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***Translation of C. R. Acad. Sci. Paris t. 276, A, p. 1377-1380 (1973)***

Henri Gouin

*Université d'Aix-Marseille & C.N.R.S. U.M.R. 6181,*
*Case 322, Av. Escadrille Normandie-Niemen, 13397 Marseille Cedex 20 France*

**Abstract**

The discontinuity of the vorticity is written as a function of the vector $T\,\mathrm{grad}\,s$, (where $T$ is the temperature and $s$ the specific entropy). The expression is obtained thanks to potential equations and independently of the mass conservation and the equation of momentum balance.

*Key words:* Instationary perfect fluids; shock waves; vorticity vector.
*PACS:* 47.40.Nm; 47.15.ki; 47.32.C-



## 1 Introduction

The aim of this note is to prove that, in the most general instationary case of perfect compressible fluids, across a shock wave we have the relations:

$$\begin{gathered}[u(\mathrm{rot}\ \mathbf{v})_{tg}] = \mathbf{n} \wedge [T\ \mathrm{grad}\ \ s] \\ [(\mathrm{rot}\ \mathbf{v})_n] = 0\end{gathered} \tag{1}$$

where $\mathbf{v}$ is the velocity vector of the fluid, $u$ is the fluid velocity with respect to the shock wave, $T$ is the temperature, $s$ is the specific entropy, the indices $tg$ and $n$ indicate the tangential and normal components to the shock wave of the vector rot $\mathbf{v}$ and the discontinuity of a tensorial quantity $\alpha$ is denoted by $[\alpha]$ (see[1]).

*Email address:* henri.gouin@univ-cezanne.fr (Henri Gouin).

*Preprint submitted to : April 9, 1973, in French version.*

We take into account the following shock conditions:

$$[\mathbf{v}_{tg}] = 0 \tag{2}$$

$$\left[\frac{1}{2}\,u^2 + h\right] = 0 \tag{3}$$

where $h$ denotes the specific enthalpy ($dh = Tds + \dfrac{dp}{\rho}$).

We do not use the following shock conditions:

$$[\rho u] = 0 \tag{4}$$

$$\left[p + \rho u^2\right] = 0 \tag{5}$$

where $\rho$ is the density and $p$ the pression of the fluid.

We will use the *potential equations* obtained by P. Casal [2] or J. Serrin [3] and expressing another form of the equations of the motions of compressible perfect fluids.

The motion of a compressible fluid is represented by a continuous mapping of a reference three-dimensional space $D_o$ in the physical space $D_t$ occupied by the fluid at time $t$:

$$\mathbf{x} = \varphi_t(\mathbf{X}),\ \mathbf{x} \in D_t,\ \mathbf{X} \in D_o,$$

or equivalently by a continuous mapping $\Phi : W_o \longrightarrow W,\ \mathbf{z} = \Phi(\mathbf{Z})$ where $W_o$ is a four-dimensional reference space and $W$ the physical time-space,

$$\mathbf{z} = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \in W \quad \text{and} \quad \mathbf{Z} = \begin{pmatrix} t \\ \mathbf{X} \end{pmatrix} \in W_o\,.$$

We assume the motion has a shock wave localized on a surface $S(t)$ propagating in $D_t$, image by $\varphi_t$ of a surface $S_o(t)$ propagating in $D_o$. We denote by $\mathbf{n_o}$ and $\mathbf{n}$ the unit normal vectors to $S_o(t)$ and $S(t)$ respectively, and $g_o$ and $g$ their respective velocities; then $u = \mathbf{n}^T\mathbf{v} - g$, where $^T$ denotes the transposition.

Equivalently, $\Sigma_o$ and $\Sigma$ are respectively the two corresponding surfaces propagating in $W_o$ and $W$; $\mathbf{N_o}$ and $\mathbf{N}$ are the associated normal vectors,

$$\mathbf{N}^T{=}(-g, \mathbf{n}^T).$$

Consequently, $\Phi$ is a differential mapping on $W_o$, except on $\Sigma_o$; its Jacobian matrix is denoted by $\partial\mathbf{z}/\partial\mathbf{Z}$, F denotes the Jacobian matrix of $\varphi_t$:

$$d\mathbf{x} = \mathbf{v}dt + Fd\mathbf{X}.$$

## 2 Exterior derivative

The covector $\mathbf{C} = \mathbf{v}^T$ has an inverse image $\mathbf{C_o}$ in $D_o$ such that

$$\mathbf{C_o} = \mathbf{C}F$$

The exterior derivative of the form $\mathbf{C}$ is a 2-form which is isomorph to the vector rot $\mathbf{v}$. It the image of the 2-form which is the exterior derivative of $\mathbf{C_o}$ isomorph to the vector $\mathrm{rot_o}\left({\mathbf{C_o}}^T\right)$:

$$\mathrm{rot}\ \mathbf{v} = \frac{F}{\det\ F}\ \mathrm{rot_o}\left({\mathbf{C_o}}^T\right) \tag{6}$$

$\mathrm{rot_o}$ is the rotational on $D_o$ (see reference[4]).

The discontinuity of the vorticity vector comes from to parts: one part comes from the discontinuity of its image $\mathrm{rot_o}\left({\mathbf{C_o}}^T\right)$, and the other part comes from the discontinuity of the Jacobian $F$.

## 3 Discontinuity of the Jacobian $F$

$\partial\mathbf{z}/\partial\mathbf{Z}$ is a linear mapping transforming any tangent vector to $S_o(t)$ in a tangent vector to $S(t)$. If we denote by $\mathbf{n_o}' = -\mathbf{n_o}/g_o$, we obtain:

$$[F] = [\mathbf{v}]\,{\mathbf{n_o}'}^T,\quad {\mathbf{n_o}'}^T = \mathbf{n}^T\frac{F_1}{u_1} = \mathbf{n}^T\frac{F_2}{u_2}$$

where indices $1, 2$ indicate quantities upstream and downstream the shock. Consequently,

$$\mathbf{n}^T\left[\frac{F}{u}\right] = 0$$

Taking into account Eq. (2) we obtain

$$[\mathbf{v}] = [u]\,\mathbf{n}, \tag{7}$$

$$[F] = [u]\,\mathbf{n}\,{\mathbf{n_o}'}^T, \tag{8}$$

$$\left[\frac{F}{\det\ F}\right] = \frac{[u]}{u_2\det\ F_1}\left(\mathbf{n}\,\mathbf{n}^T - \mathbf{I}\right)F_1. \tag{9}$$

where $\mathbf{I}$ is the identity matrix.

## 4 Discontinuity of the 1-form $\mathbf{C_o}$

Due to the fact that $\mathbf{C_o} = \mathbf{v}^T F$ and by using Eq. (7) and Eq. (8), we obtain:

$$[\mathbf{C_o}] = \left[u^2 + gu\right] \mathbf{n_o}'^T. \tag{10}$$

## 5 Discontinuity of potentials

The dot denotes the material derivative, $\Omega$ is the body force potential. We consider the two quantities $\varphi(t, \mathbf{X})$ and $\psi(t, \mathbf{X})$ (denoted potentials) such that

$$\dot{\varphi} = \beta(t, \mathbf{X}), \tag{11}$$

$$\dot{\psi} = \gamma(t, \mathbf{X}). \tag{12}$$

$\beta$ and $\gamma$ are two scalar fields defined in each point of the flow and such that

$$\beta(t, \mathbf{X}) = \frac{1}{2}\mathbf{v}^2 - h - \Omega, \quad \gamma(t, \mathbf{X}) = T.$$

There exists a covector $\mathbf{B}$ function only of $\mathbf{X}$ such that

$$\mathbf{C}_o = \frac{\partial \varphi}{\partial \mathbf{X}} + \psi \frac{\partial s}{\partial \mathbf{X}} + \mathbf{B}. \tag{13}$$

We can verify that Eq. (13) together with Eq. (11) and Eq. (12) are equivalent to *potential equations* proposed by P. Casal in [2] and J. Serrin in [3].
With the condition of adiabaticity

$$\dot{s} = 0,$$

and the equation of balance of mass

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, \mathbf{v}) = 0,$$

we obtain the complete set of motion equations.

We can choose $\varphi$ and $\psi$ null on the shock wave and continuous through the shock surface in the following manner:

$$\varphi = \int_{f(\mathbf{X})}^{t} \beta(\tau, \mathbf{X}) d\tau, \quad \psi = \int_{f(\mathbf{X})}^{t} \gamma(\tau, \mathbf{X}) d\tau, \tag{14}$$

where $t = f(\mathbf{X})$ is the equation of the shock surface $S_o(t)$ which is assumed regular.

Potential $\varphi$ being continuous through the shock surface,

$$\left[\frac{\partial \varphi}{\partial \mathbf{X}}\right] = [\dot{\varphi}] \ {\mathbf{n_o}'}^T = \left[\frac{1}{2}\mathbf{v}^2 - h\right] \ {\mathbf{n_o}'}^T.$$

By using Eq. (2), Eq. (3) and Eq. (10) we get:

$$\left[\frac{\partial \varphi}{\partial \mathbf{X}}\right] = [\mathbf{C_o}]\,. \tag{15}$$

Due to the fact that $\psi$ is null on the shock wave, Eq. (13) expresses $\mathbf{B}$ is continuous through the shock:

$$[\mathbf{B}] = 0 \tag{16}$$

## 6 Discontinuity of the image of the vorticity

Let us consider

$$\mathbf{W_o} = {\mathbf{C_o}}^T - \left(\frac{\partial \varphi}{\partial \mathbf{X}}\right)^T, \quad \text{then} \quad \operatorname{rot_o} \mathbf{W_o} = \operatorname{rot_o} {\mathbf{C_o}}^T.$$

Eq. (13) yields $\dot{\mathbf{W}}_\mathbf{o} = T \operatorname{grad_o} s$, the value of $\operatorname{grad_o} s$ being defined on $W_o$.

Due to Eq. (15), $\mathbf{W_o}$ is continuous through the shock and we get:

$$\left[\frac{\partial \mathbf{W_o}}{\partial \mathbf{X}}\right] = \left[\dot{\mathbf{W}}_\mathbf{o}\right] \ {\mathbf{n_o}'}^T, \quad [\operatorname{rot_o} \mathbf{W_o}] = \mathbf{n_o}' \wedge \left[\dot{\mathbf{W}}_\mathbf{o}\right],$$

$$\left[\operatorname{rot_o} {\mathbf{C_o}}^T\right] = \mathbf{n}'_o \wedge [T \operatorname{grad_o} s]\,. \tag{17}$$

The discontinuity of the image of the vorticity is only tangential. Using the previous results and application $\Phi$, we can verify that this property of the vorticity remains true.

## 7 Discontinuity of the vorticity

The results obtained by Eqs (6), (9) and (17) allow to obtain Formulae (1). This expression general for non stationary perfect compressible fluids is different from the result given by Hayes [5]. This is due to the fact the result is obtained thanks to Eq. (3) of conservation of energy. It neither uses Eq. (5) of the balance of the quantity of motion nor Eq. (4) of the conservation of mass.

To obtain Formulae (1), the knowledge of the enthalpy field is only necessary. In the special case of a stationary, iso-energetic, irrotational motion upstream of the shock, the relation can be expressed with the help of the curvature tensor of the shock surface [6].

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