Cluster categories for algebras of global dimension 2 and quivers with potential

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CLUSTER CATEGORIES FOR ALGEBRAS OF GLOBAL DIMENSION 2 AND QUIVERS WITH POTENTIAL

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Abstract. Let k be a field and A a finite-dimensional k-algebra of global dimension ≤ 2. We construct a triangulated category $\mathcal{C}_A$ associated to A which, if A is hereditary, is triangle equivalent to the cluster category of A. When $\mathcal{C}_A$ is Hom-finite, we prove that it is 2-CY and endowed with a canonical cluster-tilting object. This new class of categories contains some of the stable categories of modules over a preprojective algebra studied by Geiss-Leclerc-Schröer and by Buan-Iyama-Reiten-Scott. Our results also apply to quivers with potential. Namely, we introduce a cluster category $\mathcal{C}_{(Q,W)}$ associated to a quiver with potential (Q,W). When it is Jacobi-finite we prove that it is endowed with a cluster-tilting object whose endomorphism algebra is isomorphic to the Jacobian algebra $\mathcal{J}(Q,W)$.

Contents

Introduction

The cluster category associated with a finite-dimensional hereditary algebra was introduced in [BMR+06] (and in [CCS06] for the A_n case). It serves in the representation-theoretic approach to cluster algebras introduced and studied by Fomin and Zelevinsky in a series of articles (cf. [FZ02], [FZ03], [FZ07] and [BFZ05] with Berenstein). The link between cluster algebras and cluster categories is in the spirit of 'categorification'. Several articles (e.g. [MRZ03], [BMR+06], [CK08], [CC06], [BMR07], [BMR08], [BMRT07], [CK06]) deal with the categorification of the cluster algebra A_Q associated with an acyclic quiver Q using the cluster category $\mathcal{C}_Q$ associated with the path algebra of the quiver Q. Another approach consists in categorifying cluster algebras by subcategories of the category of modules over a preprojective algebra associated to an acyclic quiver (cf. [GLS07a], [GLS06a], [GLS06b], [GLS07b], [BIRS07]). In both approaches the categories $\mathcal{C}$ (or their associated stable categories) satisfy the following fundamental properties:
- $\mathcal{C}$ is a triangulated category;
- $\mathcal{C}$ is 2-Calabi-Yau (2-CY for short);
- there exist cluster-tilting objects.

It has been shown that these properties alone imply many of the most important theorems about cluster categories, respectively stable module categories over preprojective algebras (cf. [IY06], [KR06], [KR07], [Kel08a], [Pal], [Tab07]). In particular by [IY06], in a category $\mathcal{C}$ with such properties it is possible to 'mutate' the cluster-tilting objects and there exist exchange triangles. This is fundamental for categorification.

Let k be a field. In this article we want to generalize the construction of the cluster category replacing the hereditary algebra kQ by a finite-dimensional algebra A of finite global dimension. A candidate might be the orbit category $\mathcal{D}^b(A)/\nu[-2]$, where ν is the Serre functor of the derived category $\mathcal{D}^b(A)$. By [Kel05], such a category is triangulated if A is derived equivalent to an hereditary category $\mathcal{H}$. However in general, it is not triangulated. Thus a more appropiate candidate is the triangulated hull $\mathcal{C}_A$ of the orbit category $\mathcal{D}^b(A)/\nu[-2]$. It is defined in [Kel05] as the stabilization of a certain dg category and contains the orbit category as a full subcategory. More precisely the category $\mathcal{C}_A$ is a quotient of a triangulated category $\mathcal{T}$ by a thick subcategory $\mathcal{N}$ which is 3-CY. This leads us to the study of such quotients in full generality. We prove that the quotient is 2-CY if the objects of $\mathcal{T}$ are 'limits' of objects of $\mathcal{N}$ (Theorem 1.3). In particular we deduce that the cluster category $\mathcal{C}_A$ of an algebra of finite global dimension is 2-CY if it is Hom-finite (Corollary 4.5).

We study the particular case where the algebra is of global dimension ≤ 2. Since kQ is a cluster-tilting object of the category $\mathcal{C}_Q$, the canonical candidate to be a cluster-tilting object in the category $\mathcal{C}_A$ would be A itself. Using generalized tilting theory (cf. [Kel94]), we give another construction of the cluster category. We find a triangle equivalence

$$\mathcal{C}_A \xrightarrow{\sim} \mathsf{per}\,\Pi/\mathcal{D}^b\Pi$$

where Π is a dg algebra in negative degrees which is bimodule 3-CY and homologically smooth. This equivalence sends the object A onto the image of the free dg module Π in the quotient. This leads us to the study of the categories $\mathsf{per}\,\Gamma/\mathcal{D}^b\Gamma$ where Γ is a dg algebra with the above properties. We prove that if the zeroth cohomology of Γ is finite-dimensional, then the category $\mathsf{per}\,\Gamma/\mathcal{D}^b\Gamma$ is 2-CY and the image of the free dg module Γ is a cluster-tilting object (Theorem 2.1). We show that the algebra $H^0\Gamma$ is finite-dimensional if and only if the quotient $\mathsf{per}\,\Gamma/\mathcal{D}^b\Gamma$ is Hom-finite. Thus we prove the existence of a cluster-tilting object in cluster categories $\mathcal{C}_A$ associated with algebras of global dimension 2 which are Hom-finite (Theorem 4.10). Moreover, this general approach applies to the Ginzburg dg algebras [Gin06] associated with a quiver with potential. Therefore we introduce a new class of 2-CY categories $\mathcal{C}_{(Q,W)}$ endowed with a cluster-tilting object associated with a Jacobi-finite quiver with potential (Q, W) (Theorem 3.6).

In [GLS07b], Geiss, Leclerc and Schröer construct subcategories $\mathcal{C}_M$ of $\mathsf{mod}\,\Lambda$ (where $\Lambda = \Lambda_Q$ is a preprojective algebra of an acyclic quiver) associated with certain terminal kQ-modules M. We show in the last part that the stable category of such a Frobenius category $\mathcal{C}_M$ is triangle equivalent to a cluster category $\mathcal{C}_A$ where A is the endomorphism algebra of a postprojective module over an hereditary algebra (Theorem 5.15). Another approach is given by Buan, Iyama, Reiten and Scott in [BIRS07]. They construct 2-Calabi-Yau triangulated categories $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ where $\mathcal{I}_w$ is a two-sided ideal of the preprojective algebra $\Lambda = \Lambda_Q$ associated with an element w of the Weyl group of Q. For certain elements w of the Weyl group (namely those coming from preinjective tilting modules), we construct a triangle equivalence between $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ and a cluster category $\mathcal{C}_A$ where A is the endomorphism algebra of a postprojective module over a concealed algebra (Theorem 5.21).

Plan of the paper. The first section of this paper is devoted to the study of Serre functors in quotients of triangulated categories. In Section 2, we prove the existence of a cluster-tilting object in a 2-CY category coming from a bimodule 3-CY dg algebra. Section 3 is a direct application of these results to Ginzburg dg algebras associated with quivers with potential. In particular we give the definition of the cluster category $\mathcal{C}_{(Q,W)}$ of a Jacobi-finite quiver with potential (Q, W). In section 4 we define cluster categories of algebras of finite global dimension. We apply the results of Sections 1 and 2 in subsection 4.3 to the particular case of global dimension ≤ 2. The last section links the categories introduced in [GLS07b] and in [BIRS07] with these new cluster categories $\mathcal{C}_A$.

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Notations. Throughout let k be a field. By triangulated category we mean k-linear triangulated category satisfying the Krull-Schmidt property. For all triangulated categories, we will denote the shift functor by [1]. For a finite-dimensional k-algebra A we denote by $\mathsf{mod}\,A$ the category of finite-dimensional right A-modules. More generally, for an additive k-category $\mathcal{M}$ we denote by $\mathsf{mod}\,\mathcal{M}$ the category of finitely presented functors $\mathcal{M}^{op} \to \mathsf{mod}\,k$. Let D be the usual duality $\mathsf{Hom}_k(?, k)$. If A is a differential graded (=dg) k-algebra, we will denote by $\mathcal{D} = \mathcal{D}A$ the derived category of dg A-modules and by $\mathcal{D}^b A$ its full subcategory formed by the dg A-modules whose homology is of finite total dimension over k. We write $\mathsf{per}\,A$ for the category of perfect dg A-modules, *i.e.* the smallest triangulated subcategory of $\mathcal{D}A$ stable under taking direct summands and which contains A.

1. Construction of a Serre functor in a quotient category

1.1. **Bilinear form in a quotient category.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{N}$ a thick subcategory of $\mathcal{T}$ (*i.e.* a triangulated subcategory stable under taking direct summands). We assume that there is an auto-equivalence ν in $\mathcal{T}$ such that $\nu(\mathcal{N}) \subset \mathcal{N}$. Moreover we assume that there is a non degenerate bilinear form:

$$\beta_{N,X} : \mathcal{T}(N,X) \times \mathcal{T}(X,\nu N) \longrightarrow k$$

which is bifunctorial in $N \in \mathcal{N}$ and $X \in \mathcal{T}$.

Construction of a bilinear form in $\mathcal{T}/\mathcal{N}$. Let X and Y be objects in $\mathcal{T}$. The aim of this section is to construct a bifunctorial bilinear form:

$$\beta'_{X,Y} : \mathcal{T}/\mathcal{N}(X,Y) \times \mathcal{T}/\mathcal{N}(Y,\nu X[-1]) \longrightarrow k.$$

We use the calculus of left fractions [Ver77] in the triangle quotient $\mathcal{T}/\mathcal{N}$. Let $s^{-1} \circ f : X \to Y$ and $t^{-1} \circ g : Y \to \nu X[-1]$ be two morphisms in $\mathcal{T}/\mathcal{N}$. We can construct a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "s"'] \arrow[dr, "g"] & & \nu X[-1] \arrow[dl, "t"] \arrow[dll, bend left, "\nu u[-1]"] \\
& Y' \arrow[dr] & & \nu X'[-1] \arrow[dl, "s'"'] & \\
& & \nu X''[-1] & &
\end{tikzcd}$$

where the cone of s' is isomorphic to the cone of s. Denote by $N[1]$ the cone of u. It is in $\mathcal{N}$ since $\mathcal{N}$ is ν-stable. Thus we get a diagram of the form:

$$\begin{tikzcd}
N \arrow[r] \arrow[dr, bend right, "v"] & X \arrow[r, "u"] \arrow[d, "f"] & X'' \arrow[r] & N[1] \\
& Y' \arrow[d] \arrow[dr, bend left, "w"] & & \\
\nu X[-1] \arrow[r, "\nu u[-1]"'] & \nu X''[-1] \arrow[r] & \nu N \arrow[r] & \nu X,
\end{tikzcd}$$

where the two horizontal rows are triangles of $\mathcal{T}$. We define then $\beta'_{X,Y}$ as follows:

$$\beta'_{X,Y}(s^{-1} \circ f, t^{-1} \circ g) = \beta_{N,Y'}(v,w).$$

Lemma 1.1. *The form β' is well-defined, bilinear and bifunctorial.*

Proof. It is not hard to check that β' is well-defined (cf. [Ami08]). Since β is bifunctorial and bilinear, β' satisfies the same properties. □

1.2. **Non-degeneracy.** In this section, we find conditions on X and Y such that the bilinear form β'_{XY} is non-degenerate.

Definition 1.2. Let X and Y be objects in $\mathcal{T}$. A morphism $p : N \to X$ is called a *local $\mathcal{N}$-cover of X relative to Y* if N is in $\mathcal{N}$ and if it induces an exact sequence:

$$0 \longrightarrow \mathcal{T}(X,Y) \xrightarrow{p^*} \mathcal{T}(N,Y).$$

Let Y and Z be objects in $\mathcal{T}$. A morphism $i : Z \to N'$ is called a *local $\mathcal{N}$-envelope of Z relative to Y* if N' is in $\mathcal{N}$ and if it induces an exact sequence:

$$0 \longrightarrow \mathcal{T}(Y,Z) \xrightarrow{i_*} \mathcal{T}(Y,N').$$

Theorem 1.3. *Let X and Y be objects of $\mathcal{T}$. If there exists a local $\mathcal{N}$-cover of X relative to Y and a local $\mathcal{N}$-envelope of νX relative to Y, then the bilienar form β'_{XY} constructed in the previous section is non-degenerate.*

For a stronger version of this theorem see also [CR].

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{T}$ whose image in $\mathcal{T}/\mathcal{N}$ is in the kernel of β'. We have to show that it factorizes through an object of $\mathcal{N}$.

Let $p : N \to X$ be a local $\mathcal{N}$-cover of X relative to Y, and let X' be the cone of p. The morphism f is in the kernel of β', thus for each morphism $g : Y \to \nu N$ which factorizes through $\nu X'[-1]$, $\beta(fp, g)$ vanishes.

$$\begin{tikzcd}
& N \arrow[r, "p"] & X \arrow[r] \arrow[d, "f"] & X' \arrow[r] & N[1] \\
& & Y \arrow[d, dashed] \arrow[dr, "g"] & & \\
& \nu X[-1] \arrow[r] & \nu X'[-1] \arrow[r] & \nu N \arrow[r] & \nu X
\end{tikzcd}$$

This means that the linear form $\beta(fp, ?)$ vanishes on the image of the morphism $\mathcal{T}(Y, \nu X'[-1]) \longrightarrow \mathcal{T}(Y, \nu N)$. This image is canonically isomorphic to the kernel of the morphism $\mathcal{T}(Y, \nu N) \longrightarrow \mathcal{T}(Y, \nu X)$.

Let $\nu i : \nu X \to \nu N'$ be a local $\mathcal{N}$-envelope of νX relative to Y. The sequence

$$0 \longrightarrow \mathcal{T}(Y, \nu X) \longrightarrow \mathcal{T}(Y, \nu N')$$

is then exact. Therefore, the form $\beta(fp, ?)$ vanishes on $\mathsf{Ker}(\mathcal{T}(Y, \nu N) \longrightarrow \mathcal{T}(Y, \nu N'))$.

$$\begin{tikzcd}
N \arrow[r, "p"] & X \arrow[r] \arrow[dd, "f"'] \arrow[dr, "i"] & X' \arrow[r] & N[1] \\
& & N' \arrow[dl, dashed] & \\
& Y \arrow[d, "g"] & & \\
\nu X'[-1] \arrow[r] & \nu N \arrow[r] \arrow[drr, bend right] & \nu X \arrow[r] \arrow[dr, "\nu i"] & \nu X' \\
& & & \nu N'
\end{tikzcd}$$

Now β is non-degenerate on

$$\mathsf{Coker}(\mathcal{T}(N', Y) \longrightarrow \mathcal{T}(N, Y)) \times \mathsf{Ker}(\mathcal{T}(Y, \nu N) \longrightarrow \mathcal{T}(Y, \nu N')).$$

Thus the morphism fp lies in $\mathsf{Coker}(\mathcal{T}(N', Y) \longrightarrow \mathcal{T}(N, Y))$, that is to say that fp factorizes through ip. Since $p : N \to X$ is a local $\mathcal{N}$-cover of X, f factorizes through N'. $\square$

Proposition 1.4. *Let X and Y be objects in $\mathcal{T}$. If for each N in $\mathcal{N}$ the vector spaces $\mathcal{T}(N, X)$ and $\mathcal{T}(Y, N)$ are finite-dimensional, then the existence of a local $\mathcal{N}$-cover of X relative to Y is equivalent to the existence of a local $\mathcal{N}$-envelope of Y relative to X.*

Proof. Let $g : N \to X$ be a local $\mathcal{N}$-cover of X relative to Y. It induces an injection

$$0 \longrightarrow \mathcal{T}(X, Y) \xrightarrow{g^*} \mathcal{T}(N, Y).$$

The space $\mathcal{T}(N,Y)$ is finite-dimensional by hypothesis. Fix a basis $(f_1, f_2, \ldots, f_r)$ of this space. This space is in duality with the space $\mathcal{T}(Y,\nu N)$. Let $(f_1', f_2', \ldots, f_r')$ be the dual basis of the basis $(f_1, f_2, \ldots, f_r)$. We show that the morphism

$$Y \xrightarrow{(f_1',\ldots,f_r')} \bigoplus_{i=1}^r \nu N$$

is a local $\mathcal{N}$-envelope of Y relative to X. We have a commutative diagram:

$$\begin{tikzcd}
\mathcal{T}(X,Y) \arrow[r, "(f_1',\ldots,f_r')_*"] \arrow[d, "g^*"] & \bigoplus \mathcal{T}(X,\nu N) \arrow[d, "g^*"] \\
\mathcal{T}(N,Y) \arrow[r, "(f_1',\ldots,f_r')_*"] & \bigoplus \mathcal{T}(N,\nu N).
\end{tikzcd}$$

If f is in the kernel of $(f_1', \ldots, f_r')_*$, then for all $i = 1, \ldots, r$, the morphism $f_i' \circ f \circ g$ is zero. Thus $f \circ g$ is orthogonal on the vectors of the basis $f_1', \ldots, f_r'$ and therefore vanishes. Since g is a local $\mathcal{N}$-cover of X relative to Y, f is zero, and the morphism

$$\mathcal{T}(X,Y) \xrightarrow{(f_1',\ldots,f_r')_*} \bigoplus \mathcal{T}(X,\nu N)$$

is injective. Therefore, the morphism

$$Y \xrightarrow{(f_1',\ldots,f_r')} \bigoplus_{i=1}^r \nu N$$

is a local $\mathcal{N}$-envelope of Y relative to X. The proof of the converse is dual. □

Examples. Let A be a finite-dimensional self-injective k-algebra. Denote by $\mathcal{T}$ the derived category $\mathcal{D}^b(\operatorname{mod} A)$ and by $\mathcal{N}$ the triangulated category $\operatorname{per} A$. Since A is finite-dimensional, there is an inclusion $\mathcal{N} \subset \mathcal{T}$. Moreover A is self-injective so of infinite global dimension. Therefore the inclusion is strict. By [KV87], there is an exact sequence of triangulated categories:

$$0 \longrightarrow \operatorname{per} A \longrightarrow \mathcal{D}^b(\operatorname{mod} A) \longrightarrow \underline{\operatorname{mod}}\, A \longrightarrow 0.$$

The derived category $\mathcal{D}^b(\operatorname{mod} A)$ admits a Serre functor $\nu = ? \overset{L}{\otimes}_A DA$ which stabilizes $\operatorname{per} A$. Thus there is an induced functor in the quotient $\underline{\operatorname{mod}}\, A$ that we will also denote by ν. Let Σ be the suspension of the category $\underline{\operatorname{mod}}\, A$. One can easily construct (cf. [Ami08]) local $\mathcal{N}$-covers and local $\mathcal{N}$-envelopes, so we can apply theorem 1.3 and the functor $\Sigma^{-1} \circ \nu$ is a Serre functor for the stable category $\underline{\operatorname{mod}}\, A$.

An article of G. Tabuada [Tab07] gives an example of the converse construction. Let $\mathcal{C}$ be an algebraic 2-Calabi-Yau category endowed with a cluster-tilting object. The author constructs a triangulated category $\mathcal{T}$ and a triangulated 3-Calabi-Yau subcategory $\mathcal{N}$ such that the quotient category $\mathcal{T}/\mathcal{N}$ is triangle equivalent to $\mathcal{C}$. It is possible to show (cf. [Ami08]) that the categories $\mathcal{T}$ and $\mathcal{N}$ satisfy the hypotheses of theorem 1.3.

2. Existence of a cluster-tilting object

Let A be a differential graded (=dg) k-algebra. We denote by A^e the dg algebra $A^{op} \otimes A$. Suppose that A has the following properties:

- A is homologically smooth (*i.e.* the object A, viewed as an A^e-module, is perfect);
- for each $p > 0$, the space $H^p A$ is zero;
- the space $H^0 A$ is finite-dimensional;

- A is bimodule 3-CY, *i.e.* there is an isomorphism in $\mathcal{D}(A^e)$

$$R\mathsf{Hom}_{A^e}(A, A^e) \simeq A[-3].$$

Since A is homologically smooth, the category $\mathcal{D}^b A$ is a subcategory of $\mathsf{per}\, A$ (see [Kel08a], lemma 4.1). We denote by π the canonical projection functor $\pi : \mathsf{per}\, A \to \mathcal{C} = \mathsf{per}\, A/\mathcal{D}^b A$. Moreover, by the same lemma, there is a bifunctorial isomorphism

$$D\mathsf{Hom}_{\mathcal{D}}(L, M) \simeq \mathsf{Hom}_{\mathcal{D}}(M, L[3])$$

for all objects L in $\mathcal{D}^b A$ and all objects M in $\mathsf{per}\, A$. We call this property the *CY property*.

The aim of this section is to show the following result:

Theorem 2.1. *Let A be a dg k-algebra with the above properties. The category $\mathcal{C} = \mathsf{per}\, A/\mathcal{D}^b A$ is* Hom*-finite and 2-CY. Moreover, the object $\pi(A)$ is a cluster-tilting object. Its endomorphism algebra is isomorphic to $H^0 A$.*

2.1. **t-structure on $\mathsf{per}\, A$.** The main tool of the proof of theorem 2.1 is the existence of a canonical t-structure in $\mathsf{per}\, A$.

t-structure on $\mathcal{D}A$. Let $\mathcal{D}_{\leq 0}$ be the full subcategory of $\mathcal{D}$ whose objects are the dg modules X such that $H^p X$ vanishes for all $p > 0$.

Lemma 2.2. *The subcategory $\mathcal{D}_{\leq 0}$ is an aisle in the sense of Keller-Vossieck* [KV88].

Proof. The canonical morphism $\tau_{\leq 0} A \to A$ is a quasi-isomorphism of dg algebras. Thus we can assume that A^p is zero for all $p > 0$. The full subcategory $\mathcal{D}_{\leq 0}$ is stable under $X \mapsto X[1]$ and under extensions. We claim that the inclusion $\mathcal{D}_{\leq 0} \hookrightarrow \mathcal{D}$ has a right adjoint. Indeed, for each dg A-module X, the dg A-module $\tau_{\leq 0} X$ is a dg submodule of X, since A is concentrated in negative degrees. Thus $\tau_{\leq 0}$ is a well-defined functor $\mathcal{D} \to \mathcal{D}_{\leq 0}$. One can check easily that $\tau_{\leq 0}$ is the right adjoint of the inclusion. □

Proposition 2.3. *Let $\mathcal{H}$ be the heart of the t-structure, i.e. $\mathcal{H}$ is the intersection $\mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$. We have the following properties:*

(i) The functor H^0 induces an equivalence from $\mathcal{H}$ onto $\mathsf{Mod}\, H^0 A$.

(ii) For all X and Y in $\mathcal{H}$, we have an isomorphism $\mathsf{Ext}^1_{H^0 A}(X, Y) \simeq \mathsf{Hom}_{\mathcal{D}}(X, Y[1])$.

Note that it is not true for general i that $\mathsf{Ext}^i_{\mathcal{H}}(X, Y) \simeq \mathsf{Hom}_{\mathcal{D}}(X, Y[i])$.

Proof. (i) We may assume that $A^p = 0$ for all $p > 0$. We then have a canonical morphism $A \to H^0 A$. The restriction along this morphism yields a functor $\Phi : \mathsf{Mod}\, H^0 A \to \mathcal{H}$ such that $H^0 \circ \Phi$ is the identity of $\mathsf{Mod}\, H^0 A$. Thus the functor $H^0 : \mathcal{H} \to \mathsf{Mod}\, H^0 A$ is full and essentially surjective. Moreover, it is exact and an object $N \in \mathcal{H}$ vanishes if and only if $H^0 N$ vanishes. Thus if $f : L \to M$ is a morphism of $\mathcal{H}$ such that $H^0(f) = 0$, then $\mathsf{Im} H^0(f) = 0$ implies that $H^0(\mathsf{Im} f) = 0$ and $\mathsf{Im} f = 0$, so $f = 0$. Thus $H^0 : \mathcal{H} \to \mathsf{Mod}\, H^0 A$ is also faithful.

(ii) By section 3.1.7 of [BBD82] there exists a triangle functor $\mathcal{D}^b(\mathcal{H}) \to \mathcal{D}$ which yields for X and Y in $\mathcal{H}$ and for $n \leq 1$ an isomorphism (remark (ii) section 3.1.17 p.85)

$$\mathsf{Hom}_{\mathcal{D}\mathcal{H}}(X, Y[n]) \simeq \mathsf{Hom}_{\mathcal{D}}(X, Y[n]).$$

Applying this for $n = 1$ and using (i), we get the result. □

Hom-*finiteness.*

Proposition 2.4. *The category* $\mathsf{per}\, A$ *is* Hom*-finite.*

Lemma 2.5. *For each p, the space H^pA is finite-dimensional.*

Proof. By hypothesis, H^pA is zero for $p > 0$. We prove by induction on n the following statement: *The space $H^{-n}A$ is finite-dimensional, and for all $p \geq n+1$ the space $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n}A, M[p])$ is finite-dimensional for each M in $\mathsf{mod}\, H^0A$.*

For $n = 0$, the space H^0A is finite-dimensional by hypothesis. Let M be in $\mathsf{mod}\, H^0A$. Since $\tau_{\leq 0}A$ is ismorphic to A, $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq 0}A, M[p])$ is isomorphic $H^0(M[p])$, and so is zero for $p \geq 1$.

Suppose that the property holds for n. Form the triangle:

$$(H^{-n}A)[n-1] \longrightarrow \tau_{\leq -n-1}A \longrightarrow \tau_{\leq -n}A \longrightarrow (H^{-n}A)[n]$$

Let p be an integer $\geq n+1$. Applying the functor $\mathsf{Hom}_{\mathcal{D}}(?, M[p])$ we get the long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n}A, M[p]) \to \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n-1}A, M[p]) \to \mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[p]) \to \cdots.$$

By induction the space $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n}A, M[p])$ is finite-dimensional.

Since $M[p]$ is in $\mathcal{D}^bA$ we can apply the CY property. If p is $\geq n+3$, we have isomorphisms:

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[p]) &\simeq \mathsf{Hom}_{\mathcal{D}}((H^{-n}A), M[p-n+1]) \\ &\simeq D\mathsf{Hom}_{\mathcal{D}}(M[p-n-2], H^{-n}A). \end{aligned}$$

Since $p-n-2$ is ≥ 1, this space is zero.

If $p = n+2$, we have the following isomorphisms.

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[n+2]) &\simeq \mathsf{Hom}_{\mathcal{D}}((H^{-n}A), M[3]) \\ &\simeq D\mathsf{Hom}_{\mathcal{D}}(M, H^{-n}A) \\ &\simeq D\mathsf{Hom}_{H^0A}(M, H^{-n}A). \end{aligned}$$

The last isomorphism comes from lemma 2.3 (i). By induction, the space $H^{-n}A$ is finite-dimensional. Thus for $p \geq n+2$, the space $\mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[p])$ is finite-dimensional.

If $p = n+1$ we have the following isomorphisms:

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[n+1]) &\simeq \mathsf{Hom}_{\mathcal{D}}((H^{-n}A), M[2]) \\ &\simeq D\mathsf{Hom}_{\mathcal{D}}(M, H^{-n}A[1]) \\ &\simeq D\mathsf{Ext}^1_{H^0A}(M, H^{-n}A) \end{aligned}$$

The last isomorphism comes from lemma 2.3 (ii). By induction, since $H^{-n}A$ is finite-dimensional, the space $\mathsf{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[n+1])$ is finite-dimensional and so is $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n-1}A, M[n+1])$.

Now, look at the triangle

$$\begin{tikzcd} \tau_{\leq -n-2}A \arrow[r] \arrow[dr, dotted, "0"'] & \tau_{\leq -n-1}A \arrow[r] \arrow[d] & (H^{-n-1}A)[n+1] \arrow[r] \arrow[dl] & (\tau_{\leq -n-2}A)[1] \arrow[dll, dotted, "0"] . \\ & M[n+1] & & \end{tikzcd}$$

The spaces $\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n-2}A, M[n+1])$ and $\mathsf{Hom}_{\mathcal{D}}((\tau_{\leq -n-2}A)[1], M[n+1])$ vanish since $M[n+1]$ is in $\mathcal{D}_{\geq -n-1}$. Thus we have

$$\begin{aligned}\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -n-1}A[n-1], M[n+1]) &\simeq \mathsf{Hom}_{\mathcal{D}}((H^{-n-1}A)[n+1], M[n+1]) \\ &\simeq \mathsf{Hom}_{\mathcal{D}}(H^{-n-1}A, M) \\ &\simeq \mathsf{Hom}_{H^0A}(H^{-n-1}A, M).\end{aligned}$$

This holds for all finite-dimensional H^0A-modules M. Thus it holds for the compact cogenerator $M = DH^0A$. The space $\mathsf{Hom}_{H^0A}(H^{-n-1}A, DH^0A) \simeq DH^{-n-1}A$ is finite-dimensional, and therefore $H^{-(n+1)}A$ is finite-dimensional. $\square$

Proof. (of proposition 2.4) For each integer p, the space $\mathsf{Hom}_{\mathcal{D}}(A, A[p]) \simeq H^p(A)$ is finite-dimensional by lemma 2.5. The subcategory of $(\mathsf{per}\, A)^{op} \times \mathsf{per}\, A$ whose objects are the pairs (X, Y) such that $\mathsf{Hom}_{\mathcal{D}}(X, Y)$ is finite-dimensional is stable under extensions and passage to direct factors. $\square$

Restriction of the t-structure to $\mathsf{per}\, A$.

Lemma 2.6. *For each X in $\mathsf{per}\, A$, there exist integers N and M such that X belongs to $\mathcal{D}_{\leq N}$ and ${}^{\perp}\mathcal{D}_{\leq M}$.*

Proof. The object A belongs to $\mathcal{D}_{\leq 0}$. Moreover, since for X in $\mathcal{D}A$, the space $\mathsf{Hom}_{\mathcal{D}}(A, X)$ is isomorphic to H^0X, the dg module A belongs to ${}^{\perp}\mathcal{D}_{\leq -1}$. Thus the property is true for A. For the same reasons, it is true for all shifts of A. Moreover, this property is clearly stable under taking direct summands and extensions. Thus it holds for all objects of $\mathsf{per}\, A$. $\square$

This lemma implies the following result:

Proposition 2.7. *The t-structure on $\mathcal{D}A$ restricts to $\mathsf{per}\, A$.*

Proof. Let X be in $\mathsf{per}\, A$, and look at the canonical triangle:

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{>0}X \longrightarrow (\tau_{\leq 0}X)[1].$$

Since $\mathsf{per}\, A$ is Hom-finite, the space $H^pX \simeq \mathsf{Hom}_{\mathcal{D}}(A, X[p])$ is finite-dimensional for all $p \in \mathbb{Z}$ and vanishes for all $p \gg 0$ by lemma 2.6. Thus the object $\tau_{>0}X$ is in $\mathcal{D}^bA$ and so in $\mathsf{per}\, A$. Since $\mathsf{per}\, A$ is a triangulated subcategory, it follows that $\tau_{\leq 0}X$ also lies in $\mathsf{per}\, A$. $\square$

Proposition 2.8. *Let π be the projection $\pi : \mathsf{per}\, A \to \mathcal{C}$. Then for X and Y in $\mathsf{per}\, A$, we have*

$$\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y) = \varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$$

Proof. Let X and Y be in $\mathsf{per}\, A$. An element of $\varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$ is an equivalence class of morphisms $\tau_{\leq n}X \to \tau_{\leq n}Y$. Two morphisms $f : \tau_{\leq n}X \to \tau_{\leq n}Y$ and $g : \tau_{\leq m}X \to \tau_{\leq m}Y$ with $m \geq n$ are equivalent if there is a commutative square:

$$\begin{tikzcd} \tau_{\leq n}X \arrow[r, "f"] \arrow[d] & \tau_{\leq n}Y \arrow[d] \\ \tau_{\leq m}X \arrow[r, "g"] & \tau_{\leq m}Y \end{tikzcd}$$

where the vertical arrows are the canonical morphisms. If f is a morphism $f : \tau_{\leq n}X \to \tau_{\leq n}Y$, we can form the following morphism from X to Y in $\mathcal{C}$:

$$\begin{tikzcd}
& \tau_{\leq n}X \arrow[dl] \arrow[dr] \arrow[r, dotted, "f"] & \tau_{\leq n}Y \arrow[d, dotted] \\
X & & Y,
\end{tikzcd}$$

where the morphisms $\tau_{\leq n}X \to X$ and $\tau_{\leq n}Y \to Y$ are the canonical morphisms. This is a morphism from πX to πY in $\mathcal{C}$ because the cone of the morphism $\tau_{\leq n}X \to X$ is in $\mathcal{D}^bA$. Moreover, if $f : \tau_{\leq n}X \to \tau_{\leq n}Y$ and $g : \tau_{\leq m}X \to \tau_{\leq m}Y$ are equivalent, there is an equivalence of diagrams:

$$\begin{tikzcd}
& \tau_{\leq n}X \arrow[dl] \arrow[dd] \arrow[dr] \arrow[r, dotted, "f"] & \tau_{\leq n}Y \arrow[d, dotted] \arrow[dd, dotted, bend left] \\
X & & Y \\
& \tau_{\leq m}X \arrow[ul] \arrow[ur] \arrow[r, dotted, "g"] & \tau_{\leq m}Y \arrow[u, dotted]
\end{tikzcd}$$

Thus we have a well-defined map from $\varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$ to $\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y)$ which is injective.

Now let $\begin{tikzcd}[sep=small] & X' \arrow[dl] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}$ be a morphism in $\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y)$. Let X'' be the cone of s. It is an object of $\mathcal{D}^bA$, and therefore lies in $\mathcal{D}_{>n}$ for some $n \ll 0$. Thus there are no morphisms from $\tau_{\leq n}X$ to X'' and there is a factorization:

$$\begin{tikzcd}
& \tau_{\leq n}X \arrow[dl, dotted] \arrow[d] \arrow[dr, "0"] & & \\
X' \arrow[r, "s"] & X \arrow[r] & X'' \arrow[r] & X'[1]
\end{tikzcd}$$

We obtain an isomorphism of diagrams:

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr] & \\
X & & Y \\
& \tau_{\leq n}X \arrow[ul] \arrow[uu] \arrow[ur, "f"'] &
\end{tikzcd}$$

The morphism $f : \tau_{\leq n}X \to Y$ induces a morphism $f' : \tau_{\leq n}X \to \tau_{\leq n}Y$ which lifts the given morphism. Thus the map from $\varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y)$ to $\mathsf{Hom}_{\mathcal{C}}(\pi X, \pi Y)$ is surjective. $\square$

2.2. **Fundamental domain.** Let $\mathcal{F}$ be the following subcategory of $\mathsf{per}\, A$:

$$\mathcal{F} = \mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -2} \cap \mathsf{per}\, A.$$

The aim of this section is to show:

Proposition 2.9. *The projection functor $\pi : \mathsf{per}\, A \to \mathcal{C}$ induces a k-linear equivalence between $\mathcal{F}$ and $\mathcal{C}$.*

add(A)-approximation for objects of the fundamental domain.

Lemma 2.10. *For each object X of $\mathcal{F}$, there exists a triangle*

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow P_1[1]$$

with P_0 and P_1 in add(A).

Proof. For X in $\mathsf{per}\, A$, the morphism

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}}(A,X) & \to & \mathsf{Hom}_{\mathcal{H}}(H^0A, H^0X) \\ f & \mapsto & H^0(f) \end{array}$$

is an isomorphism since $\mathsf{Hom}_{\mathcal{D}}(A,X) \simeq H^0X$. Thus it is possible to find a morphism $P_0 \to X$, with P_0 a free dg A-module, inducing an epimorphism $H^0P_0 \twoheadrightarrow H^0X$. Now take X in $\mathcal{F}$ and $P_0 \to X$ as previously and form the triangle

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow P_1[1].$$

Step 1: The object P_1 is in $\mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -1}$.

The objects X and P_0 are in $\mathcal{D}_{\leq 0}$, so P_1 is in $\mathcal{D}_{\leq 1}$. Moreover, since $H^0P_0 \to H^0X$ is an epimorphism, $H^1(P_1)$ vanishes and P_1 is in $\mathcal{D}_{\leq 0}$.

Let Y be in $\mathcal{D}_{\leq -1}$, and look at the long exact sequence:

$$\cdots \longrightarrow \mathsf{Hom}_{\mathcal{D}}(P_0,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(P_1,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(X[-1],Y) \longrightarrow \cdots.$$

The space $\mathsf{Hom}_{\mathcal{D}}(X[-1],Y)$ vanishes since X is in ${}^{\perp}\mathcal{D}_{\leq -2}$ and Y is in $\mathcal{D}_{\leq -1}$. The object P_0 is free, and H^0Y is zero, so the space $\mathsf{Hom}_{\mathcal{D}}(P_0,Y)$ also vanishes. Consequently, the object P_1 is in ${}^{\perp}\mathcal{D}_{\leq -1}$.

Step 2: H^0P_1 is a projective H^0A-module.

Since P_1 is in $\mathcal{D}_{\leq 0}$ there is a triangle

$$\tau_{\leq -1}P_1 \longrightarrow P_1 \longrightarrow H^0P_1 \longrightarrow (\tau_{\leq -1}P_1)[1].$$

Now take an object M in the heart $\mathcal{H}$, and look at the long exact sequence:

$$\cdots \longrightarrow \mathsf{Hom}_{\mathcal{D}}((\tau_{\leq -1}P_1)[1], M[1]) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(H^0P_1, M[1]) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(P_1, M[1]) \longrightarrow \cdots.$$

The space $\mathsf{Hom}_{\mathcal{D}}((\tau_{\leq -1}P_1)[1], M[1])$ is zero because $\mathsf{Hom}_{\mathcal{D}}(\mathcal{D}_{\leq -2}, \mathcal{D}_{\geq -1})$ vanishes in a t-structure. Moreover, the space $\mathsf{Hom}_{\mathcal{D}}(P_1, M[1])$ vanishes because P_1 is in ${}^{\perp}\mathcal{D}_{\leq -1}$. Thus $\mathsf{Hom}_{\mathcal{D}}(H^0P_1, M[1])$ is zero. But this space is isomorphic to the space $\mathsf{Ext}^1_{\mathcal{H}}(H^0P_1, M)$ by proposition 2.3. This proves that H^0P_1 is a projective H^0A-module.

Step 3: P_1 is isomorphic to an object of add(A).

As previously, since H^0P_1 is projective, it is possible to find an object P in *add(A)* and a morphism $P \to P_1$ inducing an isomorphism $H^0P \to H^0P_1$. Form the triangle

$$Q \xrightarrow{u} P \xrightarrow{v} P_1 \xrightarrow{w} Q[1]$$

Since P and P_1 are in $\mathcal{D}_{\leq 0}$ and $H^0(v)$ is surjective, the cone Q lies in $\mathcal{D}_{\leq 0}$. But then w is zero since P_1 is in ${}^{\perp}\mathcal{D}_{\leq -1}$. Thus the triangle splits, and P is isomorphic to the direct sum $P_1 \oplus Q$. Therefore we have a short exact sequence:

$$0 \longrightarrow H^0Q \longrightarrow H^0P \longrightarrow H^0P_1 \longrightarrow 0,$$

and H^0Q vanishes. The object Q is in $\mathcal{D}_{\leq -1}$, the triangle splits, and there is no morphism between P and $\mathcal{D}_{\leq -1}$, so Q is the zero object.

$\square$

Equivalence between the shifts of $\mathcal{F}$.

Lemma 2.11. *The functor $\tau_{\leq -1}$ induces an equivalence from $\mathcal{F}$ to $\mathcal{F}[1]$*

Proof. Step 1: The image of the functor $\tau_{\leq -1}$ restricted to $\mathcal{F}$ is in $\mathcal{F}[1]$.
Recall that $\mathcal{F}$ is $\mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -2} \cap \operatorname{\mathsf{per}} A$ so $\mathcal{F}[1]$ is $\mathcal{D}_{\leq -1} \cap {}^{\perp}\mathcal{D}_{\leq -3} \cap \operatorname{\mathsf{per}} A$. Let X be an object in $\mathcal{F}$. By definition, $\tau_{\leq -1}X$ lies in $\mathcal{D}_{\leq -1}$ and there is a canonical triangle:

$$\tau_{\leq -1}X \longrightarrow X \longrightarrow H^0X \longrightarrow \tau_{\leq -1}X[1]\ .$$

Now let Y be an object in $\mathcal{D}_{\leq -3}$ and form the long exact sequence

$$\cdots \longrightarrow \mathsf{Hom}_{\mathcal{D}}(X,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -1}X,Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}((H^0X)[-1],Y) \longrightarrow \cdots$$

Since X is in ${}^{\perp}\mathcal{D}_{\leq -2}$, the space $\mathsf{Hom}_{\mathcal{D}}(X,Y)$ vanishes. The object $H^0X[-1]$ is of finite total dimension, so by the CY property, we have an isomorphism

$$\mathsf{Hom}_{\mathcal{D}}(H^0X[-1],Y) \simeq D\mathsf{Hom}_{\mathcal{D}}(Y,H^0X[2]).$$

But since $\mathsf{Hom}_{\mathcal{D}}(\mathcal{D}_{\leq -3},\mathcal{D}_{\geq -2})$ is zero, the space $\mathsf{Hom}_{\mathcal{D}}((H^0X)[-1],Y)$ vanishes and $\tau_{\leq -1}X$ lies in ${}^{\perp}\mathcal{D}_{\leq -3}$.

Step 2: The functor $\tau_{\leq -1} : \mathcal{F} \to \mathcal{F}[1]$ is fully faithful.
Let X and Y be two objects in $\mathcal{F}$ and $f : \tau_{\leq -1}X \to \tau_{\leq -1}Y$ be a morphism.

$$\begin{tikzcd}
H^0X[-1] \arrow[r] & \tau_{\leq -1}X \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d, dotted] & H^0X \\
H^0Y[-1] \arrow[r] & \tau_{\leq -1}Y \arrow[r, "i"] & Y \arrow[r] & H^0Y
\end{tikzcd}$$

The space $\mathsf{Hom}_{\mathcal{D}}(H^0X[-1],Y)$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(Y,H^0X[2])$ by the CY property. Since Y is in ${}^{\perp}\mathcal{D}_{\leq -2}$, this space is zero, and the composition $i \circ f$ factorizes through the canonical morphism $\tau_{\leq -1}X \to X$. Therefore, the functor $\tau_{\leq -1}$ is full.

Let X and Y be objects of $\mathcal{F}$ and $f : X \to Y$ a morphism satisfying $\tau_{\leq -1}f = 0$. It induces a morphism of triangles:

$$\begin{tikzcd}
H^0X[-1] \arrow[r] \arrow[d] & \tau_{\leq -1}X \arrow[r, "i"] \arrow[d, "0"] & X \arrow[r] \arrow[d, "f"] & H^0X \arrow[d] \arrow[dl, dotted] \\
H^0Y[-1] \arrow[r] & \tau_{\leq -1}Y \arrow[r] & Y \arrow[r] & H^0Y
\end{tikzcd}$$

The composition $f \circ i$ vanishes, so f factorizes through H^0X. But by the CY property the space of morphisms $\mathsf{Hom}_{\mathcal{D}}(H^0X,Y)$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(Y,H^0X[3])$ which is zero since Y is in ${}^{\perp}\mathcal{D}_{\leq -2}$. Thus the functor $\tau_{\leq -1}$ restricted to $\mathcal{F}$ is faithful.

Step 3: The functor $\tau_{\leq -1} : \mathcal{F} \to \mathcal{F}[1]$ is essentially surjective.
Let X be in $\mathcal{F}[1]$. By the previous lemma there exists a triangle

$$P_1[1] \longrightarrow P_0[1] \longrightarrow X \longrightarrow P_1[2]$$

with P_0 and P_1 in $add(A)$. Denote by ν the Nakayama functor on the projectives of $\mathsf{mod}\, H^0A$. Let M be the kernel of the morphism $\nu H^0P_1 \to \nu H^0P_0$. It lies in the heart $\mathcal{H}$.

Substep (i): There is an isomorphism of functors: $\mathsf{Hom}(?, X[1])_{|\mathcal{H}} \simeq \mathsf{Hom}_{\mathcal{H}}(?, M)$
Let L be in $\mathcal{H}$. We then have a long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}}(L, P_0[2]) \to \mathsf{Hom}_{\mathcal{D}}(L, X[1]) \to \mathsf{Hom}_{\mathcal{D}}(L, P_1[3]) \to \mathsf{Hom}_{\mathcal{D}}(L, P_0[3]) \to \cdots .$$

The space $\mathsf{Hom}_{\mathcal{D}}(L, P_0[2])$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(P_0, L[1])$ by the CY property, and vanishes because P_0 is in ${}^{\perp}\mathcal{D}_{\leq -1}$. Moreover, we have the following isomorphisms:

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}(L, P_1[3]) &\simeq D\mathsf{Hom}_{\mathcal{D}}(P_1, L) \\ &\simeq D\mathsf{Hom}_{\mathcal{H}}(H^0P_1, L) \\ &\simeq \mathsf{Hom}_{\mathcal{H}}(L, \nu H^0P_1). \end{aligned}$$

Thus $\mathsf{Hom}_{\mathcal{D}}(?, X[1])_{|\mathcal{H}}$ is isomorphic to the kernel of $\mathsf{Hom}_{\mathcal{H}}(?, \nu H^0P_1) \to \mathsf{Hom}_{\mathcal{H}}(?, \nu H^0P_0)$, which is just $\mathsf{Hom}_{\mathcal{H}}(?, M)$.

Substep (ii): There is a monomorphism of functors: $\mathsf{Ext}^1_{\mathcal{H}}(?, M) \hookrightarrow \mathsf{Hom}_{\mathcal{D}}(?, X[2])_{|\mathcal{H}}$.
For L in $\mathcal{H}$, look at the following long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}}(L, P_1[3]) \to \mathsf{Hom}_{\mathcal{D}}(L, P_1[3]) \to \mathsf{Hom}_{\mathcal{D}}(L, X[2]) \to \mathsf{Hom}_{\mathcal{D}}(L, P_1[4]) \to \cdots .$$

The space $\mathsf{Hom}_{\mathcal{D}}(L, P_1[4])$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(P_1[1], L)$ which is zero since $P_1[1]$ is in $\mathcal{D}_{\leq -1}$ and L is in $\mathcal{D}_{\geq 0}$. Thus the functor $\mathsf{Hom}_{\mathcal{D}}(?, X[2])_{|\mathcal{H}}$ is isomorphic to the cokernel of $\mathsf{Hom}_{\mathcal{H}}(?, \nu H^0P_1) \to \mathsf{Hom}_{\mathcal{H}}(?, \nu H^0P_0)$. By defninition, $\mathsf{Ext}^1_{\mathcal{H}}(?, M)$ is the first homology of a complex of the form:

$$\cdots \longrightarrow 0 \longrightarrow \mathsf{Hom}_{\mathcal{H}}(?, \nu H^0P_1) \longrightarrow \mathsf{Hom}_{\mathcal{H}}(?, \nu H^0P_0) \longrightarrow \mathsf{Hom}_{\mathcal{H}}(?, I) \longrightarrow \cdots ,$$

where I is an injective H^0A-module. Thus we get the canonical injection:

$$\mathsf{Ext}^1_{\mathcal{H}}(?, M) \hookrightarrow \mathsf{Hom}_{\mathcal{D}}(?, X[2])_{|\mathcal{H}}.$$

Now form the following triangle:

$$X \longrightarrow Y \longrightarrow M \longrightarrow X[1].$$

Substep (iii): Y is in $\mathcal{F}$ and $\tau_{\leq -1}Y$ is isomorphic to X.
Since X and M are in $\mathcal{D}_{\leq 0}$, Y is in $\mathcal{D}_{\leq 0}$. Let Z be in $\mathcal{D}_{\leq -2}$ and form the following long exact sequence:

$$\cdots \mathsf{Hom}_{\mathcal{D}}(X[1], Z) \to \mathsf{Hom}_{\mathcal{D}}(M, Z) \to \mathsf{Hom}_{\mathcal{D}}(Y, Z) \to \mathsf{Hom}_{\mathcal{D}}(X, Z) \to \mathsf{Hom}_{\mathcal{D}}(M[-1], Z) \cdots .$$

By the CY property and the fact that $Z[2]$ is in $\mathcal{D}_{\leq 0}$, we have isomorphisms

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}(M[-1], Z) &\simeq D\mathsf{Hom}_{\mathcal{D}}(Z[-2], M) \\ &\simeq D\mathsf{Hom}_{\mathcal{H}}(H^{-2}Z, M). \end{aligned}$$

Moreover, since X is in ${}^{\perp}\mathcal{D}_{\leq -3}$, we have

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}}(X, Z) &\simeq \mathsf{Hom}_{\mathcal{D}}(X, (H^{-2}Z)[2]) \\ &\simeq D\mathsf{Hom}_{\mathcal{H}}(H^{-2}Z, X[1]). \end{aligned}$$

By substep (i) the functors $\mathsf{Hom}_{\mathcal{H}}(?,M)$ and $\mathsf{Hom}_{\mathcal{D}}(?,X[1])_{|\mathcal{H}}$ are isomorphic. Therefore we deduce that the morphism $\mathsf{Hom}_{\mathcal{D}}(X,Z)\to\mathsf{Hom}_{\mathcal{D}}(M[-1],Z)$ is an isomorphism.

Now look at the triangle

$$\tau_{\leq -3}Z \longrightarrow Z \longrightarrow H^{-2}Z[2] \longrightarrow (\tau_{\leq -3}Z)[1]$$

and form the commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{D}}(M,\tau_{\leq -3}Z) \arrow[r] & \mathsf{Hom}_{\mathcal{D}}(M,Z) \arrow[r] & \mathsf{Hom}_{\mathcal{D}}(M,H^{-2}Z[2]) \arrow[r] & \mathsf{Hom}_{\mathcal{D}}(M,\tau_{\leq -3}Z[1]) \\
\mathsf{Hom}_{\mathcal{D}}(X[1],\tau_{\leq -3}Z) \arrow[r] \arrow[u, "a"] & \mathsf{Hom}_{\mathcal{D}}(X[1],Z) \arrow[r] \arrow[u, "b"] & \mathsf{Hom}_{\mathcal{D}}(X[1],H^{-2}Z[2]) \arrow[r] \arrow[u, "c"] & \mathsf{Hom}_{\mathcal{D}}(X[1],\tau_{\leq -3}Z[1]) \arrow[u, "d"]
\end{tikzcd} .$$

By the CY property and the fact that $(\tau_{\leq -3}Z)[-3]$ is in $\mathcal{D}_{\leq 0}$, we have isomorphisms

$$\begin{aligned}\mathsf{Hom}_{\mathcal{D}}(M[-1],\tau_{\leq -3}Z[-1]) &\simeq D\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -3}Z[-3],M)\\ &\simeq D\mathsf{Hom}_{\mathcal{H}}(H^{-3}Z,M).\end{aligned}$$

Since X is in ${}^{\perp}\mathcal{D}_{\leq -3}$, we have

$$\begin{aligned}\mathsf{Hom}_{\mathcal{D}}(X,(\tau_{\leq -3}Z)[-1]) &\simeq \mathsf{Hom}_{\mathcal{D}}(X,H^{-3}Z[-2])\\ &\simeq D\mathsf{Hom}_{\mathcal{H}}(H^{-3}Z,X[1]).\end{aligned}$$

Now we deduce from substep (i) that $a[-1]$ is an isomorphism.

The space $\mathsf{Hom}_{\mathcal{D}}(X[1],\tau_{\leq -3}Z[1])$ is zero because X is ${}^{\perp}\mathcal{D}_{\leq -3}$. Moreover there are isomorphisms

$$\begin{aligned}\mathsf{Hom}_{\mathcal{D}}(M,H^{-2}Z[2]) &\simeq D\mathsf{Hom}_{\mathcal{D}}(H^{-2}Z,M[1])\\ &\simeq D\mathsf{Ext}^1_{\mathcal{H}}(H^{-2}Z,M).\end{aligned}$$

The space $\mathsf{Hom}_{\mathcal{D}}(X[1],H^{-2}Z[2])$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(H^{-2}Z,X[2])$. And by substep (ii), the morphism $\mathsf{Ext}^1_{\mathcal{H}}(?,M)\to\mathsf{Hom}_{\mathcal{D}}(?,X[2])_{|\mathcal{H}}$ is injective, so c is surjective. Therefore using a weak form of the five-lemma we deduce that b is surjective.

Finally, we have the following exact sequence:

$$\mathsf{Hom}_{\mathcal{D}}(X[1],Z) \twoheadrightarrow \mathsf{Hom}_{\mathcal{D}}(M,Z) \to \mathsf{Hom}_{\mathcal{D}}(Y,Z) \to \mathsf{Hom}_{\mathcal{D}}(X,Z) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(M[-1],Z)$$

Thus the space $\mathsf{Hom}_{\mathcal{D}}(M,Z)$ is zero, and Z is in ${}^{\perp}\mathcal{D}_{\leq -2}$.

It is now easy to see that there is an isomorphism of triangles:

$$\begin{tikzcd}
\tau_{\leq -1}Y \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & H^0Y \arrow[r] \arrow[d] & \tau_{\leq -1}Y[1] \arrow[d] \\
X \arrow[r] & Y \arrow[r] & M \arrow[r] & X[1].
\end{tikzcd}$$

□

Proof of proposition 2.9. Step 1: The functor π restricted to $\mathcal{F}$ is fully faithful.

Let X and Y be objects in $\mathcal{F}$. By proposition 2.3 (iii), the space $\mathsf{Hom}_{\mathcal{C}}(\pi X,\pi Y)$ is isomorphic to the direct limit $\varinjlim \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X,\tau_{\leq n}Y)$. A morphism between X and Y in $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd}
& \tau_{\leq n}X \arrow[dl] \arrow[dr] & \\
X & & Y.
\end{tikzcd}$$

The canonical triangle

$$(\tau_{>n}X)[-1] \longrightarrow \tau_{\leq n}X \longrightarrow X \longrightarrow \tau_{>n}X$$

yields a long exact sequence:

$$\cdots \to \mathsf{Hom}_{\mathcal{D}}(\tau_{>n}X, Y) \to \mathsf{Hom}_{\mathcal{D}}(X, Y) \to \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, Y) \to \mathsf{Hom}_{\mathcal{D}}((\tau_{>n}X)[-1], Y) \to \cdots$$

The space $\mathsf{Hom}_{\mathcal{D}}((\tau_{>n}X)[-1], Y)$ is isomorphic to the space $D\mathsf{Hom}_{\mathcal{D}}(Y, (\tau_{>n}X)[2])$. The object X is in $\mathcal{D}_{\leq 0}$, thus so is $\tau_{>n}X$, and the space $D\mathsf{Hom}_{\mathcal{D}}(Y, (\tau_{>n}X)[2])$ vanishes. For the same reasons, the space $\mathsf{Hom}_{\mathcal{D}}(\tau_{>n}X, Y)$ vanishes. Thus there are bijections

$$\mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, \tau_{\leq n}Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq n}X, Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(X, Y)$$

Therefore, the functor $\pi : \mathcal{F} \to \mathcal{C}$ is fully faithful.

Step 2: For X in $\mathsf{per}\,A$, there exists an integer N and an object Y of $\mathcal{F}[-N]$ such that πX and πY are isomorphic in $\mathcal{C}$.
Let X be in $\mathsf{per}\,A$. By lemma 2.6, there exists an integer N such that X is in ${}^{\perp}\mathcal{D}_{\leq N-2}$. For an object Y in $\mathcal{D}_{\leq N-2}$, the space $\mathsf{Hom}_{\mathcal{D}}((\tau_{>N}X)[-1], Y)$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(Y, (\tau_{>N}X)[2])$ and thus vanishes. Therefore, $\tau_{\leq N}X$ is still in ${}^{\perp}\mathcal{D}_{\leq N-2}$, and thus is in $\mathcal{F}[-N]$. Since $\tau_{>N}X$ is in $\mathcal{D}^b A$, the objects $\tau_{\leq N}X$ and X are isomorphic in $\mathcal{C}$.

Step 3: The functor π restricted to $\mathcal{F}$ is essentially surjective.
Let X be in $\mathsf{per}\,A$ and N such that $\tau_{\leq N}X$ is in $\mathcal{F}[-N]$. By lemma 2.11, $\tau_{\leq -1}$ induces an equivalence between $\mathcal{F}$ and $\mathcal{F}[1]$. Thus since the functor $\pi \circ \tau_{\leq -1} : \mathsf{per}\,A \to \mathcal{C}$ is isomorphic to π, there exists an object Y in $\mathcal{F}$ such that $\pi(Y)$ and $\pi(X)$ are isomorphic in $\mathcal{C}$. Therefore, the functor π restricted to $\mathcal{F}$ is essentially surjective.

Proposition 2.12. *If X and Y are objects in $\mathcal{F}$, there is a short exact sequence:*

$$0 \longrightarrow \mathsf{Ext}^1_{\mathcal{D}}(X, Y) \longrightarrow \mathsf{Ext}^1_{\mathcal{C}}(X, Y) \longrightarrow D\mathsf{Ext}^1_{\mathcal{D}}(Y, X) \longrightarrow 0.$$

Proof. Let X and Y be in $\mathcal{F}$. The canonical triangle

$$\tau_{<0}X \longrightarrow X \longrightarrow \tau_{\geq 0}X \longrightarrow (\tau_{<0}X)[1]$$

yields the long exact sequence:

$$\mathsf{Hom}_{\mathcal{D}}((\tau_{\geq 0}X)[-1], Y[1]) \leftarrow \mathsf{Hom}_{\mathcal{D}}(\tau_{<0}X, Y[1]) \leftarrow \mathsf{Hom}_{\mathcal{D}}(X, Y[1]) \leftarrow \mathsf{Hom}_{\mathcal{D}}(\tau_{\geq 0}X, Y[1])\ .$$

The space $\mathsf{Hom}_{\mathcal{D}}(X[-1], Y[1])$ is zero because X is in ${}^{\perp}\mathcal{D}_{\leq -2}$ and Y is in $\mathcal{D}_{\leq 0}$. Moreover, the space $\mathsf{Hom}_{\mathcal{D}}(\tau_{\geq 0}X, Y[1])$ is zero because of the CY property. Thus this long sequence reduces to a short exact sequence:

$$0 \longrightarrow \mathsf{Ext}^1_{\mathcal{D}}(X, Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(\tau_{<0}X, Y[1]) \longrightarrow \mathsf{Hom}_{\mathcal{D}}((\tau_{\geq 0}X)[-1], Y[1]) \longrightarrow 0\ .$$

Step 1: There is an isomorphism $\mathsf{Hom}_{\mathcal{D}}((\tau_{\geq 0}X)[-1], Y) \simeq D\mathsf{Ext}^1_{\mathcal{D}}(Y, X)$.
The space $\mathsf{Hom}_{\mathcal{D}}((\tau_{\geq 0}X)[-1], Y[1])$ is isomorphic to $D\mathsf{Hom}_{\mathcal{D}}(Y, \tau_{\geq 0}X[1])$ by the CY property.

$$\begin{tikzcd}
& Y \arrow[dl, dotted, "0"'] \arrow[d] \arrow[dr] \arrow[drr, dotted, bend left, "0"] & & \\
(\tau_{<0}X)[1] \arrow[r] & X[1] \arrow[r] & (\tau_{\geq 0}X)[1] \arrow[r] & (\tau_{<0}X)[2]
\end{tikzcd}$$

But since $\mathsf{Hom}_{\mathcal{D}}(Y,(\tau_{<0}X)[1])$ and $\mathsf{Hom}_{\mathcal{D}}(Y,(\tau_{<0}X)[2])$ are zero, we have an isomorphism

$$\mathsf{Hom}_{\mathcal{D}}(\tau_{\geq 0}X[-1],Y)\simeq D\mathsf{Ext}^1_{\mathcal{D}}(Y,X).$$

Step 2: There is an isomorphism $\mathsf{Ext}^1_{\mathcal{C}}(\pi X,\pi Y)\simeq \mathsf{Hom}_{\mathcal{D}}(\tau_{\leq -1}X,Y[1])$.

By lemma 2.11, the object $\tau_{<0}X$ belongs to $\mathcal{F}[1]$ and clearly $Y[1]$ belongs to $\mathcal{F}[1]$. By proposition 2.9 (applied to the shifted t-structure), the functor $\pi:\mathsf{per}\,A\to\mathcal{C}$ induces an equivalence from $\mathcal{F}[1]$ to $\mathcal{C}$ and clearly we have $\pi(\tau_{<0}X,Y[1])\xrightarrow{\sim}\pi(X)$. We obtain bijections

$$\mathsf{Hom}_{\mathcal{D}}(\tau_{<0}X,Y[1])\xrightarrow{\sim}\mathsf{Hom}_{\mathcal{D}}(\pi\tau_{<0}X,\pi Y[1])\xrightarrow{\sim}\mathsf{Hom}_{\mathcal{D}}(\pi X,\pi Y[1]).$$

□

Proof of theorem 2.1. Step 1: The category $\mathcal{C}$ is Hom*-finite and 2-CY.*

The category $\mathcal{F}$ is obviously Hom-finite, hence so is $\mathcal{C}$ by proposition 2.9. The categories $\mathcal{T}=\mathsf{per}\,A$ and $\mathcal{N}=\mathcal{D}^bA\subset\mathsf{per}\,A$ satisfy the hypotheses of section 1. By [Kel08a], thanks to the CY property, there is a bifunctorial non degenerate bilinear form:

$$\beta_{N,X}:\mathsf{Hom}_{\mathcal{D}}(N,X)\times\mathsf{Hom}_{\mathcal{D}}(X,N[3])\to k$$

for N in $\mathcal{D}^bA$ and X in $\mathsf{per}\,A$. Thus, by section 1, there exists a bilinear bifunctorial form

$$\beta'_{X,Y}:\mathsf{Hom}_{\mathcal{C}}(X,Y)\times\mathsf{Hom}_{\mathcal{C}}(Y,X[2])\to k$$

for X and Y in $\mathcal{C}=\mathsf{per}\,A/\mathcal{D}^bA$. We would like to show that it is non degenerate. Since $\mathsf{per}\,A$ is Hom-finite, by theorem 1.3 and proposition 1.4, it is sufficient to show the existence of local $\mathcal{N}$-envelopes. Let X and Y be objects of $\mathsf{per}\,A$. Therefore by lemma 2.6, X is in ${}^{\perp}\mathcal{D}_{\leq N}$. Thus there is an injection

$$0\longrightarrow\mathsf{Hom}_{\mathcal{D}}(X,Y)\longrightarrow\mathsf{Hom}_{\mathcal{D}}(X,\tau_{>N}Y)$$

and $Y\to\tau_{>N}Y$ is a local $\mathcal{N}$-envelope relative to X. Therefore, $\mathcal{C}$ is 2-CY.

Step 2: The object πA is a cluster-tilting object of the category $\mathcal{C}$.

Let A be the free dg A-module in $\mathsf{per}\,A$. Since H^1A is zero, the space $\mathsf{Ext}^1_{\mathcal{D}}(A,A)$ is also zero. Thus by the short exact sequence

$$0\longrightarrow\mathsf{Ext}^1_{\mathcal{D}}(A,A)\longrightarrow\mathsf{Ext}^1_{\mathcal{C}}(\pi A,\pi A)\longrightarrow D\mathsf{Ext}^1_{\mathcal{D}}(A,A)\longrightarrow 0$$

of proposition 2.12, $\pi(A)$ is a rigid object of $\mathcal{C}$. Now let X be an object of $\mathcal{C}$. By proposition 2.9, there exists an object Y in $\mathcal{F}$ such that πY is isomorphic to X. Now by lemma 2.10 , there exists a triangle in $\mathsf{per}\,A$

$$P_1\longrightarrow P_0\longrightarrow Y\longrightarrow P_1[1]$$

with P_1 and P_0 in $add(A)$. Applying the triangle functor π we get a triangle in $\mathcal{C}$:

$$\pi P_1\longrightarrow\pi P_0\longrightarrow X\longrightarrow\pi P_1[1]$$

with πP_1 and πP_0 in $add(\pi A)$. If $\mathsf{Ext}^1_{\mathcal{C}}(\pi A,X)$ vanishes, this triangle splits and X is a direct factor of πP_0. Thus, the object πA is a cluster-tilting object in the 2-CY category $\mathcal{C}$.

3. Cluster categories for Jacobi-finite quivers with potential

3.1. **Ginzburg dg algebra.** Let Q be a finite quiver. For each arrow a of Q, we define the *cyclic derivative with respect to* a ∂_a as the unique linear map

$$\partial_a : kQ/[kQ, kQ] \to kQ$$

which takes the class of a path p to the sum $\sum_{p=uav} vu$ taken over all decompositions of the path p (where u and v are possibly idempotents e_i associated to a vertex i of Q).

An element W of $kQ/[kQ, kQ]$ is called a *potential on* Q. It is given by a linear combination of cycles in Q.

Definition 3.1 (Ginzburg)**.** [Gin06](section 4.2) Let Q be a finite quiver and W a potential on Q. Let $\widehat{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q (of degree 0),
- an arrow $a^* : j \to i$ of degree -1 for each arrow $a : i \to j$ of Q,
- a loop $t_i : i \to i$ of degree -2 for each vertex i of Q.

The *Ginzburg dg algebra* $\Gamma(Q, W)$ is a dg k-algebra whose underlying graded algebra is the graded path algebra $k\widehat{Q}$. Its differential is the unique linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule

$$d(uv) = (du)v + (-1)^p u dv,$$

for all homogeneous u of degree p and all v, and takes the following values on the arrows of $\widehat{Q}$:

- $da = 0$ for each arrow a of Q,
- $d(a^*) = \partial_a W$ for each arrow a of Q,
- $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each vertex i of Q where e_i is the idempotent associated to i and the sum runs over all arrows of Q.

The strictly positive homology of this dg algebra clearly vanishes. Moreover B. Keller showed the following result:

Theorem 3.2 (Keller)**.** [Kel08b] *Let Q be a finite quiver and W a potential on Q. Then the Ginzburg dg algebra $\Gamma(Q, W)$ is homologically smooth and bimodule 3-CY.*

3.2. **Jacobian algebra.**

Definition 3.3. Let Q be a finite quiver and W a potential on Q. The *Jacobian algebra* $J(Q, W)$ is the zeroth homology of the Ginzburg algebra $\Gamma(Q, W)$. This is the quotient algebra

$$kQ/\langle \partial_a W, a \in Q_1 \rangle$$

where $\langle \partial_a W, a \in Q_1 \rangle$ is the two-sided ideal generated by the $\partial_a W$.

Remark: We follow the terminology of H. Derksen, J. Weyman and A. Zelevinsky ([DWZ07] definition 3.1).

In recent works, B. Keller [Kel08b] and A. Buan, O. Iyama, I. Reiten and D. Smith [BIRS08] have shown independently the following result:

Theorem 3.4 (Keller, Buan-Iyama-Reiten-Smith)**.** *Let T be a cluster-tilting object in the cluster category $\mathcal{C}_Q$ associated to an acyclic quiver Q. Then there exists a quiver with potential (Q', W) such that $\mathsf{End}_{\mathcal{C}_Q}(T)$ is isomorphic to $J(Q', W)$.*

3.3. **Jacobi-finite quiver with potentials.** The quiver with potential (Q, W) is called *Jacobi-finite* if the Jacobian algebra $J(Q, W)$ is finite-dimensional.

Definition 3.5. Let (Q, W) be a Jacobi-finite quiver with potential. Denote by Γ the Ginzburg dg algebra $\Gamma(Q, W)$. Let $\mathsf{per}\,\Gamma$ be the thick subcategory of $\mathcal{D}\Gamma$ generated by Γ and $\mathcal{D}^b\Gamma$ the full subcategory of $\mathcal{D}\Gamma$ of the dg Γ-modules whose homology is of finite total dimension. The *cluster category* $\mathcal{C}_{(Q,W)}$ associated to (Q, W) is defined as the quotient of triangulated categories $\mathsf{per}\,\Gamma/\mathcal{D}^b\Gamma$.

Combining theorem 2.1 and theorem 3.2 we get the result:

Theorem 3.6. *Let (Q, W) be a Jacobi-finite quiver with potential. The cluster category $\mathcal{C}_{(Q,W)}$ associated to (Q, W) is* Hom*-finite and 2-CY. Moreover the image T of the free module Γ in the quotient $\mathsf{per}\,\Gamma/\mathcal{D}^b\Gamma$ is a cluster-tilting object. Its endomorphim algebra is isomorphic to the Jacobian algebra $J(Q, W)$.*

As a direct consequence of this theorem we get the corollary:

Corollary 3.7. *Each finite-dimensional Jacobi algebra $\mathcal{J}(Q, W)$ is 2-CY-tilted in the sense of I. Reiten (cf.* [Rei07]*), i.e. it is the endomorphism algebra of some cluster-tilting object of a 2-CY category.*

Definition 3.8. Let (Q, W) and (Q', W') be two quivers with potential. A *triangular extension* between (Q, W) and (Q', W') is a quiver with potential $(\bar{Q}, \bar{W})$ where

- $\bar{Q}_0 = Q_0 \cup Q'_0$;
- $\bar{Q}_1 = Q_1 \cup Q'_1 \cup \{a_i, i \in I\}$, where for each i in the finite index set I, the source of a_i is in Q_0 and the tail of a_i is in Q'_0;
- $\bar{W} = W + W'$.

Proposition 3.9. *Denote by $\mathcal{JF}$ the class of Jacobi-finite quivers with potential. The class $\mathcal{JF}$ satisfies the properties:*

(1) *it contains all acyclic quivers (with potential 0);*
(2) *it is stable under mutation of quivers with potential defined in* [DWZ07]*;*
(3) *it is stable under triangular extensions.*

Proof. (1) This is obvious since the Jacobi algebra $J(Q, 0)$ is isomorphic to kQ.

(2) This is corollary 6.6 of [DWZ07].

(3) Let (Q, W) and (Q', W') be two quivers with potential in $\mathcal{JF}$ and $(\bar{Q}, \bar{W})$ a triangular extension. Let $\bar{Q}_1 = Q_1 \cup Q'_1 \cup F$ be the set of arrows of $\bar{Q}$. We have then

$$k\bar{Q} = kQ' \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ$$

where R is the semi-simple algebra kQ_0 and R' is kQ'_0. Let $\bar{W}$ be the potential $W + W'$ associated to the triangular extension. If a is in Q_1, then $\partial_a \bar{W} = \partial_a W$, if a is in Q'_1 then $\partial_a \bar{W} = \partial_a W'$ and if a is in F, then $\partial_a \bar{W} = 0$. Thus we have isomorphisms

$$\begin{aligned} J(\bar{Q}, \bar{W}) &= k\bar{Q}/\langle \partial_a \bar{W}, a \in \bar{Q}_1 \rangle \\ &\simeq kQ' \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ/\langle \partial_a W, a \in Q_1, \partial_b W', b \in Q'_1 \rangle \\ &\simeq kQ'/\langle \partial_b W', b \in Q'_1 \rangle \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ/\langle \partial_a W, a \in Q_1 \rangle \\ &\simeq J(Q', W') \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R J(Q, W). \end{aligned}$$

Thus if $J(Q', W')$ and $J(Q, W)$ are finite-dimensional, $J(\bar{Q}, \bar{W})$ is finite-dimensional since F is finite.

□

In a recent work [KY08], B. Keller and D. Yang proved the following:

Theorem 3.10 (Keller-Yang). *Let (Q,W) be a Jacobi-finite quiver with potential. Assume that Q has no loops nor two-cycles. For each vertex i of Q, there is a derived equivalence*

$$\mathcal{D}\Gamma(\mu_i(Q,W)) \simeq \mathcal{D}\Gamma(Q,W),$$

where $\mu_i(Q,W)$ is the mutation of (Q,W) at the vertex i in the sense of [DWZ07].

Remark: in fact Keller and Yang proved this theorem in a more general setting. This also true if (Q,W) is not Jacobi-finite, but then there is a derived equivalence between the completions of the Ginzburg dg algebras.

An other link between mutation of quivers with potential and mutations of cluster-tilting objects is given in [BIRS08] (theorem 5.1):

Theorem 3.11 (Buan-Iyama-Reiten-Smith). *Let $\mathcal{C}$ be a 2-CY triangulated category with a cluster-tilting object T. If the endomorphism algebra $\mathsf{End}_{\mathcal{C}}(T)$ is isomorphic to the Jacobian algebra $J(Q,W)$ for some quiver with potential (Q,W), and if no 2-cycles start in the vertex i of Q, then we have an isomorphism*

$$\mathsf{End}_{\mathcal{C}}(\mu_i(T)) \simeq J(\mu_i(Q,W)).$$

Combining these two theorems with theorem 3.6, we get the corollary:

Corollary 3.12.
(1) *If Q is an acyclic quiver, and $W=0$, the cluster category $\mathcal{C}_{(Q,W)}$ is canonically equivalent to the cluster category $\mathcal{C}_Q$.*
(2) *Let Q be an acyclic quiver and T a cluster-tilting object of $\mathcal{C}_Q$. If (Q',W) is the quiver with potential associated with the cluster-tilted algebra $\mathsf{End}_{\mathcal{C}_Q}(T)$ (cf.* [Kel08b] [BIRS08]*), then the cluster category $\mathcal{C}_{(Q,W)}$ is triangle equivalent to the cluster category $\mathcal{C}_{Q'}$.*

Proof.
(1) The cluster category $\mathcal{C}_{(Q,0)}$ is a 2-CY category with a cluster-tilting object whose endomorphism algebra is isomorphic to kQ. Thus by [KR07], this category is triangle equivalent to $\mathcal{C}_Q$.
(2) In a cluster category, all cluster-tilting objects are mutation equivalent. Thus there exists a sequence of mutations which links kQ to T. Moreover the quiver of a cluster-tilted algebra has no loops nor 2-cycles. Thus by theorem 5.1 of [BIRS08], the quiver with potential (Q,W) is mutation equivalent to $(Q',0)$. Then the theorem of Keller and Yang [KY08] applies and we have an equivalence

$$\mathcal{D}\Gamma(Q,W) \simeq \mathcal{D}\Gamma(Q',0).$$

Thus the categories $\mathcal{C}_{(Q,W)}$ and $\mathcal{C}_{(Q',0)}$ are triangle equivalent. By (1) we get the result. □

4. Cluster categories for non hereditary algebras

4.1. **Definition and results of Keller.** Let A be a finite-dimensional k-algebra of finite global dimension. The category $\mathcal{D}^bA$ admits a Serre functor $\nu_A = ?\overset{L}{\otimes}_A DA$ where D is the duality $\mathsf{Hom}_k(?,k)$ over the ground field. The orbit category

$$\mathcal{D}^bA/\nu_A\circ[-2]$$

is defined as follows:

- the objects are the same as those of $\mathcal{D}^bA$;
- if X and Y are in $\mathcal{D}^bA$ the space of morphisms is isomorphic to the space
$$\bigoplus_{i\in\mathbb{Z}}\mathsf{Hom}_{\mathcal{D}A}(X,(\nu_A^iY[-2i]).$$

By Theorem 1 of [Kel05], this category is triangulated if A is derived equivalent to an hereditary category. This happens if A is the endomorphism algebra of a tilting module over an hereditary algebra, or if A is a canonical algebra (cf. [HR02], [Hap01]).

In general it is not triangulated and we define its *triangulated hull* as the algebraic triangulated category $\mathcal{C}_A$ with the following universal property:

- There exists an algebraic triangulated functor $\pi:\mathcal{D}^bA\to\mathcal{C}_A$.
- Let $\mathcal{B}$ be a dg category and X an object of $\mathcal{D}(A^{op}\otimes\mathcal{B})$. If there exists an isomorphism in $\mathcal{D}(A^{op}\otimes\mathcal{B})$ between $DA\overset{L}{\otimes}_A X[-2]$ and X, then the triangulated algebraic functor $?\overset{L}{\otimes}_A X:\mathcal{D}^bA\to\mathcal{D}\mathcal{B}$ factorizes through π.

Let B be the dg algebra $A\oplus DA[-3]$. Denote by $p:B\to A$ the canonical projection. It induces a triangulated functor $p_*:\mathcal{D}^bA\to\mathcal{D}^bB$. Let $\langle A\rangle_B$ be the thick subcategory of $\mathcal{D}^bB$ generated by the image of p_*. By Theorem 2 of [Kel05] (cf. also [Kel08c]), the triangulated hull of the orbit category $\mathcal{D}^bA/\nu_A\circ[-2]$ is the category
$$\mathcal{C}_A=\langle A\rangle_B/\mathsf{per}\,B.$$
We call it the *cluster category of A*. Note that if A is the path algebra of an acyclic quiver, there is an equivalence
$$\mathcal{C}_Q=\mathcal{D}^b(kQ)/\nu\circ[-2]\simeq\langle kQ\rangle_B/\mathsf{per}\,B.$$

4.2. **2-Calabi-Yau property.** The dg B-bimodule DB is clearly isomorphic to $B[3]$, so it is not hard to check the following lemma:

Lemma 4.1. *For each X in $\mathsf{per}\,B$ and Y in $\mathcal{D}^bB$ there is a functorial isomorphism*
$$D\mathsf{Hom}_{\mathcal{D}B}(X,Y)\simeq\mathsf{Hom}_{\mathcal{D}B}(Y,X[3]).$$

So we can apply results of section 1 and construct a bilinear bifunctorial form:
$$\beta'_{XY}:\mathsf{Hom}_{\mathcal{C}_A}(X,Y)\times\mathsf{Hom}_{\mathcal{C}_A}(Y,X[2])\to k.$$

Theorem 4.2. *Let X and Y be objects in $\mathcal{D}=\mathcal{D}^bB$. If the spaces $\mathsf{Hom}_{\mathcal{D}}(X,Y)$ and $\mathsf{Hom}_{\mathcal{D}}(Y,X[3])$ are finite-dimensional, then the bilinear form*
$$\beta'_{XY}:\mathsf{Hom}_{\mathcal{C}_A}(X,Y)\times\mathsf{Hom}_{\mathcal{C}_A}(Y,X[2])\to k$$
is non-degenerate.

Before proving this theorem, we recall some results about inverse limits of sequences of vector spaces that we will use in the proof. Let $\cdots\to V_p\xrightarrow{\varphi}V_{p-1}\xrightarrow{\varphi}\cdots\to V_1\xrightarrow{\varphi}V_0$ be an inverse system of vector spaces (or vector space complexes) inverse system. We then have the following exact sequence
$$0\longrightarrow V_\infty=\varprojlim V_p\longrightarrow\prod_p V_p\xrightarrow{\Phi}\prod_q V_q\longrightarrow\varprojlim{}^1V_p\longrightarrow 0$$
where Φ is defined by $\Phi(v_p)=v_p-\varphi(v_p)\in V_p\oplus V_{p-1}$ where v_p is in V_p.

Recall two classical lemmas due to Mittag-Leffler:

Lemma 4.3. *If, for all p, the sequence of vector spaces $W_i = \mathsf{Im}(V_{p+i} \to V_p)$ is stationary, then $\varprojlim^1 V_p$ vanishes.*

This happens in particular when all vector spaces V_p are finite-dimensional.

Lemma 4.4. *Let $\cdots \longrightarrow V_p \xrightarrow{\varphi} V_{p-1} \xrightarrow{\varphi} \cdots \longrightarrow V_1 \xrightarrow{\varphi} V_0$ be an inverse system of finite-dimensional vector spaces such that $V_\infty = \varprojlim V_p$ is also finite-dimensional. Let V'_p be the image of V_∞ in V_p. The sequence V'_p is stationary and we have $V'_\infty = \varprojlim V'_p = V_\infty$.*

Proof. (of theorem 4.2) Let X and Y be objects of $\mathcal{D}^b B$ such that $\mathsf{Hom}_{\mathcal{D}^b B}(X,Y)$ is finite-dimensional. We will prove that there exists a local $\mathsf{per}\, B$-cover of X relative to Y.

Let $P_\bullet : \ldots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0$ be a projective resolution of X. The complex $P_\bullet$ has components in $\mathsf{per}\, B$, and its homology vanishes in all degrees except in degree zero, where it is X. Let $P_{\leq n}$ and $P_{>n}$ be the natural truncations, and denote by $Tot(P)$ the total complex associated to $P_\bullet$. For all $n \in \mathbb{N}$, there is an exact sequence of dg B-modules:

$$0 \longrightarrow Tot(P_{\leq n}) \longrightarrow Tot(P) \longrightarrow Tot(P_{>n}) \longrightarrow 0$$

The complex $Tot(P)$ is quasi-isomorphic to X, and the complex $Tot(P_{\leq n})$ is in $\mathsf{per}\, B$. Moreover, $Tot(P)$ is the colimit of $Tot(P_{\leq n})$. Thus by definition, we have the following equalities

$$\begin{aligned} \mathcal{H}om^\bullet_B(Tot(P),Y) &= \mathcal{H}om^\bullet_B(\varinjlim{\rm colim}\, Tot(P_{\leq n}),Y) \\ &= \varprojlim \mathcal{H}om^\bullet_B(Tot(P_{\leq n}),Y). \end{aligned}$$

Denote by V_p the complex $\mathcal{H}om^\bullet_B(Tot(P_{\leq p}),Y)$. In the inverse system

$$\cdots \longrightarrow V_p \xrightarrow{\varphi} V_{p-1} \xrightarrow{\varphi} \cdots \longrightarrow V_1 \xrightarrow{\varphi} V_0 \ ,$$

all the maps are surjective, so by lemma 4.3, there is a short exact sequence

$$0 \longrightarrow V_\infty \longrightarrow \prod_p V_p \xrightarrow{\Phi} \prod_q V_q \longrightarrow 0$$

which induces a long exact sequence in cohomology

$$\begin{tikzcd}[column sep=small] \cdots \prod_q H^{-1}V_q \arrow[rr] \arrow[dr] & & H^0(V_\infty) \arrow[rr] \arrow[dr] & & \prod H^0 V_p \arrow[r] & \prod H^0 V_q \cdots . \\ & \varprojlim^1 H^{-1}V_p \arrow[ur] & & \varprojlim H^0 V_p \arrow[ur] & & \end{tikzcd}$$

We have the equalities

$$\begin{aligned} H^0(V_\infty) &= H^0(\mathcal{H}om^\bullet_B(Tot(P),Y)) \\ &= \mathsf{Hom}_{\mathcal{H}}(Tot(P),Y) \\ &= \mathsf{Hom}_{\mathcal{D}}(X,Y). \end{aligned}$$

Denote by W_p the complex $\mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq p}),Y)$ and by U_p the complex $H^{-1}(V_p) = \mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq p}),Y[$ The spaces $(U_p)_p$ are finite-dimensional, so by lemma 4.3, $\varprojlim^1 U_p$ vanishes and we have an isomorphism

$$H^0(\varprojlim V_p) = H^0(V_\infty) \simeq \varprojlim H^0(V_p).$$

The system $(W_p)_p$ satisfies the hypothesis of lemma 4.4. In fact, for each integer p, the space $\mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq p}),Y)$ is finite-dimensional because $Tot(P_{\leq p})$ is in $\mathsf{per}\, B$. Moreover, by the last two equalities $W_\infty = \underleftarrow{\lim} W_p$ is isomorphic to $\mathsf{Hom}_{\mathcal{D}}(X,Y)$ which is finite-dimensional by hypothesis. By lemma 4.4, the system $(W'_p)_p$ formed by the image of W_∞ in W_p is stationary. More precisely, there exists an integer n such that $W'_n = \underleftarrow{\lim} W'_p$. Moreover W'_n is a subspace of $W_n = \mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq n}), Y)$ and there is an injection

$$\mathsf{Hom}_{\mathcal{D}}(X,Y) \hookrightarrow \mathsf{Hom}_{\mathcal{D}}(Tot(P_{\leq n}), Y) \ .$$

This yields a local $\mathsf{per}\, B$-cover of X relative to Y.

The spaces $\mathsf{Hom}_{\mathcal{D}}(N,X)$ and $\mathsf{Hom}_{\mathcal{D}}(X,N)$ are finite-dimensional for N in $\mathsf{per}\, B$ and X in $\mathcal{D}^bB$. Thus by proposition 1.4, there exists local $\mathsf{per}\, B$-envelopes. Therefore theorem 1.3 applies and β' is non-degenerate.

□

Corollary 4.5. *Let A be a finite-dimensional k-algebra with finite global dimension. If the cluster category $\mathcal{C}_A$ is Hom-finite, then it is 2-CY as a triangulated category.*

Proof. Denote by $p_* : \mathcal{D}^bA \to \mathcal{D}^bB$ the restriction of the projection $p : B \to A$.

Let X and Y be in $\mathcal{D}^b(A)$. By hypothesis, the vector spaces

$$\bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^bA}(X, \nu_A^p Y[-2p]) \quad \text{and} \quad \bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^bA}(Y, \nu_A^p X[-2p+3])$$

are finite-dimensional. But by [Kel05], the space $\mathsf{Hom}_{\mathcal{D}^bB}(p_*X, p_*Y)$ is isomorphic to

$$\bigoplus_{p\geq 0} \mathsf{Hom}_{\mathcal{D}^bA}(X, \nu_A^p Y[-2p]),$$

so is finite-dimensional. For the same reasons, the space $\mathsf{Hom}_{\mathcal{D}^bB}(Y, X[3])$ is also finite-dimensional. Applying theorem 4.2, we get a non-degenerate bilinear form β'_{p_*X, p_*Y}. The non-degeneracy property is extension closed, so for each M and N in $\langle A\rangle_B$, the form β'_{MN} is non-degenerate.

□

4.3. **Case of global dimension 2.** In this section we assume that A is a finite-dimensional k-algebra of global dimension ≤ 2.

Criterion for Hom*-finiteness.* The canonical t-structure on the derived category $\mathcal{D} = \mathcal{D}^bA$ satisfies the property:

Lemma 4.6. *We have the following inclusions $\nu(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq -2}$ and $\nu^{-1}(\mathcal{D}_{\leq 0}) \subset \mathcal{D}_{\leq 2}$. Moreover, the space $\mathsf{Hom}_{\mathcal{D}}(U,V)$ vanishes for all U in $\mathcal{D}_{\geq 0}$ and all V in $\mathcal{D}_{\leq -3}$.*

Proposition 4.7. *Let X be the A-A-bimodule $\mathsf{Ext}^2_A(DA, A)$. The endomorphism algebra $\widetilde{A} = \mathsf{End}_{\mathcal{C}_A}(A)$ is isomorphic to the tensor algebra T_AX of X over A.*

Proof. By definition, the endomorphism space $\mathsf{End}_{\mathcal{C}_A}(A)$ is isomorphic to

$$\bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(A, \nu^p A[-2p])$$

For $p \geq 1$, the object $\nu^pA[-2p]$ is in $\mathcal{D}_{\geq 2}$ since νA is in $\mathcal{D}_{\geq 0}$. So since A is in $\mathcal{D}_{\leq 0}$, the space $\mathsf{Hom}_{\mathcal{D}}(A, \nu^pA[-2p])$ vanishes.

The functor $\nu = ?\overset{L}{\otimes}_A DA$ admits an inverse $\nu^{-1} = -\overset{L}{\otimes}_A\ R\mathcal{H}om_A(DA, A)$. Since the global dimension of A is ≤ 2, the homology of the complex $R\mathcal{H}om_A(DA, A)$ is concentrated in degrees 0, 1 and 2 :

$$\begin{array}{rcl} H^0(R\mathcal{H}om_A(DA, A)) & = & \mathsf{Hom}_{\mathcal{D}}(DA, A) \\ H^1(R\mathcal{H}om_A(DA, A)) & = & \mathsf{Ext}^1_A(DA, A) \\ H^2(R\mathcal{H}om_A(DA, A)) & = & \mathsf{Ext}^2_A(DA, A). \end{array}$$

Let us denote by Y the complex $R\mathcal{H}om_A(DA, A)[2]$. We then have

$$\nu^{-p}A[2p] = A\overset{L}{\otimes}_A (Y^{\overset{L}{\otimes}_A p}) = Y^{\overset{L}{\otimes}_A p}.$$

Therefore we get the following equalities

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}A}(A, S^{-p}A[-2p]) & = & \mathsf{Hom}_{\mathcal{D}A}(A, Y^{\overset{L}{\otimes}_A p}) \\ & = & H^0(Y^{\overset{L}{\otimes}_A p}). \end{array}$$

Since $H^0(Y) = X$, we conclude using the following easy lemma. $\square$

Lemma 4.8. *Let M and N be two complexes of A-modules whose homology is concentrated in negative degrees. Then there is an isomorphism*

$$H^0(M\overset{L}{\otimes}_A N) \simeq H^0(M) \otimes_A H^0(N).$$

Proposition 4.9. *Let A be a finite-dimensional algebra of global dimension 2. The following properties are equivalent:*

(1) *the cluster category $\mathcal{C}_A$ is* Hom*-finite;*
(2) *the functor $? \otimes_A \mathsf{Ext}^2(DA, A)$ is nilpotent;*
(3) *the functor $\mathsf{Tor}^A_2(?, DA)$ is nilpotent;*
(4) *there exists an integer N such that there is an inclusion $\Phi^N(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq 1}$ where Φ is the autoequivalence $\nu_A[-2]$ of the category $\mathcal{D} = \mathcal{D}^bA$ and $\mathcal{D}_{\geq 0}$ is the right aisle of the natural t-structure of $\mathcal{D}^bA$.*

Proof. $1 \Rightarrow 2$: It is obvious by proposition 4.7.

$2 \Leftrightarrow 3 \Leftrightarrow 4$: Let Φ be the autoequivalence $\nu_A[-2]$ of $\mathcal{D}^bA$. The functor $\mathsf{Tor}^2_A(?, DA)$ is isomorphic to $H^0 \circ \Phi$ and $? \otimes_A \mathsf{Ext}^2_A(DA, A)$ is isomorphic to $H^0 \circ \Phi^{-1}$. Thus it is sufficient to check that there are isomorphisms

$$H^0\Phi \circ H^0\Phi \simeq H^0\Phi^2 \text{ and } H^0\Phi^{-1} \circ H^0\Phi^{-1} \simeq H^0\Phi^{-2}.$$

This is easy using Lemma 4.8 since the algebra A has global dimension ≤ 2 .

$4 \Rightarrow 1$: Suppose that there exists some $N \geq 0$ such that $\Phi^N(\mathcal{D}_{\geq 0})$ is included in $\mathcal{D}_{\leq 1}$. For each object X in $\mathcal{C}_A$, the class of the objects Y such that the space $\mathsf{Hom}_{\mathcal{C}_A}(X, Y)$ (resp. $\mathsf{Hom}_{\mathcal{C}_A}(Y, X)$) is finite-dimensional, is extension closed. Therefore, it is sufficient to show that for all simples S, S', and each integer n, the space $\mathsf{Hom}_{\mathcal{C}_A}(S, S'[n])$ is finite-dimensional.

There exists an integer p_0 such that for all $p \geq p_0$ $\Phi^p(S')$ is in $\mathcal{D}_{\geq n+1}$. Therefore, because of the defining properties of the t-structure, the space

$$\bigoplus_{p \geq p_0} \mathsf{Hom}_{\mathcal{D}}(S, \Phi^p(S')[n])$$

vanishes. Similary, there exists an integer q_0 such that for all $q \geq q_0$, we have $\Phi^q(S) \in \mathcal{D}_{\geq -n+3}$. Since the algebra A is of global dimension ≤ 2, the space

$$\bigoplus_{q \geq q_0} \mathsf{Hom}_{\mathcal{D}}(\Phi^q(S), S'[n])$$

vanishes. Thus the space

$$\bigoplus_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(S, \Phi^p(S')[n]) = \bigoplus_{p=-q_0}^{p_0} \mathsf{Hom}_{\mathcal{D}}(S, \Phi^p(S')[n])$$

is finite-dimensional. □

Cluster-tilting object. In this section we prove the following theorem:

Theorem 4.10. *Let A be a finite-dimensional k-algebra of global dimension ≤ 2. If the functor $\mathsf{Tor}_2^A(?, DA)$ is nilpotent, then the cluster category $\mathcal{C}_A$ is Hom-finite, 2-CY and the object A is a cluster-tilting object.*

We denote by Θ a cofibrant resolution of the dg A-bimodule $R\mathcal{H}om_A^\bullet(DA, A)$. Following [Kel08a] and [Kel08b], we define the 3-derived preprojective algebra as the tensor algebra

$$\Pi_3(A) = T_A(\Theta[2]).$$

The complex $R\mathcal{H}om_A^\bullet(DA, A)[2]$ has its homology concentrated in degrees -2, -1 and 0, and we have

$$H^{-2}(\Theta[2]) \simeq \mathsf{Hom}_{\mathcal{D}A}(DA, A),\ H^{-1}(\Theta[2]) \simeq \mathsf{Ext}_A^1(DA, A)$$
$$\text{and } H^0(\Theta[2]) \simeq \mathsf{Ext}_A^2(DA, A).$$

Thus the homology of the dg algebra $\Pi_3(A)$ vanishes in strictly positive degrees and we have

$$H^0\Pi_3 A = T_A \mathsf{Ext}_A^2(DA, A) = \tilde{A}.$$

By proposition 4.9, it is finite-dimensional. Moreover, Keller showed that $\Pi_3(A)$ is homologically smooth and bimodule 3-CY [Kel08b]. Thus we can apply theorem 2.1 and we have the following result:

Corollary 4.11. *The category $\mathcal{C} = \mathsf{per}\,\Pi_3 A/\mathcal{D}^b\Pi_3 A$ is 2-CY and the free dg module $\Pi_3 A$ is a cluster-tilting object in $\mathcal{C}$.*

To complete the proof of Theorem 4.10 we now construct a triangle equivalence between $\mathcal{C}_A$ and $\mathcal{C}$ sending A to $\Pi_3 A$.

Let us recall a theorem of Keller ([Kel94], or theorem 8.5, p.96 [AHHK07]):

Theorem 4.12. *[Keller] Let B be a dg algebra, and T an object of $\mathcal{D}B$. Denote by C the dg algebra $R\mathcal{H}om_B^\bullet(T, T)$. Denote by $\langle T\rangle_B$ the thick subcategory of $\mathcal{D}B$ generated by T. The functor $R\mathcal{H}om_B^\bullet(T, ?) : \mathcal{D}B \to \mathcal{D}C$ induces an algebraic triangle equivalence*

$$R\mathcal{H}om_B^\bullet(T, ?) : \langle T\rangle_B \xrightarrow{\sim} \mathsf{per}\, C.$$

Let us denote by $\mathcal{H}o(dgalg)$ the homotopy category of dg algebras, i.e. the localization of the category of dg algebras at the class of quasi-isomorphisms.

Lemma 4.13. *In $\mathcal{H}o(dgalg)$, there is an isomorphism between $\Pi_3 A$ and $R\mathsf{Hom}_B(A_B, A_B)$.*

Proof. The dg algebra B is $A \oplus (DA)[-3]$. Denote by X a cofibrant resolution of the dg A-bimodule $DA[-2]$. Now look at the dg submodule of the bar resolution of B seen as a bimodule over itself (see the proof of theorem 7.1 in [Kel05]):

$$bar(X,B) : \cdots \longrightarrow B \otimes_A X^{\otimes_A 2} \otimes_A B \longrightarrow B \otimes_A X \otimes_A B \longrightarrow B \otimes_A B \longrightarrow 0$$

This is a cofibrant resolution of the dg B-bimodule B. Thus $A \otimes_B bar(X,B)$ is a cofibrant resolution of the dg B-module A. Therefore, we have the following isomorphisms

$$\begin{aligned} R\mathcal{H}om^\bullet_B(A_B, A_B) &\simeq \mathcal{H}om^\bullet_B(A \otimes_B bar(X,B), A) \\ &\simeq \prod_{n\geq 0} \mathcal{H}om^\bullet_B(A \otimes_A X^{\otimes_A n} \otimes_A B, A_B) \\ &\simeq \prod_{n\geq 0} \mathcal{H}om^\bullet_A(X^{\otimes_A n}, \mathsf{Hom}_B(B, A_B)_A) \\ &\simeq \prod_{n\geq 0} \mathcal{H}om^\bullet_A(X^{\otimes_A n}, A_A), \end{aligned}$$

where the differential on the last complex is induced by that of $X^{\otimes_A n}$. Note that

$$\begin{aligned} \mathcal{H}om^\bullet_A(X, A) &= R\mathcal{H}om^\bullet_A(DA[-2], A) \\ &= R\mathcal{H}om^\bullet_A(DA, A)[2] = \Theta[2]. \end{aligned}$$

We can now use the following lemma:

Lemma 4.14. *Let A be a dg algebra, and L and M dg A-bimodules such that M_A is perfect as right dg A-module. There is an isomorphism in $\mathcal{D}(A^{op} \otimes A)$*

$$R\mathcal{H}om^\bullet_A(L, A) \overset{L}{\otimes}_A R\mathcal{H}om^\bullet_A(M, A) \simeq R\mathcal{H}om^\bullet_A(M \overset{L}{\otimes}_A L, A).$$

Proof. Let X and M be dg A-bimodules. The following morphism of $\mathcal{D}(A^{op} \otimes A)$

$$\begin{aligned} X \overset{L}{\otimes}_A R\mathcal{H}om_A(M, A) &\longrightarrow R\mathcal{H}om_A(M, X) \\ x \otimes \varphi &\mapsto (m \mapsto x\varphi(m)) \end{aligned}$$

is clearly an isomorphism for $M = A$. Thus it is an isomorphism if M is perfect as a right dg A-module. Applying this to the right dg A-module $R\mathcal{H}om_A(L, A)$, we get an isomorphism of dg A-bimodules

$$R\mathcal{H}om_A(L, A) \overset{L}{\otimes}_A R\mathcal{H}om_A(M, A) \simeq R\mathcal{H}om_A(M, R\mathcal{H}om_A(L, A)).$$

Finally, by adjunction we get an isomorphism of dg A-bimodules

$$R\mathcal{H}om_A(L, A) \overset{L}{\otimes}_A R\mathcal{H}om_A(M, A) \simeq R\mathcal{H}om_A(M \overset{L}{\otimes}_A L, A).$$

□

Therefore, the dg A-bimodule $\mathcal{H}om^\bullet_A(X^{\otimes_A n}, A_A)$ is isomorphic to $(\Theta[2])^{\otimes_A n}$, and there is an isomorphism of dg algebras

$$R\mathcal{H}om^\bullet_B(A_B, A_B) \simeq \bigoplus_{n\geq 0} (\theta[2])^{\overset{L}{\otimes}_A n} = \Pi_3(A)$$

because for each $p \in \mathbb{Z}$, the group $H^p(\theta[2]^{\overset{L}{\otimes}_A n})$ vanishes for all $n \gg 0$. □

By theorem 4.12, the functor $R\mathcal{H}om^{\bullet}_{B}(A_B, ?)$ induces an equivalence between the thick subcategory $\langle A\rangle_B$ of $\mathcal{D}B$ generated by A, and $\mathsf{per}\,\Pi_3(A)$. Thus we get a triangle equivalence that we will denote by F:

$$F = R\mathcal{H}om^{\bullet}_{B}(A_B, ?) : \langle A\rangle_B \xrightarrow{\quad\sim\quad} \mathsf{per}\,\Pi_3 A$$

This functor sends the object A_B of $\mathcal{D}^bB$ onto the free module Π_3A and the free B-module B onto $R\mathcal{H}om^{\bullet}_{B}(A_B, B) \simeq R\mathcal{H}om^{\bullet}_{B}(A_B, DB[-3])$, that is to say onto $(DA)[-3]_{\Pi_3 A}$. So F induces an equivalence

$$F : \mathsf{per}\,B = \langle B\rangle_B \xrightarrow{\sim} \langle DA[-3]\rangle_{\Pi_3A} = \langle A\rangle_{\Pi_3A}.$$

Lemma 4.15. *The thick subcategory $\langle A\rangle_{\Pi_3A}$ of $\mathcal{D}\Pi_3A$ generated by A is $\mathcal{D}^b\Pi_3A$.*

Proof. The algebra A is finite-dimensional, thus $\langle A\rangle_{\Pi_3A}$ is obviously included in $\mathcal{D}^b\Pi_3A$. Moreover, the category $\mathcal{D}^b\Pi_3A$ equals $\langle \mathsf{mod}\,H^0(\Pi_3A)\rangle_{\Pi_3A}$ by the existence of the t-structure. The dg algebra Π_3A is the tensor algebra $T_A(\theta[2])$ thus there is a canonical projection $\Pi_3A \to A$ which yields a restriction functor $\mathcal{D}^bA \to \mathcal{D}^b(\Pi_3A)$ respecting the t-structure:

$$\begin{tikzcd} \mathsf{mod}\,H^0\Pi_3A = \mathcal{H} \arrow[r, hook] & \mathcal{D}^b(\Pi_3A) \\ \mathsf{mod}\,A \arrow[u] \arrow[r, hook] & \mathcal{D}^bA \arrow[u] \end{tikzcd}$$

This restriction functor induces a bijection in the set of isomorphism classes of simple modules because the kernel of the map $H^0(\Pi_3A) \to A$ is a nilpotent ideal (namely the sum of the tensor powers over A of the bimodule $\mathsf{Ext}^2_A(DA, A)$). Thus each simple of $\mathsf{mod}\,H^0\Pi_3A$ is in $\langle A\rangle_{\Pi_3A}$ and we have

$$\langle A\rangle_{\Pi_3A} \simeq \langle \mathsf{mod}\,H^0(\Pi_3A)\rangle_{\Pi_3A} \simeq \mathcal{D}^b\Pi_3A.$$

□

Proof. (of theorem 4.10) By proposition 4.9 and corollary 4.5, the cluster category is Hom-finite and 2-CY. Furthermore, the functor $F = R\mathcal{H}om^{\bullet}_{B}(A_B, ?)$ induces the following commutative square:

$$\begin{tikzcd} F : \langle A\rangle_B \arrow[r, "\sim"] & \mathsf{per}\,\Pi_3A \\ \mathsf{per}\,B \arrow[u, hook] \arrow[r, "\sim"] & \mathcal{D}^b\Pi_3A \arrow[u, hook] \end{tikzcd}$$

Thus F induces a triangle equivalence

$$\mathcal{C}_A = \langle A\rangle_B/\mathsf{per}\,B \xrightarrow{\quad\sim\quad} \mathsf{per}\,\Pi_3A/\mathcal{D}^b\Pi_3A = \mathcal{C}$$

sending the object A onto the free module Π_3A. By theorem 2.1, A is therefore a cluster-tilting object of the cluster category $\mathcal{C}_A$. □

Quiver of the endomorphism algebra of the cluster-tilting object. Let $A = kQ/I$ be a finite-dimensional k-algebra of global dimension ≤ 2. Suppose that I is an admissible ideal generated by a finite set of minimal relations r_i, $i \in J$ where for each $i \in J$, the relation r_i starts at the vertex $s(r_i)$ and ends at the vertex $t(r_i)$. Let $\widetilde{Q}$ be the following quiver:

- the set of the vertices of $\widetilde{Q}$ equals that of Q;

- the set of arrows of $\widetilde{Q}$ is obtained from that of Q by adding a new arrow ρ_i with source $t(r_i)$ and target $s(r_i)$ for each i in J.

We then have the following proposition, which has essentially been proved by I. Assem, T. Brüstle and R. Schiffler [ABS06] (thm 2.6). The proposition is also proved in [Kel08b].

Proposition 4.16. *If the algebra* $\mathsf{End}_{\mathcal{C}_A}(A) = \widetilde{A}$ *is finite-dimensional, then its quiver is* $\widetilde{Q}$.

Proof. Let B be a finite-dimensional algebra. The vertices of its quiver are determined by the quotient $B/rad(B)$ and the arrows are determined by $rad(B)/rad^2(B)$. Denote by X the A-A-bimodule $\mathsf{Ext}^2_A(DA, A)$. Since $X \otimes_A X$ is in $rad^2(B)$, the quiver of $\tilde{A} = T_A X$ is the same as the quiver of the algebra $A \rtimes X$. The proof is then exactly the same as in [ABS06] (thm 2.6). □

Example. We refer to [GLS07a] for this example. Let Q be a Dynkin quiver. Let A be its Auslander algebra. The algebra A is of global dimension ≤ 2. The category $\mathsf{mod}\, A$ is equivalent to the category $\mathsf{mod}\,(\mathsf{mod}\, kQ)$ of finitely presented functors $(\mathsf{mod}\, kQ)^{op} \to \mathsf{mod}\, k$. The projective indecomposables of $\mathsf{mod}\, A$ are the representable functors $U^\wedge = \mathsf{Hom}_{kQ}(?, U)$ where U is an indecomposable kQ-module. Let S be a simple A-module. Since A is finite-dimensional, this simple is associated to an indecomposable U of $\mathsf{mod}\, kQ$. If U is not projective, then it is easy to check that in $\mathcal{D}^b(A)$ the simple S_U is isomorphic to the complex:

$$\cdots \longrightarrow \underset{-3}{0} \longrightarrow \underset{-2}{(\tau U)^\wedge} \longrightarrow \underset{-1}{E^\wedge} \longrightarrow \underset{0}{U^\wedge} \longrightarrow \underset{1}{0} \longrightarrow \cdots$$

where $0 \longrightarrow \tau U \longrightarrow E \longrightarrow U \longrightarrow 0$ is the Auslander-Reiten sequence associated to U. Thus $\Phi(S_U) = \nu S_U[-2]$ is the complex:

$$\cdots \longrightarrow \underset{-1}{0} \longrightarrow \underset{0}{(\tau U)^\vee} \longrightarrow \underset{1}{E^\vee} \longrightarrow \underset{2}{U^\vee} \longrightarrow \underset{3}{0} \longrightarrow \cdots$$

where $U^\vee$ is the injective A-module $D\mathsf{Hom}_{kQ}(U, ?)$. It follows from the Auslander-Reiten formula that this complex is quasi-isomorphic to the simple $S_{\tau U}$.

If U is projective, then S_U is isomorphic in $\mathcal{D}^b(A)$ to

$$\cdots \longrightarrow \underset{-2}{0} \longrightarrow \underset{-1}{(radU)^\wedge} \longrightarrow \underset{0}{U^\wedge} \longrightarrow \underset{1}{0} \longrightarrow \cdots,$$

and then $\Phi(S_U)$ is in $\mathcal{D}_{\geq 1}$. Since for each indecomposable U there is some N such that $\tau^N U$ is projective, there is some M such that $\Phi^M(\mathcal{D}_{\geq 0})$ is included in $\mathcal{D}_{\geq 1}$. By proposition 4.9, the cluster category $\mathcal{C}_A$ is Hom-finite, and 2-CY by corollary 4.5.

The quiver of A is the Auslander-Reiten quiver of $\mathsf{mod}\, kQ$. The minimal relations of the algebra A are given by the mesh relations. Thus the quiver of $\tilde{A}$ is the same as that of A in which arrows $\tau x \to x$ are added for each non projective indecomposable x.

For instance, if Q is A_4 with the orientation $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$, then the quiver of the algebra $\tilde{A}$ is the following

$$\begin{tikzcd}
 & & & \bullet \arrow[dr] & & & \\
 & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ll] \arrow[dr] & & \\
 & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ll] \arrow[ur] \arrow[dr] & & \bullet \arrow[ll] \arrow[dr] & \\
\bullet \arrow[ur] & & \bullet \arrow[ll] \arrow[ur] & & \bullet \arrow[ll] \arrow[ur] & & \bullet \arrow[ll]
\end{tikzcd}$$

5. Stable module categories as cluster categories

5.1. **Definition and first properties.** Let B be a *concealed* algebra [Rin84], *i.e.* the endomorphism algebra of a preinjective tilting module over a finite-dimensional hereditary algebra. Let H be a postprojective slice of $\mathsf{mod}\, B$. We denote by *add*(H) the smallest subcategory of $\mathsf{mod}\, B$ which contains H and which is stable under taking direct summands. Let Q be the quiver such that $\mathsf{End}_B(H)$ is the path algebra kQ and let $Q_0 = \{1, \cdots, n\}$ be its set of vertices. By Happel [Hap87], there is a triangle equivalence:

$$\mathcal{D}^b(B) \underset{(D?)\overset{L}{\otimes}_{kQ} H}{\overset{DR\mathsf{Hom}_B(?,H)}{\rightleftarrows}} \mathcal{D}^b(kQ).$$

Notice that these functors induce quasi-inverse equivalences between *add*(H) and the subcategory of finite-dimensional injective kQ-modules.

Define $\mathcal{M}$ as the following subcategory of $\mathsf{mod}\, B$:

$$\mathcal{M} = \{X \in \mathsf{mod}\, B \mid \mathsf{Ext}^1_B(X, H) = 0\} = \{X \in \mathsf{mod}\, B \mid X \text{ is cogenerated by } H\}$$

We denote by τ_B the AR-translation of the category $\mathsf{mod}\, B$ and by $\tau_{\mathcal{D}}$ the AR-translation of $\mathcal{D}^b B$.

The following proposition is a classical result in tilting theory (see for example [Rin84]).

Proposition 5.1. (1) *For each X in $\mathcal{M}$ there exists a triangle*

$$X \longrightarrow H_0 \longrightarrow H_1 \longrightarrow X[1]$$

in $\mathcal{D}^b(\mathsf{mod}\, B)$ functorial in X with H_0 and H_1 in add(H);

(2) *$\mathcal{M} \subset \mathsf{mod}\, B$ is closed under kernels so in particular, $\mathcal{M}$ is closed under τ_B;*
(3) *for each indecomposable X in $\mathcal{M}$ there exists a unique $q \geq 0$ such that $\tau_B^{-q} X$ is in add(H);*
(4) *the category $\mathcal{M}$ has finitely many indecomposables.*

Hom-finiteness. Let $\overline{\mathcal{M}}$ be the quotient $\mathcal{M}/add(H)$. Denote by $p : \mathcal{M} \to \overline{\mathcal{M}}$ the canonical projection. Since H is a slice, we have the following properties.

Proposition 5.2. (1) *The category $\overline{\mathcal{M}}$ is equivalent to the full subcategory of $\mathcal{M}$ whose objects do not have non zero direct factors in add(H). We denote by $i : \overline{\mathcal{M}} \to \mathcal{M}$ the associated inclusion.*
(2) *The category $\overline{\mathcal{M}} \subset \mathsf{mod}\, B$ is closed under kernels, and hence under τ_B.*
(3) *The right exact functor $i : \mathsf{mod}\,\overline{\mathcal{M}} \to \mathsf{mod}\,\mathcal{M}$ induced by $i : \overline{\mathcal{M}} \to \mathcal{M}$ is isomorphic to the restriction along p.*

Proposition 5.3. *Let A be the endomorphism algebra $\mathsf{End}_B(\bigoplus_{M \in \mathsf{ind}\,\overline{\mathcal{M}}} M)$. The global dimension of A is at most 2.*

Proof. There is an equivalence of categories between $\mathsf{mod}\, A$ and $\mathsf{mod}\,\overline{\mathcal{M}}$. Since $\overline{\mathcal{M}}$ is stable under kernels, the global dimension of A is ≤ 2. $\square$

Theorem 5.4. *The cluster category $\mathcal{C}_A$ is a* Hom*-finite, 2-CY category, and the object A is a cluster-tilting object in $\mathcal{C}_A$.*

Proof. Using corollary 4.5 and theorem 4.9, we just have to check that the functor $\mathsf{Tor}^2_A(?, DA)$ is nilpotent. Since there are finitely many indecomposables in $\overline{\mathcal{M}}$, the proof is the same as for an Auslander algebra (cf. the examples of section 4.3). $\square$

Construction of the functor $F : \mathsf{mod}\,\mathcal{M} \to f.l.\Lambda$. Denote by $\mathcal{I}(kQ)$ the subcategory of the preinjective modules of $\mathsf{mod}\,kQ$.

Proposition 5.5. *There exists a k-linear functor $P : \mathcal{I}(kQ) \to \mathcal{M}$ unique up to isomorphism such that*

- *P restricted to subcategory of the injective kQ-modules is isomorphic to the restriction of the functor $D(?) \otimes_{kQ} H$;*
- *for each indecomposable X in $\mathcal{I}(kQ)$ such that $P(X)$ is not projective, the image*

$$0 \longrightarrow P(\tau_{\mathcal{D}}X) \xrightarrow{Pi} P(E) \xrightarrow{Pp} P(X) \longrightarrow 0$$

of an Auslander-Reiten sequence in $\mathsf{mod}\,kQ$ ending at X

$$0 \longrightarrow \tau_{\mathcal{D}}X \xrightarrow{i} E \xrightarrow{p} X \longrightarrow 0$$

is an Auslander-Reiten sequence in $\mathsf{mod}\,B$ ending at $P(X)$.

Moreover, the functor P is full, essentially surjective, and satisfies $P \circ \tau_{\mathcal{D}} \simeq \tau_B \circ P$.

Proof. The Auslander-Reiten quivers $\Gamma_{\mathcal{I}}$ of $\mathcal{I}(kQ)$ and $\Gamma_{\mathcal{M}}$ of $\mathcal{M}$ are connected translation quivers. Each vertex of $\Gamma_{\mathcal{I}}$ is of the form $\tau_{\mathcal{D}}^q x$ with $q \geq 0$ and x indecomposable injective. Each vertex of $\Gamma_{\mathcal{M}}$ is of the form $\tau_B^q x$ where x is in $add(H)$ ((3) of proposition 5.1). Moreover, there is a canonical isomorphism of quivers $\bar{P} : \Gamma_{DkQ} \to \Gamma_{add(H)}$. Thus we can inductively construct a morphism of quivers (that we will still denote by $\bar{P}$) $\bar{P} : \Gamma_{\mathcal{I}} \to \Gamma_{\mathcal{M}}$ extending $\bar{P}$ such that:

- $\bar{P}(\tau_{\mathcal{D}}x) = \tau_B\bar{P}(x)$ for each vertex x of $\Gamma_{\mathcal{I}}$;
- $\bar{P}(\sigma_{\mathcal{D}}\alpha) = \sigma_B\bar{P}(\alpha)$ for each arrow $\alpha : x \to y$ of $\Gamma_{\mathcal{I}}$, where $\sigma_{\mathcal{D}}\alpha$ (resp. $\sigma_B\beta$) denotes the arrow $\tau_{\mathcal{D}}y \to x$ (resp. $\tau_B y \to x$) such that the mesh relations in $\Gamma_{\mathcal{I}}$ (resp. in $\Gamma_{\mathcal{M}}$) are of the form $\sum_{t(\alpha)=x} \sigma_{\mathcal{D}}(\alpha)\alpha$ (resp. $\sum_{t(\beta)=x} \sigma_B(\beta)\beta$).

Clearly, this morphism of translation quivers induces surjections in the sets of vertices and the sets of arrows.

The categories $\mathcal{I}(kQ)$ and $\mathcal{M}$ are standard, *i.e.* k-linearly equivalent to the mesh categories of their Auslander-Reiten quivers. Up to isomorphism, an equivalence $k(\Gamma_{\mathcal{I}}) \to \mathcal{I}(kQ)$ is uniquely determined by its restriction to a slice. Thus there exists a k-linear functor $P : \mathcal{I}(kQ) \to \mathcal{M}$ unique up to isomorphism which is equal to $D(?) \otimes_{kQ} H$ on the slice of the injectives and such that the square

$$\begin{tikzcd} k(\Gamma_{\mathcal{I}}) \arrow[r, "\sim"] \arrow[d, "\bar{P}"] & \mathcal{I}(kQ) \arrow[d, "P"] \\ k(\Gamma_{\mathcal{M}}) \arrow[r, "\sim"] & \mathcal{M} \end{tikzcd}$$

is commutative. This functor P sends Auslander-Reiten sequences

$$0 \longrightarrow \tau_{\mathcal{D}}X \xrightarrow{i} E \xrightarrow{p} X \longrightarrow 0$$

to Auslander-Reiten sequences

$$0 \longrightarrow \tau_B P(X) \xrightarrow{Pi} P(E) \xrightarrow{Pp} P(X) \longrightarrow 0$$

if $P(X)$ is not projective. Since $\bar{P}$ is surjective, P is full and essentially surjective. $\square$

Lemma 5.6. *Let X and Y be indecomposables in $\mathcal{I}(kQ)$. The kernel of the map $\mathsf{Hom}_{kQ}(X,Y) \to \mathsf{Hom}_B(PX,PY)$ is generated by compositions of the form $X \to Z \to Y$ where Z is indecomposable and $P(Z)$ is zero.*

Proof. If $P(X)$ or $P(Y)$ is zero this is obviously true. Suppose they are not. The mesh relations are minimal relations of the k-linear category $\mathcal{M}$ and P is full. Thus the kernel of the functor P is the ideal generated by the morphisms of the form $U \xrightarrow{g} V \xrightarrow{h} W$ where $0 \longrightarrow P(U) \xrightarrow{Pg} P(V) \xrightarrow{Ph} P(W) \longrightarrow 0$ is an Auslander-Reiten sequence in $\mathcal{M}$. Since $P(U)$ is isomorphic to $\tau_B P(W)$, the indecomposable U is isomorphic to $\tau_{\mathcal{D}}(W)$. By the construction of P, V is a direct factor of the middle term of the Auslander-Reiten sequence ending at W, and we can 'complete' the composition $\tau_{\mathcal{D}}W \xrightarrow{g} V \xrightarrow{h} W$ into an Auslander-Reiten sequence

$$0 \longrightarrow \tau_{\mathcal{D}}W \xrightarrow{\left(\begin{smallmatrix} g \\ g' \end{smallmatrix}\right)} V \oplus V' \xrightarrow{(h\ h')} W \longrightarrow 0$$

with $P(V') = 0$ and $P(g') = P(h') = 0$. Thus the morpism $hg = -h'g'$ factors through an object in the kernel of P. $\square$

Now let Λ be the preprojective algebra associated to the acyclic quiver Q. It is defined as the quotient $k\bar{Q}/(c)$ where $\bar{Q}$ is the double quiver of Q which is obtained from Q by adding to each arrow $a : i \to j$ an arrow $a^* : j \to i$ pointing in the opposite direction, and where (c) is the ideal generated by the element

$$c = \sum_{a \in Q_1} (a^*a + aa^*)$$

where Q_1 is the set of arrows of Q. We denote by e_i the idempotent of Λ associated with the vertex i. We then have a natural functor

$$\begin{array}{rcl} \mathsf{proj}\Lambda & \longrightarrow & \mathcal{I}^{\Pi}(kQ) \\ e_i\Lambda & \mapsto & \prod_{p\geq 0} \tau_{\mathcal{D}}^p I_i \end{array}$$

where $\mathcal{I}^{\Pi}(kQ)$ is the closure of $\mathcal{I}(kQ)$ under countable products. Composing this functor with the natural extension of P to $\mathcal{I}^{\Pi}(kQ)$, we get a functor:

$$\begin{array}{rcl} \mathsf{proj}\Lambda & \longrightarrow & \mathcal{M} \\ e_i\Lambda & \mapsto & \bigoplus_{p\geq 0} \tau_B^p H_i. \end{array}$$

Therefore the restriction along this functor yields a functor $F : \mathsf{mod}\,\mathcal{M} \to \mathsf{mod}\,\Lambda$. Moreover, since $\mathcal{M}$ has finitely many indecomposables, the functor F takes its values in the full subcategory f.l.Λ formed by the Λ-modules of finite length.

This is an exact functor since it is a restriction. If M is an $\mathcal{M}$-module, then the vector space $F(M)e_j$ is isomorphic to $\bigoplus_{p\geq 0} M(\tau_B^p H_j)$. For X in $\overline{\mathcal{M}}$, there exists $i \in Q_0$ and $q \geq 0$ such that $\tau^q H_i = X$. It is then easy to check that the image $F(S_X)$ of the simple associated to X is the simple Λ-module S_i.

Fundamental propositions.

Proposition 5.7. *For X in $\overline{\mathcal{M}}$, there exists a functorial sequence in $\mathsf{mod}\,\Lambda$ of the form*

$$0 \longrightarrow F \circ i_*(X^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow F \circ i_*(X^\vee) \longrightarrow 0$$

where $i_* : \mathsf{mod}\,\overline{\mathcal{M}} \to \mathsf{mod}\,\mathcal{M}$ *is the right exact functor induced by* $i : \overline{\mathcal{M}} \to \mathcal{M}$*, and where* H_0 *and* H_1 *are in* $add(H)$*.*

Proof. Let X be in $\overline{\mathcal{M}}$, and iX its image in $\mathcal{M}$. By (1) of proposition 5.1, there exists a triangle functorial in X:

$$iX \longrightarrow H_0 \longrightarrow H_1 \longrightarrow (iX)[1]$$

with H_0 and H_1 in $add(H)$. It yields a long exact sequence in $\mathsf{mod}\,\mathcal{M}$:

$$0 \longrightarrow (iX)^\wedge \longrightarrow H_0^\wedge \longrightarrow H_1^\wedge \longrightarrow \mathsf{Ext}^1_B(?, iX)_{|\mathcal{M}} \longrightarrow \mathsf{Ext}^1_B(?, H_0)_{|\mathcal{M}} \longrightarrow \cdots.$$

By definition, the functor $\mathsf{Ext}^1_B(?, H_0)_{|\mathcal{M}}$ is zero. The Auslander-Reiten formula gives us an isomorphism

$$\mathsf{Ext}^1_B(?, iX)_{|\mathcal{M}} \simeq D\mathsf{Hom}_B(\tau_B^{-1} iX, ?)_{|\mathcal{M}}/\mathsf{proj}\,B.$$

Since F is an exact functor, we get the following exact sequence in $f.l.\Lambda$:

$$0 \longrightarrow F((iX)^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow F((\tau_B^{-1} iX)^\vee/\mathsf{proj}\,B) \longrightarrow 0$$

By definition, we have $F((iX)^\wedge) \simeq (F \circ i_*)(X^\wedge)$. For $j = 1, \cdots, n$, we have an isomorphism:

$$F((\tau_B^{-1} iX)^\vee/\mathsf{proj}\,B)e_j \simeq \bigoplus_{p \geq 0} D\mathsf{Hom}_B(\tau_B^{-1} iX, \tau_B^p H_j)/\mathsf{proj}\,B.$$

For $p \geq 0$, we have $\tau_B^p(H_j) = \tau_B^{-1}(\tau_B^{p+1} H_j)$ if and only if $\tau_B^p H_j$ is not projective. Thus we have a vector space isomorphism

$$F((\tau_B^{-1} iX)^\vee/\mathsf{proj}\,B)e_j \simeq \bigoplus_{p \geq 0} D\mathsf{Hom}_B(\tau_B^{-1} iX, \tau_B^{-1}\tau_B^{p+1} H_j)/\mathsf{proj}\,B.$$

A morphism $f : \tau^{-1}X \to \tau^{-1}Y$ factorizes through a projective object if and only if $\tau(f) : X \to Y$ is not zero. Thus we have:

$$\begin{aligned} F((\tau_B^{-1} iX)^\vee/\mathsf{proj}\,B)e_j &\simeq \bigoplus_{p \geq 1} D\mathsf{Hom}_B(iX, \tau_B^p H_j) \\ &\simeq \bigoplus_{p \geq 0} D\mathsf{Hom}_B(X, \tau_B^p H_j)/[add(H)] \\ &\simeq (F \circ p^*)(X^\vee)e_j \simeq (F \circ i_*)(X^\vee)e_j. \end{aligned}$$

Therefore we get this exact sequence in $f.l.\Lambda$, functorial in X:

$$0 \longrightarrow (F \circ i_*)(X^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow (F \circ i_*)(X^\vee) \longrightarrow 0$$

□

Proposition 5.8. *Let* U *and* V *be indecomposables in* $\overline{\mathcal{M}}$*. We have an isomorphism*

$$\mathsf{Hom}_{\mathcal{C}_A}(U^\wedge, V^\wedge) \simeq \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p U, V)/[add\tau_B^p H]$$

where $\mathcal{M}(\tau_B^p U, V)/[add\tau_B^p H]$ *is the cokernel of the composition map*

$$\mathcal{M}(\tau_B^p U, \tau_B^p H) \otimes \mathcal{M}(\tau_B^p H, V) \longrightarrow \mathcal{M}(\tau_B^p U, V).$$

We first show the following lemma:

Lemma 5.9. *Let e_U and e_V be the idempotents of A associated to the indecomposables U and V. We have an isomorphism*

$$e_U \mathsf{Ext}^2_A(DA, A)e_V \simeq \mathcal{M}(\tau_B U, V)/[add\tau_B H]$$

where $\mathcal{M}(\tau_B U, V)/[add\tau_B H]$ is the cokernel of the composition map

$$\mathcal{M}(\tau_B U, \tau_B H) \otimes \mathcal{M}(\tau_B H, V) \longrightarrow \mathcal{M}(\tau_B U, V).$$

Proof. We have the following isomorphisms:

$$\begin{array}{rcl} e_U \mathsf{Ext}^2_A(DA, A)e_V & = & \mathsf{Ext}^2_A(D(e_U A), Ae_V) \\ & \simeq & \mathsf{Hom}_{\mathcal{D}(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(U, ?), \overline{\mathcal{M}}(?, V)[2]). \end{array}$$

Denote by $\underline{\mathcal{M}}$ the category $\mathcal{M}/\mathsf{proj}\, B$. The functor τ_B induces an equivalence of k-linear categories $\tau_B : \underline{\mathcal{M}} \to \overline{\mathcal{M}}$. Thus we get the following isomorphisms

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(U, ?), \overline{\mathcal{M}}(?, V)[2]) & \simeq & \mathsf{Hom}_{\mathcal{D}(\overline{\mathcal{M}})}(D\underline{\mathcal{M}}(\tau_B^{-1}U, \tau_B^{-1}?), \underline{\mathcal{M}}(\tau_B^{-1}?, \tau_B^{-1}V)[2]) \\ & \simeq & \mathsf{Hom}_{\mathcal{D}(\underline{\mathcal{M}})}(D\underline{\mathcal{M}}(\tau_B^{-1}U, ?), \underline{\mathcal{M}}(?, \tau_B^{-1}V)[2]) \\ & \simeq & \mathsf{Hom}_{\mathcal{D}(\mathcal{M})}(D\mathcal{M}(\tau_B^{-1}U, ?)/\mathsf{proj}\, B, \mathcal{M}(?, \tau_B^{-1}V)/\mathsf{proj}\, B[2]) \end{array}$$

But by the previous lemma, we know a projective resolution in $\mathsf{mod}\,\mathcal{M}$ of the module $D\mathcal{M}(\tau_B^{-1}U, ?)/\mathsf{proj}\, B$. Namely, there exists an exact sequence in $\mathsf{mod}\,\mathcal{M}$ of the form:

$$0 \longrightarrow \mathcal{M}(?, U) \longrightarrow \mathcal{M}(?, H_0) \longrightarrow \mathcal{M}(?, H_1) \longrightarrow D\mathcal{M}(\tau_B^{-1}U, ?)/\mathsf{proj}\, B \longrightarrow 0$$

where H_0 and H_1 are in $add(H)$. Thus we get (using Yoneda's lemma)

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{D}(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(U, ?), \overline{\mathcal{M}}(?, V)[2]) & \simeq & \mathsf{Hom}_{\mathcal{D}(\mathcal{M})}(\mathcal{M}(?, U), \mathcal{M}(?, \tau_B^{-1}V)/\mathsf{proj}\, B)/[add\mathcal{M}(?, H)] \\ & \simeq & \underline{\mathcal{M}}(U, \tau_B^{-1}V)/[addH] \\ & \simeq & \overline{\mathcal{M}}(\tau_B U, V)/[add\tau_B H]. \end{array}$$

Since V is in $\overline{\mathcal{M}}$, a non zero morphism of $\mathcal{M}(\tau_B U, V)$ cannot factorize through $add(H)$. Thus we get $\overline{\mathcal{M}}(\tau_B U, V)/[add\tau_B H] \simeq \mathcal{M}(\tau_B U, V)/[add\tau_B H]$.

□

Proof. (of proposition 5.8) In this proof, for simplicity we denote τ_B by τ. Let $\tilde{A}$ be the algebra $\mathsf{End}_{\mathcal{C}_A}(A)$. By proposition 4.7, we have a vector space isomorphism

$$e_U \tilde{A} e_V \simeq e_U A e_V \oplus e_U \mathsf{Ext}^2_A(DA, A)e_V \oplus e_U \mathsf{Ext}^2_A(DA, A)^{\otimes_A 2} e_V \oplus \ldots$$

We prove by induction that

$$e_U \mathsf{Ext}^2_A(DA, A)^{\otimes_A p} e_V \simeq \mathcal{M}(\tau^p U, V)/[add\tau^p H].$$

For $p = 0$, $e_U A e_V$ is isomorphic to $\overline{\mathcal{M}}(U, V)$ by Yoneda's lemma, and so to $\mathcal{M}(U, V)/[add(H)]$. Suppose the proposition holds for an integer $p - 1 \geq 0$. We then have

$$e_u \mathsf{Ext}^2_A(DA, A)^{\otimes_A p} e_V \simeq \sum_{W \in \mathsf{ind}\,(\overline{\mathcal{M}})} e_u \mathsf{Ext}^2_A(DA, A)^{\otimes_A p-1} e_W \otimes e_W \mathsf{Ext}^2_A(DA, A)e_V.$$

The sum means here the direct sum modulo the mesh relations of the category $\overline{\mathcal{M}}$. Thus this vector space is the sum over the indecomposables W of $\overline{\mathcal{M}}$ of

$$\mathcal{M}(\tau^{p-1}U, W)/[add(\tau^{p-1}H)] \otimes \mathcal{M}(\tau W, V)/[add(\tau H)]$$

modulo the mesh relations of $\overline{\mathcal{M}}$. This is isomorphic to the cokernel of the map $\varphi^{p-1}_{\tau^{p-1}U,W}\otimes 1_{\tau W,V}+1_{\tau^{p-1}U,W}\otimes\varphi^1_{\tau W,V}$ where

$$\varphi^j_{X,Y}:\mathcal{M}(X,\tau^jH)\otimes\mathcal{M}(\tau^jH,Y)\longrightarrow\mathcal{M}(X,Y)$$

is the composition map and where

$$1_{X,Y}:\mathcal{M}(X,Y)\longrightarrow\mathcal{M}(X,Y)$$

is the identity. The cokernel of this map is isomorphic to the cokernel of the map $\varphi^p_{\tau^pU,\tau W}\otimes 1_{\tau W,V}+1_{U,\tau W}\otimes\varphi^1_{\tau W,V}$. But we have an isomorphism

$$\sum_{W\in\mathsf{ind}\,\overline{\mathcal{M}}}\mathcal{M}(\tau^pU,\tau W)\otimes\mathcal{M}(\tau W,V)\simeq\mathcal{M}(\tau^pU,V).$$

Finally we get

$$\mathsf{Coker}\left(\sum_{W\in\mathsf{ind}\,\overline{\mathcal{M}}}\varphi^p_{\tau^pU,\tau W}\otimes 1_{\tau W,V}+1_{U,\tau W}\otimes\varphi^1_{\tau W,V}\right)\simeq\mathsf{Coker}(\varphi^p_{\tau^pU,V}+\varphi^1_{\tau^pU,V}).$$

Furthermore, a morphism in $\mathcal{M}(\tau^pU,V)$ which factorizes through τH factorizes through τ^pH since H is a slice and U is in $\overline{\mathcal{M}}$. Thus this cokernel is in fact isomorphic to the cokernel of $\varphi^p_{\tau^pU,V}$ that is to say to the space

$$\mathcal{M}(\tau^pU,V)/[add\tau^pH].$$

□

5.2. **Case where B is hereditary.**

Results of Geiss, Leclerc and Schröer. Let Q be a finite connected quiver without oriented cycles with n vertices. Denote by $\mathcal{P}$ the postprojective component of the Auslander-Reiten quiver of $\mathsf{mod}\,kQ$, and by $P_1,\ldots,P_n$ the indecomposable projectives.

Definition 5.10 (Geiss-Leclerc-Schröer, [GLS07b]). A kQ-module $M=M_1\oplus\cdots\oplus M_r$, where the M_i are pairwise non isomorphic indecomposables, is called *initial* if the following conditions hold:

- for all $i=1,\ldots,r$, M_i is postprojective;
- if X is an indecomposable kQ-module with $\mathsf{Hom}_{kQ}(X,M)\neq 0$, then X is in $add(M)$;
- and $P_i\in add(M)$ for each indecomposable projective kQ-module P_i.

We define the integers t_i as

$$t_i=max\{j\geq 0|\tau^{-j}(P_i)\in add(M)-\{0\}\}.$$

Denote by Λ the preprojective algebra associated to Q. There is a canonical embedding of algebras $kQ\hookrightarrow\Lambda$. Denote by $\pi_Q:\mathsf{mod}\,\Lambda\to\mathsf{mod}\,kQ$ the corresponding restriction functor.

Theorem 5.11 (Geiss-Leclerc-Schröer, [GLS07b]). *Let M be an initial kQ-module, and let $\mathcal{C}_M=\pi_Q^{-1}(add(M))$ be the subcategory of all Λ-modules X with $\pi_Q(X)\in add(M)$. The following holds:*

(i) the category $\mathcal{C}_M$ is a Frobenius category with n projective-injectives;

(ii) the stable category $\underline{\mathcal{C}}_M$ is a 2-CY triangulated category.

Recall from Ringel [Rin98] that the category $\mathsf{mod}\,\Lambda$ can be seen as $\mathsf{mod}\,kQ(\tau^{-1},1)$. The objects are pairs (X,f) where X is in $\mathsf{mod}\,kQ$ and $f:\tau^{-1}X\to X$ is a morphism in $\mathsf{mod}\,kQ$. The morphisms φ between (X,f) and (Y,g) are commutative squares:

$$\begin{tikzcd} \tau^{-1}X \arrow[r, "f"] \arrow[d, "\tau^{-1}\varphi"'] & X \arrow[d, "\varphi"] \\ \tau^{-1}Y \arrow[r, "g"] & Y \end{tikzcd}$$

The image of an object (X,f) under $\pi_Q:\mathsf{mod}\,\Lambda\to\mathsf{mod}\,kQ$ is then the module X.

Let $X=\tau^{-l}P_i$ be an indecomposable summand of an initial module M. Let $R_X=(Y,f)$ be the following object in $\mathsf{mod}\,kQ(\tau^{-1},1)\simeq\mathsf{mod}\,\Lambda$:

$$Y=\bigoplus_{j=0}^{l}\tau^{-j}P_i \quad\text{and}\quad f:\bigoplus_{j=1}^{l+1}\tau^{-j}P_i\longrightarrow\bigoplus_{j=0}^{l}\tau^{-j}P_i$$

is given by the matrix

$$f=\begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Proposition 5.12 (Geiss-Leclerc-Schröer,[GLS07b])**.** *The category $\mathcal{C}_M$ has a canonical maximal rigid object $R=\bigoplus_{X\in\mathsf{ind}\,add(M)}R_X$. The projective-injectives of $\mathcal{C}_M$ are the $R_{\tau^{-t_i}P_i}$, $i=1,\ldots,n$. Therefore, R is a cluster-tilting object in $\underline{\mathcal{C}}_M$.*

Endomorphism algebra of the cluster-tilting object. Let Q be a connected quiver without oriented cycles and denote by B the path algebra kQ. Let M be an initial B-module. Let H be the following postprojective slice $H=\bigoplus_{i=1}^{n}\tau^{-t_i}P_i$ of $\mathsf{mod}\,B$. Let Q' be the quiver such that $\mathsf{End}_B(H)$ is isomorphic to kQ'.

Let us define, as in the previous section, the subcategory $\mathcal{M}$ of $\mathcal{D}^b(\mathsf{mod}\,kQ)$ as

$$\mathcal{M}=\{X\in\mathsf{mod}\,kQ\ /\mathsf{Ext}^1_B(X,H)=0\}.$$

It is then obvious that $\mathcal{M}=add(M)$. As previously, we denote by Λ the preprojective algebra associated with Q'. It is isomorphic to the one associated with Q because Q and Q' have the same underlying graph. Recall that we have $\overline{\mathcal{M}}=\mathcal{M}/add(H)$, and that $A=\mathsf{End}_B(\overline{\mathcal{M}})$ is an algebra of global dimension 2. Note that in this case τ_B and $\tau_{\mathcal{D}}$ coincide on the objects of $\mathsf{mod}\,B$ which have no projective direct summands since B is hereditary. We will denote it by τ in this section.

Lemma 5.13. *Let U and V be indecomposables in $\overline{\mathcal{M}}$. We have*

$$\mathsf{Hom}_\Lambda(R_U,R_V)\simeq\bigoplus_{j\geq 0}\mathcal{M}(\tau^jU,V).$$

Proof. Let P and Q be projective indecomposables such that $U=\tau^{-q}Q$ and $V=\tau^{-p}P$.

Case 1: $p \leq q$
An easy computation gives the following equalities

$$\begin{aligned}\mathsf{Hom}_\Lambda(R_U,R_V) &\simeq \bigoplus_{j=0}^{p}\mathcal{M}(Q,\tau^{-j}P)\simeq\bigoplus_{j=0}^{p}\mathcal{M}(\tau^{-p+j}Q,\tau^{-p}P)\\ &\simeq \bigoplus_{j=0}^{p}\mathcal{M}(\tau^{-p+j+q}(\tau^{-q}Q),\tau^{-p}P)\simeq\bigoplus_{j=q-p}^{q}\mathcal{M}(\tau^{j}U,V).\end{aligned}$$

Since $\mathcal{M}(\tau^k U,V)$ vanishes for $k\leq q-p+1$ and since $\tau^k U$ vanishes for $k\geq q+1$ we get an isomorphism

$$\mathsf{Hom}_\Lambda(R_U,R_V)\simeq\bigoplus_{j\geq 0}\mathcal{M}(\tau^j U,V).$$

Case 2: $p > q$
In this case, a morphism from R_U to R_V is given by morphisms $a_j\in\mathcal{M}(Q,\tau^{-j}P)$, with $j=0,\ldots,p$ such that $\tau^{-q+1}a_j=0$ for $j=0,\ldots,p-q-1$. But since $\tau^{-q+1-j}P$ is not zero for $j=0,\ldots,p-q-1$, the morphism $\tau^{-q+1}a_j:\tau^{-q+1}Q\to\tau^{-q+1-j}P$ vanishes if and only if a_j vanishes. Thus we get

$$\begin{aligned}\mathsf{Hom}_\Lambda(R_U,R_V) &\simeq \bigoplus_{j=p-q}^{p}\mathcal{M}(Q,\tau^{-j}P)\simeq\bigoplus_{j=p-q}^{p}\mathcal{M}(\tau^{-p+j}Q,\tau^{-p}P)\\ &\simeq \bigoplus_{j=p-q}^{p}\mathcal{M}(\tau^{-p+j+q}(\tau^{-q}Q),\tau^{-p}P)\simeq\bigoplus_{j=0}^{q}\mathcal{M}(\tau^{j}U,V).\end{aligned}$$

Since $\tau^j U$ vanishes for $j\geq q+1$ we get

$$\mathsf{Hom}_\Lambda(R_U,R_V)\simeq\bigoplus_{j\geq 0}\mathcal{M}(\tau^j U,V).$$

$\square$

Corollary 5.14. *Let U and V be indecomposable objects in $\overline{\mathcal{M}}$. We have*

$$\mathsf{Hom}_{\underline{\mathcal{C}}_M}(R_U,R_V)\simeq e_U\tilde{A}e_V$$

and therefore the algebras $\tilde{A}$ and $\mathsf{End}_{\underline{\mathcal{C}}_M}(R)$ are isomorphic.

Proof. The projective-injectives in the category $\mathcal{C}_M$ are the R_{H_i} with $i=1,\ldots,n$. Denote by R_H the sum $\bigoplus_{i=1}^n R_{H_i}$. Thus $\mathsf{Hom}_{\underline{\mathcal{C}}_M}(R_U,R_V)$ is the cokernel of the composition map

$$\mathsf{Hom}_{\mathcal{C}_M}(R_U,R_H)\otimes\mathsf{Hom}_{\mathcal{C}_M}(R_H,R_V)\longrightarrow\mathsf{Hom}_{\mathcal{C}_M}(R_U,R_V).$$

By the previous lemma this map is isomorphic to the following

$$\textstyle\bigoplus_{i,j\geq 0}\mathcal{M}(\tau^i U,H)\otimes\mathcal{M}(\tau^j H,V)\xrightarrow{\ \ \Phi\ \ }\bigoplus_{p\geq 0}\mathcal{M}(\tau^p U,V)$$

Given two morphisms $f\in\mathcal{M}(\tau^i U,H)$ and $\mathcal{M}(\tau^j H,V)$, $\Phi(f\otimes g)$ is the composition $\tau^j f\circ g\in\mathcal{M}(\tau^{i+j}U,V)$. Thus the cokernel of this map is the cokernel of the map

$$\textstyle\bigoplus_{p\geq 0}\bigoplus_{i=0}^{p}\mathcal{M}(\tau^p U,\tau^i H)\otimes\mathcal{M}(\tau^i H,V)\xrightarrow{\ \ \Phi\ \ }\bigoplus_{p\geq 0}\mathcal{M}(\tau^p U,V)\ .$$

Since H is a slice and since U is in $\overline{\mathcal{M}}$, a morphism in $\mathcal{M}(\tau^p U, V)$ which factorizes through $\tau^i H$ with $i \leq p$ factorizes through $\tau^p H$. Finally we get

$$\mathsf{Hom}_{\underline{\mathcal{C}}_M}(R_U, R_V) \simeq \bigoplus_{p\geq 0} \mathcal{M}(\tau^p U, V)/[add\tau^p H],$$

and we conclude using proposition 5.8. $\square$

Triangle equivalence.

Theorem 5.15. *The functor $F\circ i_* : \mathsf{mod}\,\overline{\mathcal{M}} \to f.l.\Lambda$ yields a triangle equivalence between $\mathcal{C}_{\overline{\mathcal{M}}}$ and $\underline{\mathcal{C}}_M$.*

Proof. Let $X = \tau_B^{-l} P_i$ be an indecomposable of $\mathcal{M}$. Let $X^\wedge$ be the projective $\mathcal{M}$-module $\mathsf{Hom}_B(?, X)_{|\mathcal{M}}$. The underlying vector space of $F(X^\wedge)$ is

$$\begin{aligned} F(X^\wedge) &\simeq \bigoplus_{q\geq 0} \mathsf{Hom}_B(\tau_B^q H, \tau_B^{-l} P_i) \simeq \bigoplus_{q\geq 0} \mathsf{Hom}_B(\tau_B^{-q} B, \tau_B^{-l} P_i) \\ &\simeq \bigoplus_{q\geq 0} \mathsf{Hom}_B(B, \tau_B^{q-l} P_i) \simeq \bigoplus_{q=0}^{l} \tau_B^{-q} P_i. \end{aligned}$$

It is then not hard to see that $F(X^\wedge)$ is equal to R_X. Thus each projective $X^\wedge$ is sent onto an object of $\mathcal{C}_M$. Therefore F induces a functor $F : \mathcal{D}^b(\mathcal{M}) \to \mathcal{D}^b(\mathcal{C}_M)$. Moreover for $i = 1, \ldots, n$, $F(H_i^\wedge)$ is equal to $R_{\tau^{-t_i} P_i}$,*i.e.* a projective-injective of $\mathcal{C}_M$. We have the following composition:

$$\begin{tikzcd} \mathcal{D}^b(\overline{\mathcal{M}}) \simeq \mathcal{D}^b(A) \arrow[loop below, "?\overset{L}{\otimes}_A DA[-2]"] \arrow[r, "i_*"] & \mathcal{D}^b(\mathcal{M}) \arrow[r, "F"] & \mathcal{D}^b(\mathcal{C}_M) \arrow[r, "\pi"] & \mathcal{D}^b(\mathcal{C}_M)/\mathsf{per}\,\mathcal{C}_M \simeq \underline{\mathcal{C}}_M \end{tikzcd}$$

The functor $F\circ i_*$ is clearly isomorphic to the left derived tensor product with the A-Λ-bimodule $R = F\circ i_*(A)$. By proposition 5.7, for X in $\overline{\mathcal{M}}$, we have the following exact sequence, functorial in X:

$$0 \longrightarrow F\circ i_*(X^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow F\circ i_*(X^\vee) \longrightarrow 0$$

with H_0 and H_1 in $add(H)$. It yields a morphism

$$F\circ i_*(DA) \to F\circ i_*(A)[2]$$

in the derived category of A-Λ-bimodules. Since the objects $F(H_0^\wedge)$ and $F(H_1^\wedge)$ vanish in the stable category $\underline{\mathcal{C}}_M$, the image

$$F\circ i_*(DA) \to F\circ i_*(A)[2]$$

of this morphism in the category of A-$\mathcal{B}$-bimodules is invertible, where $\mathcal{B}$ is a dg category whose perfect derived category is algebraically equivalent to the stable category $\underline{\mathcal{C}}_M$. In other words, in the derived category $\mathcal{D}(A^{op}\otimes \mathcal{B})$, we have an isomorphism

$$DA \overset{L}{\otimes}_A \pi F i_*(A) \simeq \pi F i_*(A)[-2].$$

By the universal property of the orbit category, we have the factorization

$$\begin{tikzcd} \mathcal{D}^b(\overline{\mathcal{M}}) \arrow[rr, "?\overset{L}{\otimes}_A \underline{R}"] \arrow[dr] & & \underline{\mathcal{C}}_M. \\ & \mathcal{C}_{\overline{\mathcal{M}}} \arrow[ur, dotted] & \end{tikzcd}$$

This factorization is an algebraic functor between 2-CY categories which sends the cluster-tilting object A onto the cluster-tilting object $\underline{R}$. Moreover by corollary 5.14, it yields an equivalence between the categories $add(A)$ and $add(\underline{R})$. Thus it is an algebraic triangle equivalence.

□

Note that if M is the initial module $kQ \oplus \tau^{-1}kQ$, Geiss, Leclerc and Schröer proved, using a result of Keller and Reiten [KR06], that the 2-CY category $\underline{\mathcal{C}}_M$ is triangle equivalent to the cluster category $\mathcal{C}_Q$. Here, H is $\tau^{-1}kQ$ and then $\overline{\mathcal{M}}$ is kQ, so we get another proof of this fact.

5.3. Relation with categories $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$.

Results of Buan, Iyama, Reiten and Scott. Let Q be a finite connected quiver without oriented cycles and Λ the associated preprojective algebra. We denote by $\{1,\dots,n\}$ the set of vertices of Q. For a vertex i of Q, we denote by $\mathcal{I}_i$ the ideal $\Lambda(1-e_i)\Lambda$ of Λ. We denote by W the *Coxeter group* associated to the quiver Q. The group W is defined by the generators $1,\dots,n$ and the relations:

- $i^2 = 1$ for all i in $\{1,\dots,n\}$;
- $ij = ji$ if there are no arrows between the vertices i and j;
- $iji = jij$ if there is exactly one arrow between i and j.

Let $w = i_1 i_2 \dots i_r$ be a W-reduced word. For $m \leq r$, let $\mathcal{I}_{w_m}$ be the following ideal:

$$\mathcal{I}_{w_m} = \mathcal{I}_{i_m} \dots \mathcal{I}_{i_2}\mathcal{I}_{i_1}.$$

For simplicity we will denote $\mathcal{I}_{w_r}$ by $\mathcal{I}_w$. The category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$ is the subcategory of f.l.Λ generated by the sub-Λ-modules of $\Lambda/\mathcal{I}_w$.

Theorem 5.16 (Buan-Iyama-Reiten-Scott [BIRS07])**.** *The category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$ is a Frobenius category and its stable category $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ is 2-CY. The object $T_w = \bigoplus_{m=1}^{r} e_{i_m}\Lambda/\mathcal{I}_{w_m}$ is a cluster-tilting object.*

Note that this theorem is written only for non Dynkin quivers in [BIRS07], but the Dynkin case is an easy consequence of theorem II.2.8 and corollary II.3.5 of [BIRS07].

Construction of a reduced word. Let B be a concealed algebra, and H a postprojective slice in $\mathsf{mod}\,B$. Let Q the quiver of $\mathsf{End}_B(H)$. It is a finite quiver without oriented cycles. We denote by $\{1,\dots,n\}$ its set of vertices and by Λ its preprojective algebra. We define as previously $\mathcal{M} = \{X \in \mathsf{mod}\,B \;/ \mathsf{Ext}^1_B(X,H) = 0\}$.

Let us order the indecomposables $X_1,\dots,X_N$ of $\mathcal{M}$ in such a way: if the morphism space $\mathsf{Hom}_B(X_i,X_j)$ does not vanish, i is smaller than j. This is possible since Q has no oriented cycles.

By proposition 5.1, for $X_i \in \mathcal{M}$ there exists a unique $q \geq 0$ such that $\tau_B^{-q}X_i \simeq H_{\varphi(i)}$ for a certain integer $\varphi(i)$. So we get a function $\varphi : \{1,\dots,N\} \to \{1,\dots,n\}$. Let w be the word $\varphi(1)\varphi(2)\dots\varphi(N)$.

Proposition 5.17. *The word w is W-reduced.*

Proof. The proof is in several steps:

Step 1: For two integers $i < j$ in $\{1,\dots,N\}$, we have $\varphi(i) = \varphi(j)$ if and only if there exists a positive integer p such that $X_i = \tau_B^p X_j$.

Step 2: The element w of the Coxeter group does not depend on the order on the indecomposables of $\mathcal{M}$.

Let i be in $\{1,\ldots,N-1\}$. Assume there is an arrow $\varphi(i)\to\varphi(i+1)$ in Q. We show that there is an arrow $X_i\to X_{i+1}$ in the Auslander-Reiten quiver of $\mathcal{M}$. By proposition 5.1, there exist positive integers p and q such that $X_i=\tau_B^q H_{\varphi(i)}$ and $X_{i+1}=\tau_B^p H_{\varphi(i+1)}$. By hypothesis there is an arrow between $H_{\varphi(i)}$ and $H_{\varphi(i+1)}$. Thus we want to show that p is equal to q.

Suppose that $p\geq q+1$, then since $\mathcal{M}$ is closed under τ_B, the objects $\tau_B^q H_{\varphi(i+1)}$ and $\tau_B^{q+1}H_{\varphi(i+1)}$ are non zero and are in $\mathcal{M}$. Let l be the integer in $\{1,\ldots,N\}$ such that $X_l=\tau_B^{q+1}H_{\varphi(i+1)}$. We have an arrow

$$X_i=\tau_B^q H_{\varphi(i)}\to\tau_B^q H_{\varphi(i+1)}=\tau_B^{-1}X_l.$$

Thus, by the property of the AR-translation, there is an arrow $X_l\to X_i$. Thus i should be strictly greater than l. But by step 1, and the hypothesis $p\geq q+1$, we have $i+1\leq l$. This is a contradiction.

The cases $q\geq p+1$, and $\varphi(i+1)\to\varphi(i)$ in Q can be solved in the same way.

Step 3: It is not possible to have $\varphi(i)=\varphi(i+1)$.

Suppose we have $\varphi(i)=\varphi(i+1)$. By step 1 there exists a positive integer p such that $X_i=\tau_B^p X_{i+1}$. Suppose that p is ≥ 2, then $\tau_B X_{i+1}=\tau_B^{-p+1}X_i$ is in $\mathcal{M}$, it is isomorphic to an X_k for an integer k with $\varphi(k)=\varphi(i)$. But k must be strictly greater than i and strictly smaller than $i+1$ which is clearly impossible. Thus p is equal to 1. There should exist an X_l in $\mathcal{M}$ such that $\mathsf{Hom}(X_i,X_l)\neq 0$ and $\mathsf{Hom}(X_l,X_{i+1})\neq 0$. Thus l must be strictly between i and $i+1$ which is impossible.

Step 4: It is not possible to have $\varphi(i)=\varphi(i+2)$ and $\varphi(i+1)=\varphi(i+3)$ with exactly one arrow in Q between $\varphi(i)$ and $\varphi(i+1)$.

In this case we have, by step 1, $X_i=\tau_B^p X_{i+2}$ and $X_{i+1}=\tau_B^q X_{i+3}$. By the same argument as in step 3, p and q have to be equal to 1. Thus the AR quiver of $\mathcal{M}$ has locally the following form:

$$\begin{tikzcd}
 & X_{i+1} \arrow[dr] \arrow[rr, dotted, no head] & & X_{i+3} & \\
X_i \arrow[ur] \arrow[rr, dotted, no head] & & X_{i+2} \arrow[ur] & &
\end{tikzcd}.$$

The module X_{i+1} is the unique direct predecessor of X_{i+2}. Indeed, suppose there is an X_k with an arrow $X_k\to X_{i+2}$. Thus there is an arrow $\tau_B X_{i+2}=X_i\to X_k$ and k must be strictly between i and $i+2$. By the same argument, there is only one arrow with tail X_{i+3}, one arrow with source X_i and one arrow with source X_{i+1}. Thus we have the following AR sequences in $\mathsf{mod}\,B$:

$$0\to X_i\to X_{i+1}\to X_{i+2}\to 0\quad\text{and}\quad 0\to X_{i+1}\to X_{i+2}\to X_{i+3}\to 0$$

which is clearly impossible.

Step 5: There is no subsequence of type $jkjlkl$ in w with an arrow between j and k and an arrow between k and l

Suppose we have $\varphi(i)=\varphi(i+2)=j$, $\varphi(i+1)=\varphi(i+4)=k$ and $\varphi(i+3)=\varphi(i+5)=l$. As previously, we have $X_i=\tau_B X_{i+2}$, $X_{i+1}=\tau_B X_{i+4}$ and $X_{i+3}=\tau_B X_{i+5}$. There is an arrow $X_{i+1}\to X_{i+2}$ so there is an arrow $X_{i+2}\to X_{i+4}$. There is an arrow $X_{i+3}\to X_{i+4}$ thus there is an arrow $X_{i+1} \to X_{i+3}$. As in step 4 it is easy to see that the AR quiver of $\mathcal{M}$ locally looks like:

$$\begin{tikzcd}
 & & X_{i+3} \arrow[dr] \arrow[rr, dotted, no head] & & X_{i+5} \\
 & X_{i+1} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & X_{i+4} \arrow[ur] & \\
X_i \arrow[ur] \arrow[rr, dotted, no head] & & X_{i+2} \arrow[ur] & &
\end{tikzcd}$$

Thus we have the 3 following AR sequences in $\operatorname{mod} B$:

$$0 \to X_i \to X_{i+1} \to X_{i+2} \to 0 \qquad 0 \to X_{i+3} \to X_{i+4} \to X_{i+5} \to 0$$

$$\text{and} \quad 0 \longrightarrow X_{i+1} \longrightarrow X_{i+3} \oplus X_{i+2} \longrightarrow X_{i+4} \longrightarrow 0$$

A simple argument of dimension permits us to conclude that X_i and X_{i+5} must be zero, that is a contradiction.

By the second step, we know that using the relation of commutativity is the same as changing the order on the indecomposables of $\mathcal{M}$. Moreover we just saw that locally we can not reduce the word w. Thus it is reduced.

$\square$

Image of the cluster-tilting object. Let $F : \operatorname{mod} \mathcal{M} \to \mathrm{f.l.}\Lambda$ be the functor constructed in section 5.1.

Proposition 5.18. *For $i = 1, \ldots, N$, we have an isomorphism in f.l.Λ:*

$$F(X_i^{\wedge}) \simeq e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}$$

where w_i is the word $\varphi(1)\cdots\varphi(i)$.

Proof. The functor F is right exact and sends the simple functor S_{X_i} onto the simple $S_{\varphi(i)}$. Since $F(X_i^{\wedge})$ surjects onto $F(S_{X_i})$, there is a morphism $e_{\varphi(i)}\Lambda \to F(X_i^{\wedge})$. Explicitly, we will take the morphism given in this way:

The object X_i is of the form $\tau_B^q H_{\varphi(i)}$ for a $q \geq 0$. If j is in $\{1, \ldots, n\}$, the vector space $e_{\varphi(i)}\Lambda e_j$ is isomorphic to $\prod_{p\geq 0} \mathsf{Hom}_{kQ}(\tau_{\mathcal{D}}^p I_j, I_{\varphi(i)})$ where I_j is the injective indecomposable module of $\operatorname{mod} kQ$ corresponding to the vertex j. Let f be a morphism in $\mathsf{Hom}_{kQ}(\tau_{\mathcal{D}}^p I_j, I_{\varphi(i)})$, then $\tau_{\mathcal{D}}^q(f)$ is a morphism in $\mathsf{Hom}_{kQ}(\tau_{\mathcal{D}}^{p+q} I_j, \tau_{\mathcal{D}}^q I_{\varphi(i)})$, and then $P(\tau_{\mathcal{D}}^q f) = \tau_B^q P(f)$ is a morphism in $\mathcal{M}$ from $\tau_B^{p+q} H_j$ to $\tau_B^q H_{\varphi(i)} = X_i$, thus is in $F(X_i^{\wedge})e_j$.

Step 1: The morphism $e_{\varphi(i)}\Lambda \to F(X_i^{\wedge})$ vanishes on the ideal $\mathcal{I}_{w_i}$.

A word $j_1 j_2 \cdots j_r$ will be called a *subword* of w_i if there exist integers $1 \leq l_1 < l_2 < \cdots < l_r \leq i$ such that $j_1 j_2 \cdots j_r = \varphi(l_1)\varphi(l_2)\cdots\varphi(l_r)$. It is easy to check that the vector space $e_{\varphi(i)}\mathcal{I}_{w_i} e_j$ is generated by the paths from j to $\varphi(i)$ such that there exists a factorization

$$j \rightsquigarrow j_1 \rightsquigarrow j_2 \rightsquigarrow \cdots \rightsquigarrow j_r \rightsquigarrow \varphi(i)$$

with $j j_1 j_2 \cdots j_r \varphi(i)$ not a subword of w_i.

Let f be a morphism $\tau_{\mathcal{D}}^{p}I_j \rightarrow I_{\varphi(i)}$ in $\mathcal{I}(kQ)$ given by such a path. Assume that the image $P(\tau_{\mathcal{D}}^{q}f)$ of f in $F(X_i^{\wedge})$ is non zero. Let

$$\tau_{\mathcal{D}}^{p}I_j \xrightarrow{f_0} \tau_{\mathcal{D}}^{p_1}I_{j_1} \xrightarrow{f_1} \tau_{\mathcal{D}}^{p_2}I_{j_2} \xrightarrow{f_2} \cdots \longrightarrow \tau_{\mathcal{D}}^{p_r}I_{j_r} \xrightarrow{f_r} I_{\varphi(i)}$$

be the factorization of f given by the above factorization of the path. Therefore $P(\tau_{\mathcal{D}}^{q}f)$ is equal to the composition

$$\tau_B^{p+q}H_j \longrightarrow \tau_B^{p_1+q}H_{j_1} \longrightarrow \tau_B^{p_2+q}H_{j_2} \longrightarrow \cdots \longrightarrow \tau_B^{p_r+q}H_{j_r} \longrightarrow \tau_B^{q}H_{\varphi(i)} = X_i\ .$$

Since $P(\tau_{\mathcal{D}}^{q}f)$ is not zero, all morphisms $P(\tau_{\mathcal{D}}^{q}f_l)$ are not zero, and all objects $\tau_B^{p_l+q}H_{j_l}$ are non zero. Thus the objects $\tau_B^{p_l+q}H_{j_l}$ are of the form X_{h_l} with $h_0 < h_1 < \cdots < h_r < i$. Furthermore, we have $\varphi(h_l) = j_l$. Thus $jj_1\cdots j_r\varphi(i) = \varphi(h_0)\varphi(h_1)\cdots\varphi(h_r)\varphi(i)$ is a subword of w_i. This contradiction shows that the image of f in $F(X_i^{\wedge})$ must be zero.

Step 2: The morphism $e_{\varphi(i)}\Lambda \rightarrow F(X_i^{\wedge})$ is surjective.

Let f be a morphism $\tau_B^{p+q}H_j \rightarrow \tau_B^{q}H_{\varphi(i)} = X_i$ in $\mathcal{M}$. Hence $\tau_B^{-q}f$ is a morphism $\tau_B^{p}H_j \rightarrow H_{\varphi(i)}$ in $\mathcal{M}$. Since P is full (cf. prop. 5.5), there exists a morphism $g : \tau_{\mathcal{D}}^{p}I_i \rightarrow I_{\varphi(i)}$ such that $P(g) = \tau_B^{-q}f$. Thus we have $P(\tau_{\mathcal{D}}^{q}g) = \tau_B^{q}P(g) = f$.

Step 3: The morphism $e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i} \rightarrow F(X_i^{\wedge})$ is injective.

Let f be a non zero morphism $\tau_{\mathcal{D}}^{p}I_j \rightarrow I_{\varphi(i)}$ in $\mathcal{I}(kQ)$ such that $P(\tau_{\mathcal{D}}^{q}f)$ is zero. By lemma 5.6, we can assume that there exists a factorization of $\tau_{\mathcal{D}}^{q}f$ of the form

$$\tau_{\mathcal{D}}^{q+p}I_j \xrightarrow{h} Y \xrightarrow{g} \tau_{\mathcal{D}}^{q}I_{\varphi(i)}$$

with Y indecomposable and $P(Y) = 0$. The object Y is of the form $\tau_{\mathcal{D}}^{h}I_l$ with $h \geq q$ and we have $\tau_B^{h}H_l = 0$.

The morphism g is a sum of compositions of irreducible morphisms between indecomposables. Let

$$\tau_{\mathcal{D}}^{h}I_l \xrightarrow{g_0} Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots \longrightarrow Y_s \xrightarrow{g_s} \tau_{\mathcal{D}}^{q}I_{\varphi(i)}$$

be such a summand of g. The objects Y_k, $1 \leq k \leq s$ are indecomposable and so are of the form $\tau_{\mathcal{D}}^{r_k}I_{j_k}$, and the morphisms g_k, $0 \leq k \leq s$ are irreducible. We will show that the word $lj_1j_2\ldots j_s\varphi(i)$ is not a subword of w_i. Without loss of generality, we may assume that for $1 \leq k \leq s$, $P(Y_k)$ is not zero, so there exist integers l_k such that $P(Y_k) = X_{l_k}$. Since the morphisms g_k are irreducible, $P(g_k)$ does not vanish, and we have $1 \leq l_1 < l_2 < \cdots < l_s < i$. The word $j_1j_2\ldots j_s\varphi(i)$ is equal to the word $\varphi(l_1)\varphi(l_2)\cdots\varphi(l_s)\varphi(i)$, so $j_1j_2\ldots j_s\varphi(i)$ is a subword of w_i.

Substep 1: The sequence $1 \leq l_1 < l_2 < \cdots < l_s < i$ is the maximal element of the set $\{1 \leq i_1 < i_2 < \cdots < i_s < i_{s+1} \leq i \ \mid\ \varphi(i_1) = j_1, \ldots, \varphi(i_s) = j_s, \varphi(i_{s+1}) = \varphi(i)\}$ for the lexicographic order.

We prove by decreasing induction that l_k is the maximal integer with $l_k < l_{k+1}$ and $\varphi(l_k) = j_k$. For $k = s+1$ it is obvious. Now suppose there exists an integer i_k such that $\varphi(l_k) = \varphi(i_k) = j_k$ and $l_k < i_k < l_{k+1}$. Thus by step 1 of proposition 5.17, there exists an integer $r \geq 1$ such that $X_{l_k} = \tau_B^{r}X_{i_k}$. The morphism $P(g_k) : X_{l_k} \rightarrow X_{l_{k+1}}$ is irreducible, so there exists a non zero irreducible morphism $X_{l_{k+1}} \to \tau_B^{-1} X_{l_k}$. The object $\tau_B^{-1} X_{l_k}$ is in $\mathcal{M}$ since X_{l_k} and $\tau_B^{-r} X_{l_k} = X_{i_k}$ are in $\mathcal{M}$. It is of the form X_t, and we have $l_{k+1} < t$. Since r is ≥ 1, t is $\leq i_k$ by step 1 of proposition 5.17. This implies $l_{k+1} < i_k$ which is a contradiction.

Substep 2: l does not belong to the set $\{\varphi(1), \varphi(2), \ldots, \varphi(l_1 - 1)\}$.
Suppose that there exists an integer $1 \leq k \leq N$ such that $\varphi(k)$ is equal to l. Thus there exists an integer $r \geq 0$ such that X_k is equal to $\tau_B^r H_l$. Since $\tau_B^h H_l = P(\tau_{\mathcal{D}}^h I_l)$ is zero, r is $\leq h-1$.

Since the morphism $g_0 : \tau_{\mathcal{D}}^h I_l \to Y_1$ is an irreducible morphism of $\mathcal{I}(kQ)$, there exists an irreducible morphism $Y_1 \to \tau_{\mathcal{D}}^{h-1} I_l$ in $\mathcal{I}(kQ)$. Thus there exists an irreducible morphism $\tau_{\mathcal{D}}^{r-h+1} Y_1 \to \tau_{\mathcal{D}}^r I_l$ in $\mathcal{I}(kQ)$. The object $P(\tau_{\mathcal{D}}^r I_l) = \tau_B^r H_l = X_k$ is not zero and lies in $\mathcal{M}$, so the object $P(\tau_{\mathcal{D}}^{r-h+1} Y_1) = \tau_B^{r-h+1} X_{l_1}$ is not zero and lies in $\mathcal{M}$ since $\mathcal{M}$ is stable by kernel. Thus there is an irreducible morphism $\tau_B^{r-h+1} X_{l_1} = X_t \to X_k$ in $\mathcal{M}$. Therefore t has to be $< k$. Moreover since $r - h + 1 \leq 0$, l_1 is $\leq s$ by step 1 of proposition 5.17. Finally we get $l_1 < k$.

Combining substep 1 and substep 2, we can prove that $l j_1 j_2 \ldots j_s \varphi(i)$ can not be a subword of w_i. Indeed, assume $l j_1 j_2 \ldots j_s \varphi(i)$ is a subword of w_i. There exist $1 \leq i_0 < i_1 < \ldots < i_s < i_{s+1} \leq i$ such that $\varphi(i_0)\varphi(i_1) \ldots \varphi(i_{s+1}) = l j_1 j_2 \ldots j_s \varphi(i)$. In particular, the word $j_1 j_2 \ldots j_s \varphi(i)$ is a subword of w_i and $1 \leq i_1 < \ldots < i_s < i_{s+1} \leq i$ is in the set of substep 1. Thus by substep 1, i_1 has to be $\leq l_1$. By substep 2, i_0 can not exist.

□

Endomorphism algebra of the cluster-tilting object.

Lemma 5.19. *Let X_i and X_j be indecomposables of $\mathcal{M}$. We have an isomorphism of vector spaces*

$$\mathsf{Hom}_\Lambda(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p X_j, X_i).$$

Proof. Case 1: $j \geq i$
By [BIRS07] (lemma II.1.14) we have an isomorphism

$$\mathsf{Hom}_\Lambda(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i} e_{\varphi(j)}.$$

By proposition 5.18, this is isomorphic to the space

$$\bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i).$$

By definition of the function φ, there exists some $q \geq 1$ such that $X_j = \tau_B^q H_{\varphi(j)}$. Thus we can write the sum

$$\bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i) = \bigoplus_{p=1}^{q} \mathcal{M}(\tau_B^{-p} X_j, X_i) \oplus \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p X_j, X_i)$$

Since $j \geq i$, there is no morphism from $\tau_B^{-p} X_j$ to X_i for $p \geq 1$, and the first summand is zero. Therefore we get the result.

Case 2: $j < i$
By [BIRS07] (lemma II.1.14) we have an isomorphism

$$\mathsf{Hom}_\Lambda(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq e_{\varphi(i)}(\mathcal{I}_{\varphi(i)} \ldots \mathcal{I}_{\varphi(j+1)}/\mathcal{I}_{w_i})e_{\varphi(j)}.$$

By proposition 5.18, this space is a subspace of the space

$$\bigoplus_{p\geq 0}\mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i) \simeq \bigoplus_{p\geq 1}\mathcal{M}(\tau_B^{-p} X_j, X_i) \oplus \bigoplus_{p\geq 0}\mathcal{M}(\tau_B^p X_j, X_i).$$

Step 1: If f is a non zero morphism in $\mathcal{M}(\tau_B^{-p}X_j, X_i)$ with $p \geq 1$ then f is not in the space $e_{\varphi(i)}\mathcal{I}_{\varphi(i)}\ldots\mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$.

Let X_{l_0} be the indecomposable $\tau_B^{-p}X_j$. Since $p \geq 1$ then l_0 is $\leq j+1$. The morphism is a sum of composition of the form

$$X_{l_0} \longrightarrow X_{l_1} \longrightarrow \cdots \longrightarrow X_{l_r} \longrightarrow X_i$$

with the X_{l_k} indecomposables. Since f is not zero, we have $j+1 \leq l_0 < l_1 < \ldots < l_r < i$. Thus the word $\varphi(l_0)\varphi(l_1)\ldots\varphi(l_r)\varphi(i)$ is a subword of $\varphi(j+1)\varphi(j+2)\ldots\varphi(i)$. Since it holds for each factorization of f, the morphism f is not in the space $e_{\varphi(i)}\mathcal{I}_{\varphi(i)}\ldots\mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$.

Step 2: If f is a morphism in $\mathcal{M}(\tau_B^p X_j, X_i)$ with $p \geq 0$ then f is in the space $e_{\varphi(i)}\mathcal{I}_{\varphi(i)}\ldots\mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$.

Let X_{l_0} be the indecomposable $\tau_B^p X_j$. Since p is ≥ 0, we have $l_0 \leq j$. Let us show that if f is a composition of irreducible morphisms

$$X_{l_0} \longrightarrow X_{l_1} \longrightarrow \cdots \longrightarrow X_{l_r} \longrightarrow X_{l_{r+1}} = X_i$$

then the word $\varphi(l_0)\varphi(l_1)\cdots\varphi(l_r)\varphi(i)$ is not a subword of $\varphi(j+1)\varphi(j+2)\ldots\varphi(i)$.

We have $l_0 < l_1 < \cdots < l_r < i$. Since l_0 is $< j+1$, and i is $\leq j+1$, there exists $1 \leq k \leq r+1$ such that $l_{k-1} < j+1 \leq l_k$. Therefore $\varphi(l_k)\ldots\varphi(l_r)\varphi(i)$ is a subword of $\varphi(j+1)\varphi(j+2)\ldots\varphi(i)$, and the sequence $l_k < l_{k+1} < \cdots < l_r < i$ is the maximal element of the set

$$\{j+1 \leq i_k < \cdots < i_{r+1} \leq i \quad | \quad \varphi(i_k) = \varphi(l_k), \ldots, \varphi(i_r) = \varphi(l_r), \varphi(i_{r+1}) = \varphi(i)\}$$

for the lexicographic order (exactly for the same reasons as in substep 1 of proposition 5.18). Now we can prove exactly as in substep 2 of proposition 5.18 that $\varphi(l_{k-1})$ does not belong to the set $\{\varphi(j+1), \ldots, \varphi(l_k - 1)\}$. Thus $\varphi(l_{k-1})\varphi(l_k)\ldots\varphi(l_r)\varphi(i)$ can not be a subword of $\varphi(j+1)\varphi(j+2)\ldots\varphi(i)$.

Finally, let $f = f_1 + f_2$ be a morphism in

$$\bigoplus_{p\geq 0}\mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i) \simeq \bigoplus_{p\geq 1}\mathcal{M}(\tau_B^{-p} X_j, X_i) \oplus \bigoplus_{p\geq 0}\mathcal{M}(\tau_B^p X_j, X_i).$$

By step 2, f_2 is in the space $e_{\varphi(i)}\mathcal{I}_{\varphi(i)}\ldots\mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$. By step 1 the morphism f is in $e_{\varphi(i)}\mathcal{I}_{\varphi(i)}\ldots\mathcal{I}_{\varphi(j+1)}e_{\varphi(j)}$ if and only if $f-1$ is zero. Thus we get an isomorphism

$$\mathsf{Hom}_\Lambda(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq \bigoplus_{p\geq 0}\mathcal{M}(\tau_B^p X_j, X_i).$$

□

Corollary 5.20. *If X_i and X_j are indecomposables of $\overline{\mathcal{M}}$, then we have*

$$\mathsf{Hom}_{\underline{Sub}\Lambda/\mathcal{I}_w}(e_{\varphi(j)}\Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)}\Lambda/\mathcal{I}_{w_i}) \simeq e_{X_j}\tilde{A}e_{X_i}.$$

Proof. The proof is exactly the same as the proof of corollary 5.14. □

Triangle equivalence.

Theorem 5.21. *The functor $F \circ i_* : \mathsf{mod}\,\overline{\mathcal{M}} \to f.l.\Lambda$ induces an algebraic triangle equivalence between $\mathcal{C}_{\overline{\mathcal{M}}}$ and $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$.*

Proof. By proposition 5.18, the functor F sends the projectives of $\mathsf{mod}\,\mathcal{M}$ onto the summands of the cluster-tilting object T_w of the category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$. For $i = 1, \ldots, n$, the projective $H_i^{\wedge}$ is sent to the projective-injective $\Lambda/\mathcal{I}_w e_i$. Furthermore, by corollary 5.20, $F \circ i_*$ induces an equivalence between the subcategories $add(A)$ and $add(T_w)$. Thus we can conclude as in the proof of theorem 5.15. $\square$

5.4. **Example.** We refer to [Ami08] for more examples. Let Q be the following quiver: $1 \longrightarrow 2 \leftleftarrows 3$
The preinjective component of $\mathsf{mod}\,kQ$ looks as follows:

$$\begin{tikzcd}[column sep=small]
\cdots & {[4\ 16\ 9]} \arrow[dr, shift left] \arrow[dr, shift right] & & \boxed{[2\ 6\ 3]} \arrow[dr, shift left] \arrow[dr, shift right] & & {[0\ 2\ 1]} \arrow[dr, shift left] \arrow[dr, shift right] & \\
{} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr] & & {[3\ 11\ 6]} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr] & & {[1\ 4\ 2]} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr] & & {[0\ 1\ 0]} \\
\cdots & \boxed{[3\ 8\ 4]} \arrow[ur] & & {[0\ 3\ 2]} \arrow[ur] & & \boxed{[1\ 1\ 0]} \arrow[ur] &
\end{tikzcd}$$

Here we denote the kQ-modules by their dimension vectors in order to lighten the writing. For example the module $[1\ 4\ 2]$ has the following composition series: $\begin{smallmatrix}2&&2&&2&&2\\&3&&1&&3&\end{smallmatrix}$.

If we mutate the tilting object $[2\ 6\ 3] \oplus [1\ 4\ 2] \oplus [1\ 1\ 0]$ in the direction $[1\ 4\ 2]$, we stay in the preinjective component. We get the tilting object:

$$T = [2\ 6\ 3] \oplus [3\ 8\ 4] \oplus [1\ 1\ 0].$$

The algebra $B = \mathsf{End}_{kQ}(T)$ is a concealed algebra and is given by the quiver:

$$\begin{tikzcd}
& 2 \arrow[dr, shift left, "b"] \arrow[dr, shift right, "b'"'] & \\
1 \arrow[ur, shift left, "a"] \arrow[ur, shift right, "a'"'] & & 3
\end{tikzcd} \qquad \text{with the relation} \quad ba + b'a' = 0.$$

The functor $R\mathsf{Hom}_{kQ}(T, ?)$ yields an equivalence between $\mathcal{D}^b(kQ)$ and $\mathcal{D}^b B$. Denote by H the image of $D(kQ)$ through this equivalence. This is a postprojective slice of $\mathsf{mod}\,B$. Moreover, this equivelence restricts to an equivalence between the category $\mathcal{M} = \{X \in \mathsf{mod}\,B \mid \mathsf{Ext}^1_B(X, H) = 0\}$ and the category $\mathcal{M}' = \{X \in \mathsf{mod}\,kQ \mid \mathsf{Ext}^1_{kQ}(T, X) = 0\}$. The indecomposable objects of $\mathcal{M}'$ are

$$[3\ 8\ 4], [2\ 6\ 3], [1\ 4\ 2], [1\ 1\ 0], [0\ 2\ 1], \text{and } [0\ 1\ 0].$$

The quiver of $\mathcal{M}'$ with an admissible ordering is the following:

$$\begin{tikzcd}[column sep=small]
& 2 \arrow[dr, shift left] \arrow[dr, shift right] & & 5 \arrow[ll, dotted] \arrow[dr, shift left] \arrow[dr, shift right] & \\
1 \arrow[ur, shift left] \arrow[ur, shift right] & & 3 \arrow[ll, dotted] \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr] & & 6 \arrow[ll, dotted] \\
& & & 4 \arrow[ur] &
\end{tikzcd}$$

The dotted arrows represent the Auslander translation τ_B. The projective indecomposables of $\mathsf{mod}\,\mathcal{M}$ have the following dimension vectors:

$$\begin{bmatrix} &0&&0&\\ 1&&0&&0\\ &&&0& \end{bmatrix}, \quad \begin{bmatrix} &1&&0&\\ 2&&0&&0\\ &&&0& \end{bmatrix}, \quad \begin{bmatrix} &2&&0&\\ 3&&1&&0\\ &&&0& \end{bmatrix}, \quad \begin{bmatrix} &2&&0&\\ 3&&1&&0\\ &&&1& \end{bmatrix}, \quad \begin{bmatrix} &3&&1&\\ 4&&2&&0\\ &&&0& \end{bmatrix}, \quad \begin{bmatrix} &6&&2&\\ 8&&4&&1\\ &&&1& \end{bmatrix}$$

Now let Λ be the preprojective associated to the quiver Q. The functor $F : \mathsf{mod}\,\mathcal{M} \to \mathsf{mod}\,\Lambda$ sends the simples $\mathcal{M}$-modules $S_1 = \left[\begin{smallmatrix} & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & & 0 & \end{smallmatrix}\right]$, $S_3 = \left[\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & & & 0 & \end{smallmatrix}\right]$ and $S_6 = \left[\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 0 & & 1 \\ & & & 0 & \end{smallmatrix}\right]$ on the simple Λ-module $S_2 = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$.

$$\begin{tikzcd}
& 2 \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dotted, no head] & & 5 \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dotted, no head] & & 3\,. \arrow[d, equal] \\
1 \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dotted, no head] & & 3 \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr] \arrow[rr, dotted, no head] & & 6 \arrow[r, dotted, no head] & 2 \arrow[d, no head] \\
& & & 4 \arrow[ur] \arrow[rr, dotted, no head] & & 1
\end{tikzcd}$$

It sends the simple $\mathcal{M}$-modules $S_2 = \left[\begin{smallmatrix} & 1 & & 0 & \\ 0 & & 0 & & 0 \\ & & & 0 & \end{smallmatrix}\right]$ and $S_5 = \left[\begin{smallmatrix} & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & & & 0 & \end{smallmatrix}\right]$ on the simple Λ-module $S_1 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$, and the simple $\mathcal{M}$-module $S_4 = \left[\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & & & 1 & \end{smallmatrix}\right]$ on the simple Λ-module $S_3 = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$. Since it is exact, it preserves the composition series and then it is easy to compute the image of the indecomposable projective $\mathcal{M}$-modules. We get

$$\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ 4 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ 4 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 4 \\ 6 \\ 0 \end{smallmatrix}\right] \text{ and } \left[\begin{smallmatrix} 8 \\ 13 \\ 1 \end{smallmatrix}\right].$$

The projectives of the preprojective algebra associated to Q have the following composition series:

$$\begin{smallmatrix} & & & & 1 & & & & \\ & & & & 2 & & & & \\ & & & 3 & & 3 & & & \\ & & 2 & & 2 & & 2 & & \\ 3 & 1 & 3 & 3 & 1 & 3 & 3 & 1 & 3 \\ & & \vdots & & \vdots & & \vdots & & \end{smallmatrix}, \quad \begin{smallmatrix} & & & & 2 & & & & \\ & & 3 & & 1 & & 3 & & \\ & 2 & & 2 & & 2 & & 2 & \\ 3 & 1 & 3 & 3 & 1 & 3 & 3 & 1 & 3 \\ & & \vdots & & \vdots & & \vdots & & \end{smallmatrix}, \quad \text{and} \quad \begin{smallmatrix} & & & & & 3 & & & & & \\ & & & 2 & & & & 2 & & & \\ & 3 & & 1 & & 3 & & 1 & & 3 & \\ 2 & & 2 & & 2 & & 2 & & 2 & & 2 \\ & & & & \vdots & & \vdots & & & & \end{smallmatrix}$$

The word w associated with the ordering is $w = 232132$. Thus the maximal rigid object of the category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$ is

$$R = \begin{smallmatrix} 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 2 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & & 2 & & \\ & 3 & & 3 & \\ 2 & & 2 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ & 3 & & 3 & \\ 2 & & 2 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & & & 3 & & & \\ & & 2 & & 2 & & \\ & 3 & & 3 & & 3 & \\ 2 & & 2 & & 2 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & & & & 2 & & & & \\ & & & 3 & 1 & 3 & & & \\ & & 2 & 2 & & 2 & 2 & & \\ & 3 & 3 & 3 & & 3 & 3 & 3 & \\ 2 & 2 & 2 & 2 & & 2 & 2 & 2 & 2 \end{smallmatrix}.$$

It is easy to check that R is the image by F of the projective indecomposable $\mathcal{M}$-modules. The last three summands are the projective-injectives of the Frobenius category $\mathsf{Sub}\,\Lambda/\mathcal{I}_w$. This confirms proposition 5.18.

Now take the module $X = 1$ in $\mathcal{M}$. It corresponds to the module $[\,3\;8\;4\,]$ in $\mathsf{mod}\,kQ$. We have the injective resolution in $\mathsf{mod}\,kQ$:

$$0 \longrightarrow [\,3\;8\;4\,] \longrightarrow [\,0\;2\;1\,]^4 \oplus [\,1\;1\;0\,]^3 \longrightarrow [\,0\;1\;0\,]^3 \longrightarrow 0$$

Thus the short exact sequence in $\mathcal{M}$: $0 \longrightarrow X \longrightarrow H_0 \longrightarrow H_1 \longrightarrow 0$ is the following:

$$0 \longrightarrow 1 \longrightarrow 4^3 \oplus 5^4 \longrightarrow 6^3 \longrightarrow 0$$

Therefore, the sequence $0 \to X^\wedge \to H_0^\wedge \to H_1^\wedge \to (\tau^{-1}X)^\vee/\mathsf{proj}\,B \to 0$ in $\mathsf{mod}\,\mathcal{M}$ becomes:

$$0 \to \left[\begin{smallmatrix} & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & & 0 & \end{smallmatrix}\right] \to \left[\begin{smallmatrix} & 2 & & 0 & \\ 3 & & 1 & & 0 \\ & & & 1 & \end{smallmatrix}\right]^3 \oplus \left[\begin{smallmatrix} & 3 & & 1 & \\ 4 & & 2 & & 0 \\ & & & 0 & \end{smallmatrix}\right]^4 \to \left[\begin{smallmatrix} & 6 & & 2 & \\ 8 & & 4 & & 1 \\ & & & 1 & \end{smallmatrix}\right]^3 \to \left[\begin{smallmatrix} & 0 & & 2 & \\ 0 & & 1 & & 3 \\ & & & 0 & \end{smallmatrix}\right] \to 0$$

where $\left[\begin{smallmatrix} & 0 & & 2 & \\ 0 & & 1 & & 3 \\ & & & 0 & \end{smallmatrix}\right]$ is the quotient of $(\tau_B^{-1}1)^\vee = 3^\vee = \left[\begin{smallmatrix} & 0 & & 2 & \\ 0 & & 1 & & 4 \\ & & & 1 & \end{smallmatrix}\right]$ by the projectives. Applying the projection functor we get the exact sequence in $\mathsf{mod}\,\Lambda$:

$$0 \longrightarrow \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} 2 \\ 4 \\ 1 \end{smallmatrix}\right]^3 \oplus \left[\begin{smallmatrix} 4 \\ 6 \\ 0 \end{smallmatrix}\right]^4 \longrightarrow \left[\begin{smallmatrix} 8 \\ 13 \\ 1 \end{smallmatrix}\right]^3 \longrightarrow \left[\begin{smallmatrix} 2 \\ 4 \\ 0 \end{smallmatrix}\right] \longrightarrow 0$$

The algebra A is the endomorphism algebra of the direct sum of the indecomposables of $\overline{\mathcal{M}} = \mathcal{M}/addH \simeq \mathcal{M}'/addD(kQ)$. Thus the algebra A is given by the quiver

$$\begin{tikzcd} & 2 \arrow[dr, shift left, "b"] \arrow[dr, shift right, "b'"'] & \\ 1 \arrow[ur, shift left, "a"] \arrow[ur, shift right, "a'"'] & & 3 \end{tikzcd}$$

and the relation $\quad ba + b'a' = 0.$

By Theorem 5.4 the cluster category $\mathcal{C}_A$ associated with the algebra A is 2-Calabi-Yau, Hom-finite and $A \in \mathcal{C}_A$ is a cluster-tilting object. Moreover by proposition 4.16, the quiver of the cluster-tilted algebra $\tilde{A} = \mathsf{End}_{\mathcal{C}_A}(A)$ has the form:

$$\begin{tikzcd} & 2 \arrow[dr, shift left] \arrow[dr, shift right] & \\ 1 \arrow[ur, shift left] \arrow[ur, shift right] & & 3 \arrow[ll] \end{tikzcd}$$

The injective A-module $I_1 = 1^{\vee}_{|\overline{\mathcal{M}}}$ has dimension vector $\left[\begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix}\right] = \begin{smallmatrix} 3 & & 3 & & 3 \\ & 2 & & 2 & \\ & & 1 & & \end{smallmatrix}$. Its image by i^* is the $\mathcal{M}$-module $\left[\begin{smallmatrix} & 2 & & 0 & \\ 1 & & 3 & & 0 \\ & & & 0 & \end{smallmatrix}\right]$. Its image through F is the same as the image of the $\mathcal{M}$-module $\left[\begin{smallmatrix} & 0 & & 2 & \\ 0 & & 1 & & 3 \\ & & & 0 & \end{smallmatrix}\right]$, indeed we have $F \circ i_*(1^{\vee}_{|\overline{\mathcal{M}}}) = \left[\begin{smallmatrix} 2 \\ 4 \\ 0 \end{smallmatrix}\right]$. By the exact sequence above, there is an isomorphism in $\underline{\mathsf{Sub}}\,\Lambda/\mathcal{I}_w$ between $F \circ i_*(I_1)$ and $F \circ i_*(P_1)[2]$ where P_1 is the projective A-module with vector dimension $\left[\begin{smallmatrix} & 0 & \\ 1 & & 0 \end{smallmatrix}\right]$.

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