Central Limit Theorems for arrays of decimated linear processes

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CENTRAL LIMIT THEOREMS FOR ARRAYS OF DECIMATED LINEAR PROCESSES

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Abstract. Linear processes are defined as a discrete-time convolution between a kernel and an infinite sequence of i.i.d. random variables. We modify this convolution by introducing decimation, that is, by stretching time accordingly. We then establish central limit theorems for arrays of squares of such decimated processes. These theorems are used to obtain the asymptotic behavior of estimators of the spectral density at specific frequencies. Another application, treated elsewhere, concerns the estimation of the long-memory parameter in time-series, using wavelets.

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1. Introduction

Consider a linear process, that is, a weakly stationary sequence

$$\sum_{t\in\mathbb{Z}} v(k-t)\xi_k, \quad k\in\mathbb{Z}, \quad \text{where} \quad \sum_{t\in\mathbb{Z}} v^2(t) < \infty$$

and $\{\xi_t,\ t\in\mathbb{Z}\}$ is a centered white noise sequence, that is an uncorrelated sequence with mean zero. We shall sometimes make the following additional assumptions on $\{\xi_t,\ t\in\mathbb{Z}\}$.

Assumptions A

- (A-1) $\{\xi_t,\ t\in\mathbb{Z}\}$ is a sequence of independent and identically distributed real-valued random variables such that $\mathbb{E}[\xi_0]=0$, $\mathbb{E}[\xi_0^2]=1$.
- (A-2) $\{\xi_t,\ t\in\mathbb{Z}\}$ satisfies (A-1) and $\kappa_4 \stackrel{\text{def}}{=} \mathbb{E}[\xi_0^4]-3$ is finite.

We will allow decimation and consider, moreover, not one but N linear sequences, all using the same $\{\xi_t,\ t\in\mathbb{Z}\}$.

Definition 1. *An array of N–dimensional decimated linear processes is a process admitting the following linear representation,*

$$Z_{i,j,k} = \sum_{t\in\mathbb{Z}} v_{i,j}(\gamma_j k - t)\,\xi_t\ , \quad i=1,\dots,N,\ k\in\mathbb{Z},\ j\ge 0\ , \tag{1}$$

where $\{\xi_t,\ t\in\mathbb{Z}\}$ is a centered weak white noise, $(\gamma_j)_{j\ge0}$ is a diverging sequence of positive integers and, for all $i=1,\dots,N$ and $j\ge0$, $\{v_{i,j}(t),\ t\in\mathbb{Z}\}$ is real-valued and satisfies $\sum_{t\in\mathbb{Z}} v_{i,j}^2(t)<\infty$.

Remark 1. $Z_{i,j,k}$ involves three indices. The index $i=1,\dots,N$ is used to define an N-variate version, the index j labels the decimation factor γ_j, and the index k corresponds to time. Because of the presence of the factor γ_j in (1), $Z_{i,j,k}$ is not a usual convolution. It can be viewed as a decimated convolution of a white noise in the sense that, after convolution, one keeps only values spaced by γ_j. A typical choice of decimation is $\gamma_j=2^j$, $j\ge0$.

Our goal is to study the asymptotic behavior of the sample mean square of $Z_{i,j,k}$, namely to find conditions on the kernels $v_{i,j}$, the decimation factor γ_j and normalization n_j, so that the normalized vector

$$\left\{ n_j^{-1/2} \sum_{k=0}^{n_j-1} \left(Z_{i,j,k}^2 - \mathbb{E}\left[Z_{i,j,k}^2\right]\right),\ i=1,\dots,N \right\}$$

converges to a multivariate normal $\mathcal{N}(0,\Gamma)$ distribution. We want also to characterize the limiting covariance matrix Γ. Thus we are interested in the sum of squares of the $Z_{i,j,k}$.

Such results are useful in estimation. In section 4, for example, we apply our result to obtain a central limit theorem for the estimator $\widehat{f_n}(0)$ of the spectral density at the origin $f(0)$ of a linear process. This CLT is compared to [2], Eq (3.9), as discussed in Remark 13. Another, more involved application, which involves wavelets, can be found in [7].

The paper is structured as follows. In Section 2, we indicate the main assumptions. The central limit theorems (Theorems 1 and 2) for decimated sequences are stated in Section 3. Section 4 contains an application to the estimation of the value of spectral density at the origin (Theorem 3). Section 5 contains technical lemmas. Theorems 1, 2 and 3 are proved in Sections 6, 7 and 8 respectively.

2. Main assumptions

Our assumptions will be expressed in terms of the Fourier series of the ℓ^2 sequences $\{v_{i,j}(t),\ t \in \mathbb{Z}\}$, namely

$$v^*_{i,j}(\lambda) = (2\pi)^{-1/2} \sum_{t\in\mathbb{Z}} v_{i,j}(t)\, \mathrm{e}^{-\mathrm{i}\lambda t}\ . \tag{2}$$

We suppose that for any $i = 1, \dots, N$, as $j \to \infty$, the Fourier series $v^*_{i,j}$ concentrates around some frequency $\lambda_{i,\infty} \in [0, \pi)$. By "concentrate", we mean that when adequately normalized, translated and rescaled around these frequencies, the series $v^*_{i,j}$ converges as $j \to \infty$ to some limit functions $v^*_{i,\infty}$, with a uniform polynomial control (see Eq (7) and (8) below). Because of the particular structure of the γ_j–decimation in (1), however, in order to derive the asymptotic behavior for the processes, we need to introduce sequences of frequencies $(\lambda_{i,j})_{j\geq 0}$ that satisfy some special conditions and converge to $\lambda_{i,\infty}$ for all $i = 1, \dots, N$. We shall first specify the conditions on the Fourier series $v^*_{i,j}$, the frequencies $\lambda_{i,\infty}$ and the limit functions $v^*_{i,\infty}$, and then comment on these conditions.

Condition C

- There exist a N–dimensional array of frequencies $(\lambda_{i,j})_{i\in 1,\dots,N, j\geq 0}$ valued in $[0, \pi)$ such that, for all $i = 1, \dots, N$,

$$\gamma_j \lambda_{i,j} \in 2\pi\mathbb{Z}_+, \quad \text{for } j \text{ large enough}\ , \tag{3}$$

$$\lambda_{i,j} \to \lambda_{i,\infty}\ , \quad \text{as } j \to \infty\ , \tag{4}$$

$$\text{if} \quad \lambda_{i,\infty} = 0\ , \quad \text{then} \quad \lambda_{i,j} = 0, \quad \text{for } j \text{ large enough}\ . \tag{5}$$

and, for all $1 \leq i < i' \leq N$,

$$\text{if} \quad \lambda_{i,\infty} = \lambda_{i',\infty}\ , \quad \text{then} \quad \lambda_{i,j} = \lambda_{i',j}, \quad \text{for } j \text{ large enough}\ . \tag{6}$$

- Moreover there exist $\delta > 1/2$ and a sequence of $[-\pi, \pi)$-valued functions $\Phi_j(\lambda)$ defined on $\lambda \in \mathbb{R}$ such that

$$\sup_{j\geq 0} \sup_{\lambda\in[0,\pi)} \gamma_j^{-1/2} |v_{i,j}^*(\lambda)|(1+\gamma_j|\lambda-\lambda_{i,j}|)^\delta < \infty \,, \tag{7}$$

$$\lim_{j\to\infty} \gamma_j^{-1/2} v_{i,j}^*(\gamma_j^{-1}\lambda + \lambda_{i,j}) \mathrm{e}^{\mathrm{i}\Phi_j(\lambda)} = v_{i,\infty}^*(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{R} \,, \tag{8}$$

The following remarks provide some insight into these conditions.

Remark 2. Equations (4) and (8) imply that the spectral density $\lambda \mapsto |v_{i,j}^*|^2(\lambda)$ of the *undecimated* stationary process

$$W_{i,j,k} = \sum_{t\in\mathbb{Z}} v_{i,j}(k-t)\, \xi_t \,, \quad k \in \mathbb{Z} \,,$$

concentrates, as $j \to \infty$ around the frequency $\lambda_{i,\infty}$. In practical applications of the theorem, the limiting frequencies $\{\lambda_{i,\infty},\ i \in 1, \ldots, N\}$ are given. However, one can often easily find sequences $(\gamma_j)_{j\geq 0}$ and $(\lambda_{i,j})_{j\geq 0}$ that satstisfy Conditions (3) and (4) hold. In the particular case where the $\lambda_{i,\infty}$ are such that $q\,\lambda_{i,\infty} \in 2\pi\mathbb{Z}$ for all $i = 1, \ldots, N$ and some positive integer q, one may take $\lambda_{i,j} = \lambda_{i,\infty}$ and γ_j as a multiple of q. This happens for instance when the limiting frequencies are all at the origin, that is, $\lambda_{1,\infty} = \cdots = \lambda_{N,\infty} = 0$ and $\gamma_j = 2^j$.

Remark 3. The presence of the phase function Φ_j in (8) offers flexibility and implies that $\gamma_j^{-1/2} v_{i,j}^*(\gamma_j^{-1}\lambda + \lambda_0)$ converges to $v_{i,\infty}^*(\lambda)$ *up to a common change of phase*. Observe, however, that Φ_j should not depend on i and thus, for $N > 1$, Condition (8) is not equivalent to requiring that $\gamma_j^{-1/2}|v_{i,j}^*(\gamma_j^{-1}\lambda + \lambda_{i,j})|$ converges to $|v_{i,\infty}^*(\lambda)|$ for all i. The presence of the phase Φ is consistent with the fact that the asymptotic covariance matrix Γ defined in (24) is invariant through a common phase translation of the functions $v_{i,\infty}^*$ for all $i = 1, \ldots, N$.

Remark 4. Condition (6) states that if two limits $\lambda_{i,\infty}$ and $\lambda_{i',\infty}$ are equal, then the $\lambda_{i,j}$ and $\lambda_{i',j}$ which converge to them must coincinde for large enough j. Condition (5) has a similar interpretation.

Remark 5. Conditions (6) and (8) imply that, for all $1 \leq i \leq i' \leq N$ such that $\lambda_{i,\infty} = \lambda_{i',\infty}$,

$$\lim_{j\to\infty} \gamma_j^{-1} [v_{i,j}^* \overline{v_{i',j}^*}](\gamma_j^{-1}\lambda + \lambda_{i,j}) = [v_{i,\infty}^* \overline{v_{i',\infty}^*}](\lambda) \quad \text{for all} \quad \lambda \in \mathbb{R} \,. \tag{9}$$

Here $\overline{z}$ denotes the conjugate of the complex z.

Remark 6. Since $v_{i,j}(t)$ is real valued, since $v_{i,j}(t)$ is real-valued, we have

$$v_{i,j}^*(-\lambda) = \overline{v_{i,j}^*(\lambda)} \,. \tag{10}$$

Thus, Conditions (7) and (9) imply that

$$\sup_{j\geq 0}\sup_{\lambda\in(-\pi,\pi)} \gamma_j^{-1/2}|v_{i,j}^*(\lambda)|(1+\gamma_j\,||\lambda|-\lambda_{i,j}|)^\delta<\infty\,, \tag{11}$$

$$\lim_{j\to\infty}\gamma_j^{-1}[v_{i,j}^*\overline{v_{i',j}^*}](\gamma_j^{-1}\lambda-\lambda_{i,j})=[\overline{v_{i,\infty}^*}v_{i',\infty}^*](-\lambda)\quad\text{for all}\quad\lambda\in\mathbb{R}\,. \tag{12}$$

In particular, if $\lambda_{i,\infty}=\lambda_{i',\infty}=0$, by (5), (9) and (12), we have

$$[v_{i,\infty}^*\overline{v_{i',\infty}^*}](\lambda)=[\overline{v_{i,\infty}^*}v_{i',\infty}^*](-\lambda)\,. \tag{13}$$

Remark 7. Since (γ_j) is a diverging sequence and $\lambda_{i,j}\to\lambda_{i,\infty}\in[0,\pi)$, for any $\lambda\in\mathbb{R}$, for j large enough, we have $\gamma_j^{-1}\lambda+\lambda_{i,j}\in[0,\pi)$. Hence Conditions (7) and (8) imply that, for all $i=1,\dots,N$,

$$\sup_{\lambda\in\mathbb{R}}\left|v_{i,\infty}^*(\lambda)\right|\;(1+|\lambda|)^\delta<\infty\,. \tag{14}$$

To better understand these assumptions, we start with a result on the asymptotic behavior of the cross-covariance function for the array (1). In this proposition, we set, without loss of generality, $N=2$.

Proposition 1. *Let $\{Z_{i,j,k},\ i=1,2,j\geq 0,k\in\mathbb{Z}\}$ be an array of 2–dimensional decimated linear processes as defined by (1). Assume that Condition (C) holds for some $\lambda_{i,\infty}\in[0,\pi)$ and functions $v_{i,\infty}^*$, $i=1,2$, from $\mathbb{R}\to\mathbb{Z}$. Then, for all $k,k'\in\mathbb{Z}$, as $j\to\infty$,*

$$\mathrm{Cov}\left(Z_{1,j,k},Z_{2,j,k'}\right)\to \mathrm{C}_{1,2}\int_{-\infty}^{\infty} w_{1,2}^*(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda(k'-k)}\,\mathrm{d}\lambda\,, \tag{15}$$

where, for any $i,i'\in\{1,2\}$,

$$w_{i,i'}^*(\lambda)=\frac{1}{2}\left[\overline{v_{i,\infty}^*(-\lambda)}v_{i',\infty}^*(-\lambda)+v_{i,\infty}^*(\lambda)\overline{v_{i',\infty}^*(\lambda)}\right],\quad\lambda\in\mathbb{R}\,, \tag{16}$$

and

$$\mathrm{C}_{i,i'}=\begin{cases}0 & \text{if } \lambda_{i,\infty}\neq\lambda_{i',\infty}\\ 1 & \text{if } \lambda_{i,\infty}=\lambda_{i',\infty}=0\\ 2 & \text{if } \lambda_{i,\infty}=\lambda_{i',\infty}>0\end{cases}\,. \tag{17}$$

Proof. Using (1) and Parseval's theorem, we have

$$\mathrm{Cov}\left(Z_{1,j,k},Z_{2,j,k'}\right)=\sum_{t\in\mathbb{Z}}v_{1,j}(\gamma_jk-t)v_{2,j}(\gamma_jk'-t) \tag{18}$$

$$=\int_{-\pi}^{\pi}[v_{1,j}^*\overline{v_{2,j}^*}](\lambda)\,\mathrm{e}^{\mathrm{i}\gamma_j\lambda(k'-k)}\,\mathrm{d}\lambda\,. \tag{19}$$

We now consider separately the three cases $\lambda_{1,\infty}\neq\lambda_{2,\infty}$, $\lambda_{1,\infty}=\lambda_{2,\infty}>0$ and $\lambda_{1,\infty}=\lambda_{2,\infty}=0$.

1) Suppose $\lambda_{1,\infty} \neq \lambda_{2,\infty}$. Then by (7), there is a constant $C > 0$ such that

$$\begin{aligned}\left|\mathrm{Cov}\left(Z_{1,j,k}, Z_{2,j,k'}\right)\right| &\leq C\gamma_j \int_0^\pi (1+\gamma_j|\lambda-\lambda_{1,j}|)^{-\delta}(1+\gamma_j|\lambda-\lambda_{2,j}|)^{-\delta}\,\mathrm{d}\lambda \\ &\to 0 \quad \text{as } j \to \infty\,, \end{aligned} \tag{20}$$

since $\gamma_j \to \infty$, $\delta > 1/2$ and $|\lambda_{1,j} - \lambda_{2,j}|$ has a positive limit.

2) Suppose $\lambda_{1,\infty} = \lambda_{2,\infty} > 0$. Setting $\lambda = \gamma_j^{-1}\xi + \lambda_{1,j}$ and using (3), we have

$$\begin{aligned}\int_0^\pi [v_{1,j}^* \overline{v_{2,j}^*}](\lambda)\,\mathrm{e}^{\mathrm{i}\gamma_j\lambda(k'-k)}\,\mathrm{d}\lambda &= \int_{-\gamma_j\lambda_{1,j}}^{\gamma_j(\pi-\lambda_{1,j})} \gamma_j^{-1}[v_{1,j}^* \overline{v_{2,j}^*}](\gamma_j^{-1}\xi+\lambda_{1,j})\,\mathrm{e}^{\mathrm{i}\xi(k'-k)}\,\mathrm{d}\xi \\ &\to \int_{-\infty}^{\infty} [v_{1,\infty}^* \overline{v_{2,\infty}^*}](\xi)\,\mathrm{e}^{\mathrm{i}\xi(k'-k)}\,\mathrm{d}\xi \quad \text{as } j \to \infty, \end{aligned} \tag{21}$$

where the limit follows from Conditions (6), (4), (7), (9) and dominated convergence. Similarly we have

$$\int_{-\pi}^0 [v_{1,j}^* \overline{v_{2,j}^*}](\lambda)\,\mathrm{e}^{\mathrm{i}\gamma_j\lambda(k'-k)}\,\mathrm{d}\lambda \to \int_{-\infty}^{\infty} [\overline{v_{1,\infty}^*} v_{2,\infty}^*](-\xi)\,\mathrm{e}^{\mathrm{i}\xi(k'-k)}\,\mathrm{d}\xi \quad \text{as } j \to \infty,$$

by using (12) instead of (9). The last display, (19) and (21) yield

$$\mathrm{Cov}\left(Z_{1,j,k}, Z_{2,j,k'}\right) \to 2\int_{-\infty}^{\infty} w_{1,2}^*(\xi)\,\mathrm{e}^{\mathrm{i}\xi(k'-k)}\,\mathrm{d}\xi \quad \text{as } j \to \infty\,. \tag{22}$$

3) Suppose finally $\lambda_{1,\infty} = \lambda_{2,\infty} = 0$. Setting $\lambda = \gamma_j^{-1}\xi$ gives

$$\begin{aligned}\int_{-\pi}^\pi [v_{1,j}^* \overline{v_{2,j}^*}](\lambda)\,\mathrm{e}^{\mathrm{i}\gamma_j\lambda(k'-k)}\,\mathrm{d}\lambda &= \int_{-\gamma_j\pi}^{\gamma_j\pi} \gamma_j^{-1}[v_{1,j}^* \overline{v_{2,j}^*}](\gamma_j^{-1}\xi)\,\mathrm{e}^{\mathrm{i}\xi(k'-k)}\,\mathrm{d}\xi \\ &\to \int_{-\infty}^{\infty} [v_{1,\infty}^* \overline{v_{2,\infty}^*}](\xi)\,\mathrm{e}^{\mathrm{i}\xi(k'-k)}\,\mathrm{d}\xi \end{aligned}$$

by using Conditions (5), (7), (9) and dominated convergence. The last display, (20) and (21) yield (15).

□

3. Main results

We let $\xrightarrow{\mathcal{L}}$ denote the convergence in law. Our first result provides the asymptotic behavior of the sample mean square of an array of a decimated linear sequence under a global assumption on the behavior of the spectral density (the bound (7)). A local version of this assumption is considered in Theorem 2.

Theorem 1. *Let $\{Z_{i,j,k},\ i = 1, 2, j \geq 0, k \in \mathbb{Z}\}$ be an array of N–dimensional decimated linear processes as defined by (1). Assume (A-2) and that γ_j is even for j large enough. For*

each $i = 1, \dots, N$, we let $\lambda_{i,\infty}$ denote a frequency in $[0, \pi)$ and $v^*_{i,\infty}$ a continuous $\mathbb{R} \to \mathbb{Z}$ function such that Condition (C) holds. Then, for any diverging sequence (n_j),

$$n_j^{-1/2} \sum_{k=0}^{n_j-1} \begin{bmatrix} Z^2_{1,j,k} - \mathbb{E}[Z^2_{1,j,k}] \\ \vdots \\ Z^2_{N,j,k} - \mathbb{E}[Z^2_{N,j,k}] \end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma) \ , \tag{23}$$

where Γ is the covariance matrix defined by

$$\Gamma_{i,i'} = 4\pi \, \mathrm{C}_{i,i'} \int_{-\pi}^{\pi} \left| \sum_{p \in \mathbb{Z}} w^*_{i,i'}(\lambda + 2p\pi) \right|^2 \mathrm{d}\lambda \ , \quad 1 \le i, i' \le N \ , \tag{24}$$

where $\mathrm{C}_{i,i'}$ and $w^*_{i,i'}$ are defined in (17) and (16).

Remark 8. From (14), it follows that the doubly infinite sum in (24) is well defined and bounded on $\lambda \in \mathbb{R}$ and hence Γ is well defined.

Remark 9. The number of time indices k appearing in the centered sum in (23) is n_j and asymptotic normalization occurs as j and n_j tend to ∞.

Remark 10. The presence of the factor γ_j in (1), and hence of decimation, is essential for the Central Limit Theorem to hold in this generality because it ensures that the dependence of the $Z_{i,j,k}$'s decreases sufficiently fast as $j \to \infty$. Decimation of this type is typically encountered in settings involving wavelets, or more generally filter banks, see [5].

Remark 11. In applications, the expectations in (23), which depend on j, will be approximated by quantities that are independent of j. To see why this is possible, observe that, applying Relation (15) in Proposition 1 with $k = k' = 0$ and $i = i' = 1, \dots, N$, we get

$$\lim_{j \to \infty} \mathbb{E}\left[Z^2_{i,j,0}\right] = \int_{-\infty}^{\infty} \left| v^*_{i,\infty}(\lambda) \right|^2 \mathrm{d}\lambda < \infty \ . \tag{25}$$

Thus, when the convergence rate to this limit is fast enough, the expectations in (23) can be replaced by $\int_{-\infty}^{\infty} \left| v^*_{i,\infty}(\lambda) \right|^2 \mathrm{d}\lambda$, $i = 1, \dots, N$, which does not depend on j.

We have assumed in (7) a bound for $v^*_{i,j}(\lambda)$ for $\lambda \in (-\pi, \pi)$. This bound implies that the spectral density of the process $Z_{i,j,\bullet}$ defined in (1) is bounded on $(-\pi, \pi)$. We shall weaken this assumption by only assuming a local bound around the frequency $\lambda_{i,j}$ as follows.

Theorem 2. *Assume that all the conditions of Theorem 1 hold except that (7) is replaced by*

$$\sup_{j \ge 0} \sup_{|\lambda - \lambda_{i,\infty}| \le \varepsilon} \gamma_j^{-1/2} |v^*_{i,j}(\lambda)| (1 + \gamma_j \, |\lambda - \lambda_{i,j}|)^{\delta} < \infty \ , \tag{26}$$

where $\varepsilon > 0$ is arbitrary small. Suppose in addition that

$$n_j^{1/2} \int_0^\pi \mathbb{1}(|\lambda - \lambda_{i,\infty}| > \varepsilon)\, |v_{i,j}^*(\lambda)|^2 \, \mathrm{d}\lambda \to 0 \quad as \quad j \to \infty \, . \tag{27}$$

Then the conclusion of Theorem 1, that is, the CLT (23), still holds.

Remark 12. Since (7) is replaced by the local condition (27), we impose the additional condition (27) on the growth of n_j. This condition does not appear in the conditions of Theorem 1, where it was only required that $n_j \to \infty$.

4. Application to spectral density estimation

Let $\{X(u),\ u \in \mathbb{Z}\}$ be a standard linear process,

$$X_u = \sum_{t \in \mathbb{Z}} a(u-t)\, \xi_t \, , \tag{28}$$

where $\{\xi_t,\ t \in \mathbb{Z}\}$ is a centered weak white noise with unit variance and $\{a(t),\, t \in \mathbb{Z}\}$ is real-valued sequence such that $\sum_k a_k^2 < \infty$ with Fourier series

$$a^*(\lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} a(t)\, \mathrm{e}^{-\mathrm{i}\lambda t} \, . \tag{29}$$

Then $\{X_k,\ k \in \mathbb{Z}\}$ admits the following spectral density

$$f(\lambda) = |a^*(\lambda)|^2 \, , \quad \lambda \in (-\pi, \pi) \, .$$

For simplicity, as in Section 3 of [2], we consider the problem of estimating $f(0)$ from observations $X_1, \ldots, X_n$.

Let us denote by W a bounded $\mathbb{R} \to \mathbb{R}$ function with compact support and by $\widehat{W}$ its Fourier transform,

$$\widehat{W}(\xi) = \int_{-\infty}^{\infty} W(t) \mathrm{e}^{-\mathrm{i}\xi t} \, \mathrm{d}t \, .$$

Let (γ_j) be any diverging sequence of even integers.

We let $N = 1$, and $\lambda_{1,j} = \lambda_{1,\infty} = 0$ for all $j \geq 0$, which yields (3), (4), (5) and (6) in Condition (C).

Define

$$Z_{1,j,k} = \gamma_j^{-1/2} \sum_{u \in \mathbb{Z}} W(k - \gamma_j^{-1} u)\, X_u \, . \tag{30}$$

We assume that

(H-1) As $\lambda \to 0$,

$$f(\lambda) = f(0) + O(|\lambda|^2) \, . \tag{31}$$

(H-2) the support of W is included in $[-1,0]$, $\sup_{\xi\in\mathbb{R}} |\widehat{W}(\xi)|(1+|\xi|)^\beta < \infty$ with $\beta > 1$ and

$$\int_{-\infty}^{\infty} \left|\widehat{W}(\lambda)\right|^2 \, \mathrm{d}\lambda = 1 \tag{32}$$

Assumptions (H-1) and (H-2) are related to the standard bias control of kernel estimates of the spectral density (see Lemma 8 below).

Define

$$n_j = [\gamma_j^{-1}(n+1)] \, . \tag{33}$$

For all $k = 0, \ldots, n_j - 1$, since $W(k - \gamma_j^{-1}u)$ vanishes for $u \le 0$ and $u \ge n+1$, we have

$$Z_{1,j,k} = \gamma_j^{-1/2} \sum_{u=1}^{n} W(k - \gamma_j^{-1}u)\, X_u \, .$$

In other words, $\{Z_{1,j,k},\, k = 0, \ldots, n_j - 1\}$ can be computed from the n observations $X_1, \ldots, X_n$. Thus

$$\widehat{f}_n(0) = n_j^{-1} \sum_{k=0}^{n_j-1} Z_{1,j,k}^2$$

can be used as an estimator of $f(0)$. The following theorem provides a central limit result for $\widehat{f}_n(0)$.

Theorem 3. *Assume (H-1) and (H-2) with $\beta > 2$. Let (γ_n) be a diverging sequence of even integers such that $\gamma_n^{-1} n \to \infty$ Then, as $n \to \infty$,*

$$\mathbb{E}\left[\widehat{f}_n(0)\right] = \mathbb{E}\left[Z_{1,n,0}^2\right] = f(0) + O(\gamma_n^{-2}) \, . \tag{34}$$

If moreover (A-2) in Section 3 holds and

$$n^{1/2} \gamma_n^{1/2-2\beta} \to 0 \, , \tag{35}$$

then

$$(\gamma_n^{-1} n)^{1/2} \{\widehat{f}_n(0) - \mathbb{E}\left[Z_{1,n,0}^2\right]\} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2) \, , \tag{36}$$

where

$$\sigma^2 = 2\pi \, f(0)^2 \int_{-\pi}^{\pi} \left(\sum_{p\in\mathbb{Z}} \left|\widehat{W}(\lambda + 2p\pi)\right|^2 \right)^2 \, \mathrm{d}\lambda \, .$$

Remark 13. Our CLT (36) can be compared with [2, Eq. 3.9], although the estimators are different since ours involve a decimation and the one in [2] is expresses as a weighted integral of the standard periodogram. In (36), our γ_n has a role similar to the $q = q_n$ for their estimator. Our bias estimate (34) has a faster decrease than the corresponding one $O(q^{-1})$ in [2], see the last display in their Section 3. Our conditions also differ from those of [2]. Our conditions on the weight sequence $a(t)$ is much more general, since we

assume a polynomial decrease neither of this sequence nor of $a(t) - a(t+1)$, as assumed for the corresponding (causal) sequence (ψ_j) in Assumption 2.1 of [2]. Standard results on spectral estimation (see e.g. [1, Theorem 9.4.1]) usually assume the even stronger condition $\sum_t |a(t)| < \infty$. On the other hand we do assume that the noise sequence $\{\xi_t\}$ has fourth finite moment (see (A-2)) while only a finite $2+\beta$ moment (with $\beta > 0$ arbitrary small) is assumed in [2]. It is an open question whether similar moment condition can be used for our estimator $\widehat{f}_n(0)$.

5. Technical lemmas

The following lemma will be used several times.

Lemma 1. *Let g be a (2π)-periodic locally integrable function. Then for all positive integer γ, the function defined by*

$$g_\gamma(\lambda) = \sum_{p=0}^{\gamma-1} g(\gamma^{-1}(\lambda + 2p\pi))$$

is (2π)-periodic. Moreover, one has

$$\int_{-\pi}^{\pi} g(\lambda)\, \mathrm{d}\lambda = \gamma^{-1} \int_{-\pi}^{\pi} g_\gamma(\lambda)\, \mathrm{d}\lambda\,. \tag{37}$$

Proof. Observe that, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} g_\gamma(\lambda + 2\pi) &= \sum_{p=0}^{\gamma-1} g\left(\gamma^{-1}\left(\lambda + (p+1)\,2\pi\right)\right) \\ &= \sum_{p=1}^{\gamma-1} g\left(\gamma^{-1}\left(\lambda + 2p\pi\right)\right) + g\left(\gamma^{-1}\left(\lambda + \gamma 2\pi\right)\right) = g_\gamma(\lambda)\,, \end{aligned}$$

since $g(\gamma^{-1}(\lambda+\gamma 2\pi)) = g(\gamma^{-1}\lambda)$ by the (2π)-periodicity of g. Hence $g_\gamma(\lambda)$ is (2π)-periodic.

With a change of variable, one gets

$$\gamma^{-1} \int_0^{2\pi} g_\gamma(\lambda)\, \mathrm{d}\lambda = \sum_{p=0}^{\gamma-1} \int_0^{2\pi\gamma^{-1}} g(\xi + 2p\pi\gamma^{-1})\, \mathrm{d}\xi = \int_0^{2\pi} g(\xi)\, \mathrm{d}\xi\,.$$

Relation (37) follows by (2π)-periodicity of the integrands. □

The next lemma relates the rates of decrease of two functions with the rate of decrease of their convolution.

Lemma 2. *Let $\delta > 0$. For all $T > 0$ and $t_0 \in [0, T/2)$, we let $h_{T,t_0}(t)$, $t \in \mathbb{R}$ be the even and T-periodic function such that*

$$h_{T,t_0}(t) = (1 + |t - t_0|)^{-\delta} \quad \textit{for all} \quad t \in [0, T/2]\,.$$

Let g be an integrable non-negative function on $\mathbb{R}$ such that

$$g(t) \le c_0 |t|^{-\delta-1} \quad \textit{for } |t| \ge 1. \tag{38}$$

Then there exists a positive constant c, depending only on δ, $\|g\|_1 = \int_{-\infty}^{\infty} g(s)\, ds$ and c_0 such that, for all $T > 0$, $t_0 \in [0, T/2)$, and $t \in [0, T/2]$,

$$g * h_{T,t_0}(t) = \int_{-\infty}^{\infty} g(t-u) h_{T,t_0}(u)\, du \le c(1 + |t - t_0|)^{-\delta} \,. \tag{39}$$

Proof. Let $t_0 \in [0, T/2)$. We shall use the bound, valid for all $t \in \mathbb{R}$,

$$g * h_{T,t_0}(t) \le \|g\|_1 \,. \tag{40}$$

This bound yields (39) only for t close enough to t_0. We shall derive a different bound valid only for $t \in [0, T/2]$ with $|t - t_0| \ge 2$, namely

$$g * h_{T,t_0}(t) \le 2^{1-\delta}\, c_0\, \delta^{-1}\, |t - t_0|^{-\delta} + \|g\|_1\, (1 + |t - t_0|/2)^{-\delta} \,. \tag{41}$$

Applying (40) for $|t - t_0| \le 2$ and (41) for $|t - t_0| \ge 2$ yields (39).

Hence it only remains to establish (41) for $t \in [0, T/2]$ with $|t - t_0| \ge 2$. We shall suppose that $t \in [t_0 + 2, T/2]$ (the case $t \in [0, t_0 - 2]$ is obtained similarly). Let u such that $|t - u| \le |t - t_0|/2$. Then $t - (t - t_0)/2 \le u \le t + (t - t_0)/2$ and thus, using that $t \le T/2$ implies $t + t/2 \le T - t/2$, we get $(t + t_0)/2 \le u \le T - (t + t_0)/2$. Observe that the middle point between $(t + t_0)/2$ and $T - (t + t_0)/2$ is $T/2$. Since $h_{T,t_0}(u)$ is decreasing on $[t_0, T/2]$, and symmetric around $T/2$, we get $h_{T,t_0}(u) \le h_{T,t_0}((t + t_0)/2) = (1 + |t - t_0|/2)^{-\delta}$. Hence we may bound $h_{T,t_0}(u)$ by 1 for $|t - u| > |t - t_0|/2$ and by $(1 + |t - t_0|/2)^{-\delta}$ otherwise, which gives

$$\int_{-\infty}^{\infty} g(t-u) h_{T,t_0}(u)\, du \le \int_{|s| > |t-t_0|/2} g(s)\, ds + \|g\|_1\, (1 + |t - t_0|/2)^{-\delta} \,.$$

Since $|t - t_0| \ge 2$ we may apply the bound (38) in the integral of the RHS of the previous display. Hence we get (41), which concludes the proof. □

The following lemma is used, in particular, to bound $\widehat{f}_j$ in the proof of Theorem 2. It will be used again in the proof of Lemma 6 below. Applying it, one can bound $g_{j,\gamma_j}(\lambda)$ independently of j and λ, where g_{j,γ_j} is defined as in Lemma 1 with g replaced by g_j, and the sequence g_j satisfies a uniform bound of the form (7), namely

$$\sup_{j \ge 0} \sup_{\lambda \in [-\pi, \pi)} |g_j(\lambda)| (1 + \gamma\, ||\lambda| - \lambda_j|)^{\delta} < \infty \,,$$

with $\lambda_j \to \lambda_\infty \in [0, \pi)$ as $j \to \infty$.

Lemma 3. *Let $\delta > 1/2$. Then*

$$\sup_{t\in\mathbb{R}}\sup_{t'\in\mathbb{R}}\sum_{p\in\mathbb{Z}}(1+\big||t+2p\pi|-t'\big|)^{-2\delta}<\infty\ . \tag{42}$$

Moreover, as $u\to\infty$,

$$\sup_{t\in\mathbb{R}}\sup_{t'\in\mathbb{R}}\sum_{p\in\mathbb{Z}}(1+\big||t+2p\pi|-t'\big|)^{-\delta}(1+\big||t+2p\pi|-t'-u\big|)^{-\delta}\to 0\ . \tag{43}$$

Proof. Let $S(t,t')=\sum_{p\in\mathbb{Z}}(3\pi+||t+2p\pi|-t'|)^{-2\delta}$. Since, for any t', $t\mapsto S(t,t')$ is (2π)–periodic we have

$$\sup_{t\in\mathbb{R}}\sup_{t'\in\mathbb{R}}S(t,t')=\sup_{t'\in\mathbb{R}}\sup_{t:|t-t'|\le\pi}S(t,t')\ . \tag{44}$$

Suppose that $t,t'\in\mathbb{R}$ are such that $|t-t'|\le\pi$. Then for any $a,b\in\mathbb{R}$, we have, if $t+a\ge 0$,

$$\big||t+a|-t'-b\big|\ge|a-b|-\pi\ , \tag{45}$$

and, if $t+a\le 0$,

$$\big||t+a|-t'-b\big|=\big|t-t'+2t'+a+b\big|\ge|2t'+a+b|-\pi\ . \tag{46}$$

Adding 3π to each of the last two displays with $a=2p\pi$ and $b=0$, we get that, for all $|t-t'|\le\pi$ and $p\in\mathbb{Z}$,

$$(3\pi+\big||t+2p\pi|-t'\big|)^{-2\delta}\le(2|p|\pi+2\pi)^{-2\delta}+(|2t'+2p\pi|+2\pi)^{-2\delta}\ . \tag{47}$$

Since $\sum_{p\in\mathbb{Z}}(2|p|\pi+\pi)^{-2\delta}<\infty$ and $\sup_{t'\in\mathbb{R}}\sum_{p\in\mathbb{Z}}(|2t'+2p\pi|+\pi)^{-2\delta}<\infty$, Relation (44) gives that $\sup_{t\in\mathbb{R}}\sup_{t'\in\mathbb{R}}S(t,t')<\infty$ and (42) follows.

We now prove (43). Let

$$S(t,t',u)=\sum_{p\in\mathbb{Z}}(3\pi+\big||t+2p\pi|-t'\big|)^{-\delta}(3\pi+\big||t+2p\pi|-t'-u\big|)^{-\delta}\ .$$

As above, we have

$$\sup_{t\in\mathbb{R}}\sup_{t'\in\mathbb{R}}S_j(t,t')=\sup_{t'\in\mathbb{R}}\sup_{t:|t-t'|\le\pi}S_j(t,t')\ . \tag{48}$$

Suppose that $t,t'\in\mathbb{R}$ are such that $|t-t'|\le\pi$. Adding 3π to (45) and (46) with $a=2p\pi$ and $b=u$, we have

$$(3\pi+\big||t+2p\pi|-t'-u\big|)^{-\delta}\le(|2p\pi-u|+2\pi)^{-\delta}+(|2t'+2p\pi+u|+2\pi)^{-\delta}\ .$$

Using (48), (47) and the previous display, we obtain

$$\begin{aligned}\sup_{t\in\mathbb{R}}\sup_{t'\in\mathbb{R}} S(t,t',u) \le & \sum_{p\in\mathbb{Z}}(|2p\pi|+2\pi)^{-\delta}(|2p\pi-u|+2\pi)^{-\delta} \\ & + \sup_{t'\in\mathbb{R}}\sum_{p\in\mathbb{Z}}(|2t'+2p\pi|+2\pi)^{-\delta}(|2t'+2p\pi+u|+2\pi)^{-\delta} \\ & + \sup_{t'\in\mathbb{R}}\sum_{p\in\mathbb{Z}}(|2p\pi|+2\pi)^{-\delta}(|2t'+2p\pi+u|+2\pi)^{-\delta} \\ & + \sup_{t'\in\mathbb{R}}\sum_{p\in\mathbb{Z}}(|2t'+2p\pi|+2\pi)^{-\delta}(|2p\pi-u|+2\pi)^{-\delta} \ .\end{aligned}$$

Since the three functions in t' appearing in the right-hand side of the last display are π-periodic the $\sup_{t'\in\mathbb{R}}$ can be replaced by $\sup_{|t'|\le\pi/2}$. Since $|2t'+2p\pi| \ge |2p\pi|-\pi$ and $|2t'+2p\pi+u| \ge |2p\pi+u|-\pi$ for $|t'|\le\pi/2$, we thus obtain

$$\sup_{t\in\mathbb{R}}\sup_{t'\in\mathbb{R}} S(t,t',u) \le 4\sum_{p\in\mathbb{Z}}(|2p\pi|+\pi)^{-\delta}(|2p\pi+u|+\pi)^{-\delta} \to 0 \quad \text{as } u\to\infty \ ,$$

which conclude the proof. □

The following lemma will be used in the proof of Lemma 6.

Lemma 4. *Let p be a positive integer. For all $\mathbb{C}^p$-valued function $\mathbf{g}\in L^2(-\pi,\pi)$ and $n\ge 1$, define*

$$M_n(\mathbf{g}) \stackrel{\text{def}}{=} \left\{\sum_{k\in\mathbb{Z}}\left(1-\frac{|k|}{n}\right)_+|\mathbf{c}_k|^2\right\}^{1/2} = \left\{\sum_{k=-n+1}^{n-1}\left(1-\frac{|k|}{n}\right)|\mathbf{c}_k|^2\right\}^{1/2} , \tag{49}$$

where $\mathbf{c}_k = (2\pi)^{-1/2}\int_{-\pi}^{\pi}\mathbf{g}(\lambda)\,\mathrm{e}^{\mathrm{i}k\lambda}\,\mathrm{d}\lambda$ and $|\cdot|$ denotes the Euclidean norm in any dimension. Then, for all $\mathbf{g}_1$ and $\mathbf{g}_2$ in $L^2(-\pi,\pi)$,

$$|M_n(\mathbf{g}_1)-M_n(\mathbf{g}_2)| \le \left(\int_{-\pi}^{\pi}|\mathbf{g}_1(\lambda)-\mathbf{g}_2(\lambda)|^2\,\mathrm{d}\lambda\right)^{1/2} . \tag{50}$$

Moreover, for all $\mathbf{g}$ in $L^2(-\pi,\pi)$, as $n\to\infty$,

$$M_n(\mathbf{g}) \to \left(\int_{-\pi}^{\pi}|\mathbf{g}(\lambda)|^2\,\mathrm{d}\lambda\right)^{1/2} . \tag{51}$$

Proof. See [6, Lemma 1 (Appendix B)]. □

The following lemmas are used to compute the limiting covariances (106) and (24).

Lemma 5. *Let $\{Z_{i,j,k},\ i=1,2, j\ge 0, k\in\mathbb{Z}\}$ be an array of 2–dimensional decimated linear processes as defined by (1). Assume (A-2). Then for all $j\ge 0$ and all $n\ge 1$, one*

has

$$\frac{1}{n}\mathrm{Cov}\left(\sum_{k=0}^{n-1} Z_{1,j,k}^2, \sum_{k=0}^{n-1} Z_{2,j,k}^2\right) = 2A_j(n) + \kappa_4 B_j(n)\ , \tag{52}$$

where

$$A_j(n) = \sum_{\tau=-n+1}^{n-1} (1-|\tau|/n)\left(\sum_{u\in\mathbb{Z}} v_{1,j}(u) v_{2,j}(\gamma_j\tau+u)\right)^2 \tag{53}$$

and

$$B_j(n) = \sum_{u\in\mathbb{Z}} v_{1,j}^2(u) \sum_{\tau=-n+1}^{n-1} (1-|\tau|/n)\, v_{2,j}^2(\gamma_j\tau+u)\ . \tag{54}$$

Proof. Using a standard formula for cumulants of products, we have

$$\begin{aligned}\mathrm{Cov}\left(\sum_{k=0}^{n-1} Z_{1,j,k}^2, \sum_{k=0}^{n-1} Z_{2,j,k}^2\right) =& \sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \mathrm{Cov}\left(Z_{1,j,k}^2, Z_{2,j,k'}^2\right)\\ =& 2\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \mathrm{Cov}^2\left(Z_{1,j,k}, Z_{2,j,k'}\right)\\ &+ \sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \mathrm{cum}\left(Z_{1,j,k}, Z_{1,j,k}, Z_{2,j,k'}, Z_{2,j,k'}\right)\ .\end{aligned}$$

By definition of $\{Z_{i,j,k},\ i=1,2, k\in\mathbb{Z}\}$ the covariance and the fourth-order cumulant in the previous display read respectively

$$\mathrm{Cov}\left(Z_{1,j,k}, Z_{2,j,k'}\right) = \sum_{t\in\mathbb{Z}} v_{1,j}(\gamma_j k - t) v_{2,j}(\gamma_j k' - t)$$

and

$$\mathrm{cum}\left(Z_{1,j,k}, Z_{1,j,k}, Z_{2,j,k'}, Z_{2,j,k'}\right) = \kappa_4 \sum_{t\in\mathbb{Z}} v_{1,j}^2(\gamma_j k - t) v_{2,j}^2(\gamma_j k' - t)\ .$$

The two last displays thus give (52). □

Lemma 6. *Let $A_j(n)$ and $B_j(n)$ be defined by (53) and (54), respectively, and $v_{i,j}^*$ by (2). Then the following inequalities hold for all $j\ge 0$ and all $n\ge 1$:*

$$A_j(n) \le 2\pi \int_{-\pi}^{\pi} \left|\gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} [v_{1,j}^* \overline{v_{2,j}^*}](\gamma_j^{-1}(\lambda+2\pi p))\right|^2 \mathrm{d}\lambda \tag{55}$$

$$B_j(n) \le \int_{-\pi}^{\pi} \left|v_{1,j}^*(\lambda)\right|^2 \mathrm{d}\lambda \int_{-\pi}^{\pi} \left(\gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} \left|v_{2,j}^*(\gamma_j^{-1}(\lambda+2\pi p))\right|\right)^2 \mathrm{d}\lambda\ . \tag{56}$$

Suppose moreover that $\gamma_j \to \infty$ as $j \to \infty$, γ_j is an even integer for j large enough and that Condition (C) holds for some $\lambda_{i,\infty} \in [0,\pi)$ and $\mathbb{R} \to \mathbb{Z}$ functions $v^*_{i,\infty}$, $i = 1,2$. Then, as $(n,j) \to (\infty,\infty)$,

$$A_j(n) \to 2\pi\ \mathrm{C}_{1,2} \int_{-\pi}^{\pi} \left| \sum_{p\in\mathbb{Z}} w^*_{1,2}(\lambda + 2\pi p) \right|^2 \mathrm{d}\lambda\ , \tag{57}$$

where $\mathrm{C}_{1,2}$ and $w^*_{1,2}$ are defined in (17) and (16) respectively. Moreover,

$$\lim_{j\to\infty} \sup_{n\geq 1} |B_j(n)| = 0\ . \tag{58}$$

Proof. **Step 1.** Using properties of the convolution of square summable sequences, we have, for all $t \in \mathbb{Z}$,

$$\sum_{u\in\mathbb{Z}} v_{1,j}(u) v_{2,j}(t+u) = \int_{-\pi}^{\pi} v^*_{1,j}(\lambda)\overline{v^*_{2,j}(\lambda)}\, \mathrm{e}^{\mathrm{i}\, t\lambda}\, \mathrm{d}\lambda\ .$$

For any $\tau \in \mathbb{Z}$, applying Lemma 1 to the (2π)-periodic function $\lambda \mapsto v^*_{1,j}(\lambda)\overline{v^*_{2,j}(\lambda)}\, \mathrm{e}^{\mathrm{i}\,\gamma_j\tau\lambda}$, one gets

$$\sum_{u\in\mathbb{Z}} v_{1,j}(u) v_{2,j}(\gamma_j\tau + u) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} \left((2\pi)^{1/2}\gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} v^*_{1,j}(\xi_{j,p}(\lambda))\overline{v^*_{2,j}(\xi_{j,p}(\lambda))} \right) \mathrm{e}^{\mathrm{i}\,\tau\lambda}\, \mathrm{d}\lambda\ ,$$

where

$$\xi_{j,p}(\lambda) \stackrel{\mathrm{def}}{=} \gamma_j^{-1}(\lambda + 2\pi p)\ . \tag{59}$$

Using the notation of Lemma 4, we can express $A_j(n)$ defined in (53), as

$$A_j(n) = \left(M_n(g_j)\right)^2 \tag{60}$$

where

$$g_j(\lambda) = (2\pi)^{1/2}\gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} v^*_{1,j}(\xi_{j,p}(\lambda))\overline{v^*_{2,j}(\xi_{j,p}(\lambda))}\ . \tag{61}$$

The bound (50) in Lemma 4 with $\mathbf{g}_1 = g_j$ and $\mathbf{g}_2 = 0$ thus gives (55).

Step 2. Let us now show (56). Since

$$v_{2,j}(\gamma_j\tau + u) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} v^*_{2,j}(\lambda)\, \mathrm{e}^{\mathrm{i}\,\lambda(\gamma_j\tau+u)} \mathrm{d}\lambda\ ,$$

we can apply Lemma 1 to the (2π)-periodic function $\lambda \mapsto v^*_{2,j}(\lambda)\, \mathrm{e}^{\mathrm{i}\,\lambda(\gamma_j\tau+u)}$ for all u and τ in $\mathbb{Z}$ and get

$$v_{2,j}(\gamma_j\tau + u) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} \left(\gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} v^*_{2,j}(\xi_{j,p}(\lambda)) \mathrm{e}^{\mathrm{i}\, u\xi_{j,p}(\lambda)} \right) \mathrm{e}^{\mathrm{i}\,\tau\lambda}\, \mathrm{d}\lambda\ .$$

Using the Parseval formula, we get, for all $u \in \mathbb{Z}$,

$$\begin{aligned}\sum_{\tau=-n+1}^{n-1} (1-|\tau|/n)\, v_{2,j}^2(\gamma_j\tau+u) &\le \sum_{\tau\in\mathbb{Z}} v_{2,j}^2(\gamma_j\tau+u)\\ &= \int_{-\pi}^{\pi} \left| \gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} v_{2,j}^*(\xi_{j,p}(\lambda)) \mathrm{e}^{\mathrm{i}\, u\xi_{j,p}(\lambda)} \right|^2 \mathrm{d}\lambda\\ &\le \int_{-\pi}^{\pi} \left(\gamma_j^{-1} \sum_{p=0}^{\gamma_j-1} \left| v_{2,j}^*(\xi_{j,p}(\lambda)) \right| \right)^2 \mathrm{d}\lambda\,.\end{aligned}$$

Observing that the resulting bound is independent of $u \in \mathbb{Z}$ and using the Parseval formula $\sum_{u\in\mathbb{Z}} v_{1,j}^2(u) = \int_{-\pi}^{\pi} \left| v_{1,j}^*(\lambda) \right|^2 \mathrm{d}\lambda$, we obtain the bound (56) for $B_j(n)$ defined in (54).

Step 3. We now establish the limit (57) successively in the cases $\lambda_{1,\infty} \neq \lambda_{2,\infty}$ and $\lambda_{1,\infty} = \lambda_{2,\infty}$. The (2π)-periodicity of $v_{1,j}^* \overline{v_{2,j}^*}$ and Lemma 1 entail that g_j (defined in (61)) is (2π)-periodic. By definition of M_n in Lemma 4, it follows that, for any $j \ge 0$ and any $\tau \in \mathbb{R}$,

$$M_n(g_j) = M_n(g_j^{(\tau)}) \quad \text{with} \quad g_j^{(\tau)}(\lambda) = g_j(\lambda-\tau), \quad \lambda \in \mathbb{R}\,, \tag{62}$$

since the modulus of the Fourier coefficients of g_j and $g_j^{(\tau)}$ are equal. In the following we will take $\tau = \pi\gamma_j$. Observe that, for all $p \in \{0,\dots,\gamma_j-1\}$, $\lambda \in (0,2\pi)$ and $j \ge 0$,

$$\xi_{j,p}(\lambda - \pi\gamma_j) \in (-\pi,\pi)\,. \tag{63}$$

Consider the case where $\lambda_{1,\infty} \neq \lambda_{2,\infty}$, which, by (4), implies

$$\gamma_j |\lambda_{1,j} - \lambda_{2,j}| \to \infty \quad \text{as } j \to \infty\,. \tag{64}$$

Using (63), (61), (62), (63) and (11), we have, for some constant $C > 0$, for all $j \ge 0$,

$$\begin{aligned}\sup_{\lambda\in(0,2\pi)} \left| g_j^{(\pi\gamma_j)}(\lambda) \right| &= \sup_{\lambda\in(0,2\pi)} |g_j(\lambda-\pi\gamma_j)|\\ &\le C \sum_{p=0}^{\gamma_j-1} \prod_{i=0}^{1} (1+\gamma_j\, ||\xi_{j,p}(\lambda-\pi\gamma_j)| - \lambda_{i,j}|)^{-\delta}\\ &\le C \sup_{t,t'\in\mathbb{R}} \sum_{p\in\mathbb{Z}} (1+\left||t+2\pi p| - t'\right|)^{-\delta}\\ &\qquad\qquad \times (1+\left||t+2\pi p| - t' - \gamma_j|\lambda_{1,j}-\lambda_{2,j}|\right|)^{-\delta}\\ &\to 0 \quad \text{as } j\to\infty,\end{aligned}$$

by (64) and (43) in Lemma 3. Applying (60), (62) and the bound (50) in Lemma 4 with $\mathbf{g}_1 = g_j$ and $\mathbf{g}_2 = 0$ yields

$$A_j(n) = (M_n(g_j^{(\pi\gamma_j)}))^2 \leq \int_{-\pi}^{\pi} |g_j^{(\pi\gamma_j)}|^2 \, \mathrm{d}\lambda \, .$$

The last two displays and the (2π)-periodicity of g_j imply $A_j(n) \to 0$ as $j \to \infty$. This proves (57) since by (17), $C_{1,2} = 0$ when $\lambda_{1,\infty} \neq \lambda_{2,\infty}$.

We now consider the case $\lambda_{1,\infty} = \lambda_{2,\infty}$. By Condition (6), we have $\lambda_{1,j} = \lambda_{2,j}$ for all j large enough. Let $p_j = \gamma_j \lambda_{1,j}/(2\pi) + \gamma_j/2$ so that

$$\lambda_{1,j} = \lambda_{2,j} = 2\pi\gamma_j^{-1} p_j - \pi \, . \tag{65}$$

By Condition (3) and since γ_j is even for j large enough by assumption, we get that p_j is an integer for j large enough. Writing

$$\sum_{p=0}^{\gamma_j-1} = \sum_{p=0}^{[\gamma_j/2]-1} + \sum_{p=[\gamma_j/2]}^{\gamma_j-1} = \sum_{q=-(\gamma_j-p_j)}^{[\gamma_j/2]-(\gamma_j-p_j)-1} + \sum_{r=[\gamma_j/2]-p_j}^{\gamma_j-p_j-1} \, ,$$

where $q = p-(\gamma_j-p_j)$ and $r = p-p_j$ and observe that, with these definitions, (59) and (65), we have $\xi_{j,q}(\lambda) = \gamma_j^{-1}(\lambda + 2\pi q) = \gamma_j^{-1}[(\lambda - \pi\gamma_j) + 2\pi p] + 2\pi\gamma_j^{-1} p_j - \pi = \xi_{j,p}(\lambda - \pi\gamma_j) + \lambda_{1,j}$, and, similarly, $\xi_{j,q}(\lambda) = \xi_{j,p}(\lambda - \pi\gamma_j) - \lambda_{1,j}$, so that $g_j^{(\pi\gamma_j)}(\lambda) = g_j(\lambda - \pi\gamma_j)$ defined in (62) and (61), can be expressed as

$$g_j^{(\pi\gamma_j)}(\lambda) = (2\pi)^{1/2} \left[\sum_{q=-(\gamma_j-p_j)}^{[\gamma_j/2]-(\gamma_j-p_j)-1} \frac{v_{1,j}^* \overline{v_{2,j}^*}}{\gamma_j} (\xi_{j,q}(\lambda) - \lambda_{1,j}) \right.$$
$$\left. + \sum_{r=[\gamma_j/2]-p_j}^{\gamma_j-p_j-1} \frac{v_{1,j}^* \overline{v_{2,j}^*}}{\gamma_j} (\xi_{j,r}(\lambda) + \lambda_{1,j}) \right] \, . \tag{66}$$

Since $\lim_{j\to\infty} \gamma_j = \infty$ and, by Condition (4), $\lim_{j\to\infty} \lambda_{i,j} = \lambda_{i,\infty} \in [0, \pi)$, we have $\lim_{j\to\infty} \gamma_j^{-1} p_j \in [1/2, 1)$ and thus

$$-(\gamma_j - p_j) \to -\infty \quad \text{and} \quad \gamma_j - p_j - 1 \to \infty \, , \tag{67}$$

namely, that in (66), the upper limit of the first sum tends to ∞ and the bottom limit of the second sum tends to $-\infty$. We now consider the remaining limits. If $\lambda_{1,\infty} = \lambda_{2,\infty} > 0$, then $\lim_{j\to\infty} \gamma_j^{-1} p_j$ falls in the open interval $(1/2, 1)$ and thus

$$[\gamma_j/2] - (\gamma_j - p_j) - 1 \to \infty \quad \text{and} \quad [\gamma_j/2] - p_j \to -\infty, \, . \tag{68}$$

If $\lambda_{1,\infty} = \lambda_{2,\infty} = 0$, using (5), (65) implies $p_j = \gamma_j/2$ and thus, for j large enough so that p_j is integer–valued and γ_j even,

$$[\gamma_j/2] - (\gamma_j - p_j) - 1 = -1 \quad \text{and} \quad [\gamma_j/2] - p_j = 0 \, . \tag{69}$$

In view of (59), Conditions (6), (7) and (8) (which imply (9), (11) and (12)), (67), (68), (69) and dominated convergence yield, for all $\lambda \in (0, 2\pi)$,

$$g_j^{(\pi\gamma_j)}(\lambda) \to g_\infty(\lambda) \quad \text{as } j \to \infty \,, \tag{70}$$

where

$$g_\infty(\lambda) = (2\pi)^{1/2} \left[\sum_{q\in\mathbb{Z}} [\overline{v^*_{1,\infty}} v^*_{2,\infty}](-\lambda - 2\pi q) + \sum_{r\in\mathbb{Z}} [v^*_{1,\infty} \overline{v^*_{2,\infty}}](\lambda + 2\pi r) \right]$$

if $\lambda_{1,\infty} = \lambda_{2,\infty} > 0$, and

$$g_\infty(\lambda) = (2\pi)^{1/2} \left[\sum_{q=-\infty}^{-1} [\overline{v^*_{1,\infty}} v^*_{2,\infty}](-\lambda - 2\pi q) + \sum_{r=0}^{\infty} [v^*_{1,\infty} \overline{v^*_{2,\infty}}](\lambda + 2\pi r) \right]$$

if $\lambda_{1,\infty} = \lambda_{2,\infty} = 0$. By definition of w^* in (16) and using (13), one has $[\overline{v^*_{1,\infty}} v^*_{2,\infty}](-\lambda) = [v^*_{1,\infty} \overline{v^*_{2,\infty}}](\lambda) = w^*_{1,2}(\lambda)$, the two previous displays read

$$g_\infty(\lambda) = (2\pi)^{1/2} \, \mathrm{C}_{1,2} \sum_{p\in\mathbb{Z}} w^*_{1,2}(\lambda + 2\pi p) \,. \tag{71}$$

Conditions (7) and (8) imply (14), thus that $g_\infty(\lambda)$ is bounded, hence square integrable on $\lambda \in (-\pi, \pi)$. Moreover, applying the same dominated argument as above, one has

$$\lim_{j\to 0} \int_{-\pi}^{\pi} \left| g_j^{(\pi\gamma_j)}(\lambda) - g_\infty(\lambda) \right|^2 \mathrm{d}\lambda = 0 \,. \tag{72}$$

One gets by (62), (50) and (72)

$$|M_n(g_j) - M_n(g_\infty)|^2 \le \int_{-\pi}^{\pi} |g_j(\lambda) - g_\infty(\lambda)|^2 \, \mathrm{d}\lambda \to 0 \quad \text{as } j \to \infty. \tag{73}$$

By applying the limit (51) with $\mathbf{g} = g_\infty$, one gets

$$M_n(g_\infty)^2 \to \int_{-\pi}^{\pi} |g_\infty(\lambda)|^2 \, \mathrm{d}\lambda \tag{74}$$

as $n \to \infty$. Hence, setting $M_n(g_j) = (M_n(g_j) - M_n(g_\infty)) + M_n(g_\infty)$, the limit (57) follows from (60), (71), (73) and (74).

Step 4. We now establish the limit (58). By Condition (7), we have

$$\sup_{j\ge 0} \int_{-\pi}^{\pi} \left| v^*_{1,j}(\lambda) \right|^2 \mathrm{d}\lambda < \infty \,. \tag{75}$$

Using similar arguments as above and Condition (7), we have

$$\int_{-\pi}^{\pi}\left(\gamma_j^{-1}\sum_{p=0}^{\gamma_j-1}\left|v_{2,j}^*(\xi_{j,p}(\lambda))\right|\right)^2\mathrm{d}\lambda=\gamma_j^{-1}\int_0^{2\pi}\left(\sum_{p=0}^{\gamma_j-1}\gamma_j^{-1/2}\left|v_{2,j}^*(\xi_{j,p}(\lambda-\pi\gamma_j))\right|\right)^2\mathrm{d}\lambda$$
$$\leq C\,\gamma_j^{-1}\int_0^{2\pi}\left(\sum_{p=0}^{\gamma_j-1}(1+||\lambda+2\pi(p-\gamma_j/2)|-\gamma_j\lambda_{2,j}|)^{-\delta}\right)^2\mathrm{d}\lambda \quad (76)$$

Using that $||a+b|-c|\geq||b|-c|-|a|$ and $\lambda\in[0,2\pi]$, we have

$$||\lambda+2\pi(p-\gamma_j/2)|-\gamma_j\lambda_{2,j}|\geq|2\pi|p-\gamma_j/2|-\gamma_j\lambda_{2,j}|-2\pi$$

Take j large enough so that γ_j is even. Since $\lambda_{2,j}\in[0,\pi)$, as $p\in\{0,\ldots,\gamma_j-1\}$, $2\pi|p-\gamma_j/2|-\gamma_j\lambda_{2,j}$ is a sequence of numbers with lag 2π and belonging to $[-\gamma_j\pi,\gamma_j\pi]$ and can thus be written as a sequence $2\pi q+c$, where q belongs to $\{-\gamma_j/2,\ldots,\gamma_j/2\}$ and c to $[-\pi,\pi]$ so that

$$|2\pi|p-\gamma_j/2|-\gamma_j\lambda_{2,j}|-2\pi\geq2\pi|q|-3\pi\ .$$

From the last two displays, we have

$$(5\pi+||\lambda+2\pi(p-\gamma_j/2)|-\gamma_j\lambda_{2,j}|)^{-\delta}\leq(2\pi)^{-\delta}(1+|q|)^{-\delta}\ ,$$

with q describing $\{-\gamma_j/2,\ldots,\gamma_j/2\}$ as p describes $\{0,\ldots,\gamma_j-1\}$. Inserting this bound in (76), we get

$$\int_{-\pi}^{\pi}\left(\gamma_j^{-1}\sum_{p=0}^{\gamma_j-1}\left|v_{2,j}^*(\xi_{j,p}(\lambda))\right|\right)^2\mathrm{d}\lambda\leq C\,\gamma_j^{-1}\left(\sum_{q=0}^{\gamma_j/2}(1+q)^{-\delta}\right)^2$$

for some constant C not depending on $j\geq0$. Since the last right-hand side of the previous display tends to 0 as $j\to\infty$ for any $\delta>1/2$, with (75) and (56), we obtain (58). □

Remark 14. The factor $4\pi=2\times2\pi$ in (24) is due to the factor 2 in the right-hand side of (52) and the presence of 2π in the right-hand side of (57).

Corollary 1. *Let $\{Z_{i,j,k},\ i=1,2,j\geq0,k\in\mathbb{Z}\}$ be an array of 2–dimensional decimated linear processes as defined by (1). Assume (A-2), that γ_j is even for j large enough and that Condition (C) holds for some $\lambda_{i,\infty}\in[0,\pi)$ and $\mathbb{R}\to\mathbb{Z}$ functions $v_{i,\infty}^*$, $i=1,2$. Then, for all $k,k'\in\mathbb{Z}$, as $j\to\infty$,*

$$\mathrm{Cov}\left(Z_{1,j,k}^2,Z_{2,j,k'}^2\right)\to2\,\mathrm{C}_{1,2}^2\left(\int_{-\infty}^{\infty}w_{1,2}^*(\lambda)\mathrm{e}^{\mathrm{i}\lambda(k'-k)}\,\mathrm{d}\lambda\right)^2\ , \quad (77)$$

where $\mathrm{C}_{1,2}$ *and* $w^*_{1,2}$ *are defined in (17) and (16), and, as* $(n,j) \to (\infty,\infty)$,

$$\frac{1}{n}\mathrm{Cov}\left(\sum_{k=0}^{n-1} Z^2_{1,j,k}, \sum_{k=0}^{n-1} Z^2_{2,j,k}\right) \to 4\pi\,\mathrm{C}_{1,2}\int_{-\pi}^{\pi}\left|\sum_{p\in\mathbb{Z}} w^*_{1,2}(\lambda+2\pi p)\right|^2 \mathrm{d}\lambda\,. \tag{78}$$

Proof. Setting $n=1$ in (52) and (53) and replacing $v_{1,j}(-t)$ by $v_{1,j}(\gamma_j k - t)$ and $v_{2,j}(-t)$ by $v_{2,j}(\gamma_j k' - t)$ so that $Z_{1,j,0}$ is replaced by $Z_{1,j,k}$ and $Z_{2,j,0}$ by $Z_{2,j,k'}$, we get

$$\begin{aligned}\mathrm{Cov}\left(Z^2_{1,j,k}, Z^2_{2,j,k'}\right) &= 2A_j(1) + \kappa_4 B_j(1)\\ &= 2\left(\sum_{t\in\mathbb{Z}} v_{1,j}(\gamma_j k - t)v_{2,j}(\gamma_j k' - t)\right)^2 + \kappa_4 B_j(1)\,,\end{aligned}$$

and thus (77) follows from (58), (18) and (15).

Relation (78) is obtained by applying Lemmas 5 and 6. □

6. Proof of Theorem 1

We first establish two Central Limit Theorems which will be used in the proof of Theorem 1. The first involves a sequence of linear filters of the sequence $\{\xi_t,\, t\in\mathbb{Z}\}$.

Proposition 2. *Define, for all* $i=1,\dots,N$, $j\ge 0$, $k\in\mathbb{Z}$,

$$Z_{i,j} = \sum_{t\in\mathbb{Z}} v_{i,j}(t)\,\xi_t\,, \tag{79}$$

where for all $i=1,\dots,N$ *and* $j\ge 0$, $\{v_{i,j}(t),\, t\in\mathbb{Z}\}$ *is real-valued and satisfies* $\sum_{t\in\mathbb{Z}} v^2_{i,j}(t) < \infty$ *and* $\{\xi_t,\, t\in\mathbb{Z}\}$ *satisfies (A-1). Assume that*

$$\lim_{j\to\infty}\sup_{t\in\mathbb{Z}}|v_{i,j}(t)| = 0 \quad \textit{for all} \quad i=1,\dots,N\,, \tag{80}$$

$$\lim_{j\to\infty}\sum_{t\in\mathbb{Z}}\left(v_{i,j}(t)v_{i',j}(t)\right) = \Sigma_{i,i'} \quad \textit{for all} \quad i,i'=1,\dots,N\,, \tag{81}$$

where Σ *is a* $N\times N$ *given matrix. Then, as* $j\to\infty$,

$$\begin{bmatrix} Z_{1,j}\\ \vdots\\ Z_{N,j}\end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0,\Sigma)\,. \tag{82}$$

Remark 15. A study on the weak convergence of such sequence without assuming Assumption (A-1) can be found in [4].

Proof. This is a standard application of the Lindeberg-Feller Central Limit Theorem. Using the Cramér-Wold device for vectorial Central Limit Theorem and since for any column

vector $\zeta \in \mathbb{R}^N$, the linear combination $\sum_{i=1}^N \zeta_i Z_{i,j}$ can be written as in (79) with $v_{i,j}(t)$ replaced by $v_j(t;\zeta) = \sum_{i=1}^N \zeta_i v_{i,j}(t)$, which satisfies

$$\lim_{j\to\infty} \sup_{t\in\mathbb{Z}} |v_j(t;\zeta)| = 0 \quad \text{for all} \quad i = 1,\dots,N \ ,$$
$$\lim_{j\to\infty} \sum_{t\in\mathbb{Z}} (v_j(t;\zeta) v_j(t;\zeta)) = \zeta^T \Sigma \zeta \ ,$$

it is sufficient by (80) and (81) to prove the result for $N = 1$, in which case we simply denote $v_{1,j}(t)$ by $v_j(t)$ and Σ by σ^2. Let (m_j) be a sequence of integers tending to infinity with j, such that

$$\sum_{|t|\geq m_j} v_j^2(t) \leq 2^{-j} \ .$$

We now show that the Lindeberg conditions hold for the sequence $\sum_{|t|\leq m_j} v_j(t)\xi_t$. The first holds because, by (81),

$$\lim_{j\to\infty} \sum_{|t|\leq m_j} \mathrm{Var}(v_j(t)\xi_t) = \lim_{j\to\infty} \sum_{|t|\leq m_j} v_j^2(t) \stackrel{\mathrm{def}}{=} \sigma^2 \ .$$

The second holds because, for all $\epsilon > 0$,

$$\sum_{|t|\leq m_j} \mathbb{E}\left[v_j^2(t)\xi_t^2 \ \mathbb{1}(|v_j(t)\xi_t| \geq \epsilon)\right] \leq \left(\sum_{t\in\mathbb{Z}} v_j^2(t)\right) \mathbb{E}\left[\xi_0^2 \ \mathbb{1}(|\xi_0| \geq \epsilon/S_j)\right] \ ,$$

where $S_j = \sup_{t\in\mathbb{Z}} |v_j(t)|$, and, by (80), the right-hand side of the last display tends to 0 as $j \to \infty$. This concludes the proof. □

The second Central Limit Theorem deals with m-dependent arrays. Recall that $\{Y_k\}$ is said to be m-dependent if, for all $p \geq 1$ and all $k_1,\dots,k_p$ such that $k_1 + m \leq k_2,\dots,k_{p-1} + m \leq k_p$, $Y_{k_1},\dots,Y_{k_p}$ are independent.

Proposition 3. *Let m be a fixed integer and (n_j) a sequence of integers such that $n_j \to \infty$ as $j \to \infty$. Let $\{Y_{j,k},\ k = 0,\dots,n_j - 1,\ j \geq 1\}$ be an array of $\mathbb{R}^d$-valued random vectors, such that, for each $j \geq 0$, $\{Y_{j,k},\ k = 0,\dots,n_j - 1\}$ has zero mean, is strictly stationary and m-dependent. Assume that there exists a $d\times d$ matrix Γ and a centered stationary $\mathbb{R}^d$-valued process $\{\mathbf{Y}_k,\ k \geq 0\}$ with finite variance such that*

$$Y_{j,\bullet} \xrightarrow{\mathcal{L}} \mathbf{Y}_\bullet \quad \textit{as} \quad j \to \infty \ , \tag{83}$$
$$\lim_{j\to\infty} \mathrm{Cov}\left(Y_{j,k}, Y_{j,k'}\right) = \mathrm{Cov}\left(\mathbf{Y}_k, \mathbf{Y}_{k'}\right) \quad \textit{for all} \quad k,k' \geq 0 \tag{84}$$
$$\lim_{l\to\infty} \lim_{j\to\infty} \mathrm{Cov}\left(l^{-1/2} \sum_{k=0}^{l-1} Y_{j,k}\right) = \Gamma \ . \tag{85}$$

Then we have, as $j \to \infty$,

$$n_j^{-1/2} \sum_{k=0}^{n_j-1} Y_{j,k} \xrightarrow{\mathcal{L}} \mathcal{N}(0,\Gamma) \; . \tag{86}$$

Proof. We may suppose that $d = 1$ since the vector case follows by the Cramér-Wold device. For convenience, we set $\Gamma = \sigma^2$. Let s be a positive integer larger than m. We decompose of $\sum_{k=0}^{n_j-1} Y_{j,k}$ in sums of random variables spaced by m, as follows:

$$\sum_{k=0}^{n_j-1} Y_{j,k} = \sum_{k=0}^{p_j} S_{j,k}^{(s)} + \sum_{k=0}^{p_j} T_{j,k}^{(s)} + R_j^{(s)} \; , \tag{87}$$

where

$$S_{j,k}^{(s)} = \sum_{i=0}^{s-1} Y_{j,k(m+s)+i}, \quad T_{j,k}^{(s)} = \sum_{i=0}^{m-1} Y_{j,k(m+s)+s+i} \quad \text{and} \quad R_j^{(s)} = \sum_{i=0}^{q_j-1} Y_{j,p_j(m+s)+i} \; ,$$

and where p_j and q_j are the non-negative integers defined by the Euclidean division

$$n_j = p_j(m+s) + q_j \quad \text{with} \quad q_j \in \{0, \ldots, m+s-1\} \; . \tag{88}$$

The m-dependence and the strict stationarity of the sequences $Y_{j,\boldsymbol{.}}$ ensure that for all $j \geq 0$ and all $s \geq m$, $S_{j,\boldsymbol{.}}^{(s)}$ and $T_{j,\boldsymbol{.}}^{(s)}$ are sequences of centered independent and identically distributed random variables. Hence, by (84), we have

$$\lim_{j\to\infty} \sum_{k=0}^{p_j} \mathrm{Var}\left(p_j^{-1/2} S_{j,k}^{(s)}\right) = \lim_{j\to\infty} \mathrm{Var}\left(S_{j,0}^{(s)}\right) = \mathrm{Var}\left(\mathbf{S}^{(s)}\right), \tag{89}$$

where

$$\mathbf{S}^{(s)} \stackrel{\text{def}}{=} \sum_{i=0}^{s-1} \mathbf{Y}_i \; ,$$

$$\lim_{j\to\infty} \mathrm{Var}\left(p_j^{-1/2} \sum_{k=0}^{p_j} T_{j,k}^{(s)}\right) = \lim_{j\to\infty} \mathrm{Var}\left(T_{j,0}^{(s)}\right) = \mathrm{Var}\left(\mathbf{S}^{(m)}\right) \tag{90}$$

and

$$\limsup_{j\to\infty} \mathrm{Var}\left(R_j^{(s)}\right) \leq \max\left\{\mathrm{Var}\left(\mathbf{S}^{(t)}\right) \; : \; t = 0, 1, \ldots, m+s-1\right\} < \infty \; . \tag{91}$$

In addition, for any $\epsilon > 0$,

$$\sum_{k=0}^{p_j} \mathbb{E}\left[(p_j^{-1/2} S_{j,k}^{(s)})^2 \mathbb{1}(p_j^{-1/2}|S_{j,k}^{(s)}| > \epsilon)\right] = \mathbb{E}\left[(S_{j,0}^{(s)})^2 \mathbb{1}(p_j^{-1/2}|S_{j,k}^{(s)}| > \epsilon)\right]$$

and hence, since $p_j \to \infty$,

$$\limsup_{j\to\infty} \sum_{k=0}^{p_j} \mathbb{E}\left[(p_j^{-1/2} S_{j,k}^{(s)})^2 \mathbb{1}(p_j^{-1/2}|S_{j,k}^{(s)}| > \epsilon)\right] \leq \inf_{M>0} \limsup_{j\to\infty} \mathbb{E}\left[(S_{j,0}^{(s)})^2 \mathbb{1}(|S_{j,k}^{(s)}| > M)\right] . \quad (92)$$

But using (83), we have

$$S_{j,0}^{(s)} \xrightarrow{\mathcal{L}} \mathbf{S}^{(s)} \quad \text{as} \quad j \to \infty .$$

Hence, denoting by ϕ_M some continuous $\mathbb{R}_+ \to [0,1]$ function satisfying $\mathbb{1}(x \leq M/2) \leq \phi_M(x) \leq \mathbb{1}(x \leq M)$, so that $x^2\phi_M(x)$ is continuous and bounded, we have

$$\begin{aligned}\mathbb{E}\left[(S_{j,0}^{(s)})^2 \mathbb{1}(|S_{j,k}^{(s)}| \leq M)\right] &\geq \mathbb{E}\left[(S_{j,0}^{(s)})^2 \phi_M(S_{j,0}^{(s)})\right] \\ &\to \mathbb{E}\left[(\mathbf{S}^{(s)})^2 \phi_M(\mathbf{S}^{(s)})\right] \quad \text{as} \quad j \to \infty \\ &\to \mathbb{E}\left[(\mathbf{S}^{(s)})^2\right] \quad \text{as} \quad M \to \infty .\end{aligned}$$

Using (84), we have $\mathbb{E}\left[(S_{j,0}^{(s)})^2\right] \to \mathbb{E}\left[(\mathbf{S}^{(s)})^2\right]$ as $j \to \infty$ and hence, for any $M > 0$,

$$\limsup_{j\to\infty} \mathbb{E}\left[(S_{j,0}^{(s)})^2 \mathbb{1}(|S_{j,k}^{(s)}| > M)\right] = \mathbb{E}\left[(\mathbf{S}^{(s)})^2\right] - \liminf_{j\to\infty} \mathbb{E}\left[(S_{j,0}^{(s)})^2 \mathbb{1}(|S_{j,k}^{(s)}| \leq M)\right]$$

The two last displays and (92) imply the second Lindeberg Condition, namely,

$$\limsup_{j\to\infty} \sum_{k=0}^{p_j} \mathbb{E}\left[(p_j^{-1/2} S_{j,k}^{(s)})^2 \mathbb{1}(p_j^{-1/2}|S_{j,k}^{(s)}| > \epsilon)\right] = 0 \quad \text{for any } \epsilon > 0 .$$

Using this and (89), we may apply the Lindeberg-Feller CLT for arrays of independent r.v.'s and we obtain

$$p_j^{-1/2} \sum_{k=0}^{p_j} S_{j,k}^{(s)} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \mathrm{Var}\left(\mathbf{S}^{(s)}\right)\right) . \quad (93)$$

By (85), we have

$$\lim_{s\to\infty} s^{-1}\mathrm{Var}\left(\mathbf{S}^{(s)}\right) = \sigma^2 . \quad (94)$$

Using (93) and (88) with $j \to \infty$ and then (94) with $s \to \infty$ yields

$$n_j^{-1/2} \sum_{k=0}^{p_j} S_{j,k}^{(s)} \xrightarrow[j\to\infty]{\mathcal{L}} \mathcal{N}\left(0, (m+s)^{-1}\mathrm{Var}\left(\mathbf{S}^{(s)}\right)\right) \xrightarrow[s\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \sigma^2\right) .$$

On the other hand, by (87), (90) and (91), we have

$$\limsup_{j\to\infty} \mathbb{E}\left[\left(n_j^{-1/2} \sum_{k=0}^{n_j-1} Y_{j,k} - n_j^{-1/2} \sum_{k=0}^{p_j} S_{j,k}^{(s)}\right)^2\right] \leq (m+s)^{-1}\mathrm{Var}\left(\mathbf{S}^{(m)}\right) \to 0$$

as $s \to \infty$. Using the last two displays and [3, Theorem 3.2], we obtain (86), which concludes the proof. □

Proof of Theorem 1. The proof is in three steps. We show in a first step the convergence of the process $[Z_{i,j,k}, i = 1, \ldots, N,\ k = 0, \ldots, n_j]$ as $j \to \infty$ towards a Gaussian limit. In the second step we prove Theorem 1 under the additional assumption that the sequence

$$Y_{j,k} \stackrel{\text{def}}{=} \begin{bmatrix} \{Z_{1,j,k}^2 - \mathbb{E}[Z_{1,j,k}^2]\} \\ \vdots \\ \{Z_{N,j,k}^2 - \mathbb{E}[Z_{N,j,k}^2]\} \end{bmatrix}, \quad k = 0, \ldots, n_j - 1$$

is m-dependent. The third step exhibits an m-dependent approximation and extends the m-dependent case to the general case. The proof uses a number of auxiliary results proved in Section 5.

Step 1. We shall apply Proposition 2. By Relation (15) in Proposition 1, we get, for all $i, i' = 1, \ldots, N$ and all $k, k' \in \mathbb{Z}$, as $j \to \infty$,

$$\mathrm{Cov}\left(Z_{i,j,k}, Z_{i',j,k'}\right) \to \mathrm{C}_{i,i'} \int_{-\infty}^{\infty} w_{i,i'}^*(\lambda)\, \mathrm{e}^{\mathrm{i}\lambda(k'-k)}\, \mathrm{d}\lambda\,. \tag{95}$$

Moreover, by (7), one has, for all $i = 1, \ldots, N$,

$$\begin{aligned} \sup_{t\in\mathbb{Z}} |v_{i,j}(t)| &\le 2\,(2\pi)^{-1/2} \int_0^{\pi} \left|v_{i,j}^*(\lambda)\right|\, \mathrm{d}\lambda \\ &\le 2\,(2\pi)^{-1/2} \gamma_j^{-1/2} \int_{-\gamma_j\lambda_{i,j}}^{\gamma_j(\pi-\lambda_{i,j})} (1+|\lambda|)^{-\delta}\, \mathrm{d}\lambda\,, \end{aligned}$$

which, by (4), tends to 0 as $j \to \infty$ for any $\delta > 1/2$. Hence, by Proposition 2, for any $p \ge 1$, any $i_1, \ldots, i_p \in \{1, \ldots, N\}$ and any $k_1, \ldots, k_p \in \mathbb{Z}$, we have, as $j \to \infty$,

$$\begin{bmatrix} Z_{i_1,j,k_1} \\ \vdots \\ Z_{i_p,j,k_p} \end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma) \tag{96}$$

where Σ is the covariance matrix with entries given for all $1 \le n, n' \le p$ by

$$\Sigma_{n,n'} = \mathrm{C}_{i_n,i_{n'}} \int_{-\infty}^{\infty} w_{i_n,i_{n'}}^*(\lambda)\, \mathrm{e}^{\mathrm{i}\lambda(k_{n'}-k_n)}\, \mathrm{d}\lambda\,.$$

Expressing this integral as $\sum_p \int_{-\pi+2p\pi}^{\pi+2p\pi}$, we get

$$\Sigma_{n,n'} = \mathrm{C}_{i_n,i_{n'}} \int_{-\pi}^{\pi} \left(\sum_{p\in\mathbb{Z}} w_{i_n,i_{n'}}^*(\lambda+2p\pi)\right) \mathrm{e}^{\mathrm{i}\lambda(k_{n'}-k_n)}\, \mathrm{d}\lambda\,.$$

The convergence (96) can be written equivalently as

$$\begin{bmatrix} Z_{1,j,\bullet} \\ \vdots \\ Z_{N,j,\bullet} \end{bmatrix} \xrightarrow{\mathcal{L}} \mathbf{Z}_{\bullet} \, , \tag{97}$$

as $j \to \infty$, where $\{\mathbf{Z}_k = [Z_{1,\infty,k} \ \dots \ Z_{N,\infty,k}]^T, \ k \geq 0\}$ is a stationary Gaussian $\mathbb{R}^N$-valued process with spectral density matrix function D with entries

$$D_{i,i'}(\lambda) = \mathrm{C}_{i,i'} \sum_{p \in \mathbb{Z}} w^*_{i_n, i_{n'}}(\lambda + 2p\pi), \quad 1 \leq i, i' \leq N \, .$$

Step 2. In this step, we prove (23), assuming that for each $j \geq 1$, $\{Y_{j,k}, \ k = 0, \dots, n_j - 1\}$ is m-dependent. We shall apply Proposition 3 under this additional assumption. We thus need to show that (83)–(85) hold. Relations (25), (97) and the continuous mapping theorem imply (83) with

$$\mathbf{Y}_k = \begin{bmatrix} Z^2_{1,\infty,k} - \mathbb{E}[Z^2_{1,\infty,k}] \\ \vdots \\ Z^2_{N,\infty,k} - \mathbb{E}[Z^2_{N,\infty,k}] \end{bmatrix} .$$

Since $\mathbf{Z}_{\bullet}$ is Gaussian, we have, for all $k, k' \geq 0$ and all $i, i' = 1, \dots, N$,

$$\begin{aligned} \mathrm{Cov}\left(\mathbf{Y}_{k,i}, \mathbf{Y}_{k',i'}\right) &= 2\mathrm{Cov}^2\left(Z_{i,\infty,k}, Z_{i',\infty,k'}\right) \\ &= 2 \, \mathrm{C}^2_{i,i'} \left(\int_{-\infty}^{\infty} w^*_{i,i'}(\lambda) \, \mathrm{e}^{\mathrm{i}\lambda(k'-k)} \, \mathrm{d}\lambda \right)^2 . \end{aligned}$$

Hence Relation (77) in Corollary 1 and the previous display yield (84). The final condition (85) follows from Relation (78) of Corollary 1 with a covariance matrix Γ with entries (24). Applying Proposition 3 then yields (23), with Γ given by (24).

Step 3. Let $K(t)$ be a non-negative infinitely differentiable function defined on $t \in \mathbb{R}$ whose support is included in $[-1/2, 1/2]$ and such that $K(0) = 1$. We will denote by $\widehat{K}$ its Fourier transform,

$$\widehat{K}(\xi) = \int_{-\infty}^{\infty} K(t) \mathrm{e}^{-\mathrm{i}\xi t} \, \mathrm{d}t \, .$$

Observe that, by the assumptions on K, $\widehat{K}(\xi)$ decreases faster than any polynomial as $|\xi| \to \infty$. In particular $\widehat{K}(\xi)$ is integrable on $\xi \in \mathbb{R}$ and, for all $t \in \mathbb{R}$,

$$K(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{K}(\xi) \mathrm{e}^{\mathrm{i}\xi t} \, \mathrm{d}\xi \, . \tag{98}$$

The function K will be used to approximate the $v^{(m)}_{i,j}$ sequence by a sequence whose dependence structure can be controlled. We thus define, for any $i = 1, \dots, N$, $j \geq 0$ and

$m \geq 1$,

$$v_{i,j}^{(m)}(t) = v_{i,j}(t)\, K(t/(m\gamma_j)),$$

which vanishes for all $|t| \geq m\gamma_j/2$, and we define $Z_{i,j,k}^{(m)}$ and ${v_{i,j}^*}^{(m)}(\lambda)$ accordingly.

Let m be a fixed integer. Then, for all $j \geq 0$,

$$\{[Z_{1,j,k}^{(m)} \ \ldots \ Z_{N,j,k}^{(m)}]^T,\, k \in \mathbb{Z}\}$$

is an m-dependent sequence of vectors. We shall now show that $\{{v_{i,j}^*}^{(m)},\, j \geq 0\}$ satisfy conditions similar to (7) and (8) and then apply Step 2. Using (98) and (2) in the equation

$${v_{i,j}^*}^{(m)}(\lambda) = (2\pi)^{-1/2} \sum_{t\in\mathbb{Z}} v_{i,j}(t)\, K(t/(m\gamma_j))\mathrm{e}^{-\mathrm{i}\lambda t}\ ,$$

we get that

$${v_{i,j}^*}^{(m)}(\lambda) = \frac{m}{2\pi} \int_{-\infty}^{\infty} \widehat{K}(m\xi) v_{i,j}^*(\lambda - \gamma_j^{-1}\xi)\, \mathrm{d}\xi\ . \tag{99}$$

It follows from Condition (7) that there exists a constant $C > 0$ such that for all $j \geq 0$ and $\lambda \in [0, \pi)$,

$$\left|v_{i,j}^*(\lambda)\right| \leq C\ \gamma_j^{1/2} (1 + |\gamma_j\lambda - \gamma_j\lambda_{i,j}|)^{-\delta}\ . \tag{100}$$

Using (10) and the (2π)-periodicity of $v_{i,j}^*$, we can express (100) using the symmetric $(2\pi\gamma_j)$-periodic function $h_{2\pi\gamma_j,\gamma_j\lambda_{i,j}}(\xi)$ defined in Lemma 2 and equal to $(1 + |\xi - \gamma_j\lambda_{i,j}|)^{-\delta}$ for $0 \leq \xi \leq \pi\gamma_j$. With $\xi = \gamma_j\lambda$, one gets

$$\left|v_{i,j}^*(\lambda)\right| \leq C\ \gamma_j^{1/2}\ h_{2\pi\gamma_j,\gamma_j\lambda_{i,j}}(\gamma_j\lambda)\ , \quad j \geq 0,\ \lambda \in \mathbb{R}\ .$$

Let $g(t) = m|\widehat{K}(mt)|$ and observe that $\|g\|_1 = \|\widehat{K}\|_1 < \infty$ and

$$g(t) \leq c_0\ m\ (m|t|)^{-\delta-1} \leq c_0|t|^{-\delta-1} \quad \text{for all } |t| \geq 1 \text{ and } m \geq 1\ ,$$

where c_0 is a positive constant such that $|\widehat{K}(u)| \leq c_0|u|^{-\delta-1}$ for $|u| \geq 1$. Applying these bounds to (99) gives

$$\left|{v_{i,j}^*}^{(m)}(\lambda)\right| \leq C\gamma_j^{1/2} \int_{-\infty}^{\infty} g(\xi)\ h_{2\pi\gamma_j,\gamma_j\lambda_{i,j}}(\gamma_j\lambda - \xi)\ \mathrm{d}\xi\ .$$

Applying Lemma 2 to this convolution, we get

$$\left|{v_{i,j}^*}^{(m)}(\lambda)\right| \leq C\ \gamma_j^{1/2}\ (1 + \gamma_j|\lambda - \lambda_{i,j}|)^{-\delta}\ ,$$

for different constants C depending neither on $m \geq 1$, $j \geq 0$ nor on $\lambda \in [0, \pi)$. One has therefore the following version of (7) for ${v_{i,j}^*}^{(m)}$, uniform in $m \geq 1$:

$$\sup_{j\geq 0}\ \sup_{m\geq 1}\ \sup_{\lambda\in[0,\pi)}\ \gamma_j^{-1/2} |{v_{i,j}^*}^{(m)}(\lambda)|(1 + \gamma_j|\lambda - \lambda_{i,j}|)^{\delta} < \infty\ . \tag{101}$$

To get a version of (8) for ${v^*_{i,j}}^{(m)}$, observe that, by (99), we have

$$\gamma_j^{-1/2}{v^*_{i,j}}^{(m)}(\gamma_j^{-1}\lambda+\lambda_{i,j}) = \frac{m}{2\pi}\int_{-\infty}^{\infty}\widehat{K}(m\xi)\left[\gamma_j^{-1/2}v^*_{i,j}(\gamma_j^{-1}(\lambda-\xi)+\lambda_{i,j})\right]\,\mathrm{d}\xi.$$

Condition (7) implies that the term in brackets is bounded independently of ξ and j and hence by (8) and dominated convergence, one has

$$\lim_{j\to\infty}\gamma_j^{-1/2}{v^*_{i,j}}^{(m)}(\gamma_j^{-1}\lambda+\lambda_{i,j}) = {v^*_{i,\infty}}^{(m)}(\lambda) \quad \text{for all} \quad \lambda\in\mathbb{R}\,, \tag{102}$$

where

$${v^*_{i,\infty}}^{(m)}(\lambda) \stackrel{\text{def}}{=} \frac{m}{2\pi}\int_{-\infty}^{\infty}\widehat{K}(m\xi)v^*_{i,\infty}(\lambda-\xi)\,\mathrm{d}\xi\,. \tag{103}$$

Note that ${v^*_{i,\infty}}^{(m)}(\lambda)$ is an approximating sequence of $v^*_{i,\infty}(\lambda)$ in the sense that, since $v^*_{i,\infty}$ is bounded (by (14)) and continuous (by hypothesis), and since $(2\pi)^{-1}\int_{-\infty}^{\infty}\widehat{K}(u)\,du = K(0) = 1$, for all $\lambda\in\mathbb{R}$,

$${v^*_{i,\infty}}^{(m)}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{K}(u)v^*_{i,\infty}(\lambda-u/m)\,du \to v^*_{i,\infty}(\lambda) \quad \text{as } m\to\infty\,. \tag{104}$$

Relations (101) and (102) are the corresponding versions of Conditions (7) and (8) for ${v^*_{i,j}}^{(m)}$. and since, we are in the m-dependent case, we may apply the result proved in Step 2, and obtain, as $j\to\infty$,

$$n_j^{-1/2}\begin{bmatrix}\sum_{k=0}^{n_j-1}\{{Z^{(m)}_{1,j,k}}^2 - \mathbb{E}[{Z^{(m)}_{1,j,k}}^2]\}\\ \vdots \\ \sum_{k=0}^{n_j-1}\{{Z^{(m)}_{N,j,k}}^2 - \mathbb{E}[{Z^{(m)}_{N,j,k}}^2]\}\end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0,\Gamma^{(m)})\,, \tag{105}$$

where $\Gamma^{(m)}$ is the covariance matrix with entries

$$\Gamma^{(m)}_{i,i'} \stackrel{\text{def}}{=} 2\pi\,\mathrm{C}_{i,i'}\int_{-\pi}^{\pi}\left|\sum_{p\in\mathbb{Z}}{w^*_{i,i'}}^{(m)}(\lambda+2p\pi)\right|^2\,\mathrm{d}\lambda, \quad 1\le i,i'\le N\,, \tag{106}$$

where ${w^*_{i,i'}}^{(m)}$ is the equivalent of $w^*_{i,i'}$ in (16),

$${w^*_{i,i'}}^{(m)}(\lambda) = \frac{1}{2}\left[\overline{{v^*_{i,\infty}}^{(m)}(-\lambda)}{v^*_{i',\infty}}^{(m)}(-\lambda) + {v^*_{i,\infty}}^{(m)}(\lambda)\overline{{v^*_{i',\infty}}^{(m)}(\lambda)}\right], \quad \lambda\in\mathbb{R}\,.$$

To obtain the corresponding result (23) for the $\{Z_{i,j,k}\}$ sequence, we apply [3, Theorem 3.2] as follows. We show that

$$\lim_{m\to\infty}\Gamma^{(m)}_{i,i'} = \Gamma_{i,i'} \quad 1\le i,i'\le N \tag{107}$$

and, for all $i=1,\dots,N$,

$$\lim_{m\to\infty}\lim_{j\to\infty}\mathrm{Var}\left(n_j^{-1/2}\sum_{k=0}^{n_j-1}{Z^{(m)}_{i,j,k}}^2 - n_j^{-1/2}\sum_{k=0}^{n_j-1}Z^2_{i,j,k}\right) = 0\,. \tag{108}$$

Relation (107) says that the RHS of (105) converges in distribution to the RHS of (23) and Relation (108) says that the LHS of (105) is a good approximation to the LHS (23) by choosing m arbitrary large.

To prove (108), it is sufficient to establish the following equalities for all $i = 1, \ldots, N$.

$$\lim_{m\to\infty} \lim_{j\to\infty} \mathrm{Cov} \left(n_j^{-1/2} \sum_{k=0}^{n_j-1} Z_{i,j,k}^{(m)\,2}, n_j^{-1/2} \sum_{k=0}^{n_j-1} Z_{i,j,k}^2 \right)$$
$$= \lim_{m\to\infty} \lim_{j\to\infty} \mathrm{Var} \left(n_j^{-1/2} \sum_{k=0}^{n_j-1} Z_{i,j,k}^{(m)\,2} \right) = \lim_{j\to\infty} \mathrm{Var} \left(n_j^{-1/2} \sum_{k=0}^{n_j-1} Z_{i,j,k}^2 \right) .$$

Using Relation (78) of Corollary 1, the limits as $j \to \infty$ (an hence $n_j \to \infty$) in the previous display are, respectively,

$$\Gamma_{i,i}^{(m,\infty)} \stackrel{\mathrm{def}}{=} 2\pi\, \mathrm{C}_{i,i'} \int_{-\pi}^{\pi} \left| \sum_{p\in\mathbb{Z}} {w_{i,i'}^*}^{(m,\infty)}(\lambda + 2p\pi) \right|^2 \mathrm{d}\lambda, \quad \Gamma_{i,i}^{(m)} \quad \text{and} \quad \Gamma_{i,i} ,$$

where $\Gamma_{i,i}^{(m)}$ is defined in (106) and $\Gamma_{i,i}$ in (24) and

$${w_{i,i'}^*}^{(m,\infty)}(\lambda) = \frac{1}{2} \left[\overline{{v_{i,\infty}^*}^{(m)}(-\lambda)} v_{i',\infty}^*(-\lambda) + {v_{i,\infty}^*}^{(m)}(\lambda) \overline{v_{i',\infty}^*(\lambda)} \right] , \quad \lambda \in \mathbb{R} .$$

Hence to prove (107) and (108), it is sufficient to show that

$$\lim_{m\to\infty} \Gamma_{i,i'}^{(m,\infty)} = \lim_{m\to\infty} \Gamma_{i,i'}^{(m)} = \Gamma_{i,i'}, \quad i, i' = 1, \ldots, N . \tag{109}$$

Observe first that Relations (101) and (102) imply

$$\sup_{\lambda\in\mathbb{R}} \sup_{m\geq 1} \left| {v_{i,\infty}^*}^{(m)}(\lambda) \right| \, (1 + |\lambda|)^\delta < \infty , \tag{110}$$

which is the uniform version of (14). Eq. (109) now follows bfrom (104), (101) and (110), and dominated convergence. □

7. Proof of Theorem 2

The following proposition is the key point for proving Theorem 2 since it shows how Condition (7) in Theorem 1 can be recovered for an approximation of the sample mean square, when using the alternative Condition (26). Condition (27) in Theorem 2 can then be used to control the sharpness of the approximation.

Proposition 4. *Let $\{Z_{1,j,k},\ i = 1, 2, j \geq 0, k \in \mathbb{Z}\}$ be an array of 1–dimensional decimated linear processes as defined by (1). Assume that $\{v_{1,j}(t),\ j \geq 0,\ t \in \mathbb{Z}\}$ satisfies (26) for $\delta > 1/2$, a sequence $(\lambda_{1,j})$ taking its values in $[0, \pi)$ and some $\varepsilon > 0$. Then there exists an array*

$\{\widehat{v_{1,j}}(t),\ j\geq 0,\ t\in\mathbb{Z}\}$ whose Fourier series coincide with those of $\{v_{1,j}(t),\ j\geq 0,\ t\in\mathbb{Z}\}$ in ε-neighborhoods of the frequencies $\{\lambda_{1,j},\ j\geq 0\}$ and satisfying (7), that is, such that

$$\widehat{v^*_{1,j}}(\lambda)=v^*_{1,j}(\lambda)\quad \textit{for all } \lambda\in(-\pi,\pi) \textit{ such that } |\lambda-\lambda_{1,j}|\leq\varepsilon\ , \tag{111}$$

$$\sup_{j\geq 0}\sup_{\lambda\in[0,\pi)}\gamma_j^{-1/2}|\widehat{v^*_{1,j}}(\lambda)|(1+\gamma_j|\lambda-\lambda_{i,j}|)^{\delta}<\infty\ , \tag{112}$$

and the following approximation holds.

$$n_j^{-1/2}\left(\sum_{k=0}^{n_j-1}\{Z^2_{1,j,k}-\mathbb{E}[Z^2_{1,j,k}]\right)=n_j^{-1/2}\sum_{k=0}^{n_j-1}\{\widehat{Z}^2_{1,j,k}-\mathbb{E}[\widehat{Z}^2_{1,j,k}]\}+R_j\ , \tag{113}$$

where

$$\widehat{Z}_{1,j,k}=\sum_{t\in\mathbb{Z}}\widehat{v_{1,j}}(\gamma_jk-t)\xi_t\ , \tag{114}$$

and, for some positive constant C not depending on j,

$$\mathbb{E}\left[|R_j|\right]\leq C\ \left[n_j^{1/2}I_j+I_j^{1/2}\right]\ , \tag{115}$$

where

$$I_j\stackrel{\text{def}}{=}\int_0^{\pi}\mathbb{1}(|\lambda-\lambda_{1,\infty}|>\varepsilon)\ \left|v^*_{1,j}(\lambda)\right|^2\ \mathrm{d}\lambda\ . \tag{116}$$

Proof. Let $\mathrm{L}_0=[-\lambda_{1,\infty}-\varepsilon,-\lambda_{1,\infty}+\varepsilon]\cup[\lambda_{1,\infty}-\varepsilon,\lambda_{1,\infty}+\varepsilon]$. We write

$$v^*_{1,j}(\lambda)=\widehat{v^*_{1,j}}(\lambda)+\widetilde{v^*_{1,j}}(\lambda),\quad \lambda\in(-\pi,\pi)\ ,$$

where $\widehat{v^*_{1,j}}(\lambda)=\mathbb{1}_{\mathrm{L}_0}(\lambda)v^*_{1,j}(\lambda)$ so that (111) holds. We define $\widehat{v_{1,j}}$, $\widetilde{v_{1,j}}$ accordingly, so that $v_{1,j}(t)=\widehat{v_{1,j}}(t)+\widetilde{v_{1,j}}(t)$ and, since $\widehat{v^*_{1,j}}$ and $\widetilde{v^*_{1,j}}$ are in $L^2(-\pi,\pi)$, $\widehat{v_{1,j}}$ and $\widetilde{v_{1,j}}$ are in $l^2(\mathbb{Z})$. Hence $Z_{1,j,k}=\widehat{Z}_{1,j,k}+\widetilde{Z}_{1,j,k}$ with $\widehat{Z}_{1,j,k}$ defined by (114) and

$$\widetilde{Z}_{1,j,k}=\sum_{t\in\mathbb{Z}}\widetilde{v}_{1,j}(\gamma_jk-t)\xi_t. \tag{117}$$

Moreover, by (26) and the definition of $\widehat{v^*_{1,j}}$, Condition (112) holds.

We now show that the remainder R_j defined by (113) satisfies (115). Observe that $\widehat{Z}_{1,j,k}$ and $\widetilde{Z}_{1,j,k}$ are centered and, since

$$\mathbb{E}[\widehat{Z}_{1,j,k}\widetilde{Z}_{1,j,k}]=\int_{-\pi}^{\pi}\widehat{v^*_{1,j}}(\lambda)\widetilde{v^*_{1,j}}(\lambda)\mathrm{d}\lambda=0\ ,$$

uncorrelated. Thus we get $\mathbb{E}[Z^2_{1,j,k}]=\mathbb{E}[\widehat{Z}^2_{1,j,k}]+\mathbb{E}[\widetilde{Z}^2_{1,j,k}]$ and hence the remainder R_j defined by (113) is

$$R_j=P_j+2Q_j\quad\text{with} \tag{118}$$

$$P_j=n_j^{-1/2}\sum_{k=0}^{n_j-1}\{\widetilde{Z}^2_{1,j,k}-\mathbb{E}[\widetilde{Z}^2_{1,j,k}]\}\quad\text{and}\quad Q_j=n_j^{-1/2}\sum_{k=0}^{n_j-1}\widetilde{Z}_{1,j,k}\widehat{Z}_{1,j,k}\ . \tag{119}$$

We have

$$\mathbb{E}[|P_j|] \le 2n_j^{1/2}\mathbb{E}[\widetilde{Z}_{1,j,0}^2] = 2n_j^{1/2}\sum_{t\in\mathbb{Z}}\widetilde{v}_{1,j}^2(t) = 4n_j^{1/2}\ I_j\ . \tag{120}$$

by the Parseval Theorem and the definitions of $\widetilde{v_{1,j}^*}$ and I_j.

Using that $\widehat{Z}_{1,j,k}$ and $\widetilde{Z}_{1,j,k'}$ are centered and uncorrelated, we have, using a standard formula for cumulants of products, for all $k, k' \in \mathbb{Z}$,

$$\begin{aligned}\mathrm{Cov}\left(\widetilde{Z}_{1,j,k}\widehat{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}\widehat{Z}_{1,j,k'}\right) = \ & \mathrm{Cov}\left(\widetilde{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}\right)\mathrm{Cov}\left(\widehat{Z}_{1,j,k}, \widehat{Z}_{1,j,k'}\right)\\ & +\mathrm{cum}\left(\widetilde{Z}_{1,j,k}, \widehat{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}, \widehat{Z}_{1,j,k'}\right)\ .\end{aligned}$$

Hence, $\mathrm{Var}\,(Q_j) = A_j + B_j$ where

$$A_j = n_j^{-1}\sum_{k=0}^{n_j-1}\sum_{k'=0}^{n_j-1}\mathrm{Cov}\left(\widetilde{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}\right)\mathrm{Cov}\left(\widehat{Z}_{1,j,k}, \widehat{Z}_{1,j,k'}\right)$$

and

$$B_j = n_j^{-1}\sum_{k=0}^{n_j-1}\sum_{k'=0}^{n_j-1}\mathrm{cum}\left(\widetilde{Z}_{1,j,k}, \widehat{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}, \widehat{Z}_{1,j,k'}\right)\ .$$

Denote by $\widehat{f}_j$ and $\widetilde{f}_j$ the respective spectral densities of the weakly stationary processes $\widehat{Z}_{1,j,\bullet}$ and $\widetilde{Z}_{1,j,\bullet}$. Replacing the covariances in the defintion of A_j by their respective expressions as Fourier coefficients of the spectral density, *e.g.* $\mathrm{Cov}\left(\widetilde{Z}_{1,j,k}, \widetilde{Z}_{1,j,k'}\right) = \int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(k-k')\lambda}\widetilde{f}_j(\lambda)\mathrm{d}\lambda$, we get

$$A_j = n_j^{-1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\widehat{f}_j(\lambda)\widetilde{f}_j(\lambda')\left|\sum_{k=0}^{n_j-1}\mathrm{e}^{\mathrm{i}k(\lambda+\lambda')}\right|^2\ \mathrm{d}\lambda\mathrm{d}\lambda'\ ,$$

which implies that

$$0 \le A_j \le 2\pi\sup_{\lambda\in(-\pi,\pi)}\widehat{f}_j(\lambda)\times\int_{-\pi}^{\pi}\widetilde{f}_j(\lambda')\mathrm{d}\lambda'\ , \tag{121}$$

where, in the last inequality, we used that, for any λ', $\int_{-\pi}^{\pi}\left|\sum_{k=0}^{n_j-1}\mathrm{e}^{\mathrm{i}k(\lambda+\lambda')}\right|^2\mathrm{d}\lambda = 2\pi n_j$. Observe that, by definition of $\widehat{Z}_{1,j,k}$ in (114),

$$\mathrm{Cov}\left(\widehat{Z}_{1,j,0}, \widehat{Z}_{1,j,k}\right) = (2\pi)\int_{-\pi}^{\pi}\left|\widehat{v_{1,j}^*}(\lambda)\right|^2\mathrm{e}^{\mathrm{i}\gamma_j k\lambda}\ \mathrm{d}\lambda$$

Using Lemma 1 with the (2π)-periodic function $g(\lambda) = \left|\widehat{v_{1,j}^*}(\lambda)\right|^2\mathrm{e}^{\mathrm{i}\gamma_j k\lambda}$, we get

$$\mathrm{Cov}\left(\widehat{Z}_{1,j,0}, \widehat{Z}_{1,j,k}\right) = (2\pi)\gamma_j^{-1}\int_{-\pi}^{\pi}\left(\sum_{p=0}^{\gamma_j-1}\left|\widehat{v_{1,j}^*}(\gamma_j^{-1}(\lambda+2p\pi))\right|^2\right)\ \mathrm{e}^{\mathrm{i}k\lambda}\ \mathrm{d}\lambda\ .$$

Hence we have

$$\widehat{f}_j(\lambda) = (2\pi)^{-1}\gamma_j^{-1}\sum_{p=0}^{\gamma_j-1}\left|\widehat{v^*_{1,j}}(\gamma_j^{-1}(\lambda+2p\pi))\right|^2 .$$

Using (112), since $|\gamma_j^{-1}(\lambda+2p\pi)| < \pi$ for $p = 0,\dots,\gamma_j-1$ and $\lambda \in (-\gamma_j\pi, -\gamma_j\pi+2\pi)$, we get

$$\widehat{f}_j(\lambda) \le C\sum_{p=0}^{\gamma_j-1}(1+||\lambda+2p\pi|-\gamma_j\lambda_{1,j}|)^{-2\delta}, \quad \lambda\in(-\gamma_j\pi,-\gamma_j\pi+2\pi) .$$

Using (42) in Lemma 3 and that $\widehat{f}_j$ is (2π)-periodic, we obtain

$$\sup_{j\ge 0}\sup_{\lambda\in(-\pi,\pi)}\widehat{f}_j(\lambda) < \infty . \tag{122}$$

Moreover, we have

$$\int_{-\pi}^{\pi}\widetilde{f}_j(\lambda')\mathrm{d}\lambda' = \mathrm{Var}\left(\widetilde{Z}_{1,j,0}\right) = \sum_{t\in\mathbb{Z}}\widetilde{v}^2_{1,j}(t) = \int_{-\pi}^{\pi}\left|\widetilde{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda , \tag{123}$$

by the Parseval Theorem. Hence by (121), there is a positive constant C such that

$$|A_j| \le C\int_{-\pi}^{\pi}\left|\widetilde{v^*_{1,j}}(\lambda)\right|^2 \mathrm{d}\lambda . \tag{124}$$

We now consider B_j. Using (A-2) and the definitions of $\widetilde{Z}_{1,j,k}$ and $\widehat{Z}_{1,j,k}$ in (114) and (117), we have

$$\mathrm{cum}\left(\widetilde{Z}_{1,j,k},\widehat{Z}_{1,j,k},\widetilde{Z}_{1,j,k'},\widehat{Z}_{1,j,k'}\right) = \kappa_4\sum_{t\in\mathbb{Z}}\widetilde{v}_{1,j}(\gamma_j k-t)\widehat{v}_{1,j}(\gamma_j k-t)\widetilde{v}_{1,j}(\gamma_j k'-t)\widehat{v}_{1,j}(\gamma_j k'-t) .$$

Hence

$$\begin{aligned}
|B_j| &\le \kappa_4\sum_{t,\tau\in\mathbb{Z}}|\widetilde{v}_{1,j}(t)\widehat{v}_{1,j}(t)\widetilde{v}_{1,j}(t+\gamma_j\tau)\widehat{v}_{1,j}(t+\gamma_j\tau)| \\
&\le \kappa_4\left(\sum_{t\in\mathbb{Z}}|\widetilde{v}_{1,j}(t)\widehat{v}_{1,j}(t)|\right)^2 \\
&\le \kappa_4\sum_{t\in\mathbb{Z}}|\widetilde{v}_{1,j}(t)|^2\times\sum_{t\in\mathbb{Z}}|\widehat{v}_{1,j}(t)|^2 \\
&= \kappa_4\int_{-\pi}^{\pi}\left|\widetilde{v^*_{1,j}}(\lambda)\right|^2\mathrm{d}\lambda\times\int_{-\pi}^{\pi}\left|\widehat{v^*_{1,j}}(\lambda)\right|^2\mathrm{d}\lambda
\end{aligned} \tag{125}$$

By definition of $\widehat{v^*_{1,j}}$ and (122), we have

$$\int_{-\pi}^{\pi}\left|\widehat{v^*_{1,j}}(\lambda)\right|^2\mathrm{d}\lambda = \mathrm{Var}\left(\widehat{Z}_{1,j,0}\right) = \int_{-\pi}^{\pi}\widehat{f}_j(\lambda)\,\mathrm{d}\lambda \le C ,$$

and hence

$$B_j \le C\int_{-\pi}^{\pi}\left|\widetilde{v^*_{1,j}}(\lambda)\right|^2\mathrm{d}\lambda \tag{126}$$

where C denotes a positive constant not depending on j. The bounds (124) and (126) and the definition of $\widetilde{v^*_{1,j}}$ yield

$$\mathbb{E}\left[Q_j^2\right] = \mathrm{Var}(Q_j) = A_j + B_j \le 2C \int_{-\pi}^{\pi} \left|\widetilde{v^*_{1,j}}(\lambda)\right|^2 \,\mathrm{d}\lambda$$
$$\le 4C \int_0^{\pi} \mathbb{1}\left(|\lambda - \lambda_{1,\infty}| > \varepsilon\right) \left|v^*_{1,j}(\lambda)\right|^2 \,\mathrm{d}\lambda = 4C\, I_j$$

by the definitions of $\widetilde{v^*_{1,j}}$ and I_j. This, with (118), (120) and Jensen's inequality yields (115), which concludes the proof. □

Proof of Theorem 2. The general case can easily be adapted from the case $N = 1$, which we assume here. We apply Proposition 4. It follows from (111) and (112) that the assumptions of Theorem 1 are verified for $\widehat{Z}_{1,j,k}$ and we obtain

$$n_j^{-1/2} \left(\sum_{k=0}^{n_j - 1} \{\widehat{Z}^2_{1,j,k} - \mathbb{E}[\widehat{Z}^2_{1,j,k}]\} \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma) \ .$$

It follows from (27), that $R_j \xrightarrow{\mathbb{P}} 0$ as $j \to \infty$,. Hence (113) yields the CLT (23), which concludes the proof. □

8. Proof of Theorem 3

We shall use the following lemmas.

Lemma 7. *Assume (H-1) and (H-2). For $\varepsilon > 0$ small enough, if $2\beta - 1 > 2$, then*

$$\left| \int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j \lambda)\right|^2 f(\lambda) \,\mathrm{d}\lambda - f(0) \right| = O\left(\gamma_j^{-2}\right) \ . \tag{127}$$

Proof. Using (31), for $\varepsilon > 0$ small enough, the left-hand side of (127) is at most

$$\left| f(0) \int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j \lambda)\right|^2 \,\mathrm{d}\lambda - f(0) \right| + C \left| \int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j \lambda)\right|^2 \lambda^2 \,\mathrm{d}\lambda \right| \ , \tag{128}$$

where C is a positive constant. To evaluate the first integral in (128) we write $\int_{-\varepsilon}^{\varepsilon} = \int_{-\infty}^{\infty} - \int_{-\infty}^{-\varepsilon} - \int_{\varepsilon}^{\infty}$. Using (H-2), one gets

$$\int_{-\varepsilon}^{\varepsilon} \gamma_j \left|\widehat{W}(\gamma_j \lambda)\right|^2 \,\mathrm{d}\lambda = 1 + O\left(\gamma_j^{1-2\beta}\right) \ .$$

The second integral in (128) is bounded by

$$\int_{-\infty}^{\infty} \gamma_j \left|\widehat{W}(\gamma_j \lambda)\right|^2 \lambda^2 \,\mathrm{d}\lambda \le C' \gamma_j^{-2} \ ,$$

where $C' = \int_{-\infty}^{\infty} \left|\widehat{W}(\lambda)\right|^2 \lambda^2 \,\mathrm{d}\lambda < \infty$ by (H-2). □

Lemma 8. *Assume (H-2) and define B_j as in (138). Then there is a positive constant C, such that, for all $j \geq 0$ and $\lambda \in (-\pi, \pi)$,*

$$\left|\gamma_j^{-1/2} B_j(\lambda) - \widehat{W}(\gamma_j \lambda)\right| \leq C\, \gamma_j^{-\beta}\,, \tag{129}$$

$$\left|\,|B_j(\lambda)|^2 - \gamma_j \left|\widehat{W}(\gamma_j\lambda)\right|^2\right| \leq C\, \left[\gamma_j^{1-\beta}\, \left|\widehat{W}(\gamma_j\lambda)\right| + \gamma_j^{1-2\beta}\right]\,, \tag{130}$$

and, for any positive ϵ,

$$\sup_{\lambda\in[0,\pi)} \mathbb{1}(|\lambda| > \epsilon)\ |B_j(\lambda)| = O(\gamma_j^{1/2-\beta})\,. \tag{131}$$

Proof. By (138), we have $B_j(\lambda) = \gamma_j^{1/2}\widehat{W}(\gamma_j\lambda) + R_j(\lambda)$, where

$$R_j(\lambda) = \sum_{p\neq 0} \gamma_j^{1/2}\widehat{W}(\gamma_j(\lambda + 2p\pi))\,.$$

Using (H-2), since $\beta > 1$, we have

$$\sup_{\lambda\in(-\pi,\pi)} |R_j(\lambda)| \leq C\, \gamma_j^{1/2} \sum_{p>0} (1 + (2p-1)\gamma_j\pi)^{-\beta} = O(\gamma_j^{1/2-\beta})\,. \tag{132}$$

The bound (132) gives (129), which yield (131) by using (H-2). The bound (130) follows from (129) and

$$\left||z_1|^2 - |z_2|^2\right| \leq 2|z_2| \times |z_1 - z_2| + |z_1 - z_2|^2$$

applied with $z_1 = B_j(\lambda)$ and $z_2 = \gamma_j^{1/2}\widehat{W}(\gamma_j\lambda)$. □

Lemma 9. *Suppose that the assumptions of Theorem 3 hold. Then for some arbitrary small $\varepsilon > 0$,*

$$\sup_{j\geq 0} \sup_{|\lambda|\leq\varepsilon} \gamma_j^{-1/2} \left|v_{1,j}^*(\lambda)\right| (1 + \gamma_j|\lambda|)^\beta < \infty\,, \tag{133}$$

$$\lim_{j\to\infty} \gamma_j^{-1/2} v_{1,j}^*(\gamma_j^{-1}\lambda) = v_{1,\infty}^*(\lambda) \quad \textit{for all} \quad \lambda \in \mathbb{R}\,, \tag{134}$$

with

$$v_{1,\infty}^*(\lambda) = a^*(0)\widehat{W}(\lambda), \quad \lambda \in \mathbb{R}\,, \tag{135}$$

and, as $j \to \infty$,

$$\int_0^\pi \mathbb{1}(|\lambda| > \varepsilon)\ |v_{1,j}^*(\lambda)|^2\, \mathrm{d}\lambda = O\left(\gamma_j^{1-2\beta}\right)\,. \tag{136}$$

Proof. We have, for all $u \in \mathbb{Z}$,

$$\begin{aligned} W(\gamma_j^{-1}u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{W}(\xi)\, \mathrm{e}^{\mathrm{i}\gamma_j^{-1}u\xi}\, \mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\sum_{p\in\mathbb{Z}} \gamma_j \widehat{W}(\gamma_j(\lambda + 2p\pi))\right]\, \mathrm{e}^{\mathrm{i}u\lambda}\, \mathrm{d}\lambda\,, \end{aligned}$$

hence the term in brackets is the Fourier series of $\{W(\gamma_j^{-1}u),\ u\in\mathbb{Z}\}$ and thus

$$\sum_{u\in\mathbb{Z}} W(\gamma_j^{-1}u)\mathrm{e}^{-\mathrm{i}\lambda u} = \sum_{p\in\mathbb{Z}} \gamma_j \widehat{W}(\gamma_j(\lambda+2p\pi))\ ,$$

which is some times called the Poisson formula. Inserting this in (139), we obtain

$$v^*_{1,j}(\lambda) = a^*(\lambda)\ B_j(\lambda), \quad \lambda\in(-\pi,\pi)\ , \tag{137}$$

where B_j is a (2π)–periodic function defined by

$$B_j(\lambda) = \sum_{p\in\mathbb{Z}} \gamma_j^{1/2}\widehat{W}(\gamma_j(\lambda+2p\pi))\ . \tag{138}$$

Applying (137) and (129) in Lemma 8, (H-2) and that $|a^*(\lambda)|$ is bounded in a neighborhood of the origin by (H-1), we get, for some arbitrary small $\varepsilon>0$, if $|\lambda|\le\varepsilon$,

$$\gamma_j^{-1/2}|v^*_{1,j}(\lambda)| \le C|\widehat{W}(\gamma_j\lambda)| + O\left(\gamma_j^{-\beta}\right) \le C\left(1+\gamma_j|\lambda|\right)^{-\beta} + O\left(\gamma_j^{-\beta}\right)\ ,$$

where C and the O-term do not depend on λ, which implies (133).

Applying (137) and (129) in Lemma 8, we have, as $j\to\infty$,

$$\gamma_j^{-1/2}v^*_{1,j}(\gamma_j^{-1}\lambda) = a^*(\gamma_j^{-1}\lambda)\widehat{W}(\lambda) + O(\gamma_j^{-\beta}) \to a^*(0)\widehat{W}(\lambda), \quad \lambda\in\mathbb{R}\ ,$$

where the limit holds by (H-1). This gives (134).

Applying (137), (131) in Lemma 8 and $\int_{-\pi}^{\pi}|a^*(\lambda)|^2\mathrm{d}\lambda<\infty$, we obtain (136). □

Proof of Theorem 3. By (28) and (30), we have

$$Z_{1,j,k} = \sum_{t\in\mathbb{Z}} \gamma_j^{-1/2}\sum_{v\in\mathbb{Z}} W(k-\gamma_j^{-1}t-\gamma_j^{-1}v)a(u-t)\,\xi_t = \sum_{t\in\mathbb{Z}} v_{1,j}(\gamma_j k-t)\,\xi_t\ ,$$

where

$$v_{1,j}(s) = \gamma_j^{-1/2}\sum_{v\in\mathbb{Z}} W(\gamma_j^{-1}(s-v))\,a(v), \quad s\in\mathbb{Z}\ .$$

Thus $\{Z_{1,j,t},\ j\ge 0, t\in\mathbb{Z}\}$ is an array of one-dimensional decimated linear processes as in Definition 1. Moreover, the Fourier series (2) of $v_{1,j}(s)$ is

$$v^*_{1,j}(\lambda) = \gamma_j^{-1/2}\,a^*(\lambda)\ \sum_{u\in\mathbb{Z}} W(\gamma_j^{-1}(u))\mathrm{e}^{-\mathrm{i}\lambda u}, \quad \lambda\in(-\pi,\pi)\ . \tag{139}$$

We let $N=1$ and $\lambda_{1,j}=\lambda_{1,\infty}=0$ for all $j\ge 0$, which yields (3), (4), (5) and (6) in Condition (C). In view of Lemma 9, Relation (8) in Condition (C) holds, as well as Relations (26) and (27) in Theorem 2 (recall that in that theorem, Relation (26) replaces Relation (7) in Condition (C)). Hence we may apply Theorem 2.

We are now in a position to show first (34) then (36). Applying (19), we have

$$\mathbb{E}\left[Z^2_{1,j,0}\right] = \mathrm{Var}\left(Z_{1,j,0}\right) = \int_{-\pi}^{\pi}\left|v^*_{1,j}(\lambda)\right|^2\,\mathrm{d}\lambda\ . \tag{140}$$

Using (137), (136) and then (130), this gives for any $\epsilon > 0$ small enough

$$\begin{aligned}\mathbb{E}\left[Z_{1,j,0}^2\right] &= \int_{-\epsilon}^{\epsilon} \left|a^*(\lambda)\, B_j(\lambda)\right|^2 \,\mathrm{d}\lambda + O\left(\gamma_j^{1-2\beta}\right) \\ &= \int_{-\epsilon}^{\epsilon} \gamma_j \left|a^*(\lambda)\widehat{W}(\gamma_j\lambda)\right|^2 \,\mathrm{d}\lambda + C\int_{-\epsilon}^{\epsilon} \gamma_j^{1-\beta} \left|\widehat{W}(\gamma_j\lambda)\right| \mathrm{d}\lambda + O\left(\gamma_j^{1-2\beta}\right)\ .\end{aligned}$$

In the last line, since $|a^*(\lambda)|^2 = f(\lambda)$, by Lemma 7, the first term is $f(0) + O\left(\gamma_j^{-2}\right)$ and, by a change of variable, the second term is less than $C\gamma_j^{-\beta}\|\widehat{W}\|_1$. Hence (34) follows since $\beta > 2$.

The bound (136) yields (27) under the condition $\gamma_j^{1-2\beta} = o(n_j^{-1/2})$. Since n_j is given by (33), the assumption (35) implies that condition. Since (A-2) holds as well, we may apply Theorem 2 with $j = n$ and obtain (36). □

References

[1] T. W. Anderson. *The statistical analysis of time series*. John Wiley & Sons Inc., New York, 1994.

[2] R. J. Bhansali, L. Giraitis, and P. S. Kokoszka. Approximations and limit theory for quadratic forms of linear processes. *Stochastic Process. Appl.*, 117(1):71–95, 2007.

[3] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[4] G. Lang and P. Soulier. Convergence de mesures spectrales aléatoires et applications à des principes d'invariance. *Stat. Inference Stoch. Process.*, 3(1-2):41–51, 2000. 19th "Rencontres Franco-Belges de Statisticiens" (Marseille, 1998).

[5] S. Mallat. *A wavelet tour of signal processing*. Academic Press Inc., San Diego, CA, 1998.

[6] E. Moulines, F. Roueff, and M.S. Taqqu. On the spectral density of the wavelet coefficients of long memory time series with application to the log-regression estimation of the memory parameter. *J. Time Ser. Anal.*, 28(2), 2007. [DOI].

[7] F. Roueff and M. S. Taqqu. Asymptotic normality of wavelet estimators of the memory parameter for linear processes. Technical report, HAL/arxiv, 2008.

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