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Georges Gras, Jean-François Jaulent

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# Note on 2-rational fields

Georges GRAS & Jean-François JAULENT

**Résumé.** Nous déterminons le groupe de Galois de la pro-2-extension 2-ramifiée maximale d'un corps de nombres 2-rationnel.

**Abstract.** We compute the Galois group of the maximal 2-ramified and complexified pro-2-extension of any 2-rational number field.

**Nota.** This short Note is motivated by the paper “Galois 2-extensions unramified outside 2” of J. Jossey and, at this occasion, we bring into focus some classical techniques of abelian  $\ell$ -ramification which, unfortunately, are often ignored, especially those developed by J-F. Jaulent with the  $\ell$ -adic class field theory, and by G. Gras in his book on class field theory, and which considerably simplify proofs in such subjects; for instance, the main Theorem 2, due to J-F. Jaulent, generalizes the purpose of Jossey’s paper in such a way.

## 1 Introduction and history

The notions of  $\ell$ -rational field and  $\ell$ -regular field (for a prime number  $\ell$  and a number field  $K$ ), independently introduced by A. Movahhedi and T. Nguyen Quang Do in [MN], and by G. Gras and J-F. Jaulent in [GJ], coincide as soon as  $K$  contains the maximal real subfield of the field of  $\ell$ th roots of unity, thus especially for  $\ell = 2$ .

- The  $\ell$ -regularity expresses the triviality of the regular  $\ell$ -kernel of  $K$  (*i.e.* the kernel, in the  $\ell$ -part of the universal group  $K_2(K)$ , of Hilbert symbols attached to the non-complex places not dividing  $\ell$ ).
- The  $\ell$ -rationality traduces the pro- $\ell$ -freeness of the Galois group  $\mathcal{G}_K := \text{Gal}(M_K/K)$  of the maximal pro- $\ell$ -extension  $\ell$ -ramified  $\infty$ -split  $M_K$  of  $K$  (*i.e.* unramified at the finite places<sup>1</sup> not dividing  $\ell$  and totally split at the infinite places).

More precisely, let  $c_K$  be the number of complex places of  $K$ ; let  $\mu_K$  (resp.  $\mu_{K_\mathfrak{l}}$ ) be the  $\ell$ -group of roots of unity in  $K$  (resp. in the localization  $K_\mathfrak{l}$ ); and let

$$V_K := \{x \in K^\times \mid x \in K_\mathfrak{l}^{\times \ell} \forall \mathfrak{l} \mid \ell \ \& \ v_\mathfrak{p}(x) \equiv 0 \pmod{\ell} \ \forall \mathfrak{p} \nmid \ell \infty\}$$

be the group of  $\ell$ -hyperprimary elements in  $K^\times$ . Then, with these notations, from [JN, Th.1.2] or [G<sub>3</sub>, IV.3.5, III.4.2.3], the  $\ell$ -rationality of  $K$  may be expressed as follows:

<sup>1</sup>According to the conventions of the  $\ell$ -adic class field theory (*cf.* [G<sub>3</sub>, Ja]), we never speak of ramification at infinity but of *complexification* of real places.

**Theorem and definition 0.** *The following conditions are equivalent:*

- (i) *The Galois group  $\mathcal{G}_K$  is a free pro- $\ell$ -group on  $1 + c_K$  generators.*
- (ii) *The abelianization  $\mathcal{G}_K^{ab}$  of  $\mathcal{G}_K$  is a free  $\mathbb{Z}_\ell$ -module of dimension  $1 + c_K$ .*
- (iii) *The field  $K$  satisfies the Leopoldt conjecture (for the prime  $\ell$ ) and the torsion submodule  $\mathcal{T}_K$  of  $\mathcal{G}_K^{ab}$  is trivial.*
- (iv) *One has the equalities:  $V_K = K^{\times\ell} \quad \& \quad \text{rk}_\ell(\boldsymbol{\mu}_K) = \sum_{i|\ell} \text{rk}_\ell(\boldsymbol{\mu}_{K_i})$ .*

When any of these conditions is realized, the number field  $K$  is said to be  $\ell$ -rational.

**Remark.** In case  $\ell = 2$ , it follows from the condition (iv) above that a 2-rational field has a *single* place above 2.

The premises of the notion of  $\ell$ -regularity go back to the works of G. Gras, mainly to his note on the  $K_2$  of number fields [G<sub>2</sub>, II, § 2; III, §§ 1, 2], whereas the notion of  $\ell$ -rationality appears (in a hidden form) in the work of H. Miki [Mi] concerning the study of a sufficient condition for the Leopoldt conjecture, as well as those of K. Wingberg [W<sub>1</sub>, W<sub>2</sub>], concerning the same condition.

Movahhedi's thesis and the above papers [GJ, MN] characterised the going up for  $\ell$ -rationality in any  $\ell$ -extension in terms of  $\ell$ -primitivity of the ramification (a definition given in [G<sub>2</sub>, III, § 1] from the use of the Log function defined in [G<sub>1</sub>]), a property which was unknown in the preceding approaches.

For instance, this gives immediately that if  $K$  is a  $\ell$ -extension of  $\mathbb{Q}$ , a N.S.C. for  $K$  to be  $\ell$ -rational is that  $K/\mathbb{Q}$  be  $\ell$ -ramified, or that  $K/\mathbb{Q}$  be  $\{p, \ell\}$ -ramified, where  $p \neq \ell$  is a prime  $\equiv 1 \pmod{\ell}$  such that  $p \not\equiv \pm 1 \pmod{8}$  if  $\ell = 2$  and  $p \not\equiv 1 \pmod{\ell^2}$  if  $\ell \neq 2$  (cf. [G<sub>3</sub>, IV.3.5.1] giving Jossey's examples [Jo]).

A synthesis of these results is given in [JN] and a systematic exposition is developed in the book of G. Gras ([G<sub>3</sub>, III, § 4, (b); IV, § 3, (b); App., § 2]).

Various generalizations of these notions have been studied by O. Sauzet and J-F. Jaulent (cf. [JS<sub>1</sub>, JS<sub>2</sub>]), especially in the case  $\ell = 2$  which is, as usual, the most tricky; in particular, they introduce the notion of 2-birational fields.

Very recently, J. Jossey [Jo] introduced a notion of  $\ell$ -rationality which is incompatible with the classical one (for  $\ell = 2$ , as soon as  $K$  contains real embeddings) and is unlucky since it does not apply to the field of rationals  $\mathbb{Q}$ .

For these reasons, to avoid any confusion, we propose to speak, in his context, of *2-superrational* fields. More precisely:

**Definition 1.** Let  $K$  be a number field with  $r_K$  real places and  $c_K$  complex places,  $M'_K$  the maximal 2-ramified pro-2-extension of  $K$ , and  $M_K$  the maximal subextension of  $M'_K$  totally split at the infinite places. We say that  $K$  is:

- (i) *2-superrational*, if  $\mathcal{G}'_K := \text{Gal}(M'_K/K)$  is pro-2-free;
- (ii) *2-rational*, if its quotient  $\mathcal{G}_K := \text{Gal}(M_K/K)$  is pro-2-free.

The purpose of the next section is to determine the structure of the Galois group  $\mathcal{G}'_K$  when the number field  $K$  is 2-rational.

Our proof relies on the functorial properties of  $\ell$ -ramification theory.

## 2 Main Theorem: description of $\mathcal{G}'_K = \text{Gal}(M'_K/K)$

Our very simple result has the following statement:

**Theorem 2.** *Let  $K$  be a 2-rational number field having  $r_K$  real places and  $c_K$  complex places. The Galois group  $\mathcal{G}'_K := \text{Gal}(M'_K/K)$  of the maximal 2-ramified pro-2-extension  $M'_K$  of  $K$  is the pro-2-free product*

$$\mathcal{G}'_K \simeq \mathbb{Z}_2^{\otimes(1+c_K)} \otimes (\mathbb{Z}/2\mathbb{Z})^{\otimes r_K}$$

of  $(1 + c_K)$  copies of the procyclic group  $\mathbb{Z}_2$  and of  $r_K$  copies of  $\mathbb{Z}/2\mathbb{Z}$ .

**Corollary 3.** *The 2-rational number fields which are 2-superrational are the totally imaginary ones.*

*Proof.* Consider the quadratic extension  $L = K[i]$  generated by the 4th roots of unity. It is 2-ramified over  $K$ , thus thanks to the going up theorem of [GJ, MN] (cf. e.g. [JN, Th. 3.5] or [G<sub>3</sub>, IV.3.4.3, (iii)]), it is 2-rational, then 2-superrational since it is totally imaginary. In other words, the Galois group  $\mathcal{G}_L = \mathcal{G}'_L$  of the maximal 2-ramified pro-2-extension  $M_L$  of  $L$  is pro-2-free.

Since the quadratic extension  $L/K$  is 2-ramified,  $M_L$  is also the maximal 2-ramified pro-2-extension  $M'_K$  of  $K$ ; the Galois group  $\mathcal{G}'_K$  is *potentially free* since it contains the pro-2-free open subgroup  $\mathcal{G}_L$  of index 2 in  $\mathcal{G}'_K$ .

As in [Jo], the results of W. Herfort and P. Zalesskii (cf. [HZ, Th.0.2]) give the existence of a finite family  $(\mathcal{F}_i)_{i=0, \dots, k}$  of free pro-2-groups on respectively  $d_0, \dots, d_k$  generators (where  $k$  is the number of conjugacy classes of subgroups of order 2 in  $\mathcal{G}'_K$ ), such that:

$$\mathcal{G}'_K \simeq \mathcal{F}_0 \otimes \left( \bigotimes_{i=1}^k (\mathcal{F}_i \times \mathbb{Z}/2\mathbb{Z}) \right).$$

In particular, the abelianisation  $\mathcal{G}'_K{}^{ab}$  of  $\mathcal{G}'_K$  admits the direct decomposition:

$$\mathcal{G}'_K{}^{ab} \simeq \mathbb{Z}_2^{d_0} \oplus \left( \bigoplus_{i=1}^k (\mathbb{Z}_2^{d_i} \oplus \mathbb{Z}/2\mathbb{Z}) \right) \simeq \mathbb{Z}_2^{d_0+d_1+\dots+d_k} \oplus (\mathbb{Z}/2\mathbb{Z})^k.$$

Since the 2-rational field  $K$  satisfies the Leopoldt conjecture, we get  $\sum_{i=0}^k d_i = 1 + c_K$  as well as the isomorphism  $\mathcal{T}'_K := \text{tor}_{\mathbb{Z}_2}(\mathcal{G}'_K{}^{ab}) \simeq (\mathbb{Z}/2\mathbb{Z})^k$ . Moreover  $\mathcal{T}_K := \text{tor}_{\mathbb{Z}_2}(\mathcal{G}_K{}^{ab}) = 1$ , so that  $\mathcal{T}'_K$  is generated by the decomposition groups of the real places of  $K$  which are deployed, a key argument of class field theory (cf. [Ja] or [G<sub>3</sub>, III.4.1.5]) giving  $k = r_K$ .

Now the pro-2-decomposition of  $\mathcal{G}'_K$  clearly shows that the minimal number of generators  $d(\mathcal{G}'_K)$  and of relations  $r(\mathcal{G}'_K)$ , defining  $\mathcal{G}'_K$  as a pro-2-group, are:

$$d(\mathcal{G}'_K) = k + \sum_{i=0}^k d_i = r_K + 1 + c_K \text{ and } r(\mathcal{G}'_K) = \sum_{i=1}^k (1 + d_i) = d(\mathcal{G}'_K) - d_0.$$

It is well-known (cf. e.g. [G<sub>3</sub>, App., Th.2.2, (i)]) that one has<sup>2</sup>:

$$d(\mathcal{G}'_K) - r(\mathcal{G}'_K) = \dim_{\mathbb{F}_2}(H^1(\mathcal{G}'_K, \mathbb{F}_2)) - \dim_{\mathbb{F}_2}(H^2(\mathcal{G}'_K, \mathbb{F}_2)) = 1 + c_K.$$

Thus we obtain  $d_0 = 1 + c_K$ , giving  $d_i = 0$  for  $1 \leq i \leq k$ , then the expected result.  $\square$

<sup>2</sup>This argument is equivalent to the use of the formulas of Šafarevič (cf. [Sa] or [NSW, Th.8.7.3])

## References

- [G<sub>1</sub>] G. GRAS, *Logarithme  $p$ -adique et groupes de Galois*, J. reine angew. Math. **343** (1983), 64–80.
- [G<sub>2</sub>] G. GRAS, *Remarks on  $K_2$  of number fields*, J. Number Th. **23** (1986), 64–80 .
- [G<sub>3</sub>] G. GRAS, *Class Field Theory : from theory to practice*, SMM 2003 ; second corrected printing 2005.
- [GJ] G. GRAS & J-F. JAULENT, *Sur les corps de nombres réguliers*, Math. Z. **202** (1989), 343–365.
- [HZ] W. HERFORT & P. ZALESSKII, *Finitely generated pro-2-groups with a free subgroup of index 2*, Manuscripta Math. **93** (1997), 457–464.
- [Ja] J-F. JAULENT, *Théorie  $\ell$ -adique globale du corps de classes*, J. Théor. Nombres Bordeaux **10** (1998), 355–397.
- [JN] J-F. JAULENT & T. NGUYEN QUANG DO, *Corps  $p$ -rationnels, corps  $p$ -réguliers et ramification restreinte*, J. Théor. Nombres Bordeaux **5** (1993), 343–365.
- [JS<sub>1</sub>] J-F. JAULENT & O. SAUZET, *Pro- $\ell$ -extensions de corps de nombres  $\ell$ -rationnels*, J. Number Th. **65** (1997), 240–267 & **80** (2000), 318–319.
- [JS<sub>2</sub>] J-F. JAULENT & O. SAUZET, *Extensions quadratiques 2-birationnelles de corps totalement réels*, Pub. Mathématiques **44** (2000), 343–351.
- [Jo] J. JOSSEY *Galois 2-extensions unramified outside 2*, J. Number Th. **124** (2007), 42–76.
- [Mi] H. MIKI, *On the Leopoldt conjecture on the  $p$ -adic regulators*, J. Numb. Th. **26** (1987), 117–128.
- [MN] A. MOVAHHEDI & T. NGUYEN QUANG DO, *Sur l'arithmétique des corps de nombres  $p$ -rationnels*, Sémin. Th. Nombres Paris (1987/1988), Prog. in Math. **89** (1990), 155–200.
- [NSW] J. NEUKIRCH, A. SCHMIDT & K. WINGBERG, *Cohomology of number fields*, Grund. der Math. Wissenschaften 323, Springer-Verlag 2000 ; second edition 2008.
- [Sa] I. R. ŠAFAREVIČ, *Extensions with prescribed ramification points*, Pub. Math. I.H.E.S. **36** (1986), 71–95.
- [W<sub>1</sub>] K. WINGBERG, *On Galois groups of  $p$ -closed algebraic number fields with restricted ramification*, J. reine angew. Math. **400** (1989), 185–202.
- [W<sub>2</sub>] K. WINGBERG, *On Galois groups of  $p$ -closed algebraic number fields with restricted ramification II*, J. reine angew. Math. **416** (1991), 187–194.

Georges GRAS

Villa la Gardette  
chemin Château Gagnière  
F-38520 Le Bourg d'Oisans  
g.mn.gras@wanadoo.fr

Jean-François JAULENT

Institut de Mathématiques  
Université de BORDEAUX  
351, cours de la Libération  
F-33405 TALENCE Cedex  
jaulent@math.u-bordeaux1.fr