# Weak Finsler Structures and the Funk Metric

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# WEAK FINSLER STRUTURES AND THE FUNK METRIC

ATHANASE PAPADOPOULOS AND MARC TROYANOV

Abstract. We discuss general notions of metrics and of Finsler structures which we call *weak metrics* and *weak Finsler structures*. Any convex domain carries a canonical weak Finsler structure, which we call its *tautological weak Finsler structure*. We compute distances in the tautological weak Finsler structure of a domain and we show that these are given by the so-called *Funk weak metric*. We conclude the paper with a discussion of geodesics, of metric balls and of convexity properties of the Funk weak metric.

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Contents



## 1. Introduction

A *weak metric* on a set is a function defined on pairs of points in that set which is nonnegative, which can take the value $\infty$, which vanishes when the two points coincide and which satisfies the triangle inequality. Compared to an ordinary metric, a weak metric can thus degenerate and take infinite values. Besides, it need not be symmetric. This is a very general notion which turns out to be useful in various situations. The terminology "weak metric" is due to Ribeiro [20], but the notion can be at least traced back to the work of Hausdorff (see [14]). In the paper [18], a number of natural weak metrics are discussed. In the present paper, we are mostly interested in a class of weak metrics that is related to convex geometry and to a general notion of Finsler structures on manifolds.

A basic construction in convex geometry is the notion of *Minkowski norm*, which associates to any convex set containing the origin in a vector space $V$ a translation-invariant homogenous weak metric on $V$. Finsler geometry is an extension of this

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construction to an arbitrary manifold. We define a *weak Finsler strucure* on a differentiable manifold to be a field of convex sets on that manifold. More precisely, a weak Finsler strucure is a subset of the tangent space of the manifold whose intersection with each fiber is an convex set containing the origin. The Minkowski norm in each tangent space of a manifold endowed with a weak Finsler structure gives rise to a function defined on the total space of the tangent bundle. We call this function the *Lagrangian* of the weak Finsler structure. Integrating this Lagrangian on piecewise smooth curves in the manifold defines a length structure and thus a notion of distance on the manifold. This distance is generally a weak metric.

A case of special interest is when the manifold is a convex domain $\Omega$ in $\mathbb{R}^n$ and when the weak Finsler structure is obtained by replicating at each point of $\Omega$ the domain $\Omega$ itself. We call this the *tautological* weak Finsler structure, and we study some of its basic properties in the present paper. More precisely, we first give a formula for the distance between two points. It turns out that this distance coincides with the metric introduced by P. Funk in [13]. We then study the geometry of balls and the geodesics in the Funk weak metric.
Modern references on Finsler geometry include [9], [2], [3], and [1]. One of Herbert Busemann's major ideas, expressed in [5], [6], [7] and [8] is that Finsler geometry should be developed without local coordinates and without the use of differential calculus. This paper brings some results in that direction.

## 2. Preliminaries on convex geometry

In this section, we recall a few notions in convex geometry that will be used in the sequel.
Given a convex subset $\Omega$ of $\mathbb{R}^n$, we shall denote $\overline{\Omega}$ its closure, $\overset{o}{\Omega}$ its interior, and $\partial\Omega = \overline{\Omega} \setminus \overset{o}{\Omega}$ its boundary.
Let $\Omega \subset \mathbb{R}^n$ be a (not necessarily open) convex set and let $x$ be a point in $\Omega$.

**Definition 2.1.** The *radial function of* $\Omega$ *with respect to* $x$ is the function $r_{\Omega,x} : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ defined by

$$r_{\Omega,x}(\xi) = \sup\{t \in \mathbb{R} \mid (x + t\xi) \in \Omega\}.$$

**Definition 2.2.** The *Minkowski function of* $\Omega$ *with respect to* $x$ is the function $p_{\Omega,x} : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ defined by

$$p_{\Omega,x}(\xi) = \frac{1}{r_{\Omega,x}(\xi)}.$$

Classically, the Minkowski function is associated to an open convex subset $\Omega$ of $\mathbb{R}^n$ containing the origin 0, and taking $x = 0$. This function is sometimes called the *Minkowski weak norm* of the convex (see e.g. [10], [17], [21] and [22]).
We also recall that for any convex set $\Omega$ in $\mathbb{R}^n$, there exists a well-defined smallest affine subspace $L$ of $\mathbb{R}^n$ containing $\Omega$, and that the intersection of $\Omega$ with $L$ has nonempty interior in $L$. We denote by $\mathrm{RelInt}(\Omega)$ this interior, called the *relative interior of the convex set* $\Omega$.
The following proposition collects a few basic properties of the Minkowski function. In particular, Property (8) tells us that we can reconstruct the relative interior of $\Omega$ from the Minkowski function of $\Omega$ at any point. The proofs are easy.

**Proposition 2.3.** *Let $\Omega$ be a convex subset of $\mathbb{R}^n$. For every $x$ in $\Omega$ and for every $\xi$ and $\eta$ in $\mathbb{R}^n$, we have*

*(1)* $p_{\Omega,x}(\xi) = \inf\{t \geq 0 \mid \xi \in t(\Omega - x)\}$. *(Here, $\Omega - x$ denotes the Minkowski sum of $\Omega$ and $-x$.)*
*(2) If the ray $\{x + t\xi \mid t \geq 0\}$ is contained in $\Omega$, then $p_{\Omega,x}(\xi) = 0$.*
*(3)* $p_{\Omega,x}(\lambda\xi) = \lambda p_{\Omega,x}(\xi)$ *for* $\lambda \geq 0$.
*(4)* $p_{\Omega,x}(\xi + \eta) \leq p_{\Omega,x}(\xi) + p_{\Omega,x}(\eta)$.
*(5) The Minkowski function $p_{\Omega,x}$ is convex.*
*(6) If $x$ is in $\overset{o}{\Omega}$, then $p_{\Omega,x}$ is continuous.*
*(7) If $\Omega$ is closed, then* $\Omega = \{y \in \mathbb{R}^n \mid y = x + \xi, p_{\Omega,x}(\xi) \leq 1\}$.
*(8)* $\mathrm{RelInt}(\Omega) = \{y = x + \xi |\ p_{\Omega,x}(\xi) < 1\}$.
*(9) If* $\Omega_1 = \mathrm{RelInt}(\Omega)$, *then* $p_{\Omega_1,x} = p_{\Omega,x}$.

In some cases, we can give explicit formulas for the Minkowski function $p_{\Omega,x}$. For instance, the Minkowski function of the closed ball $B = B(0, R)$ in $\mathbb{R}^n$ of radius $R$ and center 0 with respect to any point $x$ in $B$ is given by

$$p_{B,x}(\xi) = \frac{\sqrt{\langle \xi, x\rangle^2 + (R^2 - |x|^2)|\xi|^2} + \langle \xi, x\rangle}{(R^2 - |x|^2)}.$$

The Minkowski function of a half-space $H = \{x \in \mathbb{R}^n \,\big|\, \langle \nu, x\rangle \leq s\}$, where $\nu$ is a vector in $\mathbb{R}^n$ (which is orthogonal to the hyperplane bounding $H$) and where $s$ is a real number, with respect to a point $x$ in $H$, is given by

$$p_{H,x}(\xi) = \max\left(\frac{\langle \nu, \xi\rangle}{s - \langle \nu, x\rangle}, 0\right).$$

We shall use this formula later on in this paper. We also recall the following:

**Definition 2.4** (Support hyperplane)**.** Let $\Omega$ be a nonempty subset of $\mathbb{R}^n$. An affine hyperplane $A$ in $\mathbb{R}^n$ is called a *support hyperplane* for $\Omega$ if the relative interior of $\Omega$ is contained in one of the two closed half-spaces bounded by $A$ and if $\overline{\Omega} \cap A \neq \emptyset$. If $A$ is a support hyperplane for $\Omega$ and if $x$ is a point in $\overline{\Omega} \cap A$, then $A$ is called a *support hyperplane for $\Omega$ at $x$*. When $\Omega \subset \mathbb{R}^2$, then $A$ is called a *support line*.

Suppose now that $\Omega$ a convex subset of $\mathbb{R}^n$. It is known that any point on the boundary of $\Omega$ is contained in at least one of its support hyperplanes (see e.g. [10] p. 20). The intersection of $\Omega$ with any of its support hyperplanes is a convex set which is nonempty if $\Omega$ is closed. This intersection is not always reduced to a point. We recall the notion of a strictly convex subset in $\mathbb{R}^n$, and before that we note the following classical proposition:

**Proposition 2.5.** *Let $\Omega$ be an open convex subset of $\mathbb{R}^n$. Then, the following are equivalent:*

*(1) $\partial\Omega$ does not contain any nonempty open affine segment;*
*(2) each support hyperplane of $\Omega$ intersects $\partial\Omega$ in exactly one point;*
*(3) support hyperplanes at distinct points of $\partial\Omega$ are distinct;*
*(4) any linear function on $\mathbb{R}^n$ has exactly one maximum on $\partial\Omega$.*

**Definition 2.6** (Strictly convex subset)**.** Let $\Omega$ be an open convex subset of $\mathbb{R}^n$. Then, $\Omega$ is said to be *strictly convex* if one (or, equivalently, all) the properties of Proposition 2.5 are satisfied.

## 3. The notion of weak metric

**Definition 3.1.** A *weak metric* on a set $X$ is a function $\delta : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ satisfying

(1) $\delta(x, x) = 0$ for all $x$ in $X$;
(2) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$ for all $x$, $y$ and $z$ in $X$.

We say that such a weak metric $\delta$ is *symmetric* if $\delta(x, y) = \delta(y, x)$ for all $x$ and $y$ in $X$; that it is *finite* if $\delta(x, y) < \infty$ for every $x$ and $y$ in $X$; that $\delta$ is *strongly separating* if we have the equivalence

$$\min(\delta(x, y), \delta(y, x)) = 0 \iff x = y;$$

and that $\delta$ is *weakly separating* if we have the equivalence

$$\max(\delta(x, y), \delta(y, x)) = 0 \iff x = y.$$

We recall that the notion of weak metric already appears in the work of Hausdorff (cf. [14], in which Hausdorff defines asymmetric distances on various sets of subsets of a metric space).

**Definition 3.2** (Geodesic)**.** Let $(X, \delta)$ be a weak metric space and let $I \subset \mathbb{R}$ be an interval. We say that a map $\gamma : I \to X$ is *geodesic* if for every $t_1$, $t_2$ and $t_3$ in $I$ satisfying $t_1 \leq t_2 \leq t_3$ we have

$$\delta(\gamma(t_1), \gamma(t_2)) + \delta(\gamma(t_2), \gamma(t_3)) = \delta(\gamma(t_1), \gamma(t_3)).$$

Weak metrics were extensively studied by Busemann, cf. [5], [6], [7] & [8]. A basic example of a weak metric defined on a convex set in $\mathbb{R}^n$ is the following:

**Example 3.3.** Let $\Omega \subset \mathbb{R}^n$ be a convex set such that $0 \in \overline{\Omega}$ and let $p(\xi) = p_{\Omega,0}(\xi) = \inf\{t > 0 \mid \xi \in t\,\Omega\}$ be the Minkowski weak norm centered at 0 of $\Omega$. Then, the function $\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ defined by

$$\delta(x, y) = p(y - x)$$

is a weak metric on $\mathbb{R}^n$. For this weak metric, we have the following equivalences:

(1) $\delta$ is finite $\iff$ $0 \in \overset{o}{\Omega}$;
(2) $\delta$ is symmetric $\iff$ $\Omega = -\Omega$;
(3) $\delta$ is strongly separating $\iff$ $\Omega$ is bounded;
(4) $\delta$ is weakly separating $\iff$ $\Omega$ does not contain any Euclidean line.

The weak metric on $\mathbb{R}^n$ defined in Example 3.3 is called the *Minkowski weak metric associated to* $\Omega$. The associated weak metric space $(\mathbb{R}^n, \delta)$ is called a *weak Minkowski space*.

## 4. Weak length spaces

Let $X$ be a set and let $\Gamma$ be a groupoid of paths in $X$. Concatenation of paths is denoted by the symbol $*$. The inverse $\gamma^{-1}$ of a path $\gamma : [a, b] \to X$ is the path obtained by pre-composing $\gamma$ with the unique affine sense-reversing self-homeomorphism of $[a, b]$.

**Definition 4.1.** A *weak length structure* on $(X, \Gamma)$ is a function $\ell : \Gamma \to \mathbb{R}_+ \cup \{\infty\}$ which satisfies the following properties:

(1) *Invariance under reparametrization*: if $[a,b]$ and $[c,d]$ are intervals of $\mathbb{R}$, if $\gamma : [a,b] \to X$ is a path in $X$ that belongs to $\Gamma$ and if $f : [c,d] \to [a,b]$ is an increasing homeomorphism such that $\gamma \circ f \in \Gamma$, then $\ell(\gamma) = \ell(\gamma \circ f)$.
(2) *Additivity*: for every $\gamma_1$ and $\gamma_2$ in $\Gamma$, we have $\ell(\gamma_1 * \gamma_2) = \ell(\gamma_1) + \ell(\gamma_2)$.

A weak length structure $\Gamma$ is said to be *reversible* if for every $\gamma$ in $\Gamma$, $\gamma^{-1}$ is also in $\Gamma$ and we have $\ell(\gamma^{-1}) = \ell(\gamma)$.
A weak length structure $\Gamma$ is said to be *separating* if we have the equivalence: $\ell(\gamma) = 0 \iff \gamma$ is a unit in $\Gamma$ (i.e. $\gamma$ is a constant path).
Let $(X, \Gamma, \ell)$ be a set equipped with a groupoid of paths and with a weak length structure. We set

$$\delta_\ell(x,y) = \inf_{\gamma \in \Gamma_{x,y}} \ell(\gamma), \tag{4.1}$$

where

$$\Gamma_{x,y} = \{\gamma \in \Gamma \mid \gamma \text{ joins } x \text{ to } y\ \}.$$

It is easy to see that the function $\delta_\ell$ is a weak metric on $X$.

**Definition 4.2.** Let $(X, \Gamma, \ell)$ be a set equipped with a groupoid of paths and with a weak length structure. The weak metric $\delta_\ell$ defined in (4.1) is called the *weak metric associated to the weak length structure* $\ell$. A *weak length metric space* is a weak metric space $X$ obtained from such a triple $(X, \Gamma, \ell)$ by equipping $X$ with the associated weak metric $\delta_\ell$.

## 5. Weak Finsler structures

We introduce a general notion of Finsler structure, which we call *weak Finsler structure*, and which can be considered as an infinitesimal notion of weak length structure.

**Definition 5.1.** Let $M$ be a $C^1$ manifold and let $TM$ be its tangent bundle. A *weak Finsler structure* on $M$ is a subset $\widetilde{\Omega} \subset TM$ such that for each $x$ in $M$, the subset $\Omega_x = \widetilde{\Omega} \cap T_xM$ of the tangent space $T_xM$ of $M$ at $x$ is convex and contains the origin.

We provide the set of all weak Finsler structures on $M$ with the order relation $\preceq$ defined as follows:

$$\widetilde{\Omega_1} \preceq \widetilde{\Omega_2} \ \Leftrightarrow\ \widetilde{\Omega_1} \supset \widetilde{\Omega_2}.$$

**Examples 5.2.** In the following examples, $M$ is a $C^1$ manifold.

(1) $\widetilde{\Omega} = TM$ is a weak Finsler structure, which we call the *minimal* weak Finsler structure.
(2) $\widetilde{\Omega} = M \subset TM$, embedded as the zero section, is a weak Finsler structure which we call the *maximal* weak Finsler structure.
(3) If $\widetilde{\Omega}$ and $\widetilde{\Omega'}$ are two Finlser structures on $M$, then $\widetilde{\Omega} \cap \widetilde{\Omega'} \subset TM$ is also a Finsler structure.
(4) If $\widetilde{\Omega}$ and $\widetilde{\Omega'}$ are two Finlser structures on $M$, then, taking the union of the Minkowski sums $\Omega_x + \Omega'_x$ of the convex sets in each tangent space $T_xM$, we obtain the *Minkowski sum Finsler structure* $\widetilde{\Omega} + \widetilde{\Omega'} \subset TM$.
(5) If $\omega$ is a differential 1-form on $M$, then

$$\widetilde{\Omega}_\omega = \{(x,\xi) \in TM \mid \omega_x(\xi) \leq 1\}$$

and

$$\widetilde{\Omega}_{|\omega|} = \{(x,\xi) \in TM \mid |\omega_x|(\xi) \leq 1\}$$

are weak Finsler structures on $M$.

(6) If $\omega$ and $\omega'$ are two 1-froms on $M$, then $\max(\omega, \omega')$ defines a weak Finlser structure on $M$.

(7) If $\widetilde{\Omega}$ is a weak Finlser structure on $M$ and if $N \subset M$ is a $C^1$ submanifold, then $\widetilde{\Omega}_N = \widetilde{\Omega} \cap TN$ is a weak Finlser structure on $N$, called the weak Finsler structure *induced* by the embedding $N \subset M$.

(8) If $\widetilde{\Omega}$ is a weak Finlser structures on $M$, if $N$ is a $C^1$ manifold and if $f : N \to M$ is a $C^1$ map, then $(Tf)^{-1}(\widetilde{\Omega}) \subset TN$ is a Finsler structure on $N$. We denote it by $f^*(\widetilde{\Omega})$ and call it the *pull back* of $\widetilde{\Omega}$ by the map $f$.

**Definition 5.3** (Lagrangian)**.** The *Lagrangian* of a weak Finlser structure $\widetilde{\Omega}$ on a $C^1$ manifold $M$ is the function on the tangent bundle $TM$ whose restriction to each tangent space $T_x$ is the Minkowski function of $\Omega_x$. It is thus defined by

$$p(x,\xi) = p_{\widetilde{\Omega}}(x,\xi) = \inf\{t \mid t^{-1}\xi \in \Omega_x\}.$$

The quantity $p(x,\xi)$ is also called the *Finsler norm* of the vector $(x,\xi)$ relative to the given weak Finlser structure.

**Example 5.4.** Let $g$ be a Riemannian metric on $M$, let $\omega$ is a differential 1-form and let $\mu$ be a smooth function on $M$ satisfying $|\mu\omega_x| < 1$ at every point $x$ in $M$. Then, $p = \sqrt{g} + \mu\omega$ is the Lagrangian of a Finsler structure on $M$. Such a Finlser structure is usually called a *Randers metric* on $M$, and it has applications on physics (cf. e.g. [3] §11.3, and see also [4] for the relation of this metric with the Zermelo navigation problem.)

**Lemma 5.5.** *Let $\widetilde{\Omega}$ be a weak Finlser structure on $M$. Assume that $M$ (considered as a subset of $TM$ - the zero section) is contained in the interior of $\widetilde{\Omega} \subset TM$. Then the associated Lagrangian $p : TM \to \mathbb{R}$ is upper semi-continuous.*

*Proof.* The hypothesis implies that for every $x$ in $M$, the interior of each convex set $\Omega_x = \widetilde{\Omega} \cap T_xM \subset T_xM$ is nonempty. Therefore, the usual interior and the relative interior of $\Omega_x$ coincide. Property (9) of Proposition 2.3 implies then that the Lagrangian of $\widetilde{\Omega}$ coincides with the Lagrangian of its interior $\mathrm{Int}\left(\widetilde{\Omega}\right)$.

One may therefore assume without loss of generality that $\widetilde{\Omega} \subset TM$ is an open set, and in particular

$$\widetilde{\Omega} = \{(x,\xi) \in TM \,\big|\, p(x,\xi) < 1\}$$

(see Proposition 2.3 (8)). Now for any $t \in \mathbb{R}$, the sublevel set $\{p(x,\xi) < t\}$ is either empty (when $t \leq 0$) or it is homothetic to the open set $\widetilde{\Omega} \subset TM$ (when $t > 0$). In any case, it is an open subset of $TM$, and $p : TM \to \mathbb{R}$ is therefore upper semi-continuous.

□

**Proposition 5.6.** *Let $\widetilde{\Omega}$ be a Finsler structure on a $C^1$ manifold $M$ and let $p_{\widetilde{\Omega}} : TM \to \mathbb{R}$ be the associated Lagrangian. Then,*

*(1) for every $x$ in $M$, the function $\xi \mapsto p(x,\cdot)$ is a weak norm on $T_xM$;*

(2) *if* $\widetilde{\Omega'} \subset TM$ *is another Finsler structure on* $M$*, with associated Lagrangian* $p_{\widetilde{\Omega'}}$*, then we have the equivalence*

$$\widetilde{\Omega} \preceq \widetilde{\Omega'} \iff p_{\widetilde{\Omega}} \leq p_{\widetilde{\Omega'}},$$

(3) $p_{\widetilde{\Omega}} : TM \to \mathbb{R}$ *is Borel-measurable.*

*Proof.* The first two assertions are easy to check and we only prove the last one. If $M$ is contained in the interior of $\widetilde{\Omega} \subset TM$, then, by Lemma 5.5, the Lagrangian $p$ is upper semi-continuous and therefore Borel measurable. In the general case, $M$ is contained in $\widetilde{\Omega}$ but not necessarily in its interior. We consider a decreasing sequence

$$TM \preceq \widetilde{\Omega}_1 \preceq \widetilde{\Omega}_2 \preceq \cdots \preceq \widetilde{\Omega}$$

of weak Finsler structures such that $M$ is contained in the interior of $\widetilde{\Omega}_j \subset TM$ for every $j \in \mathbb{N}$ and

$$\widetilde{\Omega} = \bigcap_{j=1}^{\infty} \widetilde{\Omega}_j$$

We then have $p_{\widetilde{\Omega}_1} \leq p_{\widetilde{\Omega}_2} \leq \cdots \leq p_{\widetilde{\Omega}}$ and

$$p_{\widetilde{\Omega}} = \sup_j p_{\widetilde{\Omega}_j} = \lim_{j\to\infty} p_{\widetilde{\Omega}_j}$$

Therefore $p_{\widetilde{\Omega}}$ is the limit of a sequence of Borel measurable functions and is thus Borel measurable.

□

We shall say that the Finlser structure $\widetilde{\Omega}$ is *smooth* if $p$ is smooth.

**Definition 5.7** (The weak length structure associated to a weak Finsler structure)**.** Let $M$ be a $C^1$ manifold equipped with a weak Finlser structure $\widetilde{\Omega}$ with Lagrangian $p$. There is an associated weak length structure on $M$, defined by taking $\Gamma$ to be the groupoid of piecewise $C^1$ paths, and defining, for each $\gamma : [a, b] \to M$ in $\Gamma$,

$$\ell(\gamma) = \int_a^b p(\gamma(t), \dot{\gamma}(t))dt. \tag{5.1}$$

**Remark 5.8.** In Equation (5.1), $\gamma$ and $\dot{\gamma}$ are continuous, and since $p$ is Borel-measurable, the map $t \mapsto p(\gamma(t), \dot{\gamma}(t))$ is nonnegative and measurable. Therefore, the Lebesgue integral is well defined.

## 6. The tautological weak Finsler structure

In this section, $\Omega$ is an open convex subset of $\mathbb{R}^n$. We shall use the natural identification $T\Omega \simeq \Omega \times \mathbb{R}^n$.

**Definition 6.1** (The tautological weak Finsler structure)**.** The *tautological weak Finsler structure* on $\Omega$ is the weak Finsler structure $\widetilde{\Omega} \subset T\Omega$ defined by

$$\widetilde{\Omega} = \{(x, \xi) \in \Omega \times \mathbb{R}^n \mid x \in \ \Omega \text{ and } x + \xi \in \Omega\}.$$

This structure is called "tautological" because the fibre over each point $x$ of $\Omega$ is the set $\Omega$ itself (with the origin at $x$).
The proof of next proposition follows easily from the definitions.

**Proposition 6.2.** *Let $\Omega$ be an open convex subset of $\mathbb{R}^n$ equipped with its tautological weak Finsler structure $\widetilde{\Omega}$. Then, for every $x$ in $\Omega$, the Finsler norm of any tangent vector $\xi$ at $x$ is given by $p_{\Omega,x}(\xi)$, where $p_{\Omega,x}$ is the Minkowski function of $\Omega$ with respect to $x$.*

Given an open convex subset $\Omega$ of $\mathbb{R}^n$, we denote by $d_\Omega$ the weak length metric associated to the tautological weak Finsler structure on $\Omega$. This weak metric is thus defined by

$$d_\Omega(x,y) = \inf_{\gamma\in\Gamma_{x,y}} \int_\gamma p(\gamma(t),\dot\gamma(t))dt. \tag{6.1}$$

where $\Gamma_{x,y}$ is the set of piecewise $C^1$ paths joining $x$ to $y$.

**Lemma 6.3.** *Let $\Omega$ and $\Omega'$ be two convex open subsets of $\mathbb{R}^n$ satisfying $\Omega\subset\Omega'$, then $d_{\Omega'}\leq d_\Omega$.*

In the rest of this paper, we shall use the following notations: For $x$ and $y$ in $\mathbb{R}^n$, we denote by $|x-y|$ their Euclidean distance. Given two distinct points $x$ and $y$ in $\Omega$, $R(x,y)$ denote the Euclidean ray starting at $x$ and passing through $y$. In the case where $R(x,y)\not\subset\Omega$ we set $a^+=a^+(x,y)=R(x,y)\cap\partial\Omega$.

**Theorem 6.1.** *Let $\Omega$ be an open convex subset of $\mathbb{R}^n$ equipped with its tautological weak Finsler structure. Then, for every $x$ and $y$ in $\Omega$, the Euclidean segment connecting $x$ and $y$ is of minimal length, and the associated weak metric on $\Omega$ is given by*

$$d_\Omega(x,y)=\begin{cases}\log\dfrac{|x-a^+|}{|y-a^+|} & \text{if } x\neq y \text{ and } R(x,y)\not\subset\Omega\\ 0 & \text{otherwise.}\end{cases}$$

*Proof.* As before, we let $d_\Omega$ denote the weak metric defined by the tautological weak Finsler structure on $\Omega$. We also denote by $\ell(\gamma)$ the length of a path $\gamma$ for the tautological weak Finsler weak length structure.
The proof of the theorem is done in four steps.

*Step 1.*— Suppose that $R(x,y)\subset\Omega$. Consider the linear path $\gamma:[0,|x-y|]\to\Omega$ defined by

$$\gamma(t)=x+t\frac{y-x}{|y-x|}. \tag{6.2}$$

The derivative of the path $\gamma$ is the constant vector

$$\dot\gamma(t)=\frac{y-x}{|y-x|}.$$

Therefore, $p_{\widetilde\Omega}(\gamma(t),\dot\gamma(t))=\dfrac{1}{|y-x|}p_{\widetilde\Omega}(\gamma(t),y-x)$, which is equal to 0 since $R(x,y)\subset\Omega$.
Now the path $\gamma$ has length zero and satisfies $\gamma(0)=x$ and $\gamma(|y-x|)=y$. Therefore $d_\Omega(x,y)=0$.

In the rest of this proof, we suppose that $R(x,y)\not\subset\Omega$.

*Step 2.*— We show that for every distinct points $x$ and $y$ in $\Omega$ and for every Euclidean segment $\gamma$ joining $x$ to $y$, we have

$$d_\Omega(x,y) \leq \ell(\gamma) = \log \frac{|x-a^+|}{|y-a^+|}. \tag{6.3}$$

Using the radial function $r_{\Omega,x}$ introduced in §2, we can write

$$a^+ = a(x, y-x) = x + r_{\Omega,x}(y-x)\cdot(y-x).$$

To compute the Finsler length of the Euclidean segment $[x,y]$, we parametrize it as the path $\gamma$ defined in (6.2).
For $0 \leq t \leq |x-y|$, let $r(t) = |x-\gamma(t)|$. Then, $r(t) = r_{\Omega,x}(\gamma(t), \dot\gamma(t))$, and it is easy to see that

$$r(t) = |x-a^+| - t.$$

Then, we have $r'(t) = -1$ and therefore

$$\ell(\gamma) = \int_0^{|y-x|} \frac{dt}{r(t)} = -\int_0^{|y-x|} \frac{r'(t)dt}{r(t)} = -\log\big(r(t)\big)\Big|_{t=0}^{t=|y-x|} = \log\frac{|x-a^+|}{|y-a^+|}.$$

This gives the desired inequality (6.3).

*Step 3.*— We complete the proof of the theorem in the particular case where $\Omega$ is a half-space. By the invariance of the tautological Finsler structure under the group of affine transformations, it suffices to consider the case where $\Omega$ is the half-space $H \subset \mathbb{R}^n$ defined by the equation

$$H = \{x \in \mathbb{R}^n \,\big|\, \langle \nu, x\rangle \leq s\},$$

for some vector $\nu$ in $\mathbb{R}^n$ (which is orthogonal to the hyperplane bounding $H$) and for some $s$ in $\mathbb{R}$. Recall that the Minkowski function associated to $H$ is given by the formula

$$p_H(x,\xi) = \max\left\{\frac{\langle \nu, \xi\rangle}{s - \langle \nu, x\rangle}, 0\right\}.$$

Consider now an arbitrary piecewise $C^1$ path $\alpha : [0,1] \to H$ such that $x = \alpha(0)$ and $y = \alpha(1)$. Then,

$$\ell(\alpha) = \int_0^1 \max\left\{\frac{\langle \nu, \dot\alpha(t)\rangle}{s - \langle \nu, \alpha(t)\rangle}, 0\right\} dt \geq \int_0^1 \frac{\langle \nu, \dot\alpha(t)\rangle}{s - \langle \nu, \alpha(t)\rangle} dt.$$

We have

$$\frac{\langle \nu, \dot\alpha(t)\rangle}{s - \langle \nu, \alpha(t)\rangle} = -\frac{d}{dt}\big(\log\big(s - \langle \nu, \alpha(t)\rangle\big)\big).$$

Therefore,

$$\ell(\alpha) \geq -\log\big(s - \langle \nu, \alpha(1)\rangle\big) + \log\big(s - \langle \nu, \alpha(0)\rangle\big) = \log\frac{s - \langle \nu, x\rangle}{s - \langle \nu, y\rangle}.$$

Now we note that

$$s - \langle \nu, x\rangle = s - \langle \nu, x - a^+\rangle - \langle \nu, a^+\rangle = \langle x - a^+, -\nu\rangle = \langle \nu, a^+ - x\rangle.$$

Likewise,

$$s - \langle \nu, y\rangle = \langle \nu, a - y\rangle.$$

Thus, we obtain

$$\ell(\alpha) \geq \log\frac{\langle \nu, a^+ - x\rangle}{\langle \nu, a^+ - y\rangle}.$$

Now using the fact that the three points $x, y, a^+$ are aligned in that order and that $\nu$ is not parallel to the vector $x - y$, we easily see that

$$\frac{\langle \nu, a - x\rangle}{\langle \nu, a - y\rangle} = \frac{|x - a^+|}{|y - a^+|},$$

which gives

$$\ell(\alpha) \geq \log \frac{|x - a^+|}{|y - a^+|}.$$

Since $\alpha$ was arbitrary, we have

$$d_H(x, y) \geq \log \frac{|x - a^+|}{|y - a^+|}.$$

Combining this inequality and the inequality (6.3), we obtain, in the case where $\Omega = H$ is a half-space,

$$d_H(x, y) = \log \frac{|x - a^+|}{|y - a^+|}.$$

In particular any Euclidean segment is length minimizing.

*Step 4.*— Now we prove the proposition for a general open convex set $\Omega$.
Let $x$ and $y$ be two elements in $\Omega$ and consider the Euclidean ray $R(x, y)$.
By hypothesis, we have $L \not\subset \Omega$, and as before, we set $a^+ = R(x, y) \cap \partial\Omega$. We let $A$ denote a support hyperplane to $\Omega$ through $a^+$, and we let $H$ be the open half-space containing $\Omega$ and whose boundary is equal to $A$. Using Lemma 6.3 and Step 3, we have

$$d_\Omega(x, y) \geq d_H(x, y) = \log \frac{|x - a^+|}{|y - a^+|}.$$

Combining this with the inequality (6.3) we obtain $d_\Omega(x, y) = \log \frac{|x-a^+|}{|y-a^+|}$. The argument also proves that any Euclidean segment $\gamma$ is length minimizing. This completes the proof of Theorem 6.1.

□

## 7. The Funk weak metric

In this and the following section, we give a quick overview of the Funk weak metric, of its geodesics, of its balls and of its topology.
The Funk weak metric is a nice example of a weak metric, and a geometric study of this weak metric is something which seems missing in the literature. We study this weak metric in more detail in [19].
In this section, $\Omega$ is a nonempty open convex subset of $\mathbb{R}^n$. We use the notations $a^+$, $R(x, y)$, etc. established in the preceding section.

**Definition 7.1** (The Funk weak metric)**.** The *Funk weak metric* of $\Omega$, denoted by $F_\Omega$, is defined, for $x$ and $y$ in $\Omega$, by the formula

$$F_\Omega(x, y) = \begin{cases} \log \dfrac{|x - a^+|}{|y - a^+|} & \text{if } x \neq y \text{ and } R(x, y) \not\subset \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Observe that Theorem 6.1 says that the Funk weak metric is the weak metric associated to the tautological Finsler structure in $\Omega$. In particular the triangle inequality is verified. Another proof of the triangle inequality is given in [23] p. 85. This proof is not trivial and uses arguments similar to those of the classical proof of the triangle inequality for the Hilbert metric, as given by D. Hilbert in [16].
If $\Omega = \mathbb{R}^n$, then $F \equiv 0$. We shall henceforth assume that $\Omega \neq \mathbb{R}^n$ whenever we shall deal with the Funk weak metric of an nonempty open convex subset $\Omega$ of $\mathbb{R}^n$. The Funk weak metric is always unbounded. Indeed, if $x$ is any point in $\Omega$ and if $x_n$ is any sequence of points in that space converging to a point on $\partial\Omega$ (convergence here is with respect to the Euclidean metric), then $F_\Omega(x, x_n) \to \infty$. Notice that on the other hand $F_\Omega(x_n, x)$ is bounded.

**Example 7.2** (The upper half-plane)**.** Let $\Omega = H \subset \mathbb{R}^2$ be the upper half-plane, that is,

$$H = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}.$$

Then, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $H$, we have

$$F_H(x, y) = \max\left\{\log\frac{x_2}{y_2}, 0\right\}.$$

The following three propositions are easy consequences of the definitions and they will be used below. We take $\Omega$ to be again a nonempty open subset of $\mathbb{R}^n$.

**Proposition 7.3.** *Let $\Omega' \subset \Omega$ be the intersection of $\Omega$ with an affine subspace of $\mathbb{R}^n$, and suppose that $\Omega' \neq \emptyset$. Then, $F_{\Omega'}$ is the weak metric induced by $F_\Omega$ on $\Omega'$.*

**Proposition 7.4.** *In the case where $\Omega$ is bounded, the Funk weak metric $F_\Omega$ is strongly separating, and we have the following equivalences:*

$$F_\Omega(x, x_n) \to 0 \iff F_\Omega(x_n, x) \to 0 \iff |x - x_n| \to 0. \tag{7.1}$$

**Proposition 7.5.** *Let $\Omega_1$ and $\Omega_2$ be two open convex subsets of $\mathbb{R}^n$. Then,*

$$F_{\Omega_1 \cap \Omega_2} = \max\{F_{\Omega_1}, F_{\Omega_1}\}.$$

## 8. On the geometry of the Funk weak metric

In this section, we study the geodesics, and then, the geometric balls of the Funk weak metric.

**Proposition 8.1.** *Let $x$, $y$ and $z$ be three points in $\Omega$ lying in that order on a Euclidean line. Then, we have $F(x, y) + F(y, z) = F(x, z)$.*

This results follows from Theorem 6.1, but it is also quite simple to prove it directly.

*Proof.* We can assume that the three points are distinct, otherwise the proof is trivial. We have $R(x, y) \subset \Omega \iff R(x, z) \subset \Omega \iff R(y, z) \subset \Omega$, and this holds if and only if the three quantities $F(x, y)$, $F(y, z)$ and $F(x, z)$ are equal to 0. Thus, the conclusion also holds trivially in this case. Therefore, we can assume that $R(x, y) \not\subset \Omega$. In this case, we have $a^+(x, y) = a^+(x, z) = a^+(y, z)$. Denoting this common point by $a^+$, we have

$$\frac{|x - a^+|}{|y - a^+|}\frac{|y - a^+|}{|z - a^+|} = \frac{|x - a^+|}{|z - a^+|},$$

which implies

$$\log \frac{|x-a^+|}{|y-a^+|} + \log \frac{|y-a^+|}{|z-a^+|} = \log \frac{|x-a^+|}{|z-a^+|},$$

which completes the proof. □

**Corollary 8.2.** *The Euclidean segments in $\Omega$ are geodesic segments for the Funk weak metric on $\Omega$.*

Since the open set $\Omega$ is convex, Corollary 8.2 implies that $(\Omega, F_\Omega)$ is a geodesic weak metric space (any two points can be joined by a geodesic segment). It also says that $(\Omega, F_\Omega)$ is a Desarguesian space in the sense of H. Busemann (see [7]).
Notice that in general, the Euclidean segments are not the only geodesic segments for a Funk weak metric. In fact, the following proposition implies that there exist other types of geodesic segments in $\Omega$, provided there exists a Euclidean segment of nonempty interior contained in the boundary of $\Omega$.

**Proposition 8.3.** *Let $\Omega$ be an open convex subset of $\mathbb{R}^n$ such that $\partial\Omega$ contains a Euclidean segment $[p,q]$ and let $x$ and $z$ be two points in $\Omega$ such that $R(x,z) \cap [p,q] \neq \emptyset$. Let $\Omega'$ be the intersection of $\Omega$ with the affine subspace of $\mathbb{R}^n$ spanned by $\{x\} \cup [p,q]$. Then, for any point $y$ in $\Omega'$ satisfying $R(x,y) \cap [p,q] \neq \emptyset$ and $R(y,z) \cap [p,q] \neq \emptyset$, we have $F(x,y) + F(y,z) = F(x,z)$.*

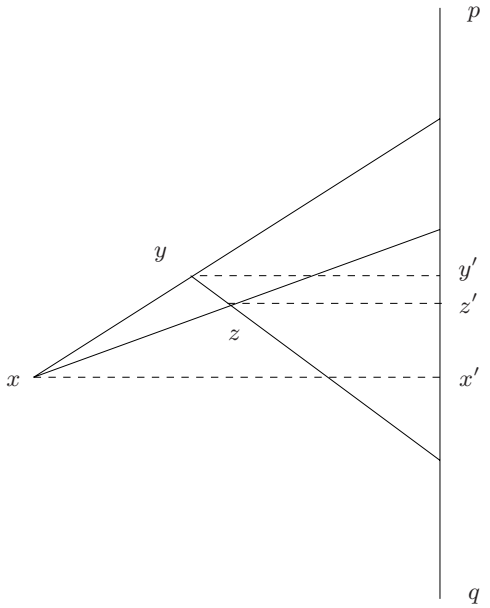


FIGURE 1.

*Proof.* It suffices to work in the space $\Omega'$. Let $x'$, $y'$ and $z'$ denote the feet of the perpendiculars from $x$ and $z$ respectively on the Euclidean line joining the points $p$ and $q$ (see Figure 1). Let $b = R(x,z) \cap [p,q]$. Since the triangles $bxx'$ and $bzz'$ are similar, we have

$$F(x,z) = \log \frac{|x-b|}{|z-b|} = \log \frac{|x-x'|}{|z-z'|}.$$

Similar formulas hold for $F(x,y)$ and $F(y,z)$. Therefore,

$$\begin{aligned} F(x,z) &= \log \frac{|x-x'|}{|z-z'|} \\ &= \log \left( \frac{|x-x'|}{|y-y'|} \frac{|y-y'|}{|z-z'|} \right) \\ &= \log \left( \frac{|x-x'|}{|y-y'|} \right) + \log \left( \frac{|y-y'|}{|z-z'|} \right) \\ &= F(x,y) + F(y,z). \end{aligned}$$

□

**Remark 8.4.** By taking limits of polygonal paths, we can easily construct, from Proposition 8.3, smooth paths which are not Euclidean paths and which are geodesic for the Funk weak metric.

**Proposition 8.5.** *Let $\Omega$ be an open convex subset of $\mathbb{R}^n$. Let $x$ and $z$ be two distinct points in $\Omega$ such that $R(x,z) \cap \partial\Omega \neq \emptyset$ and such that at the point $b = R(x,z) \cap \partial\Omega$, there is a support hyperplane whose intersection with $\partial\Omega$ is reduced to $b$. Let $y$ be a point in $\Omega$ such that the three points $x, y, z$ in $\Omega$ do not lie on the same affine line. Then, $F(x,z) < F(x,y) + F(y,z)$.*

*Proof.* To prove the proposition, we work in the affine plane spanned by $x$, $y$ and $z$ and therefore we can assume without loss of generality that $n = 2$.
We assume that the intersection points of $R(x,y)$ and $R(y,z)$ with $\partial\Omega$ are not empty, and we let $a$ and $c$ be respectively these points. From the hypothesis, there is a support line of $\Omega$ (which we call $D$) at $b$ whose intersection with $\partial\Omega$ is reduced to the point $b$.
For the proof, we distinguish three cases.

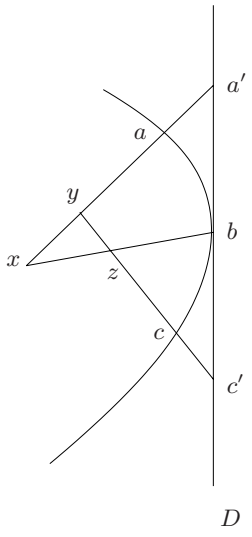


FIGURE 2.

Case 1.— The two rays $R(x,y)$ and $R(y,z)$ intersect the line $D$ (see Figure 2). Let $a'$ and $c'$ be respectively these intersection points. Note that the three points $a'$, $b$ and $c'$ are in that order on $D$. By reasoning with projections on the line $D$ and arguing as we did in the proof of Proposition 8.3, we have

$$\frac{|x-b|}{|z-b|} = \frac{|x-a'|}{|y-a'|}\frac{|y-c'|}{|z-c'|}.$$

Since we have

$$\frac{|x-a'|}{|y-a'|} < \frac{|x-a|}{|y-a|}$$

and

$$\frac{|y-c'|}{|z-c'|} < \frac{|y-c|}{|z-c|},$$

we obtain

$$\frac{|x-b|}{|z-b|} < \frac{|x-a|}{|y-a|}\frac{|y-c|}{|z-c|}$$

which gives, by taking logarithms, $F(x,z) < F(x,y) + F(y,z)$.

Case 2.— The ray $R(x,y)$ intersects $D$ and the ray $R(y,z)$ does not intersect $D$ (Figure 3). We let as before $a'$ denote the point $R(x,y) \cap D$.
Let $D'$ be the Euclidean line passing through $z$ and parallel to $D$. The hypotheses in the case considered imply that the line $D'$ intersects the segment $[x,y]$. Let $y'$ be this intersection point. The point $y'$ is contained in $\Omega$.
We have, as in Case 1,

$$F(x,z) = \log\frac{|x-b|}{|z-b|}$$

and

$$F(x,y) = \log\frac{|x-a|}{|y-a|} > \log\frac{|x-a'|}{|y-a'|}.$$

Now we have

$$\frac{|x-b|}{|z-b|} = \frac{|x-a'|}{|y'-a'|} < \frac{|x-a'|}{|y-a'|},$$

that is, $F(x,z) < F(x,y)$, which implies the desired result.

Case 3.— The ray $R(x,y)$ does not intersect the line $D$. This case can be treated as Case 2, and we have in this case $F(x,y) < F(y,z)$, which implies the desired result.

□

The following is a direct consequence of Proposition 8.5.

**Corollary 8.6.** *Let $\Omega$ be an open bounded strictly convex subset of $\mathbb{R}^n$ and let $x$, $y$ and $z$ be three points in $\Omega$ that are not contained in an affine segment. Then, $F(x,z) < F(x,y) + F(y,z)$.*

**Corollary 8.7.** *Let $\Omega$ be an open bounded strictly convex subset of $\mathbb{R}^n$. Then, the affine segments in $\Omega$ are the only geodesic segments for the Funk weak metric of $\Omega$.*

*Proof.* This follows from the previous Corollary and Corollary 8.2, which says that the affine segments are geodesic segments for the Funk weak metric.

□

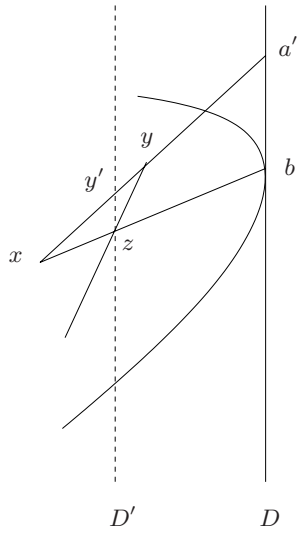


FIGURE 3.

We recall that a subset $Y$ in a (weak) metric space $X$ is said to be *geodesically convex* if for any two points $x$ and $y$ in $Y$, any geodesic segment in $X$ joining $x$ and $y$ is contained in $Y$.

**Corollary 8.8.** *Let $\Omega$ be an open bounded strictly convex subset of $\mathbb{R}^n$ and let $\Omega'$ be a subset of $\Omega$. Then, $\Omega'$ is convex with respect to the affine structure of $\mathbb{R}^n$ if and only if $\Omega'$ is a geodesically convex subset of $\Omega$ with respect to the Funk metric $F_\Omega$.*

**Remark 8.9.** Note the formal analogy between Corollary 8.7 and the following well known result on the geodesic segments of a Minkowski metric on $\mathbb{R}^n$: if the unit ball of a Minkowski metric is strictly convex, then the only geodesic segments of this metric are the affine segments.

We now consider spheres and balls in a Funk weak metric space $(\Omega, F)$. As this weak metric is non-symmetric, we have to distinguish between right and left spheres, and we use the following notations. For any point $x$ in $\Omega$ and any nonnegative real number $\delta$, we set

- $B(x,\delta) = \{y \in \Omega \mid F_\Omega(x,y) < \delta\}$ (the *right open ball of center $x$ and radius $\delta$*);
- $B'(x,\delta) = \{y \in \Omega \mid F_\Omega(y,x) < \delta\}$ (the *left open ball of center $x$ and radius $\delta$*);
- $S(x,\delta) = \{y \in \Omega \mid F_\Omega(x,y) = \delta\}$ (the *right sphere of center $x$ and radius $\delta$*);
- $S'(x,\delta) = \{y \in \Omega \mid F_\Omega(y,x) = \delta\}$ (the *left sphere of center $x$ and radius $\delta$*).

In [6] p. 20, H. Busemann discusses topologies for general weak metric spaces. In the case of a genuine metric space, the open balls are used to define the topology of that space. In general, the collections of left and of right open balls in a weak metric space generate two different topologies. For the Funk weak metric, we have the following

If $\Omega$ is a bounded convex open set of $\mathbb{R}^n$ equipped with its Funk weak metric; then, the collections of left and of right open balls are sub-bases of the same topology on

$\Omega$, and this topology coincides with the topology induced from the inclusion of $\Omega$ in $\mathbb{R}^n$.
In the case where the convex open set $\Omega$ is unbounded, the left and the right open balls of the Funk weak metric are always noncompact. In the next proposition, we study these balls in the case where $\Omega$ is bounded. We recall that a convex subset of $\mathbb{R}^n$ is unbounded if and only if it contains a Euclidean ray.

**Proposition 8.10.** *Let $\Omega$ be a bounded convex open subset of $\mathbb{R}^n$, let $x$ be a point in $\Omega$ and let $\delta$ be a nonnegative real number. Then,*

*(1) The right sphere $S(x,\delta)$ is convex as a subset of $\mathbb{R}^n$, and it is compact. Furthermore, this sphere is the image of $\partial\Omega$ by the Euclidean homothety $\sigma$ of center $x$ and factor $(1-e^{-\delta})$.*
*(2) The left sphere $S'(x,\delta)$ is convex as a subset of $\mathbb{R}^n$, and it is equal to the intersection with $\Omega$ of the image of $\partial\Omega$ by the Euclidean homothety of center $x$ and of factor $(e^{\delta}-1)$, followed by the Euclidean central symmetry of center $x$. The sphere $S'(x,\delta)$ is not necessarily compact.*

*Proof.* Let $y$ be a point in $\Omega$ and let us set, as before, $a^+ = R(x,y)\cap\partial\Omega$. We have the following equivalences:

$$y \in S(x,\delta) \iff \log\frac{|x-a^+|}{|y-a^+|} = \delta \iff \frac{|x-a^+|}{|y-a^+|} = e^{\delta},$$

which is easily seen to be equivalent to $|y-x| = |x-a^+|(1-e^{-\delta})$. From this fact Property (1) follows easily.
To prove Property (2), let $a^- = R(y,x)\cap\partial\Omega$. We have the following equivalences:

$$\log\frac{|y-a^-|}{|x-a^-|} = \delta \iff |y-a^-| = e^{\delta}|x-a^-|,$$

which is also equivalent to

$$|y-x| = (e^{\delta}-1)|x-a^-|.$$

Thus, $y\in S'(x,\delta)$ if and only if $y$ is in the intersection of $\Omega$ with the image $\sigma(\partial\Omega)$ of $\partial\Omega$ by the Euclidean homothety with center $x$ and of factor $(e^{\delta}-1)$, followed by the Euclidean central symmetry of center $x$. This intersection is convex as a subset of $\mathbb{R}^n$ but it is not necessarily a compact subset of $(\Omega,F)$. Thus, $S'(x,\delta)$ is compact if and only if $\sigma(\partial\Omega)$ is contained in $\Omega$. □

We note the following "local-implies-global" property of Funk weak metrics. The meaning of the statement is clear, and it follows directly from Proposition 8.10 (1).

**Corollary 8.11.** *We can reconstruct the boundary $\partial\Omega$ of $\Omega$ from the local geometry at any point of $\Omega$.*

**Corollary 8.12.** *Let $\Omega$ be a bounded open strictly convex subset of $\mathbb{R}^n$. Then, the left and right open balls of $\Omega$ are geodesically convex with respect to the Funk weak metric $F_\Omega$.*

*Proof.* This follows from Proposition 8.10 and from Corollary 8.8. □

We also deduce from Proposition 8.10 that for any $x$ and $x'$ in $\Omega$ and for any two positive real numbers $\delta$ and $\delta'$, the right spheres $S(x,\delta)$ and $S(x',\delta')$ are homothetic.

Thus, for instance, if $\Omega$ is the interior of a Euclidean sphere (respectively, of a Euclidean ellipsoid) in $\mathbb{R}^n$, then any right sphere $S(x,\delta)$ is a Euclidean sphere (respectively, an ellipsoid).
Note that the proof of Proposition 8.10 shows that for a fixed $x$, any two right spheres $S(x,\delta)$ and $S(x,\delta')$ are homothetic by a Euclidean homothety of center $x$, but that in general, a homothety which sends a sphere $S(x,\delta)$ to a sphere $S(x',\delta')$ does not necessarily send the center $x$ of $S(x,\delta)$ to the center $x'$ of $S(x',\delta')$. One can see this fact on the following example: Let $\Omega$ be an open Euclidean disk in $\mathbb{R}^n$, and let us take $x$ to be the Euclidean center of that disk. Then, by symmetry, for any $\delta > 0$, the right sphere $S(x,\delta)$ is a Euclidean sphere whose Euclidean and whose metric centers are both at $x$. Now let $x'$ be a point which is close to the boundary of $\Omega$. Obviously, the Euclidean homothety that sends $\partial\Omega$ to $S(x',\delta)$ does not send the center of $\partial\Omega$ to the (Funk-)geometric center of the sphere $S(x',\delta)$ (recall that the center of this homothety is the point $x$). Now taking a composition of two homotheties, we obtain a Euclidean homothety that sends the geometric sphere $S(x,\delta)$ to the geometric sphere $S(x',\delta)$, and that does not preserve the geometric centers of these spheres.

**Remark 8.13.** The property for a weak metric on a subset $\Omega$ of $\mathbb{R}^n$ that all the right spheres are homothetic is also shared by the metrics induced by Minkowski weak metrics on $\mathbb{R}^n$.

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