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UNCONDITIONAL BASIC SEQUENCES IN SPACES OF LARGE DENSITY

PANDELIS DODOS, JORDI LOPEZ-ABAD AND STEVO TODORCEVIC

Abstract. We study the problem of the existence of unconditional basic sequences in Banach spaces of high density. We show, in particular, the relative consistency with GCH of the statement that every Banach space of density $\aleph_\omega$ contains an unconditional basic sequence.

1. Introduction

In this paper we study particular instances of the general unconditional basic sequence problem asking under which conditions a given Banach space must contain an infinite unconditional basic sequence (see [LT, page 27]). We chose to study instances of the problem for Banach spaces of large densities exposing thus its connections with large-cardinal axioms of set theory. The first paper on this line of research is a well-known paper of J. Ketonen [Ke] which shows that if a density of a given Banach space E is greater or equal to the ω-Erdős cardinal (usually denoted as $\kappa(\omega)$, see Section 2.2), then E contains an infinite unconditional basic sequence. More precisely, let $\mathfrak{nc}$ be the minimal cardinal λ such that every Banach space of density at least λ contains an infinite unconditional basic sequence. Then Ketonen's result can be restated as follows.

Theorem 1 ([Ke]). $\kappa(\omega) \geq \mathfrak{nc}$.

Since $\kappa(\omega)$ is a considerably large cardinal (strongly inaccessible and more) one would like to determine is $\mathfrak{nc}$ really a large cardinal or not, and, of course at some point one would also like to determine the exact value of this cardinal. Unfortunately, there are not too many results in the literature that would point out towards lower bounds for this cardinal. In fact, the largest known lower bound for $\mathfrak{nc}$ is given by S. A. Argyros and A. Tolias

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[2]*Key words*: unconditional basic sequence, non-separable Banach spaces, separable quotient problem, forcing, polarized Ramsey, strongly compact cardinal.

[AT] who showed that $\mathfrak{nc} > 2^{\aleph_0}$. So in particular the following problem is widely open.

Question 1. Is $\exp_\omega(\aleph_0)$, any of the finite-tower exponents $\exp_n(\aleph_0)$, or any of their ω-successors $\exp_n(\aleph_0)^{+\omega}$ an upper bound of $\mathfrak{nc}$? In particular, does $(2^{\aleph_0})^{+\omega} \geq \mathfrak{nc}$ hold?

Our first result shows that $\exp_\omega(\aleph_0)$ is not such a bad candidate for an upper bound of $\mathfrak{nc}$.

Theorem 2. *The inequality* $\exp_\omega(\aleph_0) \geq \mathfrak{nc}$ *is a statement that is consistent relative to the consistency of infinitely many strongly compact cardinals.*

The consistency proof relies heavily on a Ramsey-theoretic property of $\exp_\omega(\aleph_0)$ established in a previous work of S. Shelah [Sh2] (see also [Mi]). One can also arrange the joint consistency of GCH and the inequality $\exp_\omega(\aleph_0) = \aleph_\omega \geq \mathfrak{nc}$. Combining this with a well known result of J. N. Hagler and W. B. Johnson [HJ], we get the following information about the famous separable quotient problem.

Corollary 3. *It is relatively consistent that every Banach space of density at least* $\aleph_\omega$ *has a separable quotient with an unconditional basis.*

The analysis given in this paper together with some known results from Banach space theory suggest, in particular, that by restricting the class of Banach spaces to, say, reflexive, or more generally weakly compactly generated Banach spaces, one might get different answers about the size of the corresponding cardinal numbers $\mathfrak{nc}_{\mathrm{rfl}}$ and $\mathfrak{nc}_{\mathrm{wcg}}$, respectively. To describe this difference it will be convenient to introduce yet another natural cardinal characteristic $\mathfrak{nc}_{\mathrm{seq}}$, the minimal cardinal θ such that every normalized weakly null sequence $(x_\alpha : \alpha < \theta)$ in some Banach space E has a subsequence which is unconditional. Clearly $\mathfrak{nc}_{\mathrm{rfl}} \leq \mathfrak{nc}_{\mathrm{wcg}}$ while by the Amir-Lindenstrauss theorem [AL] we see that $\mathfrak{nc}_{\mathrm{wcg}} \leq \mathfrak{nc}_{\mathrm{seq}}$. The first known lower bound on these cardinal is due to B. Maurey and H. P. Rosenthal [MR] who showed that $\mathfrak{nc}_{\mathrm{seq}} > \aleph_0$, though considerably deeper is the lower bound of W. T. Gowers and B. Maurey [GM] who showed that in fact $\mathfrak{nc}_{\mathrm{rfl}} > \aleph_0$. The largest known lower bound on these cardinals is given in [ALT] who showed that $\mathfrak{nc}_{\mathrm{rfl}} > \aleph_1$. This suggests the following question.

Question 2. Is $\aleph_\omega$ or any of the finite successors $\aleph_n$ $(n \geq 2)$ an upper bound on any of the three cardinals $\mathfrak{nc}_{\mathrm{seq}}$, $\mathfrak{nc}_{\mathrm{rfl}}$, or $\mathfrak{nc}_{\mathrm{wcg}}$?

That $\aleph_\omega$ is not such a bad choice for an upper bound of $\mathfrak{nc}_{\mathrm{seq}}$ may be seen from our second result.

Theorem 4. *The inequality $\aleph_\omega \geq \mathfrak{nc}_{\mathrm{seq}}$ is a statement that is consistent relative to the consistency of a single measurable cardinal.*

Thus, the consistency proof uses a considerably weaker assumption from that used in Theorem 2. It relies on two Ramsey-theoretic principles, one established by P. Koepke [Ko] and the other by C. A. Di Prisco and S. Todorcevic [DT]. It also gives the joint consistency of the GCH and the cardinal inequality $\aleph_\omega \geq \mathfrak{nc}_{\mathrm{seq}}$.

2. Preliminaries

Our Banach space theoretic and set theoretic terminology and notation are standard and follow [LT] and [Ku], respectively. We will consider only real Banach spaces though, using essentially the same arguments, one notices that all our results are valid for complex Banach spaces as well.

Since in this note we are concerned with the problem of the existence of unconditional basic sequences in Banach spaces of high density, let us introduce the following cardinal invariants related to the version of the unconditional basic sequence problem that we study here.

Definition 5. *Let $\mathfrak{nc}$, $\mathfrak{nc}_{\mathrm{wcg}}$, $\mathfrak{nc}_{\mathrm{rfl}}$ and $\mathfrak{nc}_{\mathrm{seq}}$ be defined as follows.*

(1) *$\mathfrak{nc}$ is the minimal cardinal λ such that every Banach space of density λ contains an unconditional basic sequence.*
(2) *$\mathfrak{nc}_{\mathrm{wcg}}$ (respectively, $\mathfrak{nc}_{\mathrm{rfl}}$) is the minimal cardinal λ such that every weakly compactly generated (respectively, reflexive) Banach space of density λ contains an unconditional basic sequence.*
(3) *$\mathfrak{nc}_{\mathrm{seq}}$ is the minimal cardinal λ such that every normalized weakly null sequence $(x_\alpha : \alpha < \lambda)$ in a Banach space E has a subsequence which is unconditional.*

Let us now recall some standard set theoretic notions that will be used throughout the paper.

2.1. **Ideals on Fields of Sets.** Let X be a non-empty set. An *ideal* $\mathcal{I}$ on X is a collection of subsets of X satisfying the following conditions.

(i) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

If $\mathcal{I}$ is an ideal on X and κ is a cardinal, then we say that $\mathcal{I}$ is *κ-complete* if for every $\lambda < \kappa$ and every sequence $(A_\xi : \xi < \lambda)$ in $\mathcal{I}$ we have $\bigcup_{\xi<\lambda} A_\xi \in \mathcal{I}$.

A subset A of X is said to be *positive* with respect to an ideal $\mathcal{I}$ if $A \notin \mathcal{I}$. The set of all positive sets with respect to $\mathcal{I}$ is denoted by $\mathcal{I}^+$. If $\mathcal{D}$ is a subset of $\mathcal{I}^+$ and κ is a cardinal, then we say that $\mathcal{D}$ is *κ-closed in $\mathcal{I}^+$* if for every $\lambda < \kappa$ and every decreasing sequence $(D_\xi : \xi < \lambda)$ in $\mathcal{D}$ we have $\bigcap_{\xi<\kappa} D_\xi \in \mathcal{I}^+$. We also say that such a set $\mathcal{D}$ is *dense* in $\mathcal{I}^+$ if for every $A \in \mathcal{I}^+$ there exists $D \in \mathcal{D}$ with $D \subseteq A$.

If $\mathcal{F}$ is a filter on X, then the family $\{X \setminus A : A \in \mathcal{F}\}$ is an ideal. Having in mind this correspondence, we will continue to use the above terminology for the filter $\mathcal{F}$. Notice that if the given filter is actually an ultrafilter $\mathcal{U}$, then, setting $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{U}$, we have that $\mathcal{I}^+ = \mathcal{U}$.

2.2. **Large Cardinals.** Let θ be a cardinal.

(a) θ is said to be *inaccessible* if it is regular and strong limit; that is, $2^\lambda < \theta$ for every $\lambda < \theta$.
(b) θ is said to be 0-*Mahlo* if it is inaccessible. In general, for an ordinal α, θ is said to be *α-Mahlo* if for every $\beta < \alpha$ and every closed and unbounded subset C of θ there is a β-Mahlo cardinal λ in C.
(c) An *α-Erdős* cardinal, usually denoted by $\kappa(\alpha)$ if exists, is the minimal cardinal λ such that $\lambda \to (\alpha)_2^{<\omega}$; that is, λ is the least cardinal with the property that for every coloring $c : [\lambda]^{<\omega} \to 2$ there is $H \subseteq \lambda$ of order-type α such that c is constant on $[H]^n$ for every $n < \omega$. A cardinal λ that is λ-Erdős (in other words, a cardinal λ which has the partition property $\lambda \to (\lambda)_2^{<\omega}$) is called a *Ramsey* cardinal.
(d) θ is said to be *measurable* if there exists a κ-complete normal ultrafilter $\mathcal{U}$ on κ. Looking at the ultrapower of the universe using $\mathcal{U}$ one can observe that the set $\{\lambda < \theta : \lambda \text{ is inaccessible}\}$ belongs to $\mathcal{U}$. Similarly, one shows that sets $\{\lambda < \theta : \lambda \text{ is } \lambda\text{-Mahlo}\}$ and $\{\lambda < \theta : \lambda \text{ is Ramsey}\}$ belong to $\mathcal{U}$.
(e) θ is said to be *strongly compact* if every κ-complete filter can be extended to a κ-complete ultrafilter.

Finally, for every cardinal κ and every $n \in \omega$ we define recursively the cardinal $\exp_n(\kappa)$ by the rule $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

2.3. **The Lévy Collapse.** Let λ be a regular infinite cardinal and let $\kappa > \lambda$ be an inaccessible cardinal. By $\mathrm{Col}(\lambda, < \kappa)$ we shall denote the set of all partial mappings p satisfying the following.

(i) $\mathrm{dom}(p) \subseteq \lambda \times \kappa$ and $\mathrm{range}(p) \subseteq \kappa$.
(ii) $|p| < \lambda$.
(iii) For every $(\alpha, \beta) \in \mathrm{dom}(p)$ with $\beta > 0$ we have $p(\alpha, \beta) < \beta$.

We equip the set $\mathrm{Col}(\lambda, < \kappa)$ with the partial order $\leq$ defined by

$$p \leq q \Leftrightarrow \mathrm{dom}(q) \subseteq \mathrm{dom}(p) \text{ and } p \restriction \mathrm{dom}(q) = q.$$

If p and q is a pair in $\mathrm{Col}(\lambda, < \kappa)$, then by $p \parallel q$ we denote the fact that p and q are *compatible* (i.e. there exists r in $\mathrm{Col}(\lambda, < \kappa)$ with $r \leq p$ and $r \leq q$), while by $p \perp q$ we denote the fact that p and q are *incompatible*.

We will need the following well-known properties of the Lévy collapse (see, for instance, [Ka]). In what follows, G will be a $\mathrm{Col}(\lambda, < \kappa)$-generic filter.

(a) The generic filter G is λ-complete (this is a consequence of the fact that the forcing $\mathrm{Col}(\lambda, < \kappa)$ is λ-closed).
(b) $\mathrm{Col}(\lambda, < \kappa)$ has the κ-cc (this follows from the fact that the cardinal κ is inaccessible).
(c) In $V[G]$, we have $\kappa = \lambda^+$.
(d) In $V[G]$, the sets ${}^{\kappa}2$ and ${}^{\kappa}2 \cap V$ are equipotent.

Finally, let us introduce some pieces of notation (actually, these pieces of notation will be used only in §5). For every $p \in \mathrm{Col}(\lambda, < \kappa)$ and every $\alpha < \kappa$ by $p \restriction \alpha$ we shall denote the restriction of the partial map p to $\mathrm{dom}(p) \cap (\lambda \times \alpha)$. Moreover, for every $p \in \mathrm{Col}(\lambda, < \kappa)$ we let $(\mathrm{dom}(p))_1 = \{\alpha < \kappa : \exists \xi < \lambda \text{ with } (\xi, \alpha) \in \mathrm{dom}(p)\}$.

3. A polarized partition relation

The purpose of this section is to analyze the following partition property, a variation of a partition property originally appearing in the problem lists of P. Erdős and A. Hajnal [EH1], [EH2, Problem 29] (see also [Sh2]).

Definition 6. *Let κ be a cardinal and $d \in \omega$ with $d \geq 1$. By $\mathrm{Pl}_d(\kappa)$ we shall denote the combinatorial principle asserting that for every coloring $c : \big[[\kappa]^d\big]^{<\omega} \to \omega$ there exists a sequence $(\mathbf{x}_n)$ of infinite disjoint subsets of κ such that for every $m \in \omega$ the restriction $c \restriction \prod_{n=0}^{m} [\mathbf{x}_n]^d$ is constant.*

Clearly property $\mathrm{Pl}_d(\kappa)$ implies property $\mathrm{Pl}_{d'}(\kappa)$ for any cardinal κ and any pair $d, d' \in \omega$ with $d \geq d' \geq 1$. From known results one can easily deduce that the principle $\mathrm{Pl}_d(\exp_{d-1}(\aleph_0)^{+n})$ is false for every $n \in \omega$ and every integer $d \geq 1$ (see, for instance, [EHMR], [CDPM] and [DT]). Thus, the minimal cardinal κ for which $\mathrm{Pl}_d(\kappa)$ could possibly be true is $\exp_{d-1}(\aleph_0)^{+\omega}$. Indeed, C. A. Di Prisco and S. Todorcevic [DT] have established the consistency of $\mathrm{Pl}_1(\aleph_\omega)$ relative the consistency of a single measurable cardinal, an assumption that also happens to be optimal. On the other hand, S. Shelah [Sh2] was able to establish that GCH and principles $\mathrm{Pl}_d(\aleph_\omega)$ $(d \geq 1)$ are jointly consistent, relative to the consistency of GCH and the existence of an infinite sequence of strongly compact cardinals.

Our aim in this section is to present a consistency proof of $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$. We shall treat the colorings in Definition 6 using an iteration of the following lemma whose proof (given in §5), while it relies heavily on an idea of S. Shelah [Sh2], it exposes certain features (the ideal $\mathcal{I}$ and the sufficiently complete dense subset $\mathcal{D}$ of its quotient), not explicitly found in [Sh2], that are likely to find application beyond the scope of our present paper.

Lemma 7. *Suppose that κ is a strongly compact cardinal and that $\lambda < \kappa$ is an infinite regular cardinal. Let G be a $\mathrm{Col}(\lambda, < \kappa)$-generic filter over V. Then, in $V[G]$, for every integer $d \geq 1$ there exists an ideal $\mathcal{I}_d$ on $[(\exp_d(\kappa))^+]^\omega$ and a subset $\mathcal{D}_d$ of $\mathcal{I}_d^+$ such that the following are satisfied.*

1. *$\mathcal{I}_d$ is κ-complete.*
2. *$\mathcal{D}_d$ is dense in $\mathcal{I}_d^+$ and is λ-closed in $\mathcal{I}_d^+$.*
3. *For every $\mu < \kappa$, every coloring $c : [(\exp_d(\kappa))^+]^{d+1} \to \mu$ and every set $A \in \mathcal{I}_d^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_d$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \restriction [\mathbf{x}]^{d+1}$ is constantly equal to ξ.*

To state our next result it is convenient to introduce a sequence (Θ_n) of cardinals, defined recursively by the rule

$$\Theta_0 = \aleph_0 \ \text{ and } \ \Theta_{n+1} = \big(2^{(2^{\Theta_n})^+}\big)^{++}. \tag{1}$$

Notice that the sequence (Θ_n) is strictly increasing and that

$$\exp_n(\aleph_0) < \Theta_n \ \leq \exp_{5n}(\aleph_0)$$

for every $n \in \omega$ with $n \geq 1$. Hence, $\sup\{\Theta_n : n \in \omega\} = \exp_\omega(\aleph_0)$. In particular, if GCH holds, then $\Theta_n = \aleph_{5n}$ for every $n \in \omega$.

Theorem 8. *Suppose that (κ_n) is a strictly increasing sequence of strongly compact cardinals with $\kappa_0 = \aleph_0$. For every $n \in \omega$ set $\lambda_n = \big(2^{(2^{\kappa_n})^+}\big)^+$. Let*

$$\mathbb{P} = \bigotimes_{n\in\omega} \mathrm{Col}(\lambda_n, < \kappa_{n+1})$$

be the iteration of the sequence of Lévy collapses. Let G be a $\mathbb{P}$-generic filter over V. Then, in $V[G]$, for every $n \in \omega$ we have $\kappa_n = \Theta_n$ and there exist an ideal $\mathcal{I}_n$ on $[(2^{\Theta_{n+1}})^+]^\omega$ and a subset $\mathcal{D}_n$ of $\mathcal{I}_n^+$ such that the following are satisfied.

(P1) *$\mathcal{I}_n$ is Θ_{n+1}-complete.*
(P2) *$\mathcal{D}_n$ is $(< \Theta_{n+1})$-closed in $\mathcal{I}_n^+$; that is, $\mathcal{D}_n$ is μ-closed in $\mathcal{I}_n^+$ for every $\mu < \Theta_{n+1}$.*
(P3) *For every $\mu < \Theta_{n+1}$, every coloring $c : [(2^{\Theta_{n+1}})^+]^2 \to \mu$ and every $A \in \mathcal{I}_n^+$ there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_n$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ the restriction $c \restriction [\mathbf{x}]^2$ is constantly equal to ξ.*

Moreover, if GCH holds in V, then GCH also holds in $V[G]$.

Proof. Let $n \in \omega$ arbitrary. Let G_n be the restriction of G to the finite iteration

$$\mathbb{P}_n = \bigotimes_{m<n} \mathrm{Col}(\lambda_m, < \kappa_{m+1}).$$

Notice, first, that the small forcing extension $V[G_n]$ preserves the strong compactness of κ_{n+1}. This fact follows immediately from the elementary-embedding characterization of strong compactness (see [Ka, Theorem 22.17]). Working in $V[G_n]$ and applying Lemma 7 for $d = 1$, we see that the intermediate forcing extension $V[G_{n+1}]$ has the ideal $\mathcal{I}_n$ whose quotient has properties (P1), (P2) and (P3) described in Lemma 7. Working still in the intermediate forcing extension $V[G_{n+1}]$, we see that the rest of the forcing

$$\mathbb{P}^{n+1} = \bigotimes_{n<m<\omega} \mathrm{Col}(\lambda_m, < \kappa_{m+1})$$

is λ_{n+1}-closed, and so, in particular, it adds no new subsets to the index set on which the ideal $\mathcal{I}_n$ lives. Therefore, properties (P1), (P2) and (P3) of the quotient of $\mathcal{I}_n$ are preserved in $V[G]$. Since n was arbitrary, the proof is completed. □

The final step towards our proof of the consistency of $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$ is included in the following, purely combinatorial, result.

Proposition 9. *Let (Θ_n) be the sequence of cardinals defined in (1) above. Suppose that for every $n \in \omega$ there exist an ideal $\mathcal{I}_n$ on $[(2^{\Theta_{n+1}})^+]^\omega$ and a subset $\mathcal{D}_n$ of $\mathcal{I}_n^+$ which satisfy properties (P1), (P2) and (P3) described in Theorem 8. Then, the principle $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$ holds.*

Proof. The proof is based on the following claim.

Claim 10. *Let $n \in \omega$. Let also $c : \prod_{i=0}^n [(2^{\Theta_{i+1}})^+]^2 \to \omega$ be a coloring and $(D_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i$. Then, there exist $(E_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i \restriction D_i$ and a color $m_0 \in \omega$ such that for every $(\mathbf{x}_i)_{i=0}^n \in \prod_{i=0}^n E_i$ the restriction $c \restriction \prod_{i=0}^n [\mathbf{x}_i]^2$ is constantly equal to m_0.*

Proof of Claim 10. By induction on n. The case $n = 0$ is an immediate consequence of property (P3) in Theorem 8. So, let $n \in \omega$ with $n \geq 1$ and assume that the result has been proved for all $k \in \omega$ with $k < n$. Fix a coloring $c : \prod_{i=0}^n [(2^{\Theta_{i+1}})^+]^2 \to \omega$. Fix also $(D_i)_{i=0}^n \in \prod_{i=0}^n \mathcal{D}_i$ and let

$$\mathcal{F} = \{f : \prod_{i=0}^{n-1} [(2^{\Theta_{i+1}})^+]^2 \to \omega : f \text{ is a coloring}\}.$$

Notice that $|\mathcal{F}| = 2^{(2^{\Theta_n})^+}$, and so, $|\mathcal{F}| < \Theta_{n+1}$. We define a coloring $d : [(2^{\Theta_{n+1}})^+]^2 \to \mathcal{F}$ by the rule $d\big(\{\alpha, \beta\}\big)(\bar{s}) = c\big(\bar{s}^\frown\{\alpha, \beta\}\big)$ for every $\bar{s} \in \prod_{i=0}^{n-1} [(2^{\Theta_{i+1}})^+]^2$. By (P3) in Theorem 8, there exist $E_n \in \mathcal{D}_n \restriction D_n$ and $f_0 \in \mathcal{F}$ such that for every $\mathbf{x} \in E_n$ the restriction $d \restriction [\mathbf{x}]^2$ is constantly equal to f_0. The result now follows by applying our inductive hypothesis to the coloring f_0. □

By Claim 10 and the fact that every $\mathcal{D}_n$ is σ-closed (property (P2) in Theorem 8), the proof of Proposition 9 is completed. □

As a consequence of the previous analysis we get the following.

Corollary 11 ([Sh2])**.** *Suppose that in our universe V there exists a strictly increasing sequence (κ_n) of strongly compact cardinals with $\kappa_0 = \aleph_0$. Then, there is a forcing extension of V in which the principle $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$ holds. Moreover, if GCH holds in V, then GCH also holds in the extension.*

Proof. Follows by Theorem 8 and Proposition 9. □

Clearly, in the forcing extension obtained above the combinatorial principle $\mathrm{Pl}_1\big(\exp_\omega(\aleph_0)\big)$ holds as well. However, as we have already indicated, one can obtain the consistency of $\mathrm{Pl}_1\big(\exp_\omega(\aleph_0)\big)$ using a considerably weaker (and, in fact, optimal) large-cardinal assumption from the one used for $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$. More precisely, we have the following.

Theorem 12 ([DT]). *Assume the existence of a measurable cardinal. Then, there is a forcing extension in which GCH and* $\mathrm{Pl}_1(\aleph_\omega)$ *hold.*

In our applications in Banach space theory we will use only the combinatorial principles $\mathrm{Pl}_2\big(\exp_\omega(\aleph_0)\big)$ and $\mathrm{Pl}_1(\aleph_\omega)$. We would like, however, to record the following higher-dimensional analogues of Theorem 8, Proposition 9 and Corollary 11 respectively. It appears that these analogues can be used in a variety of problems of combinatorial flavor. Their proofs are straightforward adaptations of our previous arguments (we leave the details to the interested reader).

Theorem 13. *Suppose that* (κ_n) *is a strictly increasing sequence of strongly compact cardinals with* $\kappa_0 = \aleph_0$. *For every* $n \in \omega$ *we can choose cardinals* $\lambda_n, \theta_n \in \big[\kappa_n, \exp_\omega(\kappa_n)\big)$ *in such a way that, if we let*

$$\mathbb{P} = \bigotimes_{n\in\omega} \mathrm{Col}(\lambda_n, < \kappa_{n+1})$$

be the iteration of the sequence of Lévy collapses and if we choose G *to be a* $\mathbb{P}$*-generic filter over* V*, then, in* $V[G]$*, we have*

$$\sup_{n\in\omega} \kappa_n = \sup_{n\in\omega} \lambda_n = \sup_{n\in\omega} \theta_n = \exp_\omega(\aleph_0)$$

and for every $n \in \omega$ *there exist an ideal* $\mathcal{I}_n$ *on* $[\theta_{n+1}]^\omega$ *and a subset* $\mathcal{D}_n$ *of* $\mathcal{I}_n^+$ *such that the following are satisfied.*

(P1) $\mathcal{I}_n$ *is* κ_{n+1}*-complete.*
(P2) $\mathcal{D}_n$ *is* $(< \lambda_n)$*-closed in* $\mathcal{I}_n^+$*; that is,* $\mathcal{D}_n$ *is* μ*-closed in* $\mathcal{I}_n^+$ *for every* $\mu < \lambda_n$.
(P3) *For every* $\mu < \kappa_{n+1}$*, every coloring* $c : [\theta_{n+1}]^{n+1} \to \mu$ *and every* $A \in \mathcal{I}_n^+$ *there exist a color* $\xi < \mu$ *and an element* $D \in \mathcal{D}_n$ *with* $D \subseteq A$ *and such that for every* $\mathbf{x} \in D$ *the restriction* $c \restriction [\mathbf{x}]^{n+1}$ *is constantly equal to* ξ.

Moreover, if GCH holds in V*, then GCH also holds in the* $V[G]$.

Proposition 14. *Suppose* (κ_n)*,* (λ_n) *and* (θ_n) *are strictly increasing sequences of regular cardinal that all converge to* $\exp_\omega(\aleph_0)$*. Suppose further that for every* $n \in \omega$ *there exist an ideal* $\mathcal{I}_n$ *on* $[\theta_{n+1}]^\omega$ *and a subset* $\mathcal{D}_n$ *of* $\mathcal{I}_n^+$ *which satisfy properties (P1), (P2) and (P3) of Theorem 13. Then the principle* $\mathrm{Pl}_d\big(\exp_\omega(\aleph_0)\big)$ *holds for every integer* $d \geq 1$.

Corollary 15 ([Sh2]). *Suppose that in our universe* V *there exists a strictly increasing sequence* (κ_n) *of strongly compact cardinals with* $\kappa_0 = \aleph_0$*. Then,*

there is a forcing extension of V in which the principle $\mathrm{Pl}_d\big(\exp_\omega(\aleph_0)\big)$ *holds for every integer $d \geq 1$. Moreover, if GCH holds in V, then GCH also holds in the forcing extension.*

4. Banach space implications

Let us recall that a sequence (x_n) in a Banach space E is said to be C-*unconditional*, where $C \geq 1$, if for every pair F and G of non-empty finite subsets of ω with $F \subseteq G$ and every choice $(a_n)_{n\in G}$ of scalars we have

$$\|\sum_{n\in F} a_n x_n\| \leq C \cdot \|\sum_{n\in G} a_n x_n\|.$$

This main result in this section is the following.

Theorem 16. *Let κ be a cardinal and assume that property* $\mathrm{Pl}_2(\kappa)$ *holds (see Definition 6). Then every Banach space E not containing ℓ_1 and of density κ contains an* 1*-unconditional basic sequence.*

In particular, if E is any Banach space of density κ, then for every $\varepsilon > 0$ the space E contains an $(1+\varepsilon)$-unconditional basic sequence.

Combining Corollary 11 with Theorem 16, we get the following corollaries.

Corollary 17. *It is consistent relative the existence of an infinite sequence of strongly compact cardinals that for every $\varepsilon > 0$ and every Banach space E of density at least* $\exp_\omega(\aleph_0)$*, the space E contains an $(1+\varepsilon)$-unconditional basic sequence. Moreover, this statement is consistent with GCH.*

Proof. Follows immediately by Corollary 11 and Theorem 16. □

Corollary 18. *It is consistent relative to the existence of an infinite sequence of strongly compact cardinals that every Banach space of density at least* $\exp_\omega(\aleph_0)$ *has a separable quotient with an unconditional basis. Moreover, this statement is consistent with GCH.*

Proof. A well-known consequence of a result due to J. N. Hagler and W. B. Johnson [HJ] asserts that if E is a Banach space such that E^* has an unconditional basic sequence, then E has a separable quotient with an unconditional basis (see also [ADK, Proposition 16]). Noticing that the density of the dual E^* of a Banach space E is at least as big as the density of E, the result follows by Corollary 17. □

The section is organized as follows. In §4.1 we give the proof of Theorem 16, while in §4.2 we present its "sequential" version. Two proofs of

this version are given, each of which is based on a different combinatorial principle.

4.1. **Proof of Theorem 16.** We start with the following lemma, which is essentially a multi-dimensional version of Odell's Schreier unconditionality theorem [O2].

Lemma 19. *Let E be a Banach space, $m \in \omega$ with $m \geq 1$ and $\varepsilon > 0$. For every $i \in \{0, \dots, m\}$ let (x_n^i) be a normalized weakly null sequence in the space E. Then, there exists an infinite subset L of ω such that for every $\{n_0 < \cdots < n_m\} \subseteq L$ the sequence $(x_{n_i}^i)_{i=0}^m$ is $(1+\varepsilon)$-unconditional.*

Proof. The first step towards the proof of the lemma is included in the following claim. It shows that, by passing to an infinite subset of ω, we may assume that for every $\{n_0 < \cdots < n_m\} \in [\mathbb{N}]^{m+1}$ the finite sequence $(x_{n_i}^i)_{i=0}^m$ is a particularly well behaved Schauder basic sequence.

Claim 20. *For every $\varepsilon > 0$ there exists an infinite subset M of ω such that for every $\{n_0 < \cdots < n_m\} \subseteq M$ the sequence $(x_{n_i}^i)_{i=0}^m$ is an $(1+\varepsilon)$-Schauder basic sequence.*

Proof of Claim 20. We define a coloring $\mathcal{B} : [\mathbb{N}]^{m+1} \to 2$ as follows. Let $s = \{n_0 < \cdots < n_m\} \in [\mathbb{N}]^{m+1}$ arbitrary. If $(x_{n_i}^i)_{i=0}^m$ is an $(1+\varepsilon)$-Schauder basic sequence, then we set $\mathcal{B}(s) = 0$; otherwise we set $\mathcal{B}(s) = 1$. By Ramsey's theorem, there exist an infinite subset M of ω and $c \in \{0, 1\}$ such that $\mathcal{B} \restriction [M]^{m+1}$ is constantly equal to c. Using Mazur's classical procedure for selecting Schauder basic sequences (see, for instance, [LT, Lemma 1.a.6]), we find $t = \{k_0 < \cdots < k_m\} \in [M]^{m+1}$ such that the sequence $(x_{k_i}^i)_{i=0}^m$ is basic with basis constant $(1+\varepsilon)$. Therefore, $\mathcal{B}(t) = 0$, and by homogeneity, $\mathcal{B} \restriction [M]^{m+1} = 0$. The claim is proved. □

Applying Claim 20 for $\varepsilon = 1$, we get an infinite subset M of ω as described above. Observe that for every $\{n_0 < \cdots < n_m\} \in [M]^{m+1}$ and every choice $(a_i)_{i=0}^m$ of scalars we have

$$\|\sum_{i=0}^m a_i x_{n_i}^i\| \geq \frac{1}{4} \max\{|a_i| \,:\, i = 0, \dots, m\}. \tag{2}$$

The desired subset L of ω will be an infinite subset of M obtained after another application of Ramsey's theorem. Specifically, consider the coloring $\mathcal{U} : [M]^{m+1} \to 2$ defined as follows. Let $s = \{n_0 < \cdots < n_m\} \in [M]^{m+1}$ and assume that the sequence $(x_{n_i}^i)_{i=0}^m$ is $(1+\varepsilon)$-unconditional. In such a case,

we set $\mathcal{U}(s) = 0$; otherwise we set $\mathcal{U}(s) = 1$. Let L be an infinite subset of M be such $\mathcal{U}$ is constant on $[L]^{m+1}$. It is enough to find some $s \in [L]^{m+1}$ such that $\mathcal{U}(s) = 0$.

To this end, fix $\delta > 0$ such that $(1+\delta)\cdot(1-\delta)^{-1} \leq (1+\varepsilon)$. Notice that there exists a finite family $\mathcal{D}$ of normalized Schauder basic sequences of length $m+1$ such that any normalized Schauder basic sequence $(y_i)_{i=0}^m$, in some Banach space Y, is $\sqrt{1+\delta}$-equivalent to some sequence in the family $\mathcal{D}$. Hence, by a further application of Ramsey's theorem and by passing to an infinite subset of L if necessary, we may assume that

(∗) for every $\{n_0 < \cdots < n_m\}, \{k_0 < \cdots < k_m\} \in [L]^{m+1}$ the sequences $(x^i_{n_i})_{i=0}^m$ and $(x^i_{k_i})_{i=0}^m$ are $(1+\delta)$-equivalent.

Now, for every $i \in \{0,\dots,m\}$ and every $\rho > 0$ let

$$\mathcal{K}_i(\rho) = \big\{\{n \in \omega \,:\, |x^*(x^i_n)| \geq \rho\} \,:\, x^* \in B_{E^*}\big\}.$$

Every sequence (x^i_n) is weakly null, and so, each $\mathcal{K}_i(\rho)$ is a pre-compact[1] family of finite subsets of ω. Hence, we may select a sequence $(F_i)_{i=0}^m$ of finite subsets of L such that

(a) $\max(F_i) < \min(F_{i+1})$ for every $i \in \{0,\dots,m-1\}$, and
(b) $F_i \notin \mathcal{K}_i(\delta\cdot 8^{-1}\cdot(m+1)^{-1})$ for every $i \in \{0,\dots,m\}$.

We set $n_i = \min(F_i)$ for all $i \in \{0,\dots,m\}$. Property (a) above implies that $n_0 < \cdots < n_m$. We claim that the sequence $(x^i_{n_i})_{i=0}^m$ is $(1+\varepsilon)$-unconditional. Indeed, let $F \subseteq \{0,\dots,m\}$ and $(a_i)_{i=0}^m$ be a choice of scalars. We want to prove that

$$\|\sum_{i\in F} a_i x^i_{n_i}\| \leq (1+\varepsilon)\|\sum_{i=0}^m a_i x^i_{n_i}\|.$$

Clearly we may assume that $\|\sum_{i\in F} a_i x^i_{n_i}\| = 1$. If $\|\sum_{i\notin F} a_i x^i_{n_i}\| \geq 2$, then

$$\|\sum_{i=0}^m a_i x^i_{n_i}\| \geq \|\sum_{i\notin F} a_i x^i_{n_i}\| - \|\sum_{i\in F} a_i x^i_{n_i}\| \geq 1 = \|\sum_{i\in F} a_i x^i_{n_i}\|.$$

So, suppose that $\|\sum_{i\notin F} a_i x^i_{n_i}\| \leq 2$. By (2), we see that

$$\max\{|a_i| \,:\, i \notin F\} \leq 8. \tag{3}$$

We select $x^*_0 \in S_{E^*}$ such that $x^*_0\big(\sum_{i\in F} a_i x^i_{n_i}\big) = \|\sum_{i\in F} a_i x^i_{n_i}\|$. We define a sequence $(k_i)_{i=0}^m$ in L as follows. If $i \notin F$, then let k_i be any member of F_i satisfying $|x^*_0(x^i_{k_i})| < \delta\cdot 8^{-1}\cdot(m+1)^{-1}$ (such a selection is possible by

[1]Recall that a family $\mathcal{F}$ of finite subsets of ω is said to be pre-compact if, identifying $\mathcal{F}$ with a subset of the Cantor set 2^ω, the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ in 2^ω consists only of finite sets.

(b) above); if $i \in F$, then we set $k_i = n_i$. By (a), we have $k_0 < \cdots < k_m$. Moreover,

$$\begin{aligned} \|\sum_{i=0}^{m} a_i x_{k_i}^i\| &= x_0^*\big(\sum_{i=0}^{m} a_i x_{k_i}^i\big) = x_0^*\big(\sum_{i\in F} a_i x_{k_i}^i\big) + x_0^*\big(\sum_{i\notin F} a_i x_{k_i}^i\big) \\ &\geq x_0^*\big(\sum_{i\in F} a_i x_{k_i}^i\big) - \sum_{i\notin F} |a_i| \cdot |x_0^*(x_{k_i}^i)| \geq 1-\delta. \end{aligned}$$

Invoking $(*)$, we conclude that

$$\|\sum_{i=0}^{m} a_i x_{n_i}^i\| \geq \frac{1}{1+\delta}\|\sum_{i=0}^{m} a_i x_{k_i}^i\| \geq \frac{1-\delta}{1+\delta} \geq \frac{1}{1+\varepsilon}\|\sum_{i\in F} a_i x_{n_i}^i\|.$$

The proof is completed. □

We are ready to proceed to the proof of Theorem 16.

Proof of Theorem 16. Let κ be a cardinal such that $\mathrm{Pl}_2(\kappa)$ holds. By a classical result of R. C. James (see [LT, Proposition 2.e.3]), it is enough to show that if E is a Banach space of density κ not containing an isomorphic copy of ℓ_1, then E has an 1-unconditional basic sequence. So, let E be one. By Rosenthal's ℓ_1 theorem [Ro] and our assumptions on the space E, we see that every bounded sequence in E has a weakly Cauchy subsequence. Let $(x_\alpha : \alpha < \kappa)$ be a normalized sequence such that $\|x_\alpha - x_\beta\| \geq 1$ for every $\alpha < \beta < \kappa$. We define a coloring $c_{\mathrm{un}} : \big[\,[\kappa]^2\,\big]^{<\omega} \to \omega$ as follows. Let $s = (\{\alpha_0 < \beta_0\}, \ldots, \{\alpha_m < \beta_m\}) \in \big[\,[\kappa]^2\,\big]^{<\omega}$ arbitrary. Assume that there exists $l \in \omega$ with $l > 0$ and such that the sequence $(x_{\beta_i} - x_{\alpha_i})_{i=0}^m$ is *not* $(1+1/l)$-unconditional. In such a case, setting l_s to be the least $l \in \omega$ with the above property, we define $c_{\mathrm{un}}(s) = l_s$. If such an l does not exist, then we set $c_{\mathrm{un}}(s) = 0$. By $\mathrm{Pl}_2(\kappa)$, there exist a sequence $(\mathbf{x}_i)$ of infinite subsets of κ and a sequence (l_m) in ω such that for every $m \in \omega$ the restriction $c_{\mathrm{un}} \restriction \prod_{i=0}^m [\mathbf{x}_i]^2$ of the coloring c_{un} on the product $\prod_{i=0}^m [\mathbf{x}_i]^2$ is constant with value l_m.

Claim 21. *For every $m \in \omega$ we have $l_m = 0$.*

Grating the claim, the proof of the theorem is completed. Indeed, observe that for every infinite sequence of pairs $\big(\{\alpha_i < \beta_i\}\big) \in \prod_{i\in\omega} [\mathbf{x}_i]^2$ the sequence $(x_{\beta_i} - x_{\alpha_i})$ is a semi-normalized 1-unconditional basic sequence in the Banach space E.

It only remains to prove Claim 21. To this end we argue by contradiction. So, assume that there exists $m \in \omega$ such that $l_m > 0$. Our definition of the

coloring c_{un} implies that $m \geq 1$. For every $i \in \{0, \ldots, m\}$ we may select an infinite subset $\{\alpha_0^i < \alpha_1^i < \cdots\}$ of $\mathbf{x}_i$ such that the sequence (x_{α_i}) is weakly Cauchy. We set

$$y_n^i = \frac{x_{\alpha_{2n}^i} - x_{\alpha_{2n+1}^i}}{\|x_{\alpha_{2n}^i} - x_{\alpha_{2n+1}^i}\|}$$

for every $i \in \{0, \ldots, m\}$ and every $n \in \omega$. Then each (y_n^i) is a normalized weakly null sequence in E. Moreover, for every $\{n_0 < \cdots < n_m\} \subseteq [\mathbb{N}]^{m+1}$ the sequence $(y_{n_i}^i)_{i=0}^m$ is not $(1 + 1/l_m)$-unconditional. This clearly contradicts Lemma 19. The proof is completed. □

4.2. **Unconditional subsequences of weakly null sequences.** This subsection is devoted to the proof of the following "sequential" version of Theorem 16.

Theorem 22. *Let κ be a cardinal and assume that property $\mathrm{Pl}_1(\kappa)$ holds (see Definition 6). Then $\mathfrak{nc}_{\mathrm{seq}} \leq \kappa$. In fact, every normalized weakly null sequence $(x_\alpha : \alpha < \kappa)$ has an 1-unconditional subsequence.*

Proof. The proof is very similar to the one of Theorem 16. Indeed, consider the coloring $c_{\mathrm{un}} : [\kappa]^{<\omega} \to \omega$ defined as follows. Let $s = (\alpha_0 < \cdots < \alpha_m) \in [\kappa]^{<\omega}$. Assume that there exists $l \in \omega$ with $l > 0$ such that the sequence $(x_{\alpha_i})_{i=0}^m$ is *not* $(1 + 1/l)$-unconditional. In such a case, let $c_{\mathrm{un}}(s)$ be the least l with this property. Otherwise, we set $c_{\mathrm{un}}(s) = 0$. Using $\mathrm{Pl}_1(\kappa)$ and Lemma 19, the result follows. □

Corollary 23. *It is consistent relative to the existence of a single measurable cardinal that every normalized weakly null sequence $(x_\alpha : \alpha < \aleph_\omega)$ has an 1-unconditional subsequence. Moreover, this statement is consistent with GCH.*

Proof. Follows immediately by Theorem 12 and Theorem 22. □

There is another well-known combinatorial property of a cardinal κ which is implied by $\mathrm{Pl}_1(\kappa)$and which is in turn sufficient for the estimate $\mathfrak{nc}_{\mathrm{seq}} \leq \kappa$. This property is in the literature called the *free set property* of κ (see [Sh1], [Ko], [DT] and the references therein).

Definition 24. *By a* structure *on κ we mean a first order structure $\mathcal{M} = (\kappa, (f_i)_{i\in\omega})$, where $n_i \in \omega$ and $f_i : \kappa^{n_i} \to \kappa$ for all $i \in \omega$.*

The free set property of κ, denoted by $\mathrm{Fr}_\omega(\kappa, \omega)$, is the assertion that every structure $\mathcal{M} = (\kappa, (f_i)_{i\in\omega})$ has a free infinite set. That is, there exists an

infinite subset L of κ such that every element x of L does not belong to the substructure of $\mathcal{M}$ generated by $L \setminus \{x\}$.

We need the following fact (its proof is left to the interested reader).

Fact 25. *Let κ be a cardinal. Then the following are equivalent.*

(a) $\mathrm{Fr}_\omega(\kappa, \omega)$ *holds.*

(b) *For every structure $\mathcal{M} = (\kappa, (f_i)_{i\in\omega})$ there exists an infinite subset L of κ such that for every $x \in L$ we have*

$$x \notin \{f_i(s) \,:\, s \in (L \setminus \{x\})^{n_i} \text{ and } i \in \omega\}.$$

(c) *Every* extended *structure $\mathcal{N} = (\kappa, (g_i)_{i\in\omega})$, where $g_i : \kappa^{<\omega} \to [\kappa]^{\leq\omega}$ for all $i \in \omega$, has an infinite free subset. That is, there exists an infinite subset L of κ such that for every $x \in L$ we have*

$$x \notin \bigcup_{i\in\omega} \bigcup_{s\in(L\setminus\{x\})^{<\omega}} g_i(s).$$

As we have already indicated above, one can use the property $\mathrm{Fr}_\omega(\kappa, \omega)$ to derive the conclusion of Theorem 22. More precisely, we have the following.

Theorem 26. *Let κ be a cardinal and assume that $\mathrm{Fr}_\omega(\kappa, \omega)$ holds. Then every normalized weakly null sequence $(x_\alpha : \alpha < \kappa)$ has an 1-unconditional subsequence.*

Proof. Let $(x_\alpha : \alpha < \kappa)$ be a normalized weakly null sequence in a Banach space E. For every $s \in [\kappa]^{<\omega}$ we select a subset F_s of S_{E^*} which is countable and 1-norming for the finite-dimensional subspace $E_s := \mathrm{span}\{x_\alpha : \alpha \in s\}$ of E. That is, for every $x \in E_s$ we have

$$\|x\| = \sup\{x^*(x) \,:\, x \in F_s\}. \tag{4}$$

Define $g : [\kappa]^{<\omega} \to [\kappa]^{\leq\omega}$ by

$$g(s) = \{\alpha < \kappa \,:\, \text{there is some } x^* \in F_s \text{ such that } x^*(x_\alpha) \neq 0\}. \tag{5}$$

Since $(x_\alpha : \alpha < \kappa)$ is weakly null and F_s is countable, we see that $g(s)$ is also countable; i.e. g is well-defined. Consider the extended structure $\mathcal{N} = (\kappa, g)$. Since $\mathrm{Fr}_\omega(\kappa, \omega)$ holds, there exists an infinite free subset L of κ. We claim that the sequence $(x_\alpha : \alpha \in L)$ is 1-unconditional.

Indeed, let s and t be finite subsets of L with $s \subseteq t$. Fix a sequence $(a_\alpha : \alpha \in t)$ of scalars and let $\varepsilon > 0$ arbitrary. By equality (4) above, we

may select $y^* \in F_s$ such that

$$\|\sum_{\alpha\in s} a_\alpha x_\alpha\| \leq (1+\varepsilon)\cdot y^*\big(\sum_{\alpha\in s} a_\alpha x_\alpha\big). \tag{6}$$

The set L is free, and so, for every $\alpha \in t \setminus s$ we have $\alpha \notin g(s)$. This implies, in particular, that $y^*(x_\alpha) = 0$ for every $\alpha \in t \setminus s$. Hence

$$\begin{aligned} \|\sum_{\alpha\in s} a_\alpha x_\alpha\| &\leq (1+\varepsilon)\cdot y^*\big(\sum_{\alpha\in s} a_\alpha x_\alpha\big) = (1+\varepsilon)\cdot y^*\big(\sum_{\alpha\in t} a_\alpha x_\alpha\big) \\ &\leq (1+\varepsilon)\cdot \|\sum_{\alpha\in t} a_\alpha x_\alpha\|. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

5. Proof of Lemma 7

Assume that $\lambda < \kappa$ is a pair of two infinite cardinals with λ regular and κ strongly compact. Let G be a $\mathrm{Col}(\lambda, <\kappa)$-generic filter. *The generic filter G will be fixed until the end of the proof. We also fix a κ-complete normal ultrafilter $\mathcal{U}$ on κ.*

Let $d \in \omega$ with $d \geq 1$ arbitrary. Let $\{V_\alpha : \alpha \in \mathrm{Ord}\}$ be the von-Neumann hierarchy of V. As κ is inaccessible (being strongly compact), we see that $|V_\kappa| = \kappa$. For every coloring $c : [(\exp_d(\kappa))^+]^{d+1} \to V_\kappa$ we let

$$\mathrm{Sol}^\omega_{d,\kappa}(c) = \{\mathbf{x} \in [(\exp_d(\kappa))^+]^\omega : c \restriction [\mathbf{x}]^{d+1} \text{ is constant}\} \tag{7}$$

and we define

$$\mathrm{Sol}^\omega_{d,\kappa} = \{\mathrm{Sol}^\omega_{d,\kappa}(c) : c : [(\exp_d(\kappa))^+]^{d+1} \to V_\kappa \text{ is a coloring}\}. \tag{8}$$

The idea of considering the family of sets which are monochromatic with respect to a coloring is taken from Shelah's paper [Sh2] and has been also used by other authors (see, for instance, [Mi]).

Fact 27. *The following hold.*

(a) *For every coloring $c : [(\exp_d(\kappa))^+]^{d+1} \to V_\kappa$ we have $\mathrm{Sol}^\omega_{d,\kappa}(c) \neq \emptyset$.*
(b) *The family $\mathrm{Sol}^\omega_{d,\kappa}$ is κ-complete. That is, for every $\delta < \kappa$ and every sequence $(A_\xi : \xi < \delta)$ in $\mathrm{Sol}^\omega_{d,\kappa}$ we have that $\bigcap_{\xi<\delta} A_\xi \in \mathrm{Sol}^\omega_{d,\kappa}$.*

Proof. (a) By our assumptions we see that $|V_\kappa| = \kappa$. Moreover, by the classical Erdős-Rado partition Theorem (see [Ku]), we have

$$(\exp_d(\kappa))^+ \to (\kappa^+)^{d+1}_\kappa$$

and the result follows.

(b) For every $\xi < \delta$ let $c_\xi : [\exp_d(\kappa))^+]^{d+1} \to V_\kappa$ be a coloring such that

$A_\xi = \mathrm{Sol}^\omega_{d,\kappa}(c_\xi)$. Observe that $(V_\kappa)^\delta \subseteq V_\kappa$. We define the coloring $c : [\exp_d(\kappa))^+]^{d+1} \to (V_\kappa)^\delta$ by $c(s) = (c_\xi(s) : \xi < \delta)$. Noticing that

$$\bigcap_{\xi<\delta} \mathrm{Sol}^\omega_{d,\kappa}(c_\xi) = \mathrm{Sol}^\omega_{d,\kappa}(c)$$

the proof is completed. □

By Fact 27(b) and our hypothesis that κ is a strongly compact cardinal, we see that there exists a κ-complete ultrafilter $\mathcal{V}$ on $[\exp_d(\kappa))^+]^\omega$ extending the family $\mathrm{Sol}^\omega_{d,\kappa}$. *We fix such an ultrafilter* $\mathcal{V}$.

Definition 28. *A* $\mathcal{V}$–sequence of conditions *is a sequence* $\overline{p} = (p_{\mathbf{x}} : \mathbf{x} \in A)$ *in* $\mathrm{Col}(\lambda, <\kappa)$*, belonging to the ground model* V *and indexed by a member* A *of the ultrafilter* $\mathcal{V}$*. We will refer to the set* A *as the* index set *of* $\overline{p}$ *and we shall denote it by* $I(\overline{p})$*.*

Definition 29. *Let* $\overline{p} = (p_{\mathbf{x}} : \mathbf{x} \in I(\overline{p}))$ *be a* $\mathcal{V}$*–sequence of conditions. We say that a condition* r *in* $\mathrm{Col}(\lambda, <\kappa)$ *is a* root *of* $\overline{p}$ *if*

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ p_{\mathbf{x}} \restriction \alpha = r^2. \tag{9}$$

Related to the above definitions, we have the following.

Fact 30. *Every* $\mathcal{V}$*–sequence of conditions* $\overline{p}$ *has a unique root* $r(\overline{p})$*.*

Proof. For every $\alpha < \kappa$ the map $I(\overline{p}) \ni \mathbf{x} \mapsto p_{\mathbf{x}} \restriction \alpha$ has fewer than κ values. So, by the κ-completeness of $\mathcal{V}$, there exist $p_\alpha \in \mathrm{Col}(\lambda, <\kappa)$ and $I_\alpha \in \mathcal{V} \restriction I(\overline{p})$ so that $p_{\mathbf{x}} \restriction \alpha = p_\alpha$ for all $\mathbf{x} \in I_\alpha$. Hence, we can select a sequence $(p_\alpha : \alpha < \kappa)$ in $\mathrm{Col}(\lambda, <\kappa)$ and a decreasing sequence $(I_\alpha : \alpha < \kappa)$ of elements of $\mathcal{V} \restriction I(\overline{p})$ such that for every $\alpha < \kappa$ and every $\mathbf{x} \in I_\alpha$ we have that $p_{\mathbf{x}} \restriction \alpha = p_\alpha$.

Let $A \subseteq \kappa$ be the set of all limit ordinals $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \lambda$. Since $\mathcal{U}$ is normal, the set A is in $\mathcal{U}$. Consider the mapping $c : A \to \kappa$ defined by

$$c(\alpha) = \sup\{\xi \,:\, \xi \in (\mathrm{dom}(p_\alpha \restriction \alpha))_1\}$$

for every $\alpha \in A$. As $\mathrm{cf}(\alpha) > \lambda$, we get that c is a regressive mapping. The ultrafilter $\mathcal{U}$ is normal, and so, there exist $A' \in \mathcal{U} \restriction A$ and $\gamma_0 < \kappa$ such that $c(\alpha) = \gamma_0$ for every $\alpha \in A'$. Now consider the map

$$A' \ni \alpha \mapsto p_\alpha \restriction \alpha = p_\alpha \restriction \gamma_0 \subseteq (\lambda \times \gamma_0) \times \gamma_0.$$

Noticing that $|\mathcal{P}\big((\lambda \times \gamma_0) \times \gamma_0\big)| < \kappa$ and recalling that $\mathcal{U}$ is κ-complete, we see that there exist $A'' \in \mathcal{U} \restriction A'$ and $r(\overline{p})$ in $\mathrm{Col}(\lambda, <\kappa)$ such that

[2]This is an abbreviation of the statement that $\{\alpha : \{\mathbf{x} : p_{\mathbf{x}} \restriction \alpha = r\} \in \mathcal{V}\} \in \mathcal{U}$.

$p_\alpha \restriction \alpha = r(\overline{p})$ for every $\alpha \in A''$. It follows that for every $\alpha \in A''$ the set $\{\mathbf{x} \in [\exp_d(\kappa))^+]^\omega : p_{\mathbf{x}} \restriction \alpha = r(\overline{p})\}$ contains the set I_α, and so

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ p_{\mathbf{x}} \restriction \alpha = r(\overline{p}).$$

The uniqueness of $r(\overline{p})$ is an immediate consequence of property (9) in Definition 29. The proof is completed. □

We are ready to introduce the ideal $\mathcal{I}_d$.

Definition 31. *In $V[G]$ we define*

$$\mathcal{I}_d = \{I \subseteq [\exp_d(\kappa))^+]^\omega : \text{ there is some } A \in \mathcal{V} \text{ such that } I \cap A = \emptyset\}.$$

We isolate, for future use, the following (easily verified) properties of $\mathcal{I}_d$.

(P1) $\mathcal{I}_d$ is an ideal; in fact, $\mathcal{I}_d$ is a κ-complete ideal.
(P2) $\mathcal{V} \subseteq \mathcal{I}_d^+$.
(P3) If $A \in \mathcal{V}$ and $B \in \mathcal{I}_d^+$, then $A \cap B \in \mathcal{I}_d^+$.

For every $\mathcal{V}$–sequence of conditions $\overline{p}$ we let

$$D_{\overline{p}} = \{\mathbf{x} \in I(\overline{p}) : p_{\mathbf{x}} \in G\}. \tag{10}$$

Now we are ready to introduce the set $\mathcal{D}_d$.

Definition 32. *In $V[G]$ we define*

$$\mathcal{D}_d = \{D_{\overline{p}} : \overline{p} \text{ is a } \mathcal{V}\text{–sequence of conditions in the ground model } V\} \cap \mathcal{I}_d^+.$$

By definition, we have that $\mathcal{D}_d \subseteq \mathcal{I}_d^+$. The rest of the proof will be devoted to the verification that the ideal $\mathcal{I}_d$ and the set $\mathcal{D}_d$ satisfy the requirements of Lemma 7. To this end, we need the following.

Lemma 33. *Let $\overline{p} = (p_{\mathbf{x}} : \mathbf{x} \in I(\overline{p}))$ be a $\mathcal{V}$–sequence of conditions. Then the following are equivalent.*

(1) $D_{\overline{p}} \in \mathcal{D}_d$.
(2) $r(\overline{p}) \in G$.

Proof. (1)⇒(2) Assume that $D_{\overline{p}} \in \mathcal{D}_d$. We use the fact that $D_{\overline{p}} \in \mathcal{I}_d^+$ and that

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ p_{\mathbf{x}} \restriction \alpha = r(\overline{p})$$

to find $\mathbf{x} \in D_{\overline{p}}$ such that $p_{\mathbf{x}} \leq r(\overline{p})$. By the definition of $D_{\overline{p}}$, we see that $p_{\mathbf{x}} \in G$, and so, $r(\overline{p}) \in G$ as well.
(2)⇒(1) Suppose that $r(\overline{p}) \in G$. Fix a ground model set A which is in $\mathcal{V}$. It is enough to show that $D_{\overline{p}} \cap A \neq \emptyset$. To this end, let

$$E = \{q \in \mathrm{Col}(\lambda, <\kappa) : q \perp r(\overline{p}) \text{ or there is } \mathbf{x} \in I(\overline{p}) \cap A \text{ with } q \leq p_{\mathbf{x}}\}.$$

We claim that E is a dense subset of $\mathrm{Col}(\lambda, < \kappa)$. To see this, let $r \in \mathrm{Col}(\lambda, < \kappa)$ arbitrary. If $r \perp r(\overline{p})$, then $r \in E$. So, suppose that $r \parallel r(\overline{p})$. Using this and the fact that

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ \ p_{\mathbf{x}} \restriction \alpha = r(\overline{p})$$

we may find $\mathbf{x} \in I(\overline{p}) \cap A$ such that $p_{\mathbf{x}} \parallel r$. So, there exist $q \in \mathrm{Col}(\lambda, < \kappa)$ and $\mathbf{x} \in I(\overline{p}) \cap A$ such that $q \leq p_{\mathbf{x}}$ and $q \leq r$. In other words, there exists $q \in E$ with $q \leq r$. This establishes our claim that E is a dense subset of $\mathrm{Col}(\lambda, < \kappa)$.

It follows by the above discussion that there exists $q \in G$ with $q \in E$. Since $r(\overline{p}) \in G$ we have that $r(\overline{p}) \parallel q$. Hence, by the definition of the set E, there exists $\mathbf{x} \in I(\overline{p}) \cap A$ with $q \leq p_{\mathbf{x}}$. It follows that $p_{\mathbf{x}} \in G$, and so, $\mathbf{x} \in D_{\overline{p}} \cap A$. The proof is completed. □

Lemma 34. *$\mathcal{D}_d$ is dense in $\mathcal{I}_d^+$.*

Proof. Fix $J \in \mathcal{I}_d^+$. We will prove that there exists a $\mathcal{V}$–sequence of conditions $\overline{q}$ in the ground model V satisfying $D_{\overline{q}} \in \mathcal{D}_d$ and $D_{\overline{q}} \subseteq J$. This will finish the proof.

To this end, we fix a $\mathrm{Col}(\lambda, < \kappa)$-name $\dot{J}$ for J. Let $p \in \mathrm{Col}(\lambda, < \kappa)$ be an arbitrary condition such that $p \Vdash \dot{J} \notin \mathcal{I}_d$. Define, in the ground model V, the set

$$A_p = \{\mathbf{x} \in [\exp_d(\kappa))^+]^\omega \ : \text{ there is } q \leq p \text{ such that } q \Vdash \check{\mathbf{x}} \in \dot{J}\}.$$

First we claim that $A_p \in \mathcal{V}$. Suppose, towards a contradiction, that the set $C := [\exp_d(\kappa))^+]^\omega \setminus A_p$ is in $\mathcal{V}$. Since $J \in \mathcal{I}_d^+$ we see that $J \cap C \neq \emptyset$ in $V[G]$. Using the fact that $p \Vdash \dot{J} \notin \mathcal{I}_d$ and that the forcing $\mathrm{Col}(\lambda, < \kappa)$ is σ-closed, we may find $\mathbf{x} \in C$ and a condition $q \leq p$ such that $q \Vdash \check{\mathbf{x}} \in \dot{J}$. But this implies that $\mathbf{x} \in A_p$, a contradiction.

It follows that we may select a $\mathcal{V}$–sequence of conditions $\overline{q} = (q_{\mathbf{x}} : \mathbf{x} \in A_p)$ such that $q_{\mathbf{x}} \leq p$ and $q_{\mathbf{x}} \Vdash \check{\mathbf{x}} \in \dot{J}$ for every $\mathbf{x} \in A_p$. By Fact 30, let $r(\overline{q})$ be the root of $\overline{q}$. Clearly $r(\overline{q}) \leq p$.

Now, fix a condition r such that $r \Vdash \dot{J} \notin \mathcal{I}_d$. What we have just proved is that the set of conditions $r(\overline{q})$ such that

(∗) $r(\overline{q})$ is the root of a $\mathcal{V}$–sequence of conditions $\overline{q} = (q_{\mathbf{x}} : \mathbf{x} \in I(\overline{q}))$ with the property that $q_{\mathbf{x}} \Vdash \check{\mathbf{x}} \in \dot{J}$ for every $\mathbf{x} \in I(\overline{q})$

is dense below r. As G is generic, we see that there exists a $\mathcal{V}$–sequence of conditions $\overline{q}$ as in (∗) above such that $r(\overline{q}) \in G$. On the one hand, by Lemma 33, we see that $D_{\overline{q}} \in \mathcal{D}_d$. On the other hand, property (∗) above

implies that $D_{\overline{q}} \subseteq J$; indeed, if $\mathbf{x} \in D_{\overline{q}}$, then $q_{\mathbf{x}} \in G$ and, by $(*)$, $q_{\mathbf{x}} \Vdash \check{\mathbf{x}} \in \dot{J}$. The proof is completed. $\square$

Lemma 35. *$\mathcal{D}_d$ is λ-closed in $\mathcal{I}_d^+$.*

Proof. Fix $\mu < \lambda$ and a decreasing sequence $(D_\xi : \xi < \mu)$ in $\mathcal{D}_d$. For every $\xi < \mu$ let $\overline{p}_\xi = (p_{\mathbf{x}}^\xi : \mathbf{x} \in I(\overline{p}_\xi))$ be a $\mathcal{V}$–sequence of conditions in V such that $D_\xi = D_{\overline{p}_\xi}$. Our forcing $\mathrm{Col}(\lambda, < \kappa)$ is λ-closed, and so, the sequence $(\overline{p}_\xi : \xi < \mu)$ is in the ground model V as well. Applying Fact 30 to every $\overline{p}_\xi$, we find a sequence $(r_\xi : \xi < \mu)$ in $\mathrm{Col}(\lambda, < \kappa)$ such that r_ξ is the root of $\overline{p}_\xi$ for every $\xi < \mu$. By Lemma 33, we get that $r_\xi \in G$ for all $\xi < \mu$.

We claim, first, that for every $\xi < \zeta < \mu$ we have

$$(\mathcal{V}\mathbf{x}) \;\; p_{\mathbf{x}}^\xi \parallel p_{\mathbf{x}}^\zeta. \tag{11}$$

Suppose, towards a contradiction, that there exist $\xi < \zeta < \mu$ such that the set $L := \{\mathbf{x} \in A : p_{\mathbf{x}}^\xi \perp p_{\mathbf{x}}^\zeta\}$ is in $\mathcal{V}$. As $D_{\overline{p}_\zeta} \in \mathcal{D}_d \subseteq \mathcal{I}_d^+$ and $L \in \mathcal{V}$, there exists $\mathbf{x} \in D_{\overline{p}_\zeta} \cap L$. And since $D_{\overline{p}_\zeta} = D_\zeta \subseteq D_\xi = D_{\overline{p}_\xi}$ we have $\mathbf{x} \in D_{\overline{p}_\xi}$ as well. But this implies that both $p_{\mathbf{x}}^\xi$ and $p_{\mathbf{x}}^\zeta$ are in G and at the same time $p_{\mathbf{x}}^\xi \perp p_{\mathbf{x}}^\zeta$, a contradiction.

Invoking (11) above, we may find $A \in \mathcal{V}$ such that for every $\xi < \zeta < \mu$ and every $\mathbf{x} \in A$ we have that $p_{\mathbf{x}}^\xi \parallel p_{\mathbf{x}}^\zeta$. We set

$$p_{\mathbf{x}} = \bigcup_{\xi < \mu} p_{\mathbf{x}}^\xi \quad \text{for every } \mathbf{x} \in A$$

and we define $\overline{p} = (p_{\mathbf{x}} : \mathbf{x} \in A)$. It is clear that $\overline{p}$ is a well-defined $\mathcal{V}$–sequence of conditions. Also observe that $D_{\overline{p}} \subseteq D_\xi$ for every $\xi < \mu$. We are going to show that $D_{\overline{p}} \in \mathcal{D}_d$. This will finish the proof.

To this end, let r be the root of $\overline{p}$. By Lemma 33, it is enough to show that $r \in G$. Notice, first, that

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x}) \;\; \bigcup_{\xi < \mu} p_{\mathbf{x}}^\xi \restriction \alpha = p_{\mathbf{x}} \restriction \alpha = r. \tag{12}$$

On the other hand, as r_ξ is the root of $\overline{p}_\xi$, we have

$$(\forall \xi < \mu)\ (\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x}) \;\; p_{\mathbf{x}}^\xi \restriction \alpha = r_\xi. \tag{13}$$

Both $\mathcal{U}$ and $\mathcal{V}$ are κ-complete, and so, (13) is equivalent to

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x})\ (\forall \xi < \mu) \;\; p_{\mathbf{x}}^\xi \restriction \alpha = r_\xi. \tag{14}$$

Combining (12) and (14) we get that

$$(\mathcal{U}\alpha)\ (\mathcal{V}\mathbf{x}) \;\; r = \bigcup_{\xi < \mu} p_{\mathbf{x}}^\xi \restriction \alpha = \bigcup_{\xi < \mu} r_\xi. \tag{15}$$

Summing up, we see that the root r of $\overline{p}$ is the union $\bigcup_{\xi<\mu} r_\xi$ of the roots of the $\overline{p}_\xi$'s. Since the generic filter G is λ-complete, we conclude that $r \in G$. The proof is completed. $\square$

Lemma 36. *Work in $V[G]$. Let $\mu < \kappa$ and let $c : [\exp_d(\kappa))^+]^{d+1} \to \mu$ be a coloring. Let also $A \in \mathcal{I}_d^+$ arbitrary. Then there exist a color $\xi < \mu$ and an element $D \in \mathcal{D}_d$ with $D \subseteq A$ and such that for every $\mathbf{x} \in D$ and every $\{\alpha_0, \dots, \alpha_d\} \in [\mathbf{x}]^{d+1}$ we have $c(\{\alpha_0, \dots, \alpha_d\}) = \xi$.*

Proof. Fix a coloring $c : [\exp_d(\kappa))^+]^{d+1} \to \mu$ and let $A \in \mathcal{I}_d^+$. Let also $\dot{c}$ be a $\mathrm{Col}(\lambda, <\kappa)$-name for the coloring c. In V, let $\mathrm{RO}(\mathrm{Col}(\lambda, <\kappa))$ be the collection of all regular-open subsets of $\mathrm{Col}(\lambda, <\kappa)$. Working in V, we define another coloring $C : [\exp_d(\kappa))^+]^{d+1} \to (\mathrm{RO}(\mathrm{Col}(\lambda, <\kappa)))^\mu$ by the rule

$$C(s) = ([\![\dot{c}(\check{s}) = \check{\xi}]\!] : \xi < \mu)$$

where $[\![\dot{c}(\check{s}) = \check{\xi}]\!] = \{p \in \mathrm{Col}(\lambda, <\kappa) \,:\, p \Vdash \dot{c}(\check{s}) = \check{\xi}\}$ is the boolean value of the formula "$c(s) = \xi$".

The forcing $\mathrm{Col}(\lambda, <\kappa)$ is κ-cc, and so, $(\mathrm{RO}(\mathrm{Col}(\lambda, <\kappa)))^\mu \subseteq V_\kappa$. Hence, $\mathrm{Sol}^\omega_{d,\kappa}(C) \in \mathcal{V}$. We set $J = A \cap \mathrm{Sol}^\omega_{d,\kappa}(C)$. Then J is in $\mathcal{I}_d^+$. Notice that for every $\mathbf{x} \in J$ and every $s, s' \in [\mathbf{x}]^{d+1}$ we have $C(s) = C(s')$. It follows that for every $\mathbf{x} \in J$ we may select a sequence $\overline{U}_\mathbf{x} = (U^\xi_\mathbf{x} : \xi < \mu)$ in $(\mathrm{RO}(\mathrm{Col}(\lambda, <\kappa)))^\mu$ such that for every $s \in [\mathbf{x}]^{d+1}$ and every $\xi < \mu$ we have $[\![\dot{c}(\check{s}) = \check{\xi}]\!] = U^\xi_\mathbf{x}$.

Now observe that for every $s \in [\exp_d(\kappa))^+]^{d+1}$ the set

$$\{[\![\dot{c}(\check{s}) = \check{\xi}]\!] \,:\, \xi < \mu\}$$

is a maximal antichain. So, we can naturally define in $V[G]$ a coloring $e : J \to \mu$ by the rule

$$e(\mathbf{x}) = \xi \ \text{ if and only if } \ U^\xi_\mathbf{x} \in G.$$

Equivalently, for every $\mathbf{x} \in J$ we have that $e(\mathbf{x}) = \xi$ if and only if $c \restriction [\mathbf{x}]^{d+1}$ is constant with value ξ. The ideal $\mathcal{I}_d$ is κ-complete and $J \in \mathcal{I}_d^+$. Hence there exists $\xi_0 < \mu$ such that $e^{-1}\{\xi_0\} \in \mathcal{I}_d^+$. By Lemma 34, we may select $D \in \mathcal{D}_d$ with $D \subseteq e^{-1}\{\xi_0\} \subseteq J \subseteq A$. Finally, notice that for every $\mathbf{x} \in D$ the restriction $c \restriction [\mathbf{x}]^{d+1}$ is constant with value ξ_0. The proof is completed. $\square$

We are ready to finish the proof of Lemma 7. As we have already mention, the ideal $\mathcal{I}_d$ will be the one defined in Definition 31, while the dense subset $\mathcal{D}_d$ of $\mathcal{I}_d^+$ will be the one defined in Definition 32. First, we notice that property (1) in Lemma 7 (i.e. the fact that $\mathcal{I}_d$ is κ-complete) follows easily

by the definition $\mathcal{I}_d$ and the fact that $\mathcal{V}$ is κ-complete (in fact, we have already isolated this property of $\mathcal{I}_d$ in (P1) above). Property (2) in Lemma 7 (i.e. the fact that $\mathcal{D}_d$ is λ-closed in $\mathcal{I}_d^+$) has been established in Lemma 35. Finally, property (3) was proved in Lemma 36. Since $d \geq 1$ was arbitrary, the proof of Lemma 7 is completed.

6. Concluding remarks

In this section we would like to discuss the possible refinements of our Theorem 2. First of all we notice that Ketonen's arguments actually give that if the density of a given Banach space E is greater or equal than the ω-Erdős cardinal, then E contains a normalized basic sequence which is equivalent to all of its subsequences, i.e. a basic sequence which is in the literature usually called a *sub-symmetric basic sequence*. Note that this is stronger than saying that the space E contains an unconditional basic sequence which can be easily seen using Rosenthal's ℓ_1 theorem [Ro].

On the other hand, we notice that our proof of the existence of an unconditional basic sequence in every Banach space of density $\exp_\omega(\aleph_0)$ *does not* guarantee the existence of a sub-symmetric basic sequence. This is mainly due to the fact that the principle $\mathrm{Pl}_2(\kappa)$ is a rectangular Ramsey property while all attempts that we have in mind for getting sub-symmetric basic sequences seem to require more classical Ramsey-type principles such as these given, for example, by the ω-Erdős cardinal. Since ω-Erdős is a large-cardinal property one might expect that there are Banach spaces of large density not containing a sub-symmetric basic sequence. So let us discuss some difficulties one encounters when trying to build such spaces.

The first example of an infinite dimensional Banach space not containing a sub-symmetric basic sequence is Tsirelson's space [Ts]. Tsirelson's space is separable; however, there do exist non-separable Banach spaces with the same property. The first such example is due to E. Odell [O1]. Odell's space is the dual of a separable one, and so, it has density $2^{\aleph_0}$. There even exist non-separable *reflexive* spaces not containing a sub-symmetric basic sequence. For example, one such a space is the space constructed in [ALT] which has density $\aleph_1$. We note that both spaces of [O1] and of [ALT] are connected in some way to the Tsirelson space. So one is led to explore generalizations of the Tsirelson construction to larger densities.

Let us comment on difficulties encountered when trying to generalize Tsirelson's construction to densities bigger than the continuum, keeping

in mind that we would like to get a space not containing a sub-symmetric basic sequence. The first natural move is to provide, for a given cardinal κ, a compact hereditary family $\mathcal{F}$ of finite subsets of κ which is sufficiently rich in the sense that for every infinite subset M of κ the restriction $\mathcal{F} \restriction M$ of the family on M has infinite rank. Notice that such a family cannot exist if κ is greater or equal the ω-Erdős cardinal. On the other hand, using a characterization of n-Mahlo cardinals due to J. H. Schmerl (see [Sch] or [Tod, Theorem 6.1.8]), we were able to show that if κ is smaller that the first ω-Mahlo cardinal, then κ carries such a family $\mathcal{F}$.

Given a compact hereditary family $\mathcal{F}$ as above, the next step is to construct the Tsirelson-like space $T(\mathcal{F})$ on $c_{00}(\kappa)$ in the natural way. Such a space always fails to contain c_0 and ℓ_p for any $1 < p < \infty$. However, there are examples of such families for which the corresponding space contains a copy of ℓ_1. The reason is that the family $\mathcal{F}$ cannot be *spreading* relative the natural well-ordering of ordinals if κ is uncountable. Recall that spreading is a crucial property of the Schreier family on ω used in the original Tsirelson construction for preventing isomorphic copies of ℓ_1. We are grateful to Spiros A. Argyros for pointing out this to us after reading a previous version of this paper containing the erroneous claim that $T(\mathcal{F})$ contains no isomorphic copy of ℓ_1.

In fact, in order to prevent the embedding of ℓ_1 inside $T(\mathcal{F})$ it suffices, beside the above requirements, to ensure that the family $\mathcal{F}$ is *weak spreading* in the sense that if $\alpha_0 \leq \beta_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n$ and $\{\alpha_0, \ldots, \alpha_n\}$ is in the family $\mathcal{F}$, then $\{\beta_0, \ldots, \beta_n\}$ is also in $\mathcal{F}$. It is unclear to us whether such a family $\mathcal{F}$ can exist on a cardinal κ greater than $\aleph_1$.

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