



HAL
open science

Note sur le comportement en l'infini d'une fonction intégrable

Emmanuel Lesigne

► **To cite this version:**

Emmanuel Lesigne. Note sur le comportement en l'infini d'une fonction intégrable. 2008. hal-00276738v2

HAL Id: hal-00276738

<https://hal.science/hal-00276738v2>

Preprint submitted on 20 Nov 2008 (v2), last revised 11 Dec 2009 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE BEHAVIOR AT INFINITY OF AN INTEGRABLE FUNCTION

EMMANUEL LESIGNE

ABSTRACT. We prove that, in a weak sense, any integrable function on the real line tends to zero at infinity. Using Khintchine's metric theorem on Diophantine approximation, we establish that this convergence to zero can be arbitrarily slow.

We denote by x a real variable and by n a positive integer variable. The reference measure on the real line \mathbb{R} is the Lebesgue measure. In this note we will use only basic properties of the Lebesgue measure and integral on \mathbb{R} .

It is well known that the fact that a function tends to zero at infinity is a condition neither necessary nor sufficient for this function to be integrable. However, we have the following result.

Theorem 1. *Let f be an integrable function on the real line \mathbb{R} . For almost all $x \in \mathbb{R}$, we have*

$$(1) \quad \lim_{n \rightarrow \infty} f(nx) = 0 .$$

Remarks

(R1) It is too much hope in Theorem 1 for a result *for all* x because we consider an integrable function f , which can take arbitrary values on a set of zero measure. Even if we consider only continuous functions, the result does not hold for all x . Indeed a classical result, using a Baire category argument, tells us that if f is a continuous function on \mathbb{R} such that for all non-zero x , $\lim_{n \rightarrow \infty} f(nx) = 0$, then $\lim_{x \rightarrow \pm\infty} f(x) = 0$. Thus for a continuous integrable function f which does not tend to zero at infinity, Property (1) is true for almost all x and not for all x .

(R2) Let f be an integrable and non-negative function on \mathbb{R} . We have $\int f(nx) dx = \frac{1}{n} \int f(x) dx$. Hence for all non-negative real sequence (ε_n) such that $\sum_n \varepsilon_n/n < +\infty$, we have

$$\sum_n \int \varepsilon_n f(nx) dx < +\infty ,$$

and the Monotone Convergence Theorem (or Fubini's Theorem) ensures that the function $x \mapsto \sum_n \varepsilon_n f(nx)$ is integrable, hence almost everywhere finite. In particular, for almost all x , we have $\lim_{n \rightarrow \infty} \varepsilon_n f(nx) = 0$.

This argument is not sufficient to prove Theorem 1.

Now we will state that, in a sense, Theorem 1 gives an optimal result. The strength of the following theorem lies in the fact that the sequence (a_n) can tend to infinity arbitrarily slowly.

Theorem 2. *Let (a_n) be a real sequence which tends to $+\infty$. There exists a continuous and integrable function f on \mathbb{R} such that, for almost all x ,*

$$\limsup_{n \rightarrow \infty} a_n f(nx) = +\infty .$$

Moreover, there exists an integrable function f on \mathbb{R} such that, for all x ,

$$\limsup_{n \rightarrow \infty} a_n f(nx) = +\infty .$$

Question and Remarks

Under the hypothesis of Theorem 2, does there exist a *continuous* and integrable f such that, for *all* x , $\limsup_{n \rightarrow \infty} a_n f(nx) = +\infty$?

We do not know the answer to this question, and we propose it to the reader. However, next remark shows that the answer is positive under a slightly more demanding hypothesis.

(R3) If the sequence (a_n) is non-decreasing and satisfies $\sum_n \frac{1}{na_n} < +\infty$, then there exists a continuous and integrable function f on \mathbb{R} such that for *all* x , $\limsup_{n \rightarrow \infty} a_n f(nx) = +\infty$.

(R4) In Theorem 2 we cannot replace the hypothesis $\lim_n a_n = +\infty$ by $\limsup_n a_n = +\infty$. Indeed, by a simple change of variable we can deduce from Theorem 1 the following result: for all integrable functions f on \mathbb{R} ,

$$\lim_{n \rightarrow \infty} n f(n^2 x) = 0 \quad \text{for almost all } x.$$

(Apply Theorem 1 to the function $x \mapsto x f(x^2)$.)

Thus the conclusion of Theorem 2 is false for the sequence (a_n) defined by

$$a_n = \begin{cases} \sqrt{n} & \text{if } n \text{ is a square of integer} \\ 0 & \text{if not} \end{cases} .$$

In the remainder of this note, we give proofs of the two theorems, and of Remark (R3).

Proof of Theorem 1. The function f is integrable on \mathbb{R} . Let us fix $\varepsilon > 0$ and denote by E the set of points $x > 0$ such that $|f(x)| \geq \varepsilon$. We know that E has finite measure. We are going to show that, for almost all $x \in [0, 1]$, we have $nx \in E$ for only finitely many n 's.

(If A is a measurable subset of \mathbb{R} , we denote by $|A|$ its Lebesgue measure.)

For each integer $m \geq 1$, we denote $E_m := E \cap (m-1, m]$. Let us fix $a \in]0, 1[$. For each integer $n \geq 1$, we consider the set

$$F_n := \left(\frac{1}{n} E \right) \cap [a, 1) = \left(\frac{1}{n} \bigcup_{m \geq 1} E_m \right) \cap [a, 1) = \frac{1}{n} \bigcup_{m \geq 1} (E_m \cap [na, n)) .$$

We have

$$\sum_{n=1}^{+\infty} |F_n| = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{n} |E_m \cap [na, n)| .$$

In this doubly indexed sum of positive numbers, we can invert the order of summation. Moreover, noticing that $E_m \cap [na, n) = \emptyset$ if $n \geq m/a$ or $n \leq m-1$, we obtain

$$\sum_{n=1}^{+\infty} |F_n| = \sum_{m=1}^{+\infty} \sum_{n=m}^{[m/a]} \frac{1}{n} |E_m \cap [na, n)| \leq \sum_{m=1}^{+\infty} |E_m| \sum_{n=m}^{[m/a]} \frac{1}{n} .$$

By comparison of the discrete sum with an integral, we see that, for all $m \geq 1$, $\sum_{n=m}^{[m/a]} \frac{1}{n} \leq (1 - \ln a)$. Thus we have

$$\sum_{n=1}^{+\infty} |F_n| \leq (1 - \ln a) \sum_{m=1}^{+\infty} |E_m| = (1 - \ln a) |E| < +\infty .$$

This implies that almost every x belongs to only finitely many sets F_n .

(This statement is the Borel-Cantelli Lemma, which has a one line proof : $\sum \mathbb{1}_{F_n} < +\infty$ almost everywhere since

$$\int \sum \mathbb{1}_{F_n}(x) dx = \sum \int \mathbb{1}_{F_n}(x) < +\infty .)$$

Returning to the definition of F_n , we conclude that, for almost all $x \in [a, 1]$, for all large enough n , $x \notin F_n$, i.e. $nx \notin E$.

Since a is arbitrary, we have in fact: for almost all $x \in [0, 1]$, for all large enough n , $nx \notin E$.

We have proved that, for all $\varepsilon > 0$, for almost all $x \in [0, 1]$, for all large enough n , $|f(nx)| \leq \varepsilon$. Since we have only to consider countably many ε 's, we can invert for all $\varepsilon > 0$ and for almost all $x \in [0, 1]$. We conclude that, for almost all $x \in [0, 1]$, $\lim_{n \rightarrow \infty} f(nx) = 0$. It is immediate, by a linear change of variable (for example), that this result extends to almost all $x \in \mathbb{R}$. \square

Proof of Theorem 2. We will utilize the following theorem, a fundamental result in the metric theory of Diophantine approximation¹.

Khinchine's Theorem. *Let (b_n) be a sequence of positive real numbers such that the sequence (nb_n) is non-increasing and the series $\sum_n b_n$ diverges. For almost all real numbers x , there are infinitely many integers n such that $\text{dist}(nx, \mathbb{Z}) < b_n$.*

We will also make use of the following lemma, that will be proved in the sequel.

Lemma 1. *Let (c_n) be a sequence of non-negative real numbers going to zero. There exists a sequence of positive real numbers (b_n) such that the sequence (nb_n) is non-increasing, $\sum_n b_n = +\infty$ and $\sum_n b_n c_n < +\infty$.*

Let us prove Theorem 2.

Replacing if necessary a_n by $\inf_{k \geq n} a_k$, we can suppose that the sequence (a_n) is non-decreasing. Applying the preceding lemma to the sequence $c_n = 1/\sqrt{a_n}$, we obtain a sequence (b_n) such that the sequence (nb_n) is non-increasing, $\sum_n b_n = +\infty$ and $\sum_n b_n/\sqrt{a_n} < +\infty$. The sequence b_n tends to zero, and we can impose that $b_n < 1/2$ for all n .

We consider the function f_1 defined on \mathbb{R} by

$$f_1(x) = \begin{cases} 1/\sqrt{a_n} & \text{if } |x - n| \leq b_n \text{ for an integer } n \geq 1 \\ 0 & \text{if not} \end{cases}.$$

This function is integrable, due to the last condition imposed on (b_n) .

By Khintchine's Theorem, for almost all $x > 0$, there exist pairs of positive integers $(n, k(n))$, with arbitrarily large n , such that

$$|nx - k(n)| \leq b_n.$$

Let us consider one fixed such x in the interval $(0, 1)$. We have $\lim_{n \rightarrow \infty} k(n) = +\infty$ and, since $\lim_{n \rightarrow +\infty} b_n = 0$, we have $k(n) \leq n$ for all large enough n . For such an n , we have

$$|nx - k(n)| \leq b_{k(n)} \quad \text{hence} \quad f_1(nx) = \frac{1}{\sqrt{a_{k(n)}}}.$$

(We used here the fact that the sequence (b_n) is non-increasing.)

Thus, for arbitrarily large n , we have

$$a_n f_1(nx) = \frac{a_n}{\sqrt{a_{k(n)}}} \geq \sqrt{a_{k(n)}}.$$

(We used here the fact that the sequence (a_n) is non-decreasing.)

We have $\limsup_{n \rightarrow \infty} a_n f_1(nx) = +\infty$.

This argument applies to almost all x between 0 and 1.

¹Theorem 32 in the book : A.Ya. Khintchine, *Continued Fractions*, original edition in Russian 1935, English translation published by University of Chicago Press in 1961, reedited by Dover in 1997.

For each integer number $m \geq 1$, let us denote by f_m the function $f_m(x) = f_1(x/m)$. This function f_m is non-negative and integrable on \mathbb{R} . It is locally a step function. For almost all x between 0 and m , we have

$$\limsup_{n \rightarrow \infty} a_n f_m(nx) = +\infty .$$

From this, it is not difficult to construct a continuous and integrable function f on \mathbb{R} such that, for all $m > 0$, there exists $A_m > 0$ with $f \geq f_m$ on $[A_m, +\infty[$. (For example, we can choose an increasing sequence of numbers (A_m) such that

$$\sum_m \int_{A_m}^{+\infty} f_1(x) + f_2(x) + \dots + f_m(x) \, dx < \infty ,$$

then we define $g = f_1 + f_2 + \dots + f_m$ on the interval $[A_m, A_{m+1}[$. This function g is integrable and is locally a step function ; it is dominated by a continuous and integrable function f .)

For almost all $x \geq 0$, we have $\limsup_{n \rightarrow \infty} a_n f(nx) = +\infty$.

A symmetrization procedure extends this property to almost all real numbers.

The first part of Theorem 2 is proved. The second part is a direct consequence. We consider the function f constructed above, and we denote by F the set of x such that the sequence $(a_n f(nx))$ is bounded. The set $\{nx \mid x \in F, n \in \mathbb{N}\}$ has zero measure. We modify the function f on this set, choosing for example the value 1. The new function is integrable and satisfies, for all x , $\limsup_{n \rightarrow \infty} a_n f(nx) = +\infty$. \square

Proof of Lemma 1. The sequence (c_n) is given, and it goes to zero. We will construct by induction an increasing sequence of integers (n_k) and a non-increasing sequence of positive numbers (d_k) and we will define $b_n = \frac{d_k}{n}$ for $n_{k-1} \leq n < n_k$.

Numbers d_k are chosen so that $\sum_{i=n_{k-1}}^{n_k-1} b_i = 1$; thus we impose

$$d_k := \left(\sum_{i=n_{k-1}}^{n_k-1} \frac{1}{i} \right)^{-1} .$$

We start from $n_0 = 1$.

We choose $n_1 > n_0$ such that, for all $n \geq n_1$, $|c_n| \leq \frac{1}{2}$.

We choose $n_2 > n_1$ such that $d_2 \leq d_1$ and, for all $n \geq n_2$, $|c_n| \leq \frac{1}{4}$.

More generally, if n_1, n_2, \dots, n_{k-1} are constructed, we choose $n_k > n_{k-1}$ such that $d_k \leq d_{k-1}$ and, for all $n \geq n_k$, $|c_n| \leq 2^{-k}$.

(Of course, this is possible because $\lim_{n \rightarrow +\infty} \left(\sum_{i=n_{k-1}}^n \frac{1}{i} \right)^{-1} = 0$.)

This defines the sequence (b_n) by blocks. The sequence (nb_n) is non-increasing

and, for all $k \geq 1$, we have

$$\sum_{i=n_{k-1}}^{n_k-1} b_i = 1 \quad \text{et} \quad \sum_{i=n_{k-1}}^{n_k-1} b_i c_i \leq 2^{1-k} .$$

This guarantees that $\sum_n b_n = +\infty$ and $\sum_n b_n c_n < +\infty$. The Lemma is proved. \square

About Remark (R3). Dirichlet's Lemma in Diophantine approximation (based on the pigeon-hole principle) concerns the particular case $b_n = 1/n$ in Khintchine's Theorem and it gives a result for all x .

Lemma 2 (Dirichlet's Lemma). *For all real numbers x , there exist infinitely many integers n such that $\text{dist}(nx, \mathbb{Z}) \leq \frac{1}{n}$.*

Now, we justify Remark (R3).

We consider a non-decreasing sequence of positive real numbers (a_n) such that

$$\sum_n \frac{1}{na_n} < +\infty .$$

We claim that there exists a sequence of positive real numbers (b_n) such that

$$b_n a_n \rightarrow +\infty \quad \text{and} \quad \sum \frac{b_n}{n} < +\infty .$$

Here is a proof of this claim: for each $k \geq 1$, there exists $n(k)$ such that

$$\sum_{n \geq n(k)} \frac{1}{na_n} \leq \frac{1}{k^2} .$$

We have

$$\sum_n \text{card}\{k \mid n(k) \leq n\} \frac{1}{na_n} = \sum_{k \geq 1} \sum_{n \geq n(k)} \frac{1}{na_n} < +\infty ,$$

and we can define $b_n := \frac{1}{a_n} \text{card}\{k \mid n(k) \leq n\}$.

Given this sequence (b_n) , we consider the function f defined on \mathbb{R} by

$$f(x) = \begin{cases} b_k & \text{if } |x - k| \leq 1/k, \quad k \text{ integer } \geq 2 \\ 0 & \text{if not} \end{cases}$$

This function is integrable.

Using Dirichlet's Lemma, we have the following: for each fixed x in $]0, 1[$, there exists pairs of positive integers $(n, k(n))$, with n arbitrarily large, such that $|nx - k(n)| \leq 1/n$. We have $\lim_{n \rightarrow \infty} k(n) = +\infty$ and, for all large enough n , $k(n) \leq n$. Hence there exist infinitely many n 's such that

$$|nx - k(n)| \leq \frac{1}{k(n)} \quad \text{and so} \quad f(nx) = b_{k(n)} .$$

For such an n , we have

$$a_n f(nx) = a_n b_{k(n)} \geq a_{k(n)} b_{k(n)} .$$

(We used here the fact that the sequence (a_n) is non-decreasing.)

This proves that $\limsup_{n \rightarrow \infty} a_n f(nx) = +\infty$. This result obtained for all numbers x between 0 and 1 extends to all real numbers by the same argument as the one used in the proof of Theorem 2. We can also replace the local step function by a continuous one as we did before. \square

Theorem 1 answers a question asked to the author by by Aris Danilidis.

Emmanuel Lesigne

Laboratoire de Mathématiques et Physique Théorique

Université François-Rabelais Tours

Fédération Denis Poisson - CNRS

Parc de Grandmont, 37200 Tours. France

emmanuel.lesigne@lmpt.univ-tours.fr