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# WEAK FINSLER STRUCTURES AND THE FUNK METRIC

ATHANASE PAPADOPOULOS AND MARC TROYANOV

ABSTRACT. We discuss general notions of metrics and of Finsler structures which we call *weak metrics* and *weak Finsler structures*. Any convex domain carries a canonical weak Finsler structure, which we call its *tautological weak Finsler structure*. We compute distances in the tautological weak Finsler structure of a domain and we show that these are given by the so-called *Funk weak metric*. We conclude the paper with a discussion of geodesics, of metric balls and of convexity properties of the Funk weak metric.

AMS Mathematics Subject Classification: 52A, 53C60, 58B20 Keywords: Finsler structure, weak metric, Funk weak metric.

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## 1. INTRODUCTION

A *weak metric* on a set is a function defined on pairs of points in that set which is nonnegative, which can take the value  $\infty$ , which vanishes when the two points coincide and which satisfies the triangle inequality. Compared to an ordinary metric, a weak metric can thus degenerate and take infinite values. Besides, it need not be symmetric. This is a very general notion which turns out to be useful in various situations. The terminology “weak metric” is due to Ribeiro [20], but the notion can be at least traced back to the work of Hausdorff (see [14]). In the paper [18], a number of natural weak metrics are discussed. In the present paper, we are mostly interested in a class of weak metrics that is related to convex geometry and to a general notion of Finsler structures on manifolds.

A basic construction in convex geometry is the notion of *Minkowski norm*, which associates to any convex set containing the origin in a vector space  $V$  a translation-invariant homogenous weak metric on  $V$ . Finsler geometry is an extension of this

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construction to an arbitrary manifold. We define a *weak Finsler structure* on a differentiable manifold to be a field of convex sets on that manifold. More precisely, a weak Finsler structure is a subset of the tangent space of the manifold whose intersection with each fiber is an convex set containing the origin. The Minkowski norm in each tangent space of a manifold endowed with a weak Finsler structure gives rise to a function defined on the total space of the tangent bundle. We call this function the *Lagrangian* of the weak Finsler structure. Integrating this Lagrangian on piecewise smooth curves in the manifold defines a length structure and thus a notion of distance on the manifold. This distance is generally a weak metric.

A case of special interest is when the manifold is a convex domain  $\Omega$  in  $\mathbb{R}^n$  and when the weak Finsler structure is obtained by replicating at each point of  $\Omega$  the domain  $\Omega$  itself. We call this the *tautological* weak Finsler structure, and we study some of its basic properties in the present paper. More precisely, we first give a formula for the distance between two points. It turns out that this distance coincides with the metric introduced by P. Funk in [13]. We then study the geometry of balls and the geodesics in the Funk weak metric.

Modern references on Finsler geometry include [9], [2], [3], and [1]. One of Herbert Busemann's major ideas, expressed in [5], [6], [7] and [8] is that Finsler geometry should be developed without local coordinates and without the use of differential calculus. This paper brings some results in that direction.

## 2. PRELIMINARIES ON CONVEX GEOMETRY

In this section, we recall a few notions in convex geometry that will be used in the sequel.

Given a convex subset  $\Omega$  of  $\mathbb{R}^n$ , we shall denote  $\overline{\Omega}$  its closure,  $\overset{\circ}{\Omega}$  its interior, and  $\partial\Omega = \overline{\Omega} \setminus \overset{\circ}{\Omega}$  its boundary.

Let  $\Omega \subset \mathbb{R}^n$  be a (not necessarily open) convex set and let  $x$  be a point in  $\Omega$ .

**Definition 2.1.** The *radial function of  $\Omega$  with respect to  $x$*  is the function  $r_{\Omega,x} : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$  defined by

$$r_{\Omega,x}(\xi) = \sup\{t \in \mathbb{R} \mid (x + t\xi) \in \Omega\}.$$

**Definition 2.2.** The *Minkowski function of  $\Omega$  with respect to  $x$*  is the function  $p_{\Omega,x} : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$  defined by

$$p_{\Omega,x}(\xi) = \frac{1}{r_{\Omega,x}(\xi)}.$$

Classically, the Minkowski function is associated to an open convex subset  $\Omega$  of  $\mathbb{R}^n$  containing the origin 0, and taking  $x = 0$ . This function is sometimes called the *Minkowski weak norm* of the convex (see e.g. [10], [17], [21] and [22]).

We also recall that for any convex set  $\Omega$  in  $\mathbb{R}^n$ , there exists a well-defined smallest affine subspace  $L$  of  $\mathbb{R}^n$  containing  $\Omega$ , and that the intersection of  $\Omega$  with  $L$  has nonempty interior in  $L$ . We denote by  $\text{RelInt}(\Omega)$  this interior, called the *relative interior of the convex set  $\Omega$* .

The following proposition collects a few basic properties of the Minkowski function. In particular, Property (8) tells us that we can reconstruct the relative interior of  $\Omega$  from the Minkowski function of  $\Omega$  at any point. The proofs are easy.

**Proposition 2.3.** *Let  $\Omega$  be a convex subset of  $\mathbb{R}^n$ . For every  $x$  in  $\Omega$  and for every  $\xi$  and  $\eta$  in  $\mathbb{R}^n$ , we have*

- (1)  $p_{\Omega,x}(\xi) = \inf\{t \geq 0 \mid \xi \in t(\Omega - x)\}$ . (Here,  $\Omega - x$  denotes the Minkowski sum of  $\Omega$  and  $-x$ .)
- (2) If the ray  $\{x + t\xi \mid t \geq 0\}$  is contained in  $\Omega$ , then  $p_{\Omega,x}(\xi) = 0$ .
- (3)  $p_{\Omega,x}(\lambda\xi) = \lambda p_{\Omega,x}(\xi)$  for  $\lambda \geq 0$ .
- (4)  $p_{\Omega,x}(\xi + \eta) \leq p_{\Omega,x}(\xi) + p_{\Omega,x}(\eta)$ .
- (5) The Minkowski function  $p_{\Omega,x}$  is convex.
- (6) If  $x$  is in  $\overset{\circ}{\Omega}$ , then  $p_{\Omega,x}$  is continuous.
- (7) If  $\Omega$  is closed, then  $\Omega = \{y \in \mathbb{R}^n \mid y = x + \xi, p_{\Omega,x}(\xi) \leq 1\}$ .
- (8)  $\text{RelInt}(\Omega) = \{y = x + \xi \mid p_{\Omega,x}(\xi) < 1\}$ .
- (9) If  $\Omega_1 = \text{RelInt}(\Omega)$ , then  $p_{\Omega_1,x} = p_{\Omega,x}$ .

In some cases, we can give explicit formulas for the Minkowski function  $p_{\Omega,x}$ . For instance, the Minkowski function of the closed ball  $B = B(0, R)$  in  $\mathbb{R}^n$  of radius  $R$  and center 0 with respect to any point  $x$  in  $B$  is given by

$$p_{B,x}(\xi) = \frac{\sqrt{\langle \xi, x \rangle^2 + (R^2 - |x|^2)|\xi|^2} + \langle \xi, x \rangle}{(R^2 - |x|^2)}.$$

The Minkowski function of a half-space  $H = \{x \in \mathbb{R}^n \mid \langle \nu, x \rangle \leq s\}$ , where  $\nu$  is a vector in  $\mathbb{R}^n$  (which is orthogonal to the hyperplane bounding  $H$ ) and where  $s$  is a real number, with respect to a point  $x$  in  $H$ , is given by

$$p_{H,x}(\xi) = \max\left(\frac{\langle \nu, \xi \rangle}{s - \langle \nu, x \rangle}, 0\right).$$

We shall use this formula later on in this paper. We also recall the following:

**Definition 2.4** (Support hyperplane). Let  $\Omega$  be a nonempty subset of  $\mathbb{R}^n$ . An affine hyperplane  $A$  in  $\mathbb{R}^n$  is called a *support hyperplane* for  $\Omega$  if the relative interior of  $\Omega$  is contained in one of the two closed half-spaces bounded by  $A$  and if  $\overline{\Omega} \cap A \neq \emptyset$ . If  $A$  is a support hyperplane for  $\Omega$  and if  $x$  is a point in  $\overline{\Omega} \cap A$ , then  $A$  is called a *support hyperplane for  $\Omega$  at  $x$* . When  $\Omega \subset \mathbb{R}^2$ , then  $A$  is called a *support line*.

Suppose now that  $\Omega$  a convex subset of  $\mathbb{R}^n$ . It is known that any point on the boundary of  $\Omega$  is contained in at least one of its support hyperplanes (see e.g. [10] p. 20). The intersection of  $\Omega$  with any of its support hyperplanes is a convex set which is nonempty if  $\Omega$  is closed. This intersection is not always reduced to a point. We recall the notion of a strictly convex subset in  $\mathbb{R}^n$ , and before that we note the following classical proposition:

**Proposition 2.5.** *Let  $\Omega$  be an open convex subset of  $\mathbb{R}^n$ . Then, the following are equivalent:*

- (1)  $\partial\Omega$  does not contain any nonempty open affine segment;
- (2) each support hyperplane of  $\Omega$  intersects  $\partial\Omega$  in exactly one point;
- (3) support hyperplanes at distinct points of  $\partial\Omega$  are distinct;
- (4) any linear function on  $\mathbb{R}^n$  has exactly one maximum on  $\partial\Omega$ .

**Definition 2.6** (Strictly convex subset). Let  $\Omega$  be an open convex subset of  $\mathbb{R}^n$ . Then,  $\Omega$  is said to be *strictly convex* if one (or, equivalently, all) the properties of Proposition 2.5 are satisfied.

## 3. THE NOTION OF WEAK METRIC

**Definition 3.1.** A *weak metric* on a set  $X$  is a function  $\delta : X \times X \rightarrow \mathbb{R}_+ \cup \{\infty\}$  satisfying

- (1)  $\delta(x, x) = 0$  for all  $x$  in  $X$ ;
- (2)  $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$  for all  $x, y$  and  $z$  in  $X$ .

We say that such a weak metric  $\delta$  is *symmetric* if  $\delta(x, y) = \delta(y, x)$  for all  $x$  and  $y$  in  $X$ ; that it is *finite* if  $\delta(x, y) < \infty$  for every  $x$  and  $y$  in  $X$ ; that  $\delta$  is *strongly separating* if we have the equivalence

$$\min(\delta(x, y), \delta(y, x)) = 0 \iff x = y;$$

and that  $\delta$  is *weakly separating* if we have the equivalence

$$\max(\delta(x, y), \delta(y, x)) = 0 \iff x = y.$$

We recall that the notion of weak metric already appears in the work of Hausdorff (cf. [14], in which Hausdorff defines asymmetric distances on various sets of subsets of a metric space).

**Definition 3.2** (Geodesic). Let  $(X, \delta)$  be a weak metric space and let  $I \subset \mathbb{R}$  be an interval. We say that a map  $\gamma : I \rightarrow X$  is *geodesic* if for every  $t_1, t_2$  and  $t_3$  in  $I$  satisfying  $t_1 \leq t_2 \leq t_3$  we have

$$\delta(\gamma(t_1), \gamma(t_2)) + \delta(\gamma(t_2), \gamma(t_3)) = \delta(\gamma(t_1), \gamma(t_3)).$$

Weak metrics were extensively studied by Busemann, cf. [5], [6], [7] & [8]. A basic example of a weak metric defined on a convex set in  $\mathbb{R}^n$  is the following:

**Example 3.3.** Let  $\Omega \subset \mathbb{R}^n$  be a convex set such that  $0 \in \overline{\Omega}$  and let  $p(\xi) = p_{\Omega, 0}(\xi) = \inf\{t > 0 \mid \xi \in t\Omega\}$  be the Minkowski weak norm centered at 0 of  $\Omega$ . Then, the function  $\delta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$  defined by

$$\delta(x, y) = p(y - x)$$

is a weak metric on  $\mathbb{R}^n$ . For this weak metric, we have the following equivalences:

- (1)  $\delta$  is finite  $\iff 0 \in \overset{\circ}{\Omega}$ ;
- (2)  $\delta$  is symmetric  $\iff \Omega = -\Omega$ ;
- (3)  $\delta$  is strongly separating  $\iff \Omega$  is bounded;
- (4)  $\delta$  is weakly separating  $\iff \Omega$  does not contain any Euclidean line.

The weak metric on  $\mathbb{R}^n$  defined in Example 3.3 is called the *Minkowski weak metric associated to  $\Omega$* . The associated weak metric space  $(\mathbb{R}^n, \delta)$  is called a *weak Minkowski space*.

## 4. WEAK LENGTH SPACES

Let  $X$  be a set and let  $\Gamma$  be a groupoid of paths in  $X$ . Concatenation of paths is denoted by the symbol  $*$ . The inverse  $\gamma^{-1}$  of a path  $\gamma : [a, b] \rightarrow X$  is the path obtained by pre-composing  $\gamma$  with the unique affine sense-reversing self-homeomorphism of  $[a, b]$ .

**Definition 4.1.** A *weak length structure* on  $(X, \Gamma)$  is a function  $\ell : \Gamma \rightarrow \mathbb{R}_+ \cup \{\infty\}$  which satisfies the following properties:

- (1) *Invariance under reparametrization*: if  $[a, b]$  and  $[c, d]$  are intervals of  $\mathbb{R}$ , if  $\gamma : [a, b] \rightarrow X$  is a path in  $X$  that belongs to  $\Gamma$  and if  $f : [c, d] \rightarrow [a, b]$  is an increasing homeomorphism such that  $\gamma \circ f \in \Gamma$ , then  $\ell(\gamma) = \ell(\gamma \circ f)$ .
- (2) *Additivity*: for every  $\gamma_1$  and  $\gamma_2$  in  $\Gamma$ , we have  $\ell(\gamma_1 * \gamma_2) = \ell(\gamma_1) + \ell(\gamma_2)$ .

A weak length structure  $\Gamma$  is said to be *reversible* if for every  $\gamma$  in  $\Gamma$ ,  $\gamma^{-1}$  is also in  $\Gamma$  and we have  $\ell(\gamma^{-1}) = \ell(\gamma)$ .

A weak length structure  $\Gamma$  is said to be *separating* if we have the equivalence:  $\ell(\gamma) = 0 \iff \gamma$  is a unit in  $\Gamma$  (i.e.  $\gamma$  is a constant path).

Let  $(X, \Gamma, \ell)$  be a set equipped with a groupoid of paths and with a weak length structure. We set

$$(4.1) \quad \delta_\ell(x, y) = \inf_{\gamma \in \Gamma_{x,y}} \ell(\gamma),$$

where

$$\Gamma_{x,y} = \{ \gamma \in \Gamma \mid \gamma \text{ joins } x \text{ to } y \}.$$

It is easy to see that the function  $\delta_\ell$  is a weak metric on  $X$ .

**Definition 4.2.** Let  $(X, \Gamma, \ell)$  be a set equipped with a groupoid of paths and with a weak length structure. The weak metric  $\delta_\ell$  defined in (4.1) is called the *weak metric associated to the weak length structure  $\ell$* . A *weak length metric space* is a weak metric space  $X$  obtained from such a triple  $(X, \Gamma, \ell)$  by equipping  $X$  with the associated weak metric  $\delta_\ell$ .

## 5. WEAK FINSLER STRUCTURES

We introduce a general notion of Finsler structure, which we call *weak Finsler structure*, and which can be considered as an infinitesimal notion of weak length structure.

**Definition 5.1.** Let  $M$  be a  $C^1$  manifold and let  $TM$  be its tangent bundle. A *weak Finsler structure* on  $M$  is a subset  $\tilde{\Omega} \subset TM$  such that for each  $x$  in  $M$ , the subset  $\Omega_x = \tilde{\Omega} \cap T_x M$  of the tangent space  $T_x M$  of  $M$  at  $x$  is convex and contains the origin.

We provide the set of all weak Finsler structures on  $M$  with the order relation  $\preceq$  defined as follows:

$$\tilde{\Omega}_1 \preceq \tilde{\Omega}_2 \iff \tilde{\Omega}_1 \supset \tilde{\Omega}_2.$$

**Examples 5.2.** In the following examples,  $M$  is a  $C^1$  manifold.

- (1)  $\tilde{\Omega} = TM$  is a weak Finsler structure, which we call the *minimal* weak Finsler structure.
- (2)  $\tilde{\Omega} = M \subset TM$ , embedded as the zero section, is a weak Finsler structure which we call the *maximal* weak Finsler structure.
- (3) If  $\tilde{\Omega}$  and  $\tilde{\Omega}'$  are two Finsler structures on  $M$ , then  $\tilde{\Omega} \cap \tilde{\Omega}' \subset TM$  is also a Finsler structure.
- (4) If  $\tilde{\Omega}$  and  $\tilde{\Omega}'$  are two Finsler structures on  $M$ , then, taking the union of the Minkowski sums  $\Omega_x + \Omega'_x$  of the convex sets in each tangent space  $T_x M$ , we obtain the *Minkowski sum Finsler structure*  $\tilde{\Omega} + \tilde{\Omega}' \subset TM$ .
- (5) If  $\omega$  is a differential 1-form on  $M$ , then

$$\tilde{\Omega}_\omega = \{ (x, \xi) \in TM \mid \omega_x(\xi) \leq 1 \}$$

and

$$\tilde{\Omega}_{|\omega|} = \{(x, \xi) \in TM \mid |\omega_x|(\xi) \leq 1\}$$

are weak Finsler structures on  $M$ .

- (6) If  $\omega$  and  $\omega'$  are two 1-forms on  $M$ , then  $\max(\omega, \omega')$  defines a weak Finsler structure on  $M$ .
- (7) If  $\tilde{\Omega}$  is a weak Finsler structure on  $M$  and if  $N \subset M$  is a  $C^1$  submanifold, then  $\tilde{\Omega}_N = \tilde{\Omega} \cap TN$  is a weak Finsler structure on  $N$ , called the weak Finsler structure *induced* by the embedding  $N \subset M$ .
- (8) If  $\tilde{\Omega}$  is a weak Finsler structures on  $M$ , if  $N$  is a  $C^1$  manifold and if  $f : N \rightarrow M$  is a  $C^1$  map, then  $(Tf)^{-1}(\tilde{\Omega}) \subset TN$  is a Finsler structure on  $N$ . We denote it by  $f^*(\tilde{\Omega})$  and call it the *pull back* of  $\tilde{\Omega}$  by the map  $f$ .

**Definition 5.3** (Lagrangian). The *Lagrangian* of a weak Finsler structure  $\tilde{\Omega}$  on a  $C^1$  manifold  $M$  is the function on the tangent bundle  $TM$  whose restriction to each tangent space  $T_x$  is the Minkowski function of  $\Omega_x$ . It is thus defined by

$$p(x, \xi) = p_{\tilde{\Omega}}(x, \xi) = \inf\{t \mid t^{-1}\xi \in \Omega_x\}.$$

The quantity  $p(x, \xi)$  is also called the *Finsler norm* of the vector  $(x, \xi)$  relative to the given weak Finsler structure.

**Example 5.4.** Let  $g$  be a Riemannian metric on  $M$ , let  $\omega$  is a differential 1-form and let  $\mu$  be a smooth function on  $M$  satisfying  $|\mu\omega_x| < 1$  at every point  $x$  in  $M$ . Then,  $p = \sqrt{g} + \mu\omega$  is the Lagrangian of a Finsler structure on  $M$ . Such a Finsler structure is usually called a *Randers metric* on  $M$ , and it has applications on physics (cf. e.g. [3] §11.3, and see also [4] for the relation of this metric with the Zermelo navigation problem.)

**Lemma 5.5.** *Let  $\tilde{\Omega}$  be a weak Finsler structure on  $M$ . Assume that  $M$  (considered as a subset of  $TM$  - the zero section) is contained in the interior of  $\tilde{\Omega} \subset TM$ . Then the associated Lagrangian  $p : TM \rightarrow \mathbb{R}$  is upper semi-continuous.*

*Proof.* The hypothesis implies that for every  $x$  in  $M$ , the interior of each convex set  $\Omega_x = \tilde{\Omega} \cap T_x M \subset T_x M$  is nonempty. Therefore, the usual interior and the relative interior of  $\Omega_x$  coincide. Property (9) of Proposition 2.3 implies then that the Lagrangian of  $\tilde{\Omega}$  coincides with the Lagrangian of its interior  $\text{Int}(\tilde{\Omega})$ .

One may therefore assume without loss of generality that  $\tilde{\Omega} \subset TM$  is an open set, and in particular

$$\tilde{\Omega} = \{(x, \xi) \in TM \mid p(x, \xi) < 1\}$$

(see Proposition 2.3 (8)). Now for any  $t \in \mathbb{R}$ , the sublevel set  $\{p(x, \xi) < t\}$  is either empty (when  $t \leq 0$ ) or it is homothetic to the open set  $\tilde{\Omega} \subset TM$  (when  $t > 0$ ). In any case, it is an open subset of  $TM$ , and  $p : TM \rightarrow \mathbb{R}$  is therefore upper semi-continuous. □

**Proposition 5.6.** *Let  $\tilde{\Omega}$  be a Finsler structure on a  $C^1$  manifold  $M$  and let  $p_{\tilde{\Omega}} : TM \rightarrow \mathbb{R}$  be the associated Lagrangian. Then,*

- (1) *for every  $x$  in  $M$ , the function  $\xi \mapsto p(x, \cdot)$  is a weak norm on  $T_x M$ ;*

(2) if  $\tilde{\Omega}' \subset TM$  is another Finsler structure on  $M$ , with associated Lagrangian  $p_{\tilde{\Omega}'}$ , then we have the equivalence

$$\tilde{\Omega} \preceq \tilde{\Omega}' \iff p_{\tilde{\Omega}} \leq p_{\tilde{\Omega}'},$$

(3)  $p_{\tilde{\Omega}} : TM \rightarrow \mathbb{R}$  is Borel-measurable.

*Proof.* The first two assertions are easy to check and we only prove the last one. If  $M$  is contained in the interior of  $\tilde{\Omega} \subset TM$ , then, by Lemma 5.5, the Lagrangian  $p$  is upper semi-continuous and therefore Borel measurable. In the general case,  $M$  is contained in  $\tilde{\Omega}$  but not necessarily in its interior. We consider a decreasing sequence

$$TM \supseteq \tilde{\Omega}_1 \supseteq \tilde{\Omega}_2 \supseteq \cdots \supseteq \tilde{\Omega}$$

of weak Finsler structures such that  $M$  is contained in the interior of  $\tilde{\Omega}_j \subset TM$  for every  $j \in \mathbb{N}$  and

$$\tilde{\Omega} = \bigcap_{j=1}^{\infty} \tilde{\Omega}_j$$

We then have  $p_{\tilde{\Omega}_1} \leq p_{\tilde{\Omega}_2} \leq \cdots \leq p_{\tilde{\Omega}}$  and

$$p_{\tilde{\Omega}} = \sup_j p_{\tilde{\Omega}_j} = \lim_{j \rightarrow \infty} p_{\tilde{\Omega}_j}$$

Therefore  $p_{\tilde{\Omega}}$  is the limit of a sequence of Borel measurable functions and is thus Borel measurable. □

We shall say that the Finsler structure  $\tilde{\Omega}$  is *smooth* if  $p$  is smooth.

**Definition 5.7** (The weak length structure associated to a weak Finsler structure). Let  $M$  be a  $C^1$  manifold equipped with a weak Finsler structure  $\tilde{\Omega}$  with Lagrangian  $p$ . There is an associated weak length structure on  $M$ , defined by taking  $\Gamma$  to be the groupoid of piecewise  $C^1$  paths, and defining, for each  $\gamma : [a, b] \rightarrow M$  in  $\Gamma$ ,

$$(5.1) \quad \ell(\gamma) = \int_a^b p(\gamma(t), \dot{\gamma}(t)) dt.$$

**Remark 5.8.** In Equation (5.1),  $\gamma$  and  $\dot{\gamma}$  are continuous, and since  $p$  is Borel-measurable, the map  $t \mapsto p(\gamma(t), \dot{\gamma}(t))$  is nonnegative and measurable. Therefore, the Lebesgue integral is well defined.

## 6. THE TAUTOLOGICAL WEAK FINSLER STRUCTURE

In this section,  $\Omega$  is an open convex subset of  $\mathbb{R}^n$ . We shall use the natural identification  $T\Omega \simeq \Omega \times \mathbb{R}^n$ .

**Definition 6.1** (The tautological weak Finsler structure). The *tautological weak Finsler structure* on  $\Omega$  is the weak Finsler structure  $\tilde{\Omega} \subset T\Omega$  defined by

$$\tilde{\Omega} = \{(x, \xi) \in \Omega \times \mathbb{R}^n \mid x \in \Omega \text{ and } x + \xi \in \Omega\}.$$

This structure is called “tautological” because the fibre over each point  $x$  of  $\Omega$  is the set  $\Omega$  itself (with the origin at  $x$ ).

The proof of next proposition follows easily from the definitions.

**Proposition 6.2.** *Let  $\Omega$  be an open convex subset of  $\mathbb{R}^n$  equipped with its tautological weak Finsler structure  $\tilde{\Omega}$ . Then, for every  $x$  in  $\Omega$ , the Finsler norm of any tangent vector  $\xi$  at  $x$  is given by  $p_{\Omega,x}(\xi)$ , where  $p_{\Omega,x}$  is the Minkowski function of  $\Omega$  with respect to  $x$ .*

Given an open convex subset  $\Omega$  of  $\mathbb{R}^n$ , we denote by  $d_\Omega$  the weak length metric associated to the tautological weak Finsler structure on  $\Omega$ . This weak metric is thus defined by

$$(6.1) \quad d_\Omega(x, y) = \inf_{\gamma \in \Gamma_{x,y}} \int_\gamma p(\gamma(t), \dot{\gamma}(t)) dt.$$

where  $\Gamma_{x,y}$  is the set of piecewise  $C^1$  paths joining  $x$  to  $y$ .

**Lemma 6.3.** *Let  $\Omega$  and  $\Omega'$  be two convex open subsets of  $\mathbb{R}^n$  satisfying  $\Omega \subset \Omega'$ , then  $d_{\Omega'} \leq d_\Omega$ .*

In the rest of this paper, we shall use the following notations: For  $x$  and  $y$  in  $\mathbb{R}^n$ , we denote by  $|x - y|$  their Euclidean distance. Given two distinct points  $x$  and  $y$  in  $\Omega$ ,  $R(x, y)$  denote the Euclidean ray starting at  $x$  and passing through  $y$ . In the case where  $R(x, y) \not\subset \Omega$  we set  $a^+ = a^+(x, y) = R(x, y) \cap \partial\Omega$ .

**Theorem 6.1.** *Let  $\Omega$  be an open convex subset of  $\mathbb{R}^n$  equipped with its tautological weak Finsler structure. Then, for every  $x$  and  $y$  in  $\Omega$ , the Euclidean segment connecting  $x$  and  $y$  is of minimal length, and the associated weak metric on  $\Omega$  is given by*

$$d_\Omega(x, y) = \begin{cases} \log \frac{|x - a^+|}{|y - a^+|} & \text{if } x \neq y \text{ and } R(x, y) \not\subset \Omega \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* As before, we let  $d_\Omega$  denote the weak metric defined by the tautological weak Finsler structure on  $\Omega$ . We also denote by  $\ell(\gamma)$  the length of a path  $\gamma$  for the tautological weak Finsler weak length structure.

The proof of the theorem is done in four steps.

*Step 1.*— Suppose that  $R(x, y) \subset \Omega$ . Consider the linear path  $\gamma : [0, |x - y|] \rightarrow \Omega$  defined by

$$(6.2) \quad \gamma(t) = x + t \frac{y - x}{|y - x|}.$$

The derivative of the path  $\gamma$  is the constant vector

$$\dot{\gamma}(t) = \frac{y - x}{|y - x|}.$$

Therefore,  $p_{\tilde{\Omega}}(\gamma(t), \dot{\gamma}(t)) = \frac{1}{|y - x|} p_{\tilde{\Omega}}(\gamma(t), y - x)$ , which is equal to 0 since  $R(x, y) \subset \Omega$ .

Now the path  $\gamma$  has length zero and satisfies  $\gamma(0) = x$  and  $\gamma(|y - x|) = y$ . Therefore  $d_\Omega(x, y) = 0$ .

In the rest of this proof, we suppose that  $R(x, y) \not\subset \Omega$ .

*Step 2.*— We show that for every distinct points  $x$  and  $y$  in  $\Omega$  and for every Euclidean segment  $\gamma$  joining  $x$  to  $y$ , we have

$$(6.3) \quad d_{\Omega}(x, y) \leq \ell(\gamma) = \log \frac{|x - a^+|}{|y - a^+|}.$$

Using the radial function  $r_{\Omega, x}$  introduced in §2, we can write

$$a^+ = a(x, y - x) = x + r_{\Omega, x}(y - x) \cdot (y - x).$$

To compute the Finsler length of the Euclidean segment  $[x, y]$ , we parametrize it as the path  $\gamma$  defined in (6.2).

For  $0 \leq t \leq |x - y|$ , let  $r(t) = |x - \gamma(t)|$ . Then,  $r(t) = r_{\Omega, x}(\gamma(t), \dot{\gamma}(t))$ , and it is easy to see that

$$r(t) = |x - a^+| - t.$$

Then, we have  $r'(t) = -1$  and therefore

$$\ell(\gamma) = \int_0^{|y-x|} \frac{dt}{r(t)} = - \int_0^{|y-x|} \frac{r'(t)dt}{r(t)} = - \log(r(t)) \Big|_{t=0}^{t=|y-x|} = \log \frac{|x - a^+|}{|y - a^+|}.$$

This gives the desired inequality (6.3).

*Step 3.*— We complete the proof of the theorem in the particular case where  $\Omega$  is a half-space. By the invariance of the tautological Finsler structure under the group of affine transformations, it suffices to consider the case where  $\Omega$  is the half-space  $H \subset \mathbb{R}^n$  defined by the equation

$$H = \{x \in \mathbb{R}^n \mid \langle \nu, x \rangle \leq s\},$$

for some vector  $\nu$  in  $\mathbb{R}^n$  (which is orthogonal to the hyperplane bounding  $H$ ) and for some  $s$  in  $\mathbb{R}$ . Recall that the Minkowski function associated to  $H$  is given by the formula

$$p_H(x, \xi) = \max \left\{ \frac{\langle \nu, \xi \rangle}{s - \langle \nu, x \rangle}, 0 \right\}.$$

Consider now an arbitrary piecewise  $C^1$  path  $\alpha : [0, 1] \rightarrow H$  such that  $x = \alpha(0)$  and  $y = \alpha(1)$ . Then,

$$\ell(\alpha) = \int_0^1 \max \left\{ \frac{\langle \nu, \dot{\alpha}(t) \rangle}{s - \langle \nu, \alpha(t) \rangle}, 0 \right\} dt \geq \int_0^1 \frac{\langle \nu, \dot{\alpha}(t) \rangle}{s - \langle \nu, \alpha(t) \rangle} dt.$$

We have

$$\frac{\langle \nu, \dot{\alpha}(t) \rangle}{s - \langle \nu, \alpha(t) \rangle} = - \frac{d}{dt} (\log (s - \langle \nu, \alpha(t) \rangle)).$$

Therefore,

$$\ell(\alpha) \geq - \log (s - \langle \nu, \alpha(1) \rangle) + \log (s - \langle \nu, \alpha(0) \rangle) = \log \frac{s - \langle \nu, x \rangle}{s - \langle \nu, y \rangle}.$$

Now we note that

$$s - \langle \nu, x \rangle = s - \langle \nu, x - a^+ \rangle - \langle \nu, a^+ \rangle = \langle x - a^+, -\nu \rangle = \langle \nu, a^+ - x \rangle.$$

Likewise,

$$s - \langle \nu, y \rangle = \langle \nu, a - y \rangle.$$

Thus, we obtain

$$\ell(\alpha) \geq \log \frac{\langle \nu, a^+ - x \rangle}{\langle \nu, a^+ - y \rangle}.$$

Now using the fact that the three points  $x, y, a^+$  are aligned in that order and that  $\nu$  is not parallel to the vector  $x - y$ , we easily see that

$$\frac{\langle \nu, a - x \rangle}{\langle \nu, a - y \rangle} = \frac{|x - a^+|}{|y - a^+|},$$

which gives

$$\ell(\alpha) \geq \log \frac{|x - a^+|}{|y - a^+|}.$$

Since  $\alpha$  was arbitrary, we have

$$d_H(x, y) \geq \log \frac{|x - a^+|}{|y - a^+|}.$$

Combining this inequality and the inequality (6.3), we obtain, in the case where  $\Omega = H$  is a half-space,

$$d_H(x, y) = \log \frac{|x - a^+|}{|y - a^+|}.$$

In particular any Euclidean segment is length minimizing.

*Step 4.*— Now we prove the proposition for a general open convex set  $\Omega$ .

Let  $x$  and  $y$  be two elements in  $\Omega$  and consider the Euclidean ray  $R(x, y)$ .

By hypothesis, we have  $L \not\subset \Omega$ , and as before, we set  $a^+ = R(x, y) \cap \partial\Omega$ . We let  $A$  denote a support hyperplane to  $\Omega$  through  $a^+$ , and we let  $H$  be the open half-space containing  $\Omega$  and whose boundary is equal to  $A$ . Using Lemma 6.3 and Step 3, we have

$$d_\Omega(x, y) \geq d_H(x, y) = \log \frac{|x - a^+|}{|y - a^+|}.$$

Combining this with the inequality (6.3) we obtain  $d_\Omega(x, y) = \log \frac{|x - a^+|}{|y - a^+|}$ . The argument also proves that any Euclidean segment  $\gamma$  is length minimizing. This completes the proof of Theorem 6.1. □

## 7. THE FUNK WEAK METRIC

In this and the following section, we give a quick overview of the Funk weak metric, of its geodesics, of its balls and of its topology.

The Funk weak metric is a nice example of a weak metric, and a geometric study of this weak metric is something which seems missing in the literature. We study this weak metric in more detail in [19].

In this section,  $\Omega$  is a nonempty open convex subset of  $\mathbb{R}^n$ . We use the notations  $a^+$ ,  $R(x, y)$ , etc. established in the preceding section.

**Definition 7.1** (The Funk weak metric). The *Funk weak metric* of  $\Omega$ , denoted by  $F_\Omega$ , is defined, for  $x$  and  $y$  in  $\Omega$ , by the formula

$$F_\Omega(x, y) = \begin{cases} \log \frac{|x - a^+|}{|y - a^+|} & \text{if } x \neq y \text{ and } R(x, y) \not\subset \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Observe that Theorem 6.1 says that the Funk weak metric is the weak metric associated to the tautological Finsler structure in  $\Omega$ . In particular the triangle inequality is verified. Another proof of the triangle inequality is given in [23] p. 85. This proof is not trivial and uses arguments similar to those of the classical proof of the triangle inequality for the Hilbert metric, as given by D. Hilbert in [16].

If  $\Omega = \mathbb{R}^n$ , then  $F \equiv 0$ . We shall henceforth assume that  $\Omega \neq \mathbb{R}^n$  whenever we shall deal with the Funk weak metric of a nonempty open convex subset  $\Omega$  of  $\mathbb{R}^n$ . The Funk weak metric is always unbounded. Indeed, if  $x$  is any point in  $\Omega$  and if  $x_n$  is any sequence of points in that space converging to a point on  $\partial\Omega$  (convergence here is with respect to the Euclidean metric), then  $F_\Omega(x, x_n) \rightarrow \infty$ . Notice that on the other hand  $F_\Omega(x_n, x)$  is bounded.

**Example 7.2** (The upper half-plane). Let  $\Omega = H \subset \mathbb{R}^2$  be the upper half-plane, that is,

$$H = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}.$$

Then, for  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  in  $H$ , we have

$$F_H(x, y) = \max \left\{ \log \frac{x_2}{y_2}, 0 \right\}.$$

The following three propositions are easy consequences of the definitions and they will be used below. We take  $\Omega$  to be again a nonempty open subset of  $\mathbb{R}^n$ .

**Proposition 7.3.** *Let  $\Omega' \subset \Omega$  be the intersection of  $\Omega$  with an affine subspace of  $\mathbb{R}^n$ , and suppose that  $\Omega' \neq \emptyset$ . Then,  $F_{\Omega'}$  is the weak metric induced by  $F_\Omega$  on  $\Omega'$ .*

**Proposition 7.4.** *In the case where  $\Omega$  is bounded, the Funk weak metric  $F_\Omega$  is strongly separating, and we have the following equivalences:*

$$(7.1) \quad F_\Omega(x, x_n) \rightarrow 0 \iff F_\Omega(x_n, x) \rightarrow 0 \iff |x - x_n| \rightarrow 0.$$

**Proposition 7.5.** *Let  $\Omega_1$  and  $\Omega_2$  be two open convex subsets of  $\mathbb{R}^n$ . Then,*

$$F_{\Omega_1 \cap \Omega_2} = \max \{F_{\Omega_1}, F_{\Omega_2}\}.$$

## 8. ON THE GEOMETRY OF THE FUNK WEAK METRIC

In this section, we study the geodesics, and then, the geometric balls of the Funk weak metric.

**Proposition 8.1.** *Let  $x, y$  and  $z$  be three points in  $\Omega$  lying in that order on a Euclidean line. Then, we have  $F(x, y) + F(y, z) = F(x, z)$ .*

This result follows from Theorem 6.1, but it is also quite simple to prove it directly.

*Proof.* We can assume that the three points are distinct, otherwise the proof is trivial. We have  $R(x, y) \subset \Omega \iff R(x, z) \subset \Omega \iff R(y, z) \subset \Omega$ , and this holds if and only if the three quantities  $F(x, y)$ ,  $F(y, z)$  and  $F(x, z)$  are equal to 0. Thus, the conclusion also holds trivially in this case. Therefore, we can assume that  $R(x, y) \not\subset \Omega$ . In this case, we have  $a^+(x, y) = a^+(x, z) = a^+(y, z)$ . Denoting this common point by  $a^+$ , we have

$$\frac{|x - a^+|}{|y - a^+|} \frac{|y - a^+|}{|z - a^+|} = \frac{|x - a^+|}{|z - a^+|},$$

which implies

$$\log \frac{|x - a^+|}{|y - a^+|} + \log \frac{|y - a^+|}{|z - a^+|} = \log \frac{|x - a^+|}{|z - a^+|},$$

which completes the proof.  $\square$

**Corollary 8.2.** *The Euclidean segments in  $\Omega$  are geodesic segments for the Funk weak metric on  $\Omega$ .*

Since the open set  $\Omega$  is convex, Corollary 8.2 implies that  $(\Omega, F_\Omega)$  is a geodesic weak metric space (any two points can be joined by a geodesic segment). It also says that  $(\Omega, F_\Omega)$  is a Desarguesian space in the sense of H. Busemann (see [7]).

Notice that in general, the Euclidean segments are not the only geodesic segments for a Funk weak metric. In fact, the following proposition implies that there exist other types of geodesic segments in  $\Omega$ , provided there exists a Euclidean segment of nonempty interior contained in the boundary of  $\Omega$ .

**Proposition 8.3.** *Let  $\Omega$  be an open convex subset of  $\mathbb{R}^n$  such that  $\partial\Omega$  contains a Euclidean segment  $[p, q]$  and let  $x$  and  $z$  be two points in  $\Omega$  such that  $R(x, z) \cap [p, q] \neq \emptyset$ . Let  $\Omega'$  be the intersection of  $\Omega$  with the affine subspace of  $\mathbb{R}^n$  spanned by  $\{x\} \cup [p, q]$ . Then, for any point  $y$  in  $\Omega'$  satisfying  $R(x, y) \cap [p, q] \neq \emptyset$  and  $R(y, z) \cap [p, q] \neq \emptyset$ , we have  $F(x, y) + F(y, z) = F(x, z)$ .*

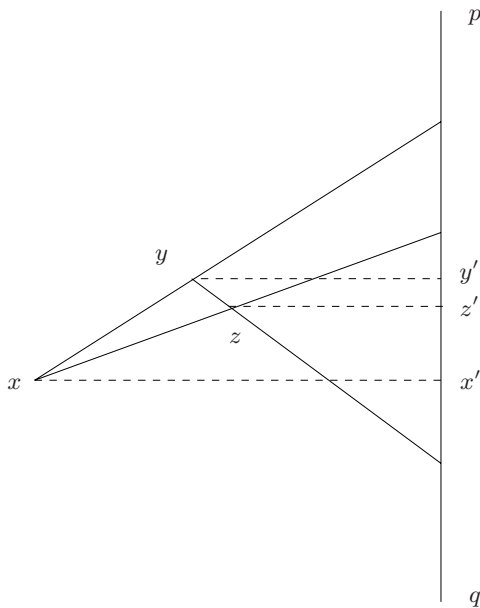


FIGURE 1.

*Proof.* It suffices to work in the space  $\Omega'$ . Let  $x'$ ,  $y'$  and  $z'$  denote the feet of the perpendiculars from  $x$  and  $z$  respectively on the Euclidean line joining the points  $p$  and  $q$  (see Figure 1). Let  $b = R(x, z) \cap [p, q]$ . Since the triangles  $bxx'$  and  $bz z'$  are similar, we have

$$F(x, z) = \log \frac{|x - b|}{|z - b|} = \log \frac{|x - x'|}{|z - z'|}.$$

Similar formulas hold for  $F(x, y)$  and  $F(y, z)$ . Therefore,

$$\begin{aligned} F(x, z) &= \log \frac{|x - x'|}{|z - z'|} \\ &= \log \left( \frac{|x - x'|}{|y - y'|} \frac{|y - y'|}{|z - z'|} \right) \\ &= \log \left( \frac{|x - x'|}{|y - y'|} \right) + \log \left( \frac{|y - y'|}{|z - z'|} \right) \\ &= F(x, y) + F(y, z). \end{aligned}$$

□

**Remark 8.4.** By taking limits of polygonal paths, we can easily construct, from Proposition 8.3, smooth paths which are not Euclidean paths and which are geodesic for the Funk weak metric.

**Proposition 8.5.** *Let  $\Omega$  be an open convex subset of  $\mathbb{R}^n$ . Let  $x$  and  $z$  be two distinct points in  $\Omega$  such that  $R(x, z) \cap \partial\Omega \neq \emptyset$  and such that at the point  $b = R(x, z) \cap \partial\Omega$ , there is a support hyperplane whose intersection with  $\partial\Omega$  is reduced to  $b$ . Let  $y$  be a point in  $\Omega$  such that the three points  $x, y, z$  in  $\Omega$  do not lie on the same affine line. Then,  $F(x, z) < F(x, y) + F(y, z)$ .*

*Proof.* To prove the proposition, we work in the affine plane spanned by  $x, y$  and  $z$  and therefore we can assume without loss of generality that  $n = 2$ .

We assume that the intersection points of  $R(x, y)$  and  $R(y, z)$  with  $\partial\Omega$  are not empty, and we let  $a$  and  $c$  be respectively these points. From the hypothesis, there is a support line of  $\Omega$  (which we call  $D$ ) at  $b$  whose intersection with  $\partial\Omega$  is reduced to the point  $b$ .

For the proof, we distinguish three cases.

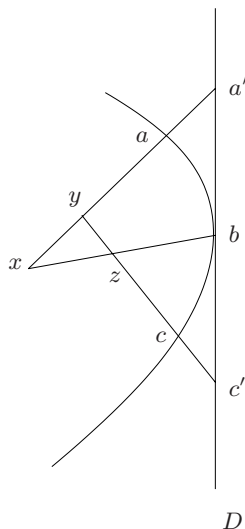


FIGURE 2.

Case 1.— The two rays  $R(x, y)$  and  $R(y, z)$  intersect the line  $D$  (see Figure 2). Let  $a'$  and  $c'$  be respectively these intersection points. Note that the three points  $a'$ ,  $b$  and  $c'$  are in that order on  $D$ . By reasoning with projections on the line  $D$  and arguing as we did in the proof of Proposition 8.3, we have

$$\frac{|x - b|}{|z - b|} = \frac{|x - a'|}{|y - a'|} \frac{|y - c'|}{|z - c'|}.$$

Since we have

$$\frac{|x - a'|}{|y - a'|} < \frac{|x - a|}{|y - a|}$$

and

$$\frac{|y - c'|}{|z - c'|} < \frac{|y - c|}{|z - c|},$$

we obtain

$$\frac{|x - b|}{|z - b|} < \frac{|x - a|}{|y - a|} \frac{|y - c|}{|z - c|}$$

which gives, by taking logarithms,  $F(x, z) < F(x, y) + F(y, z)$ .

Case 2.— The ray  $R(x, y)$  intersects  $D$  and the ray  $R(y, z)$  does not intersect  $D$  (Figure 3). We let as before  $a'$  denote the point  $R(x, y) \cap D$ .

Let  $D'$  be the Euclidean line passing through  $z$  and parallel to  $D$ . The hypotheses in the case considered imply that the line  $D'$  intersects the segment  $[x, y]$ . Let  $y'$  be this intersection point. The point  $y'$  is contained in  $\Omega$ .

We have, as in Case 1,

$$F(x, z) = \log \frac{|x - b|}{|z - b|}$$

and

$$F(x, y) = \log \frac{|x - a|}{|y - a|} > \log \frac{|x - a'|}{|y - a'|}.$$

Now we have

$$\frac{|x - b|}{|z - b|} = \frac{|x - a'|}{|y' - a'|} < \frac{|x - a'|}{|y - a'|},$$

that is,  $F(x, z) < F(x, y)$ , which implies the desired result.

Case 3.— The ray  $R(x, y)$  does not intersect the line  $D$ . This case can be treated as Case 2, and we have in this case  $F(x, y) < F(y, z)$ , which implies the desired result. □

The following is a direct consequence of Proposition 8.5.

**Corollary 8.6.** *Let  $\Omega$  be an open bounded strictly convex subset of  $\mathbb{R}^n$  and let  $x$ ,  $y$  and  $z$  be three points in  $\Omega$  that are not contained in an affine segment. Then,  $F(x, z) < F(x, y) + F(y, z)$ .*

**Corollary 8.7.** *Let  $\Omega$  be an open bounded strictly convex subset of  $\mathbb{R}^n$ . Then, the affine segments in  $\Omega$  are the only geodesic segments for the Funk weak metric of  $\Omega$ .*

*Proof.* This follows from the previous Corollary and Corollary 8.2, which says that the affine segments are geodesic segments for the Funk weak metric. □

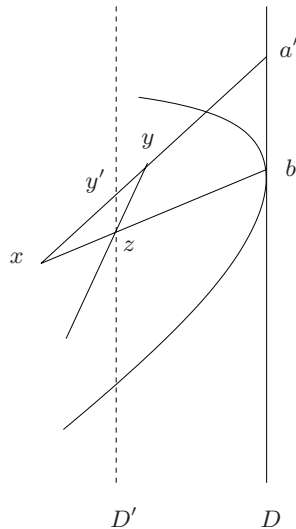


FIGURE 3.

We recall that a subset  $Y$  in a (weak) metric space  $X$  is said to be *geodesically convex* if for any two points  $x$  and  $y$  in  $Y$ , any geodesic segment in  $X$  joining  $x$  and  $y$  is contained in  $Y$ .

**Corollary 8.8.** *Let  $\Omega$  be an open bounded strictly convex subset of  $\mathbb{R}^n$  and let  $\Omega'$  be a subset of  $\Omega$ . Then,  $\Omega'$  is convex with respect to the affine structure of  $\mathbb{R}^n$  if and only if  $\Omega'$  is a geodesically convex subset of  $\Omega$  with respect to the Funk metric  $F_\Omega$ .*

**Remark 8.9.** Note the formal analogy between Corollary 8.7 and the following well known result on the geodesic segments of a Minkowski metric on  $\mathbb{R}^n$ : if the unit ball of a Minkowski metric is strictly convex, then the only geodesic segments of this metric are the affine segments.

We now consider spheres and balls in a Funk weak metric space  $(\Omega, F)$ . As this weak metric is non-symmetric, we have to distinguish between right and left spheres, and we use the following notations. For any point  $x$  in  $\Omega$  and any nonnegative real number  $\delta$ , we set

- $B(x, \delta) = \{y \in \Omega \mid F_\Omega(x, y) < \delta\}$  (the *right open ball of center  $x$  and radius  $\delta$* );
- $B'(x, \delta) = \{y \in \Omega \mid F_\Omega(y, x) < \delta\}$  (the *left open ball of center  $x$  and radius  $\delta$* );
- $S(x, \delta) = \{y \in \Omega \mid F_\Omega(x, y) = \delta\}$  (the *right sphere of center  $x$  and radius  $\delta$* );
- $S'(x, \delta) = \{y \in \Omega \mid F_\Omega(y, x) = \delta\}$  (the *left sphere of center  $x$  and radius  $\delta$* ).

In [6] p. 20, H. Busemann discusses topologies for general weak metric spaces. In the case of a genuine metric space, the open balls are used to define the topology of that space. In general, the collections of left and of right open balls in a weak metric space generate two different topologies. For the Funk weak metric, we have the following

If  $\Omega$  is a bounded convex open set of  $\mathbb{R}^n$  equipped with its Funk weak metric; then, the collections of left and of right open balls are sub-bases of the same topology on

$\Omega$ , and this topology coincides with the topology induced from the inclusion of  $\Omega$  in  $\mathbb{R}^n$ .

In the case where the convex open set  $\Omega$  is unbounded, the left and the right open balls of the Funk weak metric are always noncompact. In the next proposition, we study these balls in the case where  $\Omega$  is bounded. We recall that a convex subset of  $\mathbb{R}^n$  is unbounded if and only if it contains a Euclidean ray.

**Proposition 8.10.** *Let  $\Omega$  be a bounded convex open subset of  $\mathbb{R}^n$ , let  $x$  be a point in  $\Omega$  and let  $\delta$  be a nonnegative real number. Then,*

- (1) *The right sphere  $S(x, \delta)$  is convex as a subset of  $\mathbb{R}^n$ , and it is compact. Furthermore, this sphere is the image of  $\partial\Omega$  by the Euclidean homothety  $\sigma$  of center  $x$  and factor  $(1 - e^{-\delta})$ .*
- (2) *The left sphere  $S'(x, \delta)$  is convex as a subset of  $\mathbb{R}^n$ , and it is equal to the intersection with  $\Omega$  of the image of  $\partial\Omega$  by the Euclidean homothety of center  $x$  and of factor  $(e^\delta - 1)$ , followed by the Euclidean central symmetry of center  $x$ . The sphere  $S'(x, \delta)$  is not necessarily compact.*

*Proof.* Let  $y$  be a point in  $\Omega$  and let us set, as before,  $a^+ = R(x, y) \cap \partial\Omega$ . We have the following equivalences:

$$y \in S(x, \delta) \iff \log \frac{|x - a^+|}{|y - a^+|} = \delta \iff \frac{|x - a^+|}{|y - a^+|} = e^\delta,$$

which is easily seen to be equivalent to  $|y - x| = |x - a^+|(1 - e^{-\delta})$ . From this fact Property (1) follows easily.

To prove Property (2), let  $a^- = R(y, x) \cap \partial\Omega$ . We have the following equivalences:

$$\log \frac{|y - a^-|}{|x - a^-|} = \delta \iff |y - a^-| = e^\delta |x - a^-|,$$

which is also equivalent to

$$|y - x| = (e^\delta - 1)|x - a^-|.$$

Thus,  $y \in S'(x, \delta)$  if and only if  $y$  is in the intersection of  $\Omega$  with the image  $\sigma(\partial\Omega)$  of  $\partial\Omega$  by the Euclidean homothety with center  $x$  and of factor  $(e^\delta - 1)$ , followed by the Euclidean central symmetry of center  $x$ . This intersection is convex as a subset of  $\mathbb{R}^n$  but it is not necessarily a compact subset of  $(\Omega, F)$ . Thus,  $S'(x, \delta)$  is compact if and only if  $\sigma(\partial\Omega)$  is contained in  $\Omega$ .  $\square$

We note the following ‘‘local-implies-global’’ property of Funk weak metrics. The meaning of the statement is clear, and it follows directly from Proposition 8.10 (1).

**Corollary 8.11.** *We can reconstruct the boundary  $\partial\Omega$  of  $\Omega$  from the local geometry at any point of  $\Omega$ .*

**Corollary 8.12.** *Let  $\Omega$  be a bounded open strictly convex subset of  $\mathbb{R}^n$ . Then, the left and right open balls of  $\Omega$  are geodesically convex with respect to the Funk weak metric  $F_\Omega$ .*

*Proof.* This follows from Proposition 8.10 and from Corollary 8.8.  $\square$

We also deduce from Proposition 8.10 that for any  $x$  and  $x'$  in  $\Omega$  and for any two positive real numbers  $\delta$  and  $\delta'$ , the right spheres  $S(x, \delta)$  and  $S(x', \delta')$  are homothetic.

Thus, for instance, if  $\Omega$  is the interior of a Euclidean sphere (respectively, of a Euclidean ellipsoid) in  $\mathbb{R}^n$ , then any right sphere  $S(x, \delta)$  is a Euclidean sphere (respectively, an ellipsoid).

Note that the proof of Proposition 8.10 shows that for a fixed  $x$ , any two right spheres  $S(x, \delta)$  and  $S(x, \delta')$  are homothetic by a Euclidean homothety of center  $x$ , but that in general, a homothety which sends a sphere  $S(x, \delta)$  to a sphere  $S(x', \delta')$  does not necessarily send the center  $x$  of  $S(x, \delta)$  to the center  $x'$  of  $S(x', \delta')$ . One can see this fact on the following example: Let  $\Omega$  be an open Euclidean disk in  $\mathbb{R}^n$ , and let us take  $x$  to be the Euclidean center of that disk. Then, by symmetry, for any  $\delta > 0$ , the right sphere  $S(x, \delta)$  is a Euclidean sphere whose Euclidean and whose metric centers are both at  $x$ . Now let  $x'$  be a point which is close to the boundary of  $\Omega$ . Obviously, the Euclidean homothety that sends  $\partial\Omega$  to  $S(x', \delta)$  does not send the center of  $\partial\Omega$  to the (Funk-)geometric center of the sphere  $S(x', \delta)$  (recall that the center of this homothety is the point  $x$ ). Now taking a composition of two homotheties, we obtain a Euclidean homothety that sends the geometric sphere  $S(x, \delta)$  to the geometric sphere  $S(x', \delta)$ , and that does not preserve the geometric centers of these spheres.

**Remark 8.13.** The property for a weak metric on a subset  $\Omega$  of  $\mathbb{R}^n$  that all the right spheres are homothetic is also shared by the metrics induced by Minkowski weak metrics on  $\mathbb{R}^n$ .

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