsilon(\varepsilon + m)}$  we de-

duce the Berry's phase  $\phi_B = -\frac{\tau}{2} + \tau \left( \frac{m}{2\varepsilon} + \frac{P_z^2}{2\varepsilon(\varepsilon+m)} \right)$  where  $\tau = \pm 1$  are the eigenvalues of the Pauli matrix  $\sigma_z$ . Berry's phase is the sum of a topological part  $-\frac{\tau}{2}$  and a non-topological  $\tau \left( \frac{m}{2\varepsilon} + \frac{P_z^2}{2\varepsilon(\varepsilon+m)} \right)$  one. The contribution from the magnetic moment yields  $\frac{1}{2\pi} \oint \frac{M_z d\kappa}{[\partial\varepsilon/\partial\pi_\perp]} = -\tau \left( \frac{m}{2\varepsilon} + \frac{P_z^2}{2\varepsilon(\varepsilon+m)} \right)$  a term which exactly cancels the non-topological contribution of  $\phi_B$ , so that finally

$$\mathcal{E}_n = \sqrt{m^2 + 2\hbar B e \left( n + \frac{1}{2} + \frac{\tau}{2} \right) + P_z^2}$$

It turns out in this example that the semiclassical energy quantization coincides also with the exact result. It is usually expected that for a massless Dirac particle the Berry's phase takes the topological value  $\phi_B = \pm 1/2$ , as a consequence of the band degeneracy at zero momentum [13]. This is not the case here because the magnetic field lifts this degeneracy as  $P_z$  is not zero in, general. But it turns out that the magnetic moment contribution exactly compensates for the non-topological Berry's phase contribution. This cancellation can be easily understood from the expression Eq. (8) for the coefficient  $\gamma$ . Indeed from the equality  $[\tilde{\mathbf{v}} \times \mathbf{A} + \mathbf{M}]_z = \left[ \frac{\boldsymbol{\pi} \times \mathbf{A}}{\varepsilon} \right]_z + \frac{\tau}{2\varepsilon} - \frac{\mathbf{L}_z}{\varepsilon} = \frac{\tau}{2\varepsilon}$  we deduce the expected result  $\gamma = \frac{1}{2} + \frac{\tau}{2} = 0$  or  $1$ .

For a two-dimensional Dirac system it is therefore expected that the magnetic moment for massless particles exactly vanishes, and that the Berry's phase takes the topological value  $\phi_B = \pm 1/2$ . The electron motion in graphene is an interesting physical situation which illustrates this assertion. Indeed, graphene is a two-dimensional carbon crystalline honeycomb structure with inversion symmetry so that  $\mathbf{M} = 0$ . The hexagonal Brillouin zone has two distinct and degenerate Dirac points or valleys (labelled by  $\tau \pm 1$ ) where the conduction and valence bands meet and the electronic excitations behave like massless relativistic fermions, so that  $\phi_B = \pm 1/2$  and consequently  $\mathcal{E}_n = \pm \sqrt{2\hbar e B \left( n + \frac{1}{2} + \frac{\tau}{2} \right)}$  [14]. Therefore the ground state is not degenerate as there is only one possibility to realize it  $n = 0$ . This result explains the peculiar quantum Hall effect of graphene [15].

*Summary.* We have shown that a generalized Peierls substitution including a Berry phase term must be considered for a correct semiclassical treatment of electrons in a magnetic field. This substitution is essential for the determination of the full semiclassical equations of motion, as well as for the semiclassical Bohr-Sommerfeld

quantization condition for energy levels. Indeed, the substitution in the Bohr-Sommerfeld condition leads to an expression for the cross-sectional area which in some sort generalizes the formula found by Roth in the context of Bloch electrons in a crystal. Application of this formula to Dirac electrons shows the subtle cancellation mechanism between the magnetic moment and the non-topological part of the Berry's phase, which yields the Landau energy levels.

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- [1] W. G. Flynn and R. G. Littlejohn, Phys. Rev. Lett 66 (1991) 2839; R. G. Littlejohn and W. G. Flynn, Phys. Rev. A 44 (1991) 5239.
  - [2] P. Gosselin, A. Bérard and H. Mohrbach, Eur. Phys. J. B 58 (2007) 137;
  - [3] P. Gosselin, J. Hanssen and H. Mohrbach, Phys. Rev. D 77 (2008) 085008, P. Gosselin and H. Mohrbach, *Diagonal Representation for a Generic Matrix Valued Quantum Hamiltonian*, arXiv:0801.0940
  - [4] L. M. Roth, Phys. Rev. 145 (1966) 434.
  - [5] L.A. Fal'kovskii, JETP 22 (1966) 423; Usp. Fiz. Nauk 94 (1968) 1.
  - [6] R. Peierls, Z. Phys 80, (1933) 763.
  - [7] E. M. Lifshitz, L. P. Pitaevskii, Statistical Physics, vol 9, Pergamon Press, 1981.
  - [8] K. Y. Bliokh, Europhys. Lett. 72 (2005) 7.
  - [9] M. C. Chang and Q. Niu, Phys. Rev. Lett 75 (1995) 1348; Phys. Rev. B 53 (1996) 7010; G. Sundaram and Q. Niu, Phys. Rev. B 59 (1999) 14915.
  - [10] P. Gosselin, F. Ménas, A. Bérard and H. Mohrbach, Europhys.Lett 76 (2006) 651
  - [11] G. P. Mikitik and Yu. V. Sharlai, Phys. Rev. Lett. 82, 2147 (1999)
  - [12] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill, 1985.
  - [13] M. V. Berry, Proc. Roy. Soc. London A 392 (1984) 45; A. Shapere and F. Wilczek, *Geometric Phases in Physics*, World Scientific, 1989.
  - [14] P. Gosselin, A. Bérard, H. Mohrbach and S. Ghosh, *Berry Curvatures in Graphene: a new approach*, arXiv:0802.3565.
  - [15] K. S. Novoselov et al., Science 306 (2004) 666; V. P. Gusynin, S. G. Sharapov and J. P. Carbotte, Int. Jour. Mod. Phys. B 21 (2007) 4611.