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UNCONDITIONAL BASIC SEQUENCES IN SPACES OF LARGE DENSITY

PANDELIS DODOS, JORDI LOPEZ-ABAD AND STEVO TODORCEVIC

ABSTRACT. We study the problem of the existence of unconditional basic sequences in Banach spaces of high density. We show, in particular, the relative consistency with GCH of the statement that every Banach space of density \aleph_ω contains an unconditional basic sequence.

1. INTRODUCTION

In this paper we study particular instances of the general unconditional basic sequence problem asking under which conditions a given Banach space must contain an infinite unconditional basic sequence (see [LT, page 27]). We chose to study instances of the problem for Banach spaces of large densities exposing thus its connections with large-cardinal Axioms of Set Theory. The first paper on this line of research is a well-known paper of J. Ketonen [Ke] which shows that if a density of a given Banach space E is greater or equal to the first ω -Erdős cardinal, then E contains a basic sequence which is equivalent to all of its subsequences, i.e. a sub-symmetric basic sequence (therefore, E must also contain an unconditional basic sequence). Our first result shows that some sort of a large cardinal is necessary for getting sub-symmetric basic sequences.

Theorem 1. *If \mathfrak{sm} is the minimal cardinal θ with the property that every Banach space of density at least θ contains an infinite sub-symmetric basic sequence, then \mathfrak{sm} lies somewhere in between first ω -Mahlo and first ω -Erdős cardinal.*

Our construction falls short of proving the analogous statement about unconditional basic sequences, or more precisely, proving a similar lower bound for the cardinal \mathfrak{nc} , the minimal cardinal λ such that every Banach

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space of density at least λ contains an infinite unconditional basic sequence. The largest known lower bound for \mathfrak{nc} is that of S. A. Argyros and A. Toliás [AT] who showed that $\mathfrak{nc} > 2^{\aleph_0}$. So in particular the following problem is widely open.

Question 1. Is $\exp_\omega(\aleph_0)$, any of the finite-tower exponents $\exp_n(\aleph_0)$, or any of their ω -successors $\exp_n(\aleph_0)^{+\omega}$ an upper bound of \mathfrak{nc} ? In particular, does $(2^{\aleph_0})^{+\omega} \geq \mathfrak{nc}$ hold?

Our second result shows that $\exp_\omega(\aleph_0)$ is not such a bad candidate for an upper bound of \mathfrak{nc} .

Theorem 2. *The inequality $\exp_\omega(\aleph_0) \geq \mathfrak{nc}$ is a statement that is consistent relative to the consistency of infinitely many strongly compact cardinals.*

The consistency proof relies heavily on a Ramsey-theoretic property of $\exp_\omega(\aleph_0)$ established in a previous work of S. Shelah [Sh2] (see also [Mi]). One can also arrange the joint consistency of GCH and the inequality $\exp_\omega(\aleph_0) = \aleph_\omega \geq \mathfrak{nc}$. Combining this with a well known result of J. N. Hagler and W. B. Johnson [HJ], we get the following information about the famous separable quotient problem.

Corollary 3. *It is relatively consistent that every Banach space of density at least \aleph_ω has a separable quotient with an unconditional basis.*

The analysis given in this paper together with some known results suggest, in particular, that by restricting the class of Banach spaces to, say, reflexive, or more generally weakly compactly generated Banach spaces, one might get different answers about the size of the corresponding cardinal numbers $\mathfrak{nc}_{\text{rfl}}$ and $\mathfrak{nc}_{\text{wcg}}$ respectively. To describe this difference it will be convenient to introduce yet another natural cardinal characteristic $\mathfrak{nc}_{\text{seq}}$, the minimal cardinal θ such that every normalized weakly null sequence $(x_\alpha : \alpha < \theta)$ in some Banach space E has a subsequence which is unconditional. Clearly $\mathfrak{nc}_{\text{rfl}} \leq \mathfrak{nc}_{\text{wcg}}$ while by the Amir-Lindenstrauss Theorem [AL] we see that $\mathfrak{nc}_{\text{wcg}} \leq \mathfrak{nc}_{\text{seq}}$. The first known lower bound on these cardinal is due to B. Maurey and H. P. Rosenthal [MR] who showed that $\mathfrak{nc}_{\text{seq}} > \aleph_0$, though considerably deeper is the lower bound of W. T. Gowers and B. Maurey [GM] who showed that in fact $\mathfrak{nc}_{\text{rfl}} > \aleph_0$. The largest known lower bound on these cardinals is given in [ALT] who showed that $\mathfrak{nc}_{\text{rfl}} > \aleph_1$. This suggests the following question.

Question 2. Is \aleph_ω or any of the finite successors \aleph_n ($n \geq 2$) an upper bound on any of the three cardinals $\mathfrak{nc}_{\text{seq}}$, $\mathfrak{nc}_{\text{rfl}}$, or $\mathfrak{nc}_{\text{wcg}}$?

That \aleph_ω is not such a bad choice for an upper bound of $\mathfrak{nc}_{\text{seq}}$ may be seen from our third result.

Theorem 4. *The inequality $\aleph_\omega \geq \mathfrak{nc}_{\text{seq}}$ is a statement that is consistent relative to the consistency of a single measurable cardinal.*

Thus, the consistency proof uses a considerably weaker assumption from that used in Theorem 2. It relies on two Ramsey-theoretic principles, one established by P. Koepke [Ko] and the other by C. A. Di Prisco and S. Todorcevic [DT]. It also gives the joint consistency of the GCH and the cardinal inequality $\aleph_\omega \geq \mathfrak{nc}_{\text{seq}}$.

2. PRELIMINARIES

Our Banach space and set theoretic terminology and notation are standard and follow [LT] and [Ku] respectively. We will consider only real Banach spaces. We notice, however, that all our results are valid for complex Banach spaces too (with the same proofs).

Since in this note we are concerned with the problem of the existence of unconditional basic sequences in Banach spaces of high density, let us introduce the following cardinal invariants related to the version of the unconditional basic sequence problem that we study here.

Definition 5. *Let \mathfrak{nc} , $\mathfrak{nc}_{\text{wcg}}$, $\mathfrak{nc}_{\text{rfl}}$, \mathfrak{sm} and $\mathfrak{nc}_{\text{seq}}$ be defined as follows.*

- (1) \mathfrak{nc} (respectively $\mathfrak{nc}_{\text{wcg}}$, $\mathfrak{nc}_{\text{rfl}}$) is the minimal cardinal λ such that every Banach space (respectively, every weakly compactly generated, every reflexive Banach space) of density λ contains an unconditional basic sequence.
- (2) \mathfrak{sm} is the minimal cardinal λ such that every Banach space of density λ contains a sub-symmetric basic sequence.
- (3) $\mathfrak{nc}_{\text{seq}}$ is the minimal cardinal λ such that every normalized weakly null sequence $(x_\alpha : \alpha < \lambda)$ in a Banach space E has a subsequence which is unconditional.

For the convenience of the reader, we gather below some basic notions used throughout the paper.

2.1. Ideals. Let X be a non-empty set. An *ideal* \mathcal{I} on X is a collection of subsets of X satisfying the following conditions.

- (i) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
- (ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

If \mathcal{I} is an ideal on X and κ is a cardinal, then we say that \mathcal{I} is κ -*complete* if for every $\lambda < \kappa$ and every sequence $(A_\xi : \xi < \lambda)$ in \mathcal{I} we have $\bigcup_{\xi < \lambda} A_\xi \in \mathcal{I}$.

A subset A of X is said to be *positive* with respect to an ideal \mathcal{I} if $A \notin \mathcal{I}$. The set of all positive sets with respect to \mathcal{I} is denoted by \mathcal{I}^+ . If \mathcal{D} is a subset of \mathcal{I}^+ and κ is a cardinal, then we say that \mathcal{D} is κ -*closed in* \mathcal{I}^+ if for every $\lambda < \kappa$ and every decreasing sequence $(D_\xi : \xi < \lambda)$ in \mathcal{D} we have $\bigcap_{\xi < \lambda} D_\xi \in \mathcal{I}^+$. We also say that such a set \mathcal{D} is *dense* in \mathcal{I}^+ if for every $A \in \mathcal{I}^+$ there exists $D \in \mathcal{D}$ with $D \subseteq A$.

If \mathcal{F} is a filter on X , then the family $\{X \setminus A : A \in \mathcal{F}\}$ is an ideal. Having in mind this correspondence, we will continue to use the above terminology for the filter \mathcal{F} . Notice that if the given filter is actually an ultrafilter \mathcal{U} , then, setting $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{U}$, we have that $\mathcal{I}^+ = \mathcal{U}$.

2.2. Cardinals. Let θ be a cardinal.

- (a) θ is said to be *inaccessible* if it is regular and strong limit; that is, $2^\lambda < \theta$ for every $\lambda < \theta$.
- (b) θ is said to be *0-Mahlo* if it is inaccessible. In general, for an ordinal α , θ is said to be α -*Mahlo* if for every $\beta < \alpha$ and every closed and unbounded subset C of θ there is a β -Mahlo cardinal λ in C .
- (c) An α -*Erdős* cardinal, if exists usually denoted by $\kappa(\alpha)$, is the minimal cardinal λ such that $\lambda \rightarrow (\alpha)_2^{<\omega}$, i.e., the least cardinal λ with the property that for every coloring $c : [\lambda]^{<\omega} \rightarrow 2$ there is $H \subseteq \lambda$ of order-type α such that c is constant on $[H]^n$ for every $n < \omega$. A cardinal λ that is λ -Erdős, or in other words, it has that partition property $\lambda \rightarrow (\lambda)_2^{<\omega}$, is called a *Ramsey* cardinal.
- (d) θ is said to be *measurable* if there exists a κ -complete normal ultrafilter \mathcal{U} on κ . Looking at the ultrapower of the universe using \mathcal{U} one can observe that the set $\{\lambda < \theta : \lambda \text{ is inaccessible}\}$ belongs to \mathcal{U} . Similarly, one shows that sets $\{\lambda < \theta : \lambda \text{ is } \lambda\text{-Mahlo}\}$ and $\{\lambda < \theta : \lambda \text{ is Ramsey}\}$ belong to \mathcal{U} .
- (e) θ is said to be *strongly compact* if every κ -complete filter can be extended to a κ -complete ultrafilter.

2.3. The Lévy Collapse. Let λ be a regular infinite cardinal and let $\kappa > \lambda$ be an inaccessible cardinal. By $\text{Col}(\lambda, < \kappa)$ we shall denote the set of all partial mappings p satisfying the following.

- (i) $\text{dom}(p) \subseteq \lambda \times \kappa$ and $\text{range}(p) \subseteq \kappa$.
- (ii) $|p| < \lambda$.
- (iii) For every $(\alpha, \beta) \in \text{dom}(p)$ with $\beta > 0$ we have $p(\alpha, \beta) < \beta$.

We equip the set $\text{Col}(\lambda, < \kappa)$ with the partial order \leq defined by

$$p \leq q \Leftrightarrow \text{dom}(q) \subseteq \text{dom}(p) \text{ and } p \upharpoonright \text{dom}(q) = q.$$

If p and q is a pair in $\text{Col}(\lambda, < \kappa)$, then by $p \parallel q$ we denote the fact that p and q are *compatible* (i.e. there exists r in $\text{Col}(\lambda, < \kappa)$ with $r \leq p$ and $r \leq q$), while by $p \perp q$ we denote the fact that p and q are *incompatible*.

We will need the following well-known properties of the Lévy collapse (see, for instance, [Ka]). In what follows, G will be a $\text{Col}(\lambda, < \kappa)$ -generic filter.

- (a) The generic filter G is λ -complete (this is a consequence of the fact that the forcing $\text{Col}(\lambda, < \kappa)$ is λ -closed).
- (b) $\text{Col}(\lambda, < \kappa)$ has the κ -cc (this follows from the fact that the cardinal κ is inaccessible).
- (c) In $V[G]$, we have $\kappa = \lambda^+$.
- (d) In $V[G]$, the sets ${}^\kappa 2$ and ${}^\kappa 2 \cap V$ are equipotent.

Finally, let us introduce some pieces of notation (actually, these pieces of notation will be used only in §5). For every $p \in \text{Col}(\lambda, < \kappa)$ and every $\alpha < \kappa$ by $p \upharpoonright \alpha$ we shall denote the restriction of the partial map p to $\text{dom}(p) \cap (\lambda \times \alpha)$. Moreover, for every $p \in \text{Col}(\lambda, < \kappa)$ we let $(\text{dom}(p))_1 = \{\alpha < \kappa : \exists \xi < \lambda \text{ with } (\xi, \alpha) \in \text{dom}(p)\}$.

3. A POLARIZED PARTITION RELATION

The purpose of this section is to analyze the following partition property, a variation of a partition property originally appearing in the problem lists of P. Erdős and A. Hajnal [EH1], [EH2] (see also [Sh2]).

Definition 6. Let κ be a cardinal and $d \in \omega$ with $d \geq 1$. By $\text{Pl}_d(\kappa)$ we shall denote the combinatorial principle asserting that for every coloring $c : [[\kappa]^d]^{<\omega} \rightarrow \omega$ there exists a sequence (\mathbf{x}_n) of infinite disjoint subsets of κ such that for every $m \in \omega$ the restriction $c \upharpoonright \prod_{n=0}^m [\mathbf{x}_n]^d$ is constant.

Clearly $\text{Pl}_d(\kappa)$ implies $\text{Pl}_{d'}(\kappa)$ for any cardinal κ and any pair $d, d' \in \omega$ with $d \geq d' \geq 1$. From known results one can easily deduce that the principle $\text{Pl}_d(\exp_{d-1}(\aleph_0)^{+n})$ is false for every $n \in \omega$ and every integer $d \geq 1$ (see, for instance, [EHMR], [CDPM] and [DT]). Thus, the minimal cardinal κ for which $\text{Pl}_d(\kappa)$ could possibly be true is $\exp_{d-1}(\aleph_0)^{+\omega}$. Indeed, C. A. Di Prisco and S. Todorcevic [DT] have established the consistency of $\text{Pl}_1(\aleph_\omega)$ relative to the consistency of a single measurable cardinal, an assumption that happens to be optimal. On the other hand, S. Shelah [Sh2] was able to establish that GCH and principles $\text{Pl}_d(\aleph_\omega)$ ($d \geq 1$) are jointly consistent, relative to the consistency of GCH and the existence of an infinite sequence of strongly compact cardinals.

Our aim in this section is to present a consistency proof of $\text{Pl}_2(\exp_\omega(\aleph_0))$. We shall treat the colorings in Definition 6 using an iteration of the following lemma whose proof (given in §5), while it relies heavily on an idea of S. Shelah [Sh2], it exposes certain features (the ideal \mathcal{I} and the sufficiently complete dense subset \mathcal{D} of its quotient), not explicitly found in [Sh2], that are likely to find application beyond the scope of our present paper.

Lemma 7. *Suppose that κ is a strongly compact cardinal and that $\lambda < \kappa$ is an infinite regular cardinal. Let G be a $\text{Col}(\lambda, < \kappa)$ -generic filter over V . Then, in $V[G]$ there exists an ideal \mathcal{I} on $[(2^\kappa)^+]^\omega$ and a subset \mathcal{D} of \mathcal{I}^+ such that the following are satisfied.*

- (1) \mathcal{I} is κ -complete.
- (2) \mathcal{D} is dense in \mathcal{I}^+ and is λ -closed in \mathcal{I}^+ .
- (3) For every $\mu < \kappa$, every coloring $c : [(2^\kappa)^+]^2 \rightarrow \mu$ and every $A \in \mathcal{I}^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \upharpoonright [\mathbf{x}]^2$ is constantly equal to ξ .

It will be convenient to introduce the following sequence (Θ_n) of cardinals defined recursively by the rule

$$\Theta_0 = \aleph_0 \text{ and } \Theta_{n+1} = (2^{(2^{\Theta_n})^+})^{++}.$$

Notice that (Θ_n) is strictly increasing and $\exp_n(\aleph_0) < \Theta_n \leq \exp_{5n}(\aleph_0)$ for every $n \in \omega$ with $n \geq 1$. Hence, $\sup\{\Theta_n : n \in \omega\} = \exp_\omega(\aleph_0)$. In particular, if GCH holds, then $\Theta_n = \aleph_{5n}$ for every $n \in \omega$.

Corollary 8. *Suppose that (κ_n) is a strictly increasing sequence of strongly compact cardinals with $\kappa_0 = \aleph_0$. For every $n \in \omega$ set $\lambda_n = (2^{(2^{\kappa_n})^+})^+$. Let*

$$\mathbb{P} = \bigotimes_{n \in \omega} \text{Col}(\lambda_n, < \kappa_{n+1})$$

be the iteration of the sequence of Lévy collapses. Let G be a \mathbb{P} -generic filter over V . Then, in $V[G]$, for every $n \in \omega$ we have $\kappa_n = \Theta_n$ and there exist an ideal \mathcal{I}_n on $[(2^{\Theta_{n+1}})^+]^\omega$ and a subset \mathcal{D}_n of \mathcal{I}_n^+ such that the following are satisfied.

- (P1) \mathcal{I}_n is Θ_{n+1} -complete.
- (P2) \mathcal{D}_n is $(< \Theta_{n+1})$ -closed in \mathcal{I}_n^+ ; that is, \mathcal{D}_n is μ -closed in \mathcal{I}_n^+ for every $\mu < \Theta_{n+1}$.
- (P3) For every $\mu < \Theta_{n+1}$, every coloring $c : [(2^{\Theta_{n+1}})^+]^2 \rightarrow \mu$ and every $A \in \mathcal{I}_n^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_n$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \upharpoonright [\mathbf{x}]^2$ is constantly equal to ξ .

Moreover, if GCH holds in V , then GCH also holds in $V[G]$.

Proof. Fix $m < \omega$. Notice, first, that the small forcing extension $V[G_m]$, where G_m is the restriction of G to the finite iteration

$$\mathbb{P}_m = \bigotimes_{n < m} \text{Col}(\lambda_n, < \kappa_{n+1}),$$

preserves the strong compactness of κ_{m+1} . This fact follows immediately from the elementary-embedding characterization of strong compactness (see [Ka, Theorem 22.17]). Working in $V[G_m]$ and applying Lemma 7, we see that the intermediate forcing extension $V[G_{m+1}]$ has the ideal \mathcal{I}_m whose quotient has properties (P1), (P2) and (P3). Working still in the intermediate forcing extension $V[G_{m+1}]$, we see that the rest of the forcing

$$\mathbb{P}^{m+1} = \bigotimes_{m < n < \omega} \text{Col}(\lambda_n, < \kappa_{n+1}),$$

is λ_{m+1} -closed, and so, in particular, it adds no new subsets to the index set on which the ideal \mathcal{I}_m lives, preserving, thus, properties (P1), (P2) and (P3) of its quotient. \square

As indicated above, for our purposes here we shall only need the following result of Shelah.

Corollary 9 ([Sh2]). *Suppose that in our universe V there exists a strictly increasing sequence (κ_n) of strongly compact cardinals with $\kappa_0 = \aleph_0$. Then,*

there is a forcing extension of V in which the principle $\text{Pl}_2(\exp_\omega(\aleph_0))$ holds. Moreover, if GCH holds in V , then GCH also holds in the extension.

Proof. We shall deduce $\text{Pl}_2(\exp_\omega(\aleph_0))$ from the conclusion of Corollary 8. Indeed, for every $n \in \omega$ let \mathcal{I}_n and \mathcal{D}_n be given by Corollary 8. We need the following.

Claim 10. *Let $n \in \omega$. Let also $c : \prod_{i=0}^n [(2^{\Theta_{i+1}})^+]^2 \rightarrow \omega$ be a coloring and $(D_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i$. Then, there exist $(E_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i \upharpoonright D_i$ and a color $n_0 \in \omega$ such that for every $(\mathbf{x}_i)_{i=0}^n \in \prod_{i=0}^n E_i$ the restriction $c \upharpoonright \prod_{i=0}^n [\mathbf{x}_i]^2$ is constantly equal to n_0 .*

Proof of Claim 10. By induction on n . The case $n = 0$ is an immediate consequence of property (P3) in Corollary 8. Now let $n \in \omega$ with $n \geq 1$ and assume that the result has been proved for all $k \in \omega$ with $k < n$. Fix a coloring $c : \prod_{i=0}^n [(2^{\Theta_{i+1}})^+]^2 \rightarrow \omega$. Fix also $(D_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i$ and let

$$\mathcal{F} = \{f : \prod_{i=0}^{n-1} [(2^{\Theta_{i+1}})^+]^2 \rightarrow \omega : f \text{ is a coloring}\}.$$

Notice that $|\mathcal{F}| = 2^{(2^{\Theta_n})^+}$, and so, $|\mathcal{F}| < \Theta_{n+1}$. We define a coloring $d : [(2^{\Theta_{n+1}})^+]^2 \rightarrow \mathcal{F}$ by the rule $d(\{\alpha, \beta\})(\bar{s}) = c(\bar{s} \frown \{\alpha, \beta\})$ for every $\bar{s} \in \prod_{i=0}^{n-1} [(2^{\Theta_{i+1}})^+]^2$. By (P3) in Corollary 8, there exist $E_n \in \mathcal{D}_n \upharpoonright D_n$ and $f_0 \in \mathcal{F}$ such that for every $\mathbf{x} \in E_n$ the restriction $d \upharpoonright [\mathbf{x}]^2$ is constantly equal to f_0 . The result now follows by applying our inductive hypothesis to the coloring f_0 . \square

By Claim 10 and the fact that every \mathcal{D}_n is σ -closed (property (P2) in Corollary 8), the proof of Corollary 9 is completed. \square

Clearly, in the forcing extension obtained above the combinatorial principle $\text{Pl}_1(\exp_\omega(\aleph_0))$ holds as well. As already indicated above, one can obtain the consistency of $\text{Pl}_1(\exp_\omega(\aleph_0))$ using a considerably weaker large-cardinal assumption from the one used for $\text{Pl}_2(\exp_\omega(\aleph_0))$, an assumption which in fact happens to be optimal. More precisely, we have the following.

Theorem 11 ([DT]). *Assume the existence of a measurable cardinal. Then, there is a forcing extension in which GCH and $\text{Pl}_1(\aleph_\omega)$ hold.*

It should be clear from the proof of Lemma 7 that we also have its higher-dimensional variation.

Lemma 12. *Suppose that d is a positive integer, that κ is a strongly compact cardinal and that $\lambda < \kappa$ is an infinite regular cardinal. Let G be a $\text{Col}(\lambda, < \kappa)$ -generic filter over V . Then, in $V[G]$ there exists an ideal \mathcal{I} on $[(\exp_d(\kappa))^+]^\omega$ and a subset \mathcal{D} of \mathcal{I}^+ such that the following are satisfied.*

- (1) \mathcal{I} is κ -complete.
- (2) \mathcal{D} is dense in \mathcal{I}^+ and is λ -closed in \mathcal{I}^+ .
- (3) For every $\mu < \kappa$, every coloring $c : [(\exp_d(\kappa))^+]^{d+1} \rightarrow \mu$ and every $A \in \mathcal{I}^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \upharpoonright [\mathbf{x}]^{d+1}$ is constantly equal to ξ .

This leads to the following multi-dimensional version of Corollary 8.

Corollary 13. *Suppose that (κ_n) is a strictly increasing sequence of strongly compact cardinals with $\kappa_0 = \aleph_0$. For every $n \in \omega$ we can choose cardinals $\lambda_n, \theta_n \in [\kappa_n, \exp_\omega(\kappa_n))$ in such a way that, if we let*

$$\mathbb{P} = \bigotimes_{n \in \omega} \text{Col}(\lambda_n, < \kappa_{n+1})$$

be the iteration of the sequence of Lévy collapses and if we choose G to be a \mathbb{P} -generic filter over V , then, in $V[G]$, we have

$$\sup_{n < \omega} \kappa_n = \sup_{n < \omega} \lambda_n = \sup_{n < \omega} \theta_n = \exp_\omega(\aleph_0)$$

and for every $n \in \omega$ there exist an ideal \mathcal{I}_n on $[\theta_{n+1}]^\omega$ and a subset \mathcal{D}_n of \mathcal{I}_n^+ such that the following are satisfied.

- (P1) \mathcal{I}_n is κ_{n+1} -complete.
- (P2) \mathcal{D}_n is $(< \lambda_n)$ -closed in \mathcal{I}_n^+ ; that is, \mathcal{D}_n is μ -closed in \mathcal{I}_n^+ for every $\mu < \lambda_n$.
- (P3) For every $\mu < \kappa_{n+1}$, every coloring $c : [\theta_{n+1}]^{n+1} \rightarrow \mu$ and every $A \in \mathcal{I}_n^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_n$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \upharpoonright [\mathbf{x}]^{n+1}$ is constantly equal to ξ .

It should be also clear that using Corollary 13 instead Corollary 8 in the proof of Corollary 9 we have the following higher dimensional version of this result.

Corollary 14 ([Sh2]). *Suppose that in our universe V there exists a strictly increasing sequence (κ_n) of strongly compact cardinals with $\kappa_0 = \aleph_0$. Then, there is a forcing extension of V in which the principle $\text{Pl}_d(\exp_\omega(\aleph_0))$ holds*

for every integer $d \geq 1$. Moreover, if GCH holds in V , then GCH also holds in the extension.

4. BANACH SPACE IMPLICATIONS

Let us recall that a sequence (x_n) in a Banach space E is said to be C -unconditional, where $C \geq 1$, if for every pair F and G of non-empty finite subsets of ω with $F \subseteq G$ and every choice $(a_n)_{n \in G}$ of scalars we have

$$\left\| \sum_{n \in F} a_n x_n \right\| \leq C \cdot \left\| \sum_{n \in G} a_n x_n \right\|.$$

This section is devoted to the proof of the following result.

Theorem 15. *Let κ be a cardinal and assume that property $\text{Pl}_2(\kappa)$ holds (see Definition 6). Then every Banach space E not containing ℓ_1 and of density κ contains an 1-unconditional basic sequence.*

In particular, if E is any Banach space of density κ , then for every $\varepsilon > 0$ the space E contains an $(1 + \varepsilon)$ -unconditional basic sequence.

Combining Corollary 9 with Theorem 15, we get the following corollaries.

Corollary 16. *It is consistent relative the existence of an infinite sequence of strongly compact cardinals that for every $\varepsilon > 0$ and every Banach space E of density at least $\exp_\omega(\aleph_0)$, the space E contains an $(1 + \varepsilon)$ -unconditional basic sequence. Moreover, this is compatible with GCH.*

Proof. Follows immediately by Corollary 9 and Theorem 15. \square

Corollary 17. *It is consistent relative to the existence of an infinite sequence of strongly compact cardinals that every Banach space of density at least $\exp_\omega(\aleph_0)$ has a separable quotient with an unconditional basis. Moreover, this is compatible with GCH.*

Proof. A well-known consequence of a result due to J. N. Hagler and W. B. Johnson [HJ] asserts that if E is a Banach space such that E^* has an unconditional basic sequence, then E has a separable quotient with an unconditional basis (see also [ADK, Proposition 16]). Noticing that the density of the dual E^* of a Banach space E is at least as big as the density of E , the result follows by Corollary 16. \square

For the proof of Theorem 15 we need the following lemma, which is essentially a multi-dimensional version of Odell's Schreier unconditionality Theorem [O2].

Lemma 18. *Let E be a Banach space, $m \in \omega$ with $m \geq 1$ and $\varepsilon > 0$. For every $i \in \{0, \dots, m\}$ let (x_n^i) be a normalized weakly null sequence in the space E . Then, there exists an infinite subset L of ω such that for every $\{n_0 < \dots < n_m\} \subseteq L$ the sequence $(x_{n_i}^i)_{i=0}^m$ is $(1 + \varepsilon)$ -unconditional.*

Proof. The first step towards the proof of the lemma is included in the following claim. It shows that, by passing to an infinite subset of ω , we may assume that for every $\{n_0 < \dots < n_m\} \in [\mathbb{N}]^{m+1}$ the finite sequence $(x_{n_i}^i)_{i=0}^m$ is a particularly well behaved Schauder basic sequence.

Claim 19. *For every $\varepsilon > 0$ there exists an infinite subset M of ω such that for every $\{n_0 < \dots < n_m\} \subseteq M$ the sequence $(x_{n_i}^i)_{i=0}^m$ is an $(1 + \varepsilon)$ -Schauder basic sequence.*

Proof of Claim 19. We define a coloring $\mathcal{B} : [\mathbb{N}]^{m+1} \rightarrow 2$ as follows. Let $s = \{n_0 < \dots < n_m\} \in [\mathbb{N}]^{m+1}$ arbitrary. If $(x_{n_i}^i)_{i=0}^m$ is an $(1 + \varepsilon)$ -Schauder basic sequence, then we set $\mathcal{B}(s) = 0$; otherwise we set $\mathcal{B}(s) = 1$. By Ramsey's Theorem, there exist an infinite subset M of ω and $c \in \{0, 1\}$ such that $\mathcal{B} \upharpoonright [M]^{m+1}$ is constantly equal to c . Using Mazur's classical procedure for selecting Schauder basic sequences (see, for instance, [LT, Lemma 1.a.6]), we find $t = \{k_0 < \dots < k_m\} \in [M]^{m+1}$ such that the sequence $(x_{k_i}^i)_{i=0}^m$ is basic with basis constant $(1 + \varepsilon)$. Therefore, $\mathcal{B}(t) = 0$, and by homogeneity, $\mathcal{B} \upharpoonright [M]^{m+1} = 0$. The claim is proved. \square

Applying Claim 19 for $\varepsilon = 1$, we get an infinite subset M of ω as described above. Observe that for every $\{n_0 < \dots < n_m\} \in [M]^{m+1}$ and every choice $(a_i)_{i=0}^m$ of scalars we have

$$(1) \quad \left\| \sum_{i=0}^m a_i x_{n_i}^i \right\| \geq \frac{1}{4} \max\{|a_i| : i = 0, \dots, m\}.$$

The desired subset L of ω will be an infinite subset of M obtained after another application of Ramsey's Theorem. Specifically, consider the coloring $\mathcal{U} : [M]^{m+1} \rightarrow 2$ defined as follows. Let $s = \{n_0 < \dots < n_m\} \in [M]^{m+1}$ and assume that the sequence $(x_{n_i}^i)_{i=0}^m$ is $(1 + \varepsilon)$ -unconditional. In such a case, we set $\mathcal{U}(s) = 0$; otherwise we set $\mathcal{U}(s) = 1$. Let L be an infinite subset of M be such \mathcal{U} is constant on $[L]^{m+1}$. It is enough to find some $s \in [L]^{m+1}$ such that $\mathcal{U}(s) = 0$.

To this end, fix $\delta > 0$ such that $(1 + \delta) \cdot (1 - \delta)^{-1} \leq (1 + \varepsilon)$. Notice that there exists a finite family \mathcal{D} of normalized Schauder basic sequences of length $m + 1$ such that any normalized Schauder basic sequence $(y_i)_{i=0}^m$, in

some Banach space Y , is $\sqrt{1+\delta}$ -equivalent to some sequence in the family \mathcal{D} . Hence, by a further application of Ramsey's Theorem and by passing to an infinite subset of L if necessary, we may assume that

- (*) for every $\{n_0 < \dots < n_m\}, \{k_0 < \dots < k_m\} \in [L]^{m+1}$ the sequences $(x_{n_i}^i)_{i=0}^m$ and $(x_{k_i}^i)_{i=0}^m$ are $(1+\delta)$ -equivalent.

Now, for every $i \in \{0, \dots, m\}$ and every $\rho > 0$ let

$$\mathcal{K}_i(\rho) = \{\{n \in \omega : |x^*(x_n^i)| \geq \rho\} : x^* \in B_{E^*}\}.$$

Every sequence (x_n^i) is weakly null, and so, each $\mathcal{K}_i(\rho)$ is a pre-compact¹ family of finite subsets of ω . Hence, we may select a sequence $(F_i)_{i=0}^m$ of finite subsets of L such that

- (a) $\max(F_i) < \min(F_{i+1})$ for every $i \in \{0, \dots, m-1\}$, and
(b) $F_i \notin \mathcal{K}_i(\delta \cdot 8^{-1} \cdot (m+1)^{-1})$ for every $i \in \{0, \dots, m\}$.

We set $n_i = \min(F_i)$ for all $i \in \{0, \dots, m\}$. Property (a) above implies that $n_0 < \dots < n_m$. We claim that the sequence $(x_{n_i}^i)_{i=0}^m$ is $(1+\varepsilon)$ -unconditional. Indeed, let $F \subseteq \{0, \dots, m\}$ and $(a_i)_{i=0}^m$ be a choice of scalars. We want to prove that

$$\left\| \sum_{i \in F} a_i x_{n_i}^i \right\| \leq (1+\varepsilon) \left\| \sum_{i=0}^m a_i x_{n_i}^i \right\|.$$

Clearly we may assume that $\left\| \sum_{i \in F} a_i x_{n_i}^i \right\| = 1$. If $\left\| \sum_{i \notin F} a_i x_{n_i}^i \right\| \geq 2$, then

$$\left\| \sum_{i=0}^m a_i x_{n_i}^i \right\| \geq \left\| \sum_{i \notin F} a_i x_{n_i}^i \right\| - \left\| \sum_{i \in F} a_i x_{n_i}^i \right\| \geq 1 = \left\| \sum_{i \in F} a_i x_{n_i}^i \right\|.$$

So, suppose that $\left\| \sum_{i \notin F} a_i x_{n_i}^i \right\| \leq 2$. By (1), we see that

$$(2) \quad \max\{|a_i| : i \notin F\} \leq 8.$$

We select $x_0^* \in S_{E^*}$ such that $x_0^*(\sum_{i \in F} a_i x_{n_i}^i) = \left\| \sum_{i \in F} a_i x_{n_i}^i \right\|$. We define a sequence $(k_i)_{i=0}^m$ in L as follows. If $i \notin F$, then let k_i be any member of F_i satisfying $|x_0^*(x_{k_i}^i)| < \delta \cdot 8^{-1} \cdot (m+1)^{-1}$ (such a selection is possible by (b) above); if $i \in F$, then we set $k_i = n_i$. By (a), we have $k_0 < \dots < k_m$. Moreover,

$$\begin{aligned} \left\| \sum_{i=0}^m a_i x_{k_i}^i \right\| &= x_0^*\left(\sum_{i=0}^m a_i x_{k_i}^i\right) = x_0^*\left(\sum_{i \in F} a_i x_{k_i}^i\right) + x_0^*\left(\sum_{i \notin F} a_i x_{k_i}^i\right) \\ &\geq x_0^*\left(\sum_{i \in F} a_i x_{k_i}^i\right) - \sum_{i \notin F} |a_i| \cdot |x_0^*(x_{k_i}^i)| \geq 1 - \delta. \end{aligned}$$

¹Recall that a family \mathcal{F} of finite subsets of ω is said to be pre-compact if, identifying \mathcal{F} with a subset of the Cantor set 2^ω , the closure $\overline{\mathcal{F}}$ of \mathcal{F} in 2^ω consists only of finite sets.

Invoking (*), we conclude that

$$\left\| \sum_{i=0}^m a_i x_{n_i}^i \right\| \geq \frac{1}{1+\delta} \left\| \sum_{i=0}^m a_i x_{k_i}^i \right\| \geq \frac{1-\delta}{1+\delta} \geq \frac{1}{1+\varepsilon} \left\| \sum_{i \in F} a_i x_{n_i}^i \right\|.$$

The proof is completed. \square

We are ready to proceed to the proof of Theorem 15.

Proof of Theorem 15. Let κ be a cardinal such that $\text{Pl}_2(\kappa)$ holds. By a classical result of R. C. James (see [LT, Proposition 2.e.3]), it is enough to show that if E is a Banach space of density κ not containing an isomorphic copy of ℓ_1 , then E has an 1-unconditional basic sequence. So, let E be one. By Rosenthal's Dichotomy [Ro] and our assumptions on the space E , we see that every bounded sequence in E has a weakly Cauchy subsequence. Let $(x_\alpha : \alpha < \kappa)$ be a normalized sequence such that $\|x_\alpha - x_\beta\| \geq 1$ for every $\alpha < \beta < \kappa$. We define a coloring $c_{\text{un}} : [[\kappa]^2]^{<\omega} \rightarrow \omega$ as follows. Let $s = (\{\alpha_0 < \beta_0\}, \dots, \{\alpha_m < \beta_m\}) \in [[\kappa]^2]^{<\omega}$ arbitrary. Assume that there exists $l \in \omega$ with $l > 0$ and such that the sequence $(x_{\beta_i} - x_{\alpha_i})_{i=0}^m$ is *not* $(1 + 1/l)$ -unconditional. In such a case, setting l_s to be the least $l \in \omega$ with the above property, we define $c_{\text{un}}(s) = l_s$. If such an l does not exist, then we set $c_{\text{un}}(s) = 0$. By $\text{Pl}_2(\kappa)$, there exist a sequence (\mathbf{x}_i) of infinite subsets of κ and a sequence (l_m) in ω such that for every $m \in \omega$ the restriction $c_{\text{un}} \upharpoonright \prod_{i=0}^m [\mathbf{x}_i]^2$ of the coloring c_{un} on the product $\prod_{i=0}^m [\mathbf{x}_i]^2$ is constant with value l_m .

Claim 20. *For every $m \in \omega$ we have $l_m = 0$.*

Grating the claim, the proof of the theorem is completed. Indeed, observe that for every infinite sequence of pairs $(\{\alpha_i < \beta_i\}) \in \prod_{i \in \omega} [\mathbf{x}_i]^2$ the sequence $(x_{\beta_i} - x_{\alpha_i})$ is a semi-normalized 1-unconditional basic sequence in the Banach space E .

It only remains to prove Claim 20. To this end we argue by contradiction. So, assume that there exists $m \in \omega$ such that $l_m > 0$. Our definition of the coloring c_{un} implies that $m \geq 1$. For every $i \in \{0, \dots, m\}$ we may select an infinite subset $\{\alpha_0^i < \alpha_1^i < \dots\}$ of \mathbf{x}_i such that the sequence (x_{α_i}) is weakly Cauchy. We set

$$y_n^i = \frac{x_{\alpha_{2n}^i} - x_{\alpha_{2n+1}^i}}{\|x_{\alpha_{2n}^i} - x_{\alpha_{2n+1}^i}\|}$$

for every $i \in \{0, \dots, m\}$ and every $n \in \omega$. Then each (y_n^i) is a normalized weakly null sequence in E . Moreover, for every $\{n_0 < \dots < n_m\} \subseteq \mathbb{N}^{m+1}$

the sequence $(y_{n_i}^i)_{i=0}^m$ is not $(1 + 1/l_m)$ -unconditional. This clearly contradicts Lemma 18. The proof is completed. \square

4.1. Sub-symmetric basic sequences. A semi-normalized basic sequence (x_n) in a Banach space E is said to be *sub-symmetric* if every subsequence of (x_n) is equivalent to (x_n) itself².

In light of Corollary 16, it is natural to ask whether we can show, consistently, that there exists a small cardinal θ such that every Banach space E of density θ contains a sub-symmetric basic sequence (this would improve upon Corollary 16).

Recall, first, that Ketonen [Ke] has proved that if λ is an ω -Erdős cardinal, then any Banach space of density λ has a normalized sub-symmetric basic sequence. However, there do exist non-separable Banach spaces not containing a sub-symmetric basic sequence. The first such example is due to E. Odell [O1]. Odell's space is the dual of a separable Banach space, and so, it has density 2^{\aleph_0} (and, of course, is not reflexive). In [ALT], a reflexive Banach space \mathfrak{X}_{ω_1} was constructed which has density \aleph_1 and does not contain an unconditional basic sequence, and a fortiori neither a sub-symmetric basic sequence.

Our aim in this subsection is to show that if κ is any cardinal below the first ω -Mahlo cardinal (see §2.2), then we can construct, in ZFC, a Banach space E_κ of density κ not containing a sub-symmetric basic sequence. More precisely, we have the following.

Theorem 21. *Let θ be the first ω -Mahlo cardinal and κ be a cardinal with $\kappa < \theta$. Then there exists a reflexive Banach space E_κ of density κ not containing a sub-symmetric basic sequence. In particular, $\mathfrak{sm} \geq \theta$.*

For the proof of Theorem 21 we need a well-known construction in Banach Space Theory (see [Ts] or [LT]). Let κ be any cardinal and let \mathcal{F} be a compact and hereditary family on κ containing the singletons. This means that \mathcal{F} is a family of finite subsets of κ satisfying the following conditions.

- (i) If L is an infinite subset of κ , then there exists a non-empty finite subset G of L such that $G \notin \mathcal{F}$.

²We notice that our definition of a sub-symmetric basic sequence defers slightly from the one adopted in [LT], where a sub-symmetric sequence (x_n) is additionally assumed to be unconditional. For our purposes, however, this extra assumption is redundant, as it is easily seen, using Rosenthal's Dichotomy, that every normalized sub-symmetric sequence has a normalized block which is both sub-symmetric and unconditional.

- (ii) If $G \in \mathcal{F}$ and $F \subseteq G$, then $F \in \mathcal{F}$.
- (iii) For every $\alpha < \kappa$ we have $\{\alpha\} \in \mathcal{F}$.

Consider the Tsirelson³ space $T(\mathcal{F})$ on $c_{00}(\kappa)$ build using the family \mathcal{F} . For the convenience of the reader, we recall its definition. First, we define a norming set $K(\mathcal{F}) \subseteq c_{00}(\kappa)$ as follows.

- (1) $K(\mathcal{F})$ contains the set $\{\pm e_\gamma^* : \gamma < \kappa\}$.
- (2) Let $(\phi_i)_{i=0}^m$ be a sequence in $K(\mathcal{F})$ with the following properties.
 - (2.a) If $i, j \in \{0, \dots, m\}$ with $i \neq j$, then $\text{supp } \phi_i \cap \text{supp } \phi_j = \emptyset$.
 - (2.b) The set $\{\min(\text{supp } \phi_i) : 0 \leq i \leq m\}$ is in \mathcal{F} , i.e. the sequence $(\phi_i)_{i=0}^m$ is \mathcal{F} -admissible.
 Then $2^{-1} \sum_{i=0}^m \phi_i$ belongs to $K(\mathcal{F})$.
- (3) $K(\mathcal{F})$ is symmetric; that is, if $\phi \in K(\mathcal{F})$, then $-\phi \in K(\mathcal{F})$.
- (4) $K(\mathcal{F})$ is the minimal set of $c_{00}(\kappa)$ satisfying (1), (2) and (3) above.

The space $T(\mathcal{F})$ is defined to be the completion of $c_{00}(\kappa)$ equipped with the norm

$$(3) \quad \|x\| = \sup\{\langle \phi, x \rangle : \phi \in K(\mathcal{F})\}.$$

The following properties of the space $T(\mathcal{F})$ are almost immediate consequences of the relevant definitions.

- (P1) The standard Hamel basis $(e_\gamma : \gamma < \kappa)$ of $c_{00}(\kappa)$ defines an unconditional Schauder basis of $T(\mathcal{F})$.
- (P2) Assume that for every infinite subset M of κ the restriction

$$\mathcal{F} \upharpoonright M = \{s \in \mathcal{F} : s \subseteq M\}$$

of \mathcal{F} on M has infinite Cantor-Bendixson rank; i.e. for every $n \in \omega$ with $n \geq 1$ there exists an infinite subset L of M such that $[L]^n \subseteq \mathcal{F}$. Then the space $T(\mathcal{F})$ is reflexive. Moreover, in such a case, the space $T(\mathcal{F})$ can contain no sub-symmetric basic sequence. To see this, notice that if (x_n) is a sequence of normalized and disjoint supported vectors in $T(\mathcal{F})$, then for every $k \in \omega$ there exist $n_0 < \dots < n_k$ in ω such that⁴

$$\left\| \frac{x_{n_0} + \dots + x_{n_k}}{k+1} \right\| \geq \frac{1}{2}.$$

³The observant reader will realize that the space $T(\mathcal{F})$ actually corresponds to the modified version of Tsirelson's space, discovered by W. B. Johnson [Jo].

⁴This observation actually shows that, under the hypotheses on the family \mathcal{F} described in (P2), any normalized weakly null sequence (x_n) in the space $T(\mathcal{F})$ has a subsequence generating an ℓ_1 spreading model.

Hence, if $T(\mathcal{F})$ contained a sub-symmetric sequence, then this sequence should be equivalent to the standard basis of ℓ_1 ; this is clearly impossible by the reflexivity of the space.

The above construction reduces the problem of finding non-separable Banach spaces not containing a sub-symmetric sequence to the problem of constructing non-trivial compact and hereditary families. This is the content of the following theorem.

Theorem 22. *Let θ be the first ω -Mahlo cardinal and let κ be an infinite cardinal with $\kappa < \theta$. Then there exists a compact and hereditary family \mathcal{F} on κ containing the singletons and such that for every infinite set M of κ the restriction $\mathcal{F} \upharpoonright M$ of \mathcal{F} on M has infinite Cantor-Bendixson rank.*

Proof. It will be convenient to adopt the following terminology. We say that a family \mathcal{F} of finite subsets of a cardinal κ is *nice* if \mathcal{F} is compact, hereditary, contains all singletons and is such that for every infinite subset M of κ the restriction $\mathcal{F} \upharpoonright M$ has infinite Cantor-Bendixson rank. We also need to introduce the following notation. If s and t are non-empty finite subsets of κ we write $s < t$ if $\max(s) < \min(t)$. Finally, we say that a sequence $(t_i)_{i=0}^n$ of non-empty finite subsets of κ is *block* provided that $t_0 < t_1 < \dots < t_n$.

We proceed to the proof. The construction of the family \mathcal{F} is done recursively on an infinite cardinal κ smaller than the first ω -Mahlo cardinal θ . If $\kappa = \omega$, we let $\mathcal{F}_\omega = \{s \subseteq \omega : |s| \leq \min(s) + 1\}$ be the Schreier family. Suppose that $\kappa > \omega$. We distinguish the following cases.

CASE 1: κ is *singular*. Let $\xi = \text{cf}(\kappa) < \kappa$. We select a sequence $(X_\delta : \delta < \xi)$ of pairwise disjoint infinite subsets of κ such that $\kappa = \bigcup_{\delta < \xi} X_\delta$ and $|X_\delta| < \kappa$ for every $\delta < \xi$. By our inductive hypothesis, there exist a nice family \mathcal{F}_ξ on ξ and for every $\delta < \xi$ a nice family \mathcal{G}_δ on X_δ . We define \mathcal{F}_κ by the rule

$$s \in \mathcal{F}_\kappa \Leftrightarrow s \cap X_\delta \in \mathcal{G}_\delta \text{ for all } \delta < \xi \text{ and } \{\delta < \xi : s \cap X_\delta \neq \emptyset\} \in \mathcal{F}_\xi.$$

It is easily checked that \mathcal{F}_κ is a nice family on κ .

CASE 2: κ is *regular*. Since κ is not ω -Mahlo, it is not n -Mahlo for some $n \in \omega$. So, we may use the characterization of J. H. Schmerl (see [Sch] or [Tod, Theorem 6.1.8]) to fix two integers $m, \bar{n} \geq 3$, an unbounded subset Γ of κ and a regressive⁵ mapping $f : [\Gamma]^{\bar{n}} \rightarrow \kappa$ such that the following holds.

⁵A mapping $f : [\kappa]^{<\omega} \rightarrow \kappa$ is said to be regressive if $f(s) < \min(s)$ for every $s \in [\kappa]^{<\omega}$ with $\min(s) \neq 0$.

(*) For every $X \in [\Gamma]^m$ the restriction $f \upharpoonright [X]^{\bar{n}}$ is *not* min-homogeneous⁶.

We will define the desired family not directly on κ itself but instead on Γ . This is enough since κ is regular, and so, κ and Γ are equipotent. We use the inductive hypothesis and we fix for each $\delta \in \Gamma$ a nice family \mathcal{F}_δ on δ . Define \mathcal{F} by declaring that a finite set $s \subseteq \Gamma$ is in \mathcal{F} if and only if the following are satisfied.

- (1) Either $|s| < \bar{n}$, or
- (2) for every $\alpha \in s$ and every block sequence $(t_i)_{i=0}^k$ with $t_i \in [s]^{\bar{n}-1}$ ($i \leq k$) and $\alpha < t_0$, we have that
 - (2.a) $f(\{\alpha\} \cup t_i) < f(\{\alpha\} \cup t_j)$ if $i < j \leq k$, and
 - (2.b) $\{f(\{\alpha\} \cup t_i) : 0 \leq i \leq k\} \in \mathcal{F}_\alpha$.

The map f is regressive, and so, the family \mathcal{F} is well-defined. We claim that \mathcal{F} is a nice family on Γ . It is clear that \mathcal{F} is compact, hereditary and contains the singletons. What remains is to show that for every infinite subset M of Γ , the restriction $\mathcal{F} \upharpoonright M$ has infinite rank. So, let M be an infinite subset of Γ . It will be enough to prove that $\mathcal{F} \cap [N]^k \neq \emptyset$ for every $N \subseteq M$ of order type ω and every integer k . Let $k \geq 1$ arbitrary. We will prove that $\mathcal{F} \cap [N]^{(\bar{n}-1)k+1} \neq \emptyset$.

To this end we use, first, the Erdős-Rado canonization Theorem [ER] to find an infinite subset P of N such that $f \upharpoonright [P]^{\bar{n}}$ is *canonical*; that is, there exists some subset I of \bar{n} such that for every $s, t \in [P]^{\bar{n}}$ we have $f(s) = f(t)$ if and only if $s(i) = t(i)$ for every $i \in I$. Since $f \upharpoonright [P]^{\bar{n}}$ is canonical and not min-homogeneous, it follows that if $s, t \subseteq P$ with $|s| = |t| = \bar{n} - 1$ and $\alpha < s < t$, then $f(\{\alpha\} \cup s) \neq f(\{\alpha\} \cup t)$. We define $d : [P]^{2\bar{n}-1} \rightarrow 2$ as follows. Let $s \in [P]^{2\bar{n}-1}$ arbitrary. There exist $\alpha \in P$ and $s_0, s_1 \subseteq P$ such that $\alpha < s_0 < s_1$, $|s_0| = |s_1| = \bar{n} - 1$ and $s = \{\alpha\} \cup s_0 \cup s_1$. We set $d(s) = 0$ if $f(\{\alpha\} \cup s_0) < f(\{\alpha\} \cup s_1)$, while we set $d(s) = 1$ if $f(\{\alpha\} \cup s_0) > f(\{\alpha\} \cup s_1)$. By Ramsey's Theorem, there is some infinite subset Q of P such that d is constant on $[Q]^{2\bar{n}-1}$. If the constant value were 1, then we would be able to produce an infinite strictly decreasing sequence of ordinals. So the constant value must be 0, i.e.

- (**) for every $\alpha \in Q$ and every $s_0, s_1 \subseteq Q$ with $\alpha < s_0 < s_1$ and $|s_0| = |s_1| = \bar{n} - 1$ we have that $f(\{\alpha\} \cup s_0) < f(\{\alpha\} \cup s_1)$.

⁶Recall that X is called min-homogeneous for f if $f(s) = f(t)$ for every pair $s, t \in [X]^{\bar{n}}$ with $\min(s) = \min(t)$.

Finally, we define a coloring $e : [Q]^{(\bar{n}-1)k+1} \rightarrow 2$ as follows. Let $s \in [Q]^{(\bar{n}-1)k+1}$ arbitrary and pick $\alpha \in Q$ and $s_0, \dots, s_{k-1} \in [Q]^{\bar{n}-1}$ with $\alpha < s_0 < \dots < s_{k-1}$ and $s = \{\alpha\} \cup s_0 \cup \dots \cup s_{k-1}$. We set $e(s) = 0$ if $\{f(\{\alpha\} \cup s_i) : 0 \leq i < k\} \in \mathcal{F}_\alpha$; otherwise we set $e(s) = 1$. Let $R \subseteq Q$ infinite such that e is constant on R with value, say, $\varepsilon \in \{0, 1\}$. We claim that $\varepsilon = 0$. Suppose, towards a contradiction, that $\varepsilon = 1$. Let $\alpha = \min(R)$ and let (s_i) be a block sequence of subsets of $R \setminus \{\alpha\}$ each of which has size $\bar{n} - 1$ and is such that $\bigcup_{i \in \omega} s_i = R \setminus \{\alpha\}$. Let $T = \{f(\{\alpha\} \cup s_i) : i < \omega\}$. This is an infinite subset of α_0 . By our hypothesis on \mathcal{F}_α , we can find $t \in \mathcal{F}_\alpha \cap [T]^k$. Find integers $i_0 < \dots < i_{k-1}$ such that $t = \{f(\{\alpha\} \cup s_{i_j}) : j < k\} \in \mathcal{F}_\alpha$. Hence

$$(4) \quad e(\{\alpha\} \cup s_{i_0} \cup \dots \cup s_{i_{k-1}}) = 0,$$

a contradiction. So, the constant value of the coloring e is 0. Since every family \mathcal{F}_α is hereditary, we see that $[R]^{(\bar{n}-1)k+1} \subseteq \mathcal{F}$. This shows that the family \mathcal{F} is nice. The proof is completed. \square

We are ready to give the proof of Theorem 21.

Proof of Theorem 21. Fix an infinite cardinal κ smaller than the first ω -Mahlo cardinal θ . Let \mathcal{F} be the family obtained by Theorem 22 and consider the Tsirelson space $T(\mathcal{F})$ build using the family \mathcal{F} . The space $T(\mathcal{F})$ is as desired. \square

4.2. Unconditional subsequences of weakly null sequences. We have the following “subsequence” version of Theorem 15.

Theorem 23. *Let κ be a cardinal and assume that property $\text{Pl}_1(\kappa)$ holds (see Definition 6). Then $\text{nc}_{\text{seq}} \leq \kappa$. In fact, every normalized weakly null sequence $(x_\alpha : \alpha < \kappa)$ has an 1-unconditional subsequence.*

Proof. The proof is very similar to the one of Theorem 15. Indeed, consider the coloring $c_{\text{un}} : [\kappa]^{<\omega} \rightarrow \omega$ defined as follows. Let $s = (\alpha_0 < \dots < \alpha_m) \in [\kappa]^{<\omega}$. Assume that there exists $l \in \omega$ with $l > 0$ such that the sequence $(x_{\alpha_i})_{i=0}^m$ is not $(1 + 1/l)$ -unconditional. In such a case, let $c_{\text{un}}(s)$ be the least l with this property. Otherwise, we set $c_{\text{un}}(s) = 0$. Using $\text{Pl}_1(\kappa)$ and Lemma 18, the result follows. \square

Corollary 24. *It is relative consistent with the existence of a measurable cardinal that every normalized weakly null sequence $(x_\alpha : \alpha < \aleph_\omega)$ has an*

1-unconditional subsequence. Moreover, this statement is compatible with GCH.

Proof. Follows immediately by Theorem 11 and Theorem 23. \square

There is another well-known combinatorial property of a cardinal κ , closely related to $\text{Pl}_1(\kappa)$, which also implies the estimate $\text{nc}_{\text{seq}} \leq \kappa$. This property is the *free set property* of κ (see [Sh1], [Ko], [DT] and the references therein).

Definition 25. *By a structure on κ we mean a first order structure $\mathcal{M} = (\kappa, (f_i)_{i \in \omega})$, where $n_i \in \omega$ and $f_i : \kappa^{n_i} \rightarrow \kappa$ for all $i \in \omega$.*

The free set property of κ , denoted by $\text{Fr}_\omega(\kappa, \omega)$, is the assertion that every structure $\mathcal{M} = (\kappa, (f_i)_{i \in \omega})$ has a free infinite set. That is, there exists an infinite subset L of κ such that every element x of L does not belong to the substructure of \mathcal{M} generated by $L \setminus \{x\}$.

We need the following fact (its proof is left to the interested reader).

Fact 26. *Let κ be a cardinal. Then the following are equivalent.*

- (a) $\text{Fr}_\omega(\kappa, \omega)$ holds.
- (b) *For every structure $\mathcal{M} = (\kappa, (f_i)_{i \in \omega})$ there exists an infinite subset L of κ such that for every $x \in L$ we have*

$$x \notin \{f_i(s) : s \in (L \setminus \{x\})^{n_i} \text{ and } i \in \omega\}.$$

- (c) *Every extended structure $\mathcal{N} = (\kappa, (g_i)_{i \in \omega})$, where $g_i : \kappa^{<\omega} \rightarrow [\kappa]^{\leq \omega}$ for all $i \in \omega$, has an infinite free subset. That is, there exists an infinite subset L of κ such that for every $x \in L$ we have*

$$x \notin \bigcup_{i \in \omega} \bigcup_{s \in (L \setminus \{x\})^{<\omega}} g_i(s).$$

Theorem 27. *Let κ be a cardinal and assume that $\text{Fr}_\omega(\kappa, \omega)$ holds. Then every normalized weakly null sequence $(x_\alpha : \alpha < \kappa)$ has an 1-unconditional subsequence.*

Proof. Let $(x_\alpha : \alpha < \kappa)$ be a normalized weakly null sequence in a Banach space E . For every $s \in [\kappa]^{<\omega}$ we select a subset F_s of S_{E^*} which is countable and 1-norming for the finite-dimensional subspace $E_s := \text{span}\{x_\alpha : \alpha \in s\}$ of E . That is, for every $x \in E_s$ we have

$$(5) \quad \|x\| = \sup\{x^*(x) : x \in F_s\}.$$

Define $g : [\kappa]^{<\omega} \rightarrow [\kappa]^{\leq \omega}$ by

$$(6) \quad g(s) = \{\alpha < \kappa : \text{there is some } x^* \in F_s \text{ such that } x^*(x_\alpha) \neq 0\}.$$

Since $(x_\alpha : \alpha < \kappa)$ is weakly null and F_s is countable, we see that $g(s)$ is also countable; i.e. g is well-defined. Consider the extended structure $\mathcal{N} = (\kappa, g)$. Since $\text{Fr}_\omega(\kappa, \omega)$ holds, there exists an infinite free subset L of κ . We claim that the sequence $(x_\alpha : \alpha \in L)$ is 1-unconditional.

Indeed, let s and t be finite subsets of L with $s \subseteq t$. Fix a sequence $(a_\alpha : \alpha \in t)$ of scalars and let $\varepsilon > 0$ arbitrary. By equality (5) above, we may select $y^* \in F_s$ such that

$$(7) \quad \left\| \sum_{\alpha \in s} a_\alpha x_\alpha \right\| \leq (1 + \varepsilon) \cdot y^* \left(\sum_{\alpha \in s} a_\alpha x_\alpha \right).$$

The set L is free, and so, for every $\alpha \in t \setminus s$ we have $\alpha \notin g(s)$. This implies, in particular, that $y^*(x_\alpha) = 0$ for every $\alpha \in t \setminus s$. Hence

$$\begin{aligned} \left\| \sum_{\alpha \in s} a_\alpha x_\alpha \right\| &\leq (1 + \varepsilon) \cdot y^* \left(\sum_{\alpha \in s} a_\alpha x_\alpha \right) = (1 + \varepsilon) \cdot y^* \left(\sum_{\alpha \in t} a_\alpha x_\alpha \right) \\ &\leq (1 + \varepsilon) \cdot \left\| \sum_{\alpha \in t} a_\alpha x_\alpha \right\|. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the result follows. \square

5. PROOF OF LEMMA 7

Assume that $\lambda < \kappa$ is a pair of two infinite cardinals with λ regular and κ strongly compact. We fix a κ -complete normal ultrafilter \mathcal{U} on κ . Let $\{V_\alpha : \alpha \in \text{Ord}\}$ be the von-Neumann hierarchy of V . As κ is inaccessible (being strongly compact), we see that $|V_\kappa| = \kappa$.

For every coloring $c : [(2^\kappa)^+]^2 \rightarrow V_\kappa$ we let

$$(8) \quad \text{Sol}_\kappa^\omega(c) = \{\mathbf{x} \in [(2^\kappa)^+]^\omega : c \upharpoonright [\mathbf{x}]^2 \text{ is constant}\}$$

and we define

$$(9) \quad \text{Sol}_\kappa^\omega = \{\text{Sol}_\kappa^\omega(c) : c : [(2^\kappa)^+]^2 \rightarrow V_\kappa \text{ is a coloring}\}.$$

The idea of considering the family of sets which are monochromatic with respect to a coloring is taken from Shelah's paper [Sh2] and has been also used by other authors (see, for instance, [Mi]).

Fact 28. *The following hold.*

- (a) *For every coloring $c : [(2^\kappa)^+]^2 \rightarrow V_\kappa$ we have $\text{Sol}_\kappa^\omega(c) \neq \emptyset$.*
- (b) *The family Sol_κ^ω is κ -complete. That is, for every $\delta < \kappa$ and every sequence $(A_\xi : \xi < \delta)$ in Sol_κ^ω we have that $\bigcap_{\xi < \delta} A_\xi \in \text{Sol}_\kappa^\omega$.*

Proof. (a) By our assumptions we see that $|V_\kappa| = \kappa$. Moreover, by the classical Erdős-Rado partition Theorem (see [Ku]), we have

$$(2^\kappa)^+ \rightarrow (\kappa^+)_\kappa^2$$

and the result follows.

(b) For every $\xi < \delta$ let $c_\xi : [(2^\kappa)^+]^2 \rightarrow V_\kappa$ be a coloring such that $A_\xi = \text{Sol}_\kappa^\omega(c_\xi)$. Observe that $(V_\kappa)^\delta \subseteq V_\kappa$. We define the coloring $c : [(2^\kappa)^+]^2 \rightarrow (V_\kappa)^\delta$ by $c(s) = (c_\xi(s) : \xi < \delta)$. Noticing that

$$\bigcap_{\xi < \delta} \text{Sol}_\kappa^\omega(c_\xi) = \text{Sol}_\kappa^\omega(c),$$

the proof is completed. \square

By Fact 28(b) and our hypothesis that κ is a strongly compact cardinal, we see that there exists a κ -complete ultrafilter \mathcal{V} on $[(2^\kappa)^+]^\omega$ extending the family Sol_κ^ω . We fix such an ultrafilter \mathcal{V} .

Definition 29. A \mathcal{V} -sequence of conditions is a sequence $\bar{p} = (p_{\mathbf{x}} : \mathbf{x} \in A)$ in $\text{Col}(\lambda, < \kappa)$, belonging to the ground model V and indexed by a member A of the ultrafilter \mathcal{V} . We will refer to the set A as the index set of \bar{p} and we shall denote it by $I(\bar{p})$.

Definition 30. Let $\bar{p} = (p_{\mathbf{x}} : \mathbf{x} \in I(\bar{p}))$ be a \mathcal{V} -sequence of conditions. We say that a condition r in $\text{Col}(\lambda, < \kappa)$ is a root of \bar{p} if

$$(10) \quad (\mathcal{U}\alpha) (\mathcal{V}\mathbf{x}) \ p_{\mathbf{x}} \upharpoonright \alpha = r^\upharpoonright \alpha.$$

Related to the above definitions, we have the following.

Fact 31. Every \mathcal{V} -sequence of conditions \bar{p} has a unique root $r(\bar{p})$.

Proof. For every $\alpha < \kappa$ the map $I(\bar{p}) \ni \mathbf{x} \mapsto p_{\mathbf{x}} \upharpoonright \alpha$ has fewer than κ values. So, by the κ -completeness of \mathcal{V} , there exist $p_\alpha \in \text{Col}(\lambda, < \kappa)$ and $I_\alpha \in \mathcal{V} \upharpoonright I(\bar{p})$ so that $p_{\mathbf{x}} \upharpoonright \alpha = p_\alpha$ for all $\mathbf{x} \in I_\alpha$. Hence, we can select a sequence $(p_\alpha : \alpha < \kappa)$ in $\text{Col}(\lambda, < \kappa)$ and a decreasing sequence $(I_\alpha : \alpha < \kappa)$ of elements of $\mathcal{V} \upharpoonright I(\bar{p})$ such that for every $\alpha < \kappa$ and every $\mathbf{x} \in I_\alpha$ we have that $p_{\mathbf{x}} \upharpoonright \alpha = p_\alpha$.

Let $A \subseteq \kappa$ be the set of all limit ordinals $\alpha < \kappa$ with $\text{cf}(\alpha) > \lambda$. Since \mathcal{U} is normal, the set A is in \mathcal{U} . Consider the mapping $c : A \rightarrow \kappa$ defined by

$$c(\alpha) = \sup\{\xi : \xi \in (\text{dom}(p_\alpha \upharpoonright \alpha))_1\}$$

⁷This is an abbreviation of the statement that $\{\alpha : \{\mathbf{x} : p_{\mathbf{x}} \upharpoonright \alpha = r\} \in \mathcal{V}\} \in \mathcal{U}$.

for every $\alpha \in A$. As $\text{cf}(\alpha) > \lambda$, we get that c is a regressive mapping. The ultrafilter \mathcal{U} is normal, and so, there exist $A' \in \mathcal{U} \upharpoonright A$ and $\gamma_0 < \kappa$ such that $c(\alpha) = \gamma_0$ for every $\alpha \in A'$. Now consider the map

$$A' \ni \alpha \mapsto p_\alpha \upharpoonright \alpha = p_\alpha \upharpoonright \gamma_0 \subseteq (\lambda \times \gamma_0) \times \gamma_0.$$

Noticing that $|\mathcal{P}((\lambda \times \gamma_0) \times \gamma_0)| < \kappa$ and recalling that \mathcal{U} is κ -complete, we see that there exist $A'' \in \mathcal{U} \upharpoonright A'$ and $r(\bar{p})$ in $\text{Col}(\lambda, < \kappa)$ such that $p_\alpha \upharpoonright \alpha = r(\bar{p})$ for every $\alpha \in A''$. It follows that for every $\alpha \in A''$ the set $\{\mathbf{x} \in [(2^\kappa)^+]^\omega : p_\mathbf{x} \upharpoonright \alpha = r(\bar{p})\}$ contains the set I_α , and so

$$(\mathcal{U}\alpha) (\mathcal{V}\mathbf{x}) p_\mathbf{x} \upharpoonright \alpha = r(\bar{p}).$$

The uniqueness of $r(\bar{p})$ is an immediate consequence of property (10) in Definition 30. The proof is completed. \square

Let G be a $\text{Col}(\lambda, < \kappa)$ -generic filter (*the generic filter G will be fixed until the end of the proof*). We are ready to introduce the ideal \mathcal{I} .

Definition 32. In $V[G]$ we define

$$\mathcal{I} = \{I \subseteq [(2^\kappa)^+]^\omega : \text{there is some } A \in \mathcal{V} \text{ such that } I \cap A = \emptyset\}.$$

We isolate, for future use, the following (easily verified) properties of \mathcal{I} .

- (P1) \mathcal{I} is an ideal; in fact, \mathcal{I} is a κ -complete ideal.
- (P2) $\mathcal{V} \subseteq \mathcal{I}^+$.
- (P3) If $A \in \mathcal{V}$ and $B \in \mathcal{I}^+$, then $A \cap B \in \mathcal{I}^+$.

For every \mathcal{V} -sequence of conditions \bar{p} we let

$$(11) \quad D_{\bar{p}} = \{\mathbf{x} \in I(\bar{p}) : p_\mathbf{x} \in G\}.$$

Now we are ready to introduce the set \mathcal{D} .

Definition 33. In $V[G]$ we define

$$\mathcal{D} = \{D_{\bar{p}} : \bar{p} \text{ is a } \mathcal{V}\text{-sequence of conditions in the ground model } V\} \cap \mathcal{I}^+.$$

By definition, we have that $\mathcal{D} \subseteq \mathcal{I}^+$. The rest of the proof will be devoted to the verification that the ideal \mathcal{I} and the set \mathcal{D} satisfy the requirements of Lemma 7. To this end, we need the following.

Lemma 34. Let $\bar{p} = (p_\mathbf{x} : \mathbf{x} \in I(\bar{p}))$ be a \mathcal{V} -sequence of conditions. Then the following are equivalent.

- (1) $D_{\bar{p}} \in \mathcal{D}$.
- (2) $r(\bar{p}) \in G$.

Proof. (1) \Rightarrow (2) Assume that $D_{\bar{p}} \in \mathcal{D}$. We use the fact that $D_{\bar{p}} \in \mathcal{I}^+$ and that

$$(\mathcal{U}\alpha) (\mathcal{V}\mathbf{x}) \ p_{\mathbf{x}} \upharpoonright \alpha = r(\bar{p})$$

to find $\mathbf{x} \in D_{\bar{p}}$ such that $p_{\mathbf{x}} \leq r(\bar{p})$. By the definition of $D_{\bar{p}}$, we see that $p_{\mathbf{x}} \in G$, and so, $r(\bar{p}) \in G$ as well.

(2) \Rightarrow (1) Suppose that $r(\bar{p}) \in G$. Fix a ground model set A which is in \mathcal{V} . It is enough to show that $D_{\bar{p}} \cap A \neq \emptyset$. To this end, let

$$E = \{q \in \text{Col}(\lambda, < \kappa) : q \perp r(\bar{p}) \text{ or there is } \mathbf{x} \in I(\bar{p}) \cap A \text{ with } q \leq p_{\mathbf{x}}\}.$$

We claim that E is a dense subset of $\text{Col}(\lambda, < \kappa)$. To see this, let $r \in \text{Col}(\lambda, < \kappa)$ arbitrary. If $r \perp r(\bar{p})$, then $r \in E$. So, suppose that $r \parallel r(\bar{p})$. Using this and the fact that

$$(\mathcal{U}\alpha) (\mathcal{V}\mathbf{x}) \ p_{\mathbf{x}} \upharpoonright \alpha = r(\bar{p})$$

we may find $\mathbf{x} \in I(\bar{p}) \cap A$ such that $p_{\mathbf{x}} \parallel r$. So, there exist $q \in \text{Col}(\lambda, < \kappa)$ and $\mathbf{x} \in I(\bar{p}) \cap A$ such that $q \leq p_{\mathbf{x}}$ and $q \leq r$. In other words, there exists $q \in E$ with $q \leq r$. This establishes our claim that E is a dense subset of $\text{Col}(\lambda, < \kappa)$.

It follows by the above discussion that there exists $q \in G$ with $q \in E$. Since $r(\bar{p}) \in G$ we have that $r(\bar{p}) \parallel q$. Hence, by the definition of the set E , there exists $\mathbf{x} \in I(\bar{p}) \cap A$ with $q \leq p_{\mathbf{x}}$. It follows that $p_{\mathbf{x}} \in G$, and so, $\mathbf{x} \in D_{\bar{p}} \cap A$. The proof is completed. \square

Lemma 35. \mathcal{D} is dense in \mathcal{I}^+ .

Proof. Fix $J \in \mathcal{I}^+$. We will prove that there exists a \mathcal{V} -sequence of conditions \bar{q} in the ground model V satisfying $D_{\bar{q}} \in \mathcal{D}$ and $D_{\bar{q}} \subseteq J$. This will finish the proof.

To this end, we fix a $\text{Col}(\lambda, < \kappa)$ -name \dot{J} for J . Let $p \in \text{Col}(\lambda, < \kappa)$ be an arbitrary condition such that $p \Vdash \dot{J} \notin \mathcal{I}$. Define, in the ground model V , the set

$$A_p = \{\mathbf{x} \in [(2^\kappa)^+]^\omega : \text{there is } q \leq p \text{ such that } q \Vdash \check{\mathbf{x}} \in \dot{J}\}.$$

First we claim that $A_p \in \mathcal{V}$. Suppose, towards a contradiction, that the set $C := [(2^\kappa)^+]^\omega \setminus A_p$ is in \mathcal{V} . Since $J \in \mathcal{I}^+$ we see that $J \cap C \neq \emptyset$ in $V[G]$. Using the fact that $p \Vdash \dot{J} \notin \mathcal{I}$ and that the forcing $\text{Col}(\lambda, < \kappa)$ is σ -closed, we may find $\mathbf{x} \in C$ and a condition $q \leq p$ such that $q \Vdash \check{\mathbf{x}} \in \dot{J}$. But this implies that $\mathbf{x} \in A_p$, a contradiction.

By the above discussion, we may find \mathcal{V} -sequence of conditions $\bar{q} = (q_{\mathbf{x}} : \mathbf{x} \in A_p)$ such that $q_{\mathbf{x}} \leq p$ and $q_{\mathbf{x}} \Vdash \dot{\mathbf{x}} \in \dot{J}$ for every $\mathbf{x} \in A_p$. By Fact 31, let $r(\bar{q})$ be the root of \bar{q} . Clearly $r(\bar{q}) \leq p$.

Now fix a condition r such that $r \Vdash \dot{J} \notin \mathcal{I}$. What we have just proved is that the set of conditions $r(\bar{q})$ such that

- (*) $r(\bar{q})$ is the root of a \mathcal{V} -sequence of conditions $\bar{q} = (q_{\mathbf{x}} : \mathbf{x} \in I(\bar{q}))$ with the property that $q_{\mathbf{x}} \Vdash \dot{\mathbf{x}} \in \dot{J}$ for every $\mathbf{x} \in I(\bar{q})$

is dense below r . As G is generic, we may find such a canonical \mathcal{V} -sequence of conditions \bar{q} such that $r(\bar{q}) \in G$. On the one hand, by Lemma 34, we see that $D_{\bar{q}} \in \mathcal{D}$. On the other hand, property (*) above implies that $D_{\bar{q}} \subseteq \dot{J}$; indeed, if $\mathbf{x} \in D_{\bar{q}}$, then $q_{\mathbf{x}} \in G$ and, by (*), $q_{\mathbf{x}} \Vdash \dot{\mathbf{x}} \in \dot{J}$. The proof is completed. \square

Lemma 36. *\mathcal{D} is λ -closed in \mathcal{I}^+ .*

Proof. Fix $\mu < \lambda$ and a decreasing sequence $(D_{\xi} : \xi < \mu)$ in \mathcal{D} . For every $\xi < \mu$ let $\bar{p}_{\xi} = (p_{\mathbf{x}}^{\xi} : \mathbf{x} \in I(\bar{p}_{\xi}))$ be a \mathcal{V} -sequence of conditions in V such that $D_{\xi} = D_{\bar{p}_{\xi}}$. Our forcing $\text{Col}(\lambda, < \kappa)$ is λ -closed, and so, the sequence $(\bar{p}_{\xi} : \xi < \mu)$ is in the ground model V as well. Applying Fact 31 to every \bar{p}_{ξ} , we find a sequence $(r_{\xi} : \xi < \mu)$ in $\text{Col}(\lambda, < \kappa)$ such that r_{ξ} is the root of \bar{p}_{ξ} for every $\xi < \mu$. By Lemma 34, we get that $r_{\xi} \in G$ for all $\xi < \mu$.

We claim, first, that for every $\xi < \zeta < \mu$ we have

$$(12) \quad (\mathcal{V}\mathbf{x}) \quad p_{\mathbf{x}}^{\xi} \parallel p_{\mathbf{x}}^{\zeta}.$$

Suppose, towards a contradiction, that there exist $\xi < \zeta < \mu$ such that the set $L := \{\mathbf{x} \in A : p_{\mathbf{x}}^{\xi} \perp p_{\mathbf{x}}^{\zeta}\}$ is in \mathcal{V} . As $D_{\bar{p}_{\zeta}} \in \mathcal{D} \subseteq \mathcal{I}^+$ and $L \in \mathcal{V}$, there exists $\mathbf{x} \in D_{\bar{p}_{\zeta}} \cap L$. And since $D_{\bar{p}_{\zeta}} = D_{\zeta} \subseteq D_{\xi} = D_{\bar{p}_{\xi}}$ we have $\mathbf{x} \in D_{\bar{p}_{\xi}}$ as well. But this implies that both $p_{\mathbf{x}}^{\xi}$ and $p_{\mathbf{x}}^{\zeta}$ are in G and at the same time $p_{\mathbf{x}}^{\xi} \perp p_{\mathbf{x}}^{\zeta}$, a contradiction.

Invoking (12) above, we may find $A \in \mathcal{V}$ such that for every $\xi < \zeta < \mu$ and every $\mathbf{x} \in A$ we have that $p_{\mathbf{x}}^{\xi} \parallel p_{\mathbf{x}}^{\zeta}$. We set

$$p_{\mathbf{x}} = \bigcup_{\xi < \mu} p_{\mathbf{x}}^{\xi} \quad \text{for every } \mathbf{x} \in A$$

and we define $\bar{p} = (p_{\mathbf{x}} : \mathbf{x} \in A)$. It is clear that \bar{p} is a well-defined \mathcal{V} -sequence of conditions. Also observe that $D_{\bar{p}} \subseteq D_{\xi}$ for every $\xi < \mu$. We are going to show that $D_{\bar{p}} \in \mathcal{D}$. This will finish the proof.

To this end, let r be the root of \bar{p} . By Lemma 34, it is enough to show that $r \in G$. Notice, first, that

$$(13) \quad (\mathcal{U}\alpha) (\mathcal{V}\mathbf{x}) \bigcup_{\xi < \mu} p_{\mathbf{x}}^{\xi} \upharpoonright \alpha = p_{\mathbf{x}} \upharpoonright \alpha = r.$$

On the other hand, as r_{ξ} is the root of \bar{p}_{ξ} , we have

$$(14) \quad (\forall \xi < \mu) (\mathcal{U}\alpha) (\mathcal{V}\mathbf{x}) p_{\mathbf{x}}^{\xi} \upharpoonright \alpha = r_{\xi}.$$

Both \mathcal{U} and \mathcal{V} are κ -complete, and so, (14) is equivalent to

$$(15) \quad (\mathcal{U}\alpha) (\mathcal{V}\mathbf{x}) (\forall \xi < \mu) p_{\mathbf{x}}^{\xi} \upharpoonright \alpha = r_{\xi}.$$

Combining (13) and (15) we get that

$$(16) \quad (\mathcal{U}\alpha) (\mathcal{V}\mathbf{x}) r = \bigcup_{\xi < \mu} p_{\mathbf{x}}^{\xi} \upharpoonright \alpha = \bigcup_{\xi < \mu} r_{\xi}.$$

Summing up, we see that the root r of \bar{p} is the union $\bigcup_{\xi < \mu} r_{\xi}$ of the roots of the \bar{p}_{ξ} 's. Since the generic filter G is λ -complete, we conclude that $r \in G$. The proof is completed. \square

Lemma 37. *Work in $V[G]$. Let $\mu < \kappa$ and let $c : [(2^{\kappa})^+]^2 \rightarrow \mu$ be a coloring. Let also $A \in \mathcal{I}^+$ arbitrary. Then there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ and every $\{\alpha, \beta\} \in [\mathbf{x}]^2$ we have $c(\{\alpha, \beta\}) = \xi$.*

Proof. Fix a coloring $c : [(2^{\kappa})^+]^2 \rightarrow \mu$ and let $A \in \mathcal{I}^+$. Let also \dot{c} be a $\text{Col}(\lambda, < \kappa)$ -name for the coloring c . In V , let $\text{RO}(\text{Col}(\lambda, < \kappa))$ be the collection of all regular-open subsets of $\text{Col}(\lambda, < \kappa)$. Working in V , we define a coloring $d : [(2^{\kappa})^+]^2 \rightarrow (\text{RO}(\text{Col}(\lambda, < \kappa)))^{\mu}$ by the rule

$$d(s) = (\llbracket \dot{c}(\check{s}) = \check{\xi} \rrbracket : \xi < \mu)$$

where $\llbracket \dot{c}(\check{s}) = \check{\xi} \rrbracket = \{p \in \text{Col}(\lambda, < \kappa) : p \Vdash \dot{c}(\check{s}) = \check{\xi}\}$ is the boolean value of the formula “ $c(s) = \xi$ ”.

The forcing $\text{Col}(\lambda, < \kappa)$ is κ -cc, and so, $(\text{RO}(\text{Col}(\lambda, < \kappa)))^{\mu} \subseteq V_{\kappa}$. Hence, $\text{Sol}_{\kappa}^{\omega}(d) \in \mathcal{V}$. We set $J = A \cap \text{Sol}_{\kappa}^{\omega}(d)$. Then J is in \mathcal{I}^+ . Notice that for every $\mathbf{x} \in J$ and every $s, s' \in [\mathbf{x}]^2$ we have $d(s) = d(s')$. It follows that for every $\mathbf{x} \in J$ we may select a sequence $\bar{U}_{\mathbf{x}} = (U_{\mathbf{x}}^{\xi} : \xi < \mu)$ in $(\text{RO}(\text{Col}(\lambda, < \kappa)))^{\mu}$ such that for every $s \in [\mathbf{x}]^2$ and every $\xi < \mu$ we have $\llbracket \dot{c}(\check{s}) = \check{\xi} \rrbracket = U_{\mathbf{x}}^{\xi}$.

Now observe that for every $s \in [(2^{\kappa})^+]^2$ the set $\{\llbracket \dot{c}(\check{s}) = \check{\xi} \rrbracket : \xi < \mu\}$ is a maximal antichain. So, we can naturally define in $V[G]$ a coloring $e : J \rightarrow \mu$ by the rule

$$e(\mathbf{x}) = \xi \text{ if and only if } U_{\mathbf{x}}^{\xi} \in G.$$

Equivalently, for every $\mathbf{x} \in J$ we have that $e(\mathbf{x}) = \xi$ if and only if $c \upharpoonright [\mathbf{x}]^2$ is constant with value ξ . The ideal \mathcal{I} is κ -complete and $J \in \mathcal{I}^+$. Hence there exists $\xi_0 < \mu$ such that $e^{-1}\{\xi_0\} \in \mathcal{I}^+$. By Lemma 35, we may select $D \in \mathcal{D}$ with $D \subseteq e^{-1}\{\xi_0\} \subseteq J \subseteq A$. Finally, notice that for every $\mathbf{x} \in D$ the restriction $c \upharpoonright [\mathbf{x}]^2$ is constant with value ξ_0 . The proof is completed. \square

We are now ready to finish the proof of Lemma 7. As we have already mention, the ideal \mathcal{I} will be the one defined in Definition 32, while the dense subset \mathcal{D} of \mathcal{I}^+ will be the one defined in Definition 33. First, we notice that property (1) in Lemma 7 (i.e. the fact that \mathcal{I} is κ -complete) follows easily by the definition \mathcal{I} and the fact that \mathcal{V} is κ -complete (in fact, we have already isolated this property of \mathcal{I} in (P1) above). Property (2) in Lemma 7 (i.e. the fact that \mathcal{D} is λ -closed in \mathcal{I}^+) has been established in Lemma 36. Finally, property (3) is proved in Lemma 37. The proof of Lemma 7 is completed.

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